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A Proof-theoretical Investigation of Global Intuitionistic (Fuzzy) Logic

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Abstract. We perform a proof-theoretical investigation of two modal predicate logics: global intuitionistic logic **GI** and global intuitionistic fuzzy logic **GIF**. These logics were introduced by Takeuti and Titani to formulate an intuitionistic set theory and an intuitionistic fuzzy set theory together with their metatheories. Here we define analytic Gentzen style calculi for **GI** and **GIF**. Among other things, these calculi allows one to prove Herbrand's theorem for suitable fragments of **GI** and **GIF**.

1. Introduction

Intuitionistic fuzzy logic **IF** was introduced in [19] by Takeuti and Titani as the logic corresponding to intuitionistic fuzzy set theory ZF_{IF}. (**IF** also coincides with first-order Gödel logic [12] based on the truth-values set $[0, 1]$ – one of the basic t-norm logics, see [13]). Global intuitionistic logic **GI** and global intuitionistic fuzzy logic **GIF** are predicate modal logics extending intuitionistic logic **IL** and **IF** with an additional logical symbol $\Box$, called *globalization* ($\Box A$ equals 1 if the value of A equals 1, and 0 otherwise). **GI** and **GIF** were introduced in [22] and [20], respectively, to formulate an intuitionistic set theory and an intuitionistic fuzzy set theory together with their metatheories.

Both **GI** and **GIF** are defined by extending Gentzen's **LJ** sequent calculus for **IL** with a number of extra axioms. (An alternative sequent calculus for **GI** that makes no use of these additional axioms is contained in [22]). The resulting calculi are not analytic, i.e., their derivations do not proceed by stepwise decomposition of the formulas to be proved. As is well known, analytic calculi are a basic prerequisite for developing automated reasoning methods and a key to a profound understanding of the relation between

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the logic's syntax and its semantics. In the literature there do exist analytic calculi for *fragments* of **GI** and **GIF**. E.g. cut-free sequent calculi for some modal logics properly contained in the propositional fragment of **GI** can be found in [14] (see [1]). Moreover, a resolution-like calculus for the *prenex* fragment of **GIF** was outlined in [6] while cut-free Gentzen style calculi for **IF** and propositional **GIF** were defined in [8,7]. The latter calculi use hypersequents, a simple and natural generalization of Gentzen sequents to multisets of sequents (see [3] for an overview on hypersequent calculi).

In this paper we provide analytic hypersequent calculi for **GI** and **GIF** and use them to analyze these logics. Our calculi are defined by extending the hypersequent version of **LJ**, with the (hyper)sequent rules for $\Box$ in modal logic **S4** together with suitable rules that allow the "exchange of information" between different sequents. In the resulting calculi the cut-elimination theorem holds. Among other things, this ensures that **GI** and **GIF** are conservative extensions of **IL** and **IF** respectively. Using these calculi: (1) we prove Herbrand's theorem for suitable fragments of **GI** and **GIF** (2) we define alternative axiomatizations for **GIF** that make no use of either Barcan's axiom $\forall x \Box A(x) \Rightarrow \Box \forall x A(x)$ or of the shifting law of universal quantifier w.r.t. $\vee$, i.e. $\forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)$, where x does not occur in B (3) we prove that if P is a quantifier-free formula and $\Box \exists x P(x)$ is valid in **GIF**, then $\exists x \Box P(x)$ is valid in **GIF** too and (4) we show that Takeuti and Titani's density rule — used in [19,20] to define **IF** and **GIF** — is syntactically eliminable from derivations in **GIF**.

2. Global Intuitionistic Logic (GI)

2.1. Preliminaries

Each logic has its corresponding set theory in which each logical operation is translated into a basic operation for set theory; namely, the relations $\subseteq$ and $=$ on sets are translation of the logical operations $\supset$ and $\leftrightarrow$. In order to define an intuitionistic set theory in which global concepts such as "check sets" and "truth values" can be expressed, in [21] Takeuti and Titani extended intuitionistic logic **IL** by a new operator $\Box$. Following [1], we refer to this extended intuitionistic logic as $\mathbf{GI_{TT}}$.

A deductive system for $\mathbf{GI_{TT}}$ was defined in [21] by extending **LJ** with the following axioms[1] and inference rules:

$$(G1)\ \Box A \Rightarrow A,\ (G2)\ \forall x \Box A(x) \Rightarrow \Box \forall x A(x),\ (G3)\ \Box A \supset \Box B \Rightarrow \Box(\Box A \supset B)$$

$$\frac{\Box A_1, \ldots, \Box A_n \Rightarrow B}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}\ (G4)$$

Remark 1. As observed in [1], the propositional fragment of $\mathbf{GI_{TT}}$ is closely related to some of Ono's systems of intuitionistic modal logic considered in [14].

[1] In this paper, axioms will be always considered as schemata. (I.e. all instances are axioms too.)

However, the intuitionistic set theory corresponding to $\mathbf{GI_{TT}}$ does not have a real equality. To overcome this problem, in [22] Titani extended $\mathbf{GI_{TT}}$ with

$$(G5) \qquad \Rightarrow \Box A \vee \neg \Box A$$

She called the resulting logic Global Intuitionistic logic **GI**. In **GI** the semantics of the modality $\Box$ (called globalization) is defined as: For any interpretation v_D, on a domain D, on a complete Heyting algebra (cHa) Ω with a globalization $\Box$ ($\Omega = (H, \wedge, \vee, \supset, \neg, \Box, 0, 1, \bigwedge, \bigvee)$)

$$(*) \qquad v_D(\Box A) = \begin{cases} 1 & \text{if } v_D(A) = 1 \\ 0 & \text{if } v_D(A) \neq 1 \end{cases}$$

We say that a formula P is *valid* in **GI** iff for all such interpretations v_D, $v_D(P) = 1$.

Henceforth we abbreviate "P is derivable in the deductive system **D**", "P is valid in the logic **L**" and "**D** extended with the axiom/rule $(*)$" with $\vdash_{\mathbf{D}} P$, $\models_{\mathbf{L}} P$ and $\mathbf{D} + (*)$, respectively.

2.2. An Analytic Calculus for **GI**

A sequent formulation for **GI** that makes no use of axioms $(G1)-(G3)$ and $(G5)$ was introduced in [22]. This calculus –we refer to it as **scGI**– is obtained by adding natural rules for introducing $\Box$ to (a suitable modification of) Maehara's $\mathbf{LJ'}$. Recall that $\mathbf{LJ'}$ is an equivalent version of Gentzen's **LJ** where the restriction to at most one formula in the succedent of sequents applies not generally but only in the case of the right rules for $\supset, \neg$ and $\forall$ (see [18]). In **scGI**, these three rules can be applied when the succedent of the lower sequent contains at most one formula that is not $\Box$-closed, where a $\Box$-*closed formula* is inductively defined as:

1. If Φ is a formula, then $\Box\Phi$ is a $\Box$-closed formula.
2. If Φ and Ψ are $\Box$-closed formulas, then so are $\neg\Phi$, $\Phi \wedge \Psi$, $\Phi \vee \Psi$ and $\Phi \supset \Psi$.
3. If $\Phi(x)$ is a $\Box$-closed formula with free variable x, then so are $\forall x\Phi(x)$ and $\exists x\Phi(x)$.

Let us denote with $\overline{\Sigma}$ and $\overline{\Gamma}$ sets of $\Box$-closed formulas. The right rules for $\supset, \neg$ and $\forall$ in **scGI** are:

$$\frac{\Gamma, A \Rightarrow B, \overline{\Sigma}}{\Gamma \Rightarrow A \supset B, \overline{\Sigma}}\ (\supset, r)' \qquad \frac{\Gamma, A \Rightarrow \overline{\Sigma}}{\Gamma \Rightarrow \neg A, \overline{\Sigma}}\ (\neg, r)' \qquad \frac{\Gamma \Rightarrow A(a), \overline{\Sigma}}{\Gamma \Rightarrow \forall x A(x), \overline{\Sigma}}\ (\forall, r)'$$

where in $(\forall, r)'$ the free variable a does not occur in the lower sequent. Moreover, the following rules for introducing $\Box$ are part of **scGI**

$$\frac{\Gamma, A \Rightarrow \Sigma}{\Gamma, \Box A \Rightarrow \Sigma}\ (\Box, l)' \qquad \frac{\overline{\Gamma} \Rightarrow A, \overline{\Sigma}}{\overline{\Gamma} \Rightarrow \Box A, \overline{\Sigma}}\ (\Box, r)'$$

Proposition 1 ([1]). $\models_{\mathbf{GI}} P$ *if and only if* $\vdash_{\mathbf{scGI}} P$.

However **scGI** does not admit the elimination of cuts. E.g. the sequent $\Rightarrow \forall x \neg \Box A(x), \exists x A(x)$ is provable in **scGI** but is not provable in **scGI** without applying the cut rule.

In this section we introduce a cut-free calculus **HGI** for **GI**. This calculus uses hypersequents, a simple and natural generalization of Gentzen sequents.

Definition 1. *A* hypersequent *is an expression of the form*

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ *is an ordinary sequent.* $\Gamma_i \Rightarrow \Pi_i$ *is called a* component *of the hypersequent.*

The symbol "|" is intended to denote disjunction at the meta-level (see Definition 2 below).

Here we consider sequents (resp. hypersequents) as multisets of formulas (resp. sequents)[2]. Moreover we deal only with *single-conclusioned* hypersequents, i.e. hypersequents with at most one formula in succedents of each component.

Like ordinary sequent calculi, hypersequent calculi consist of initial hypersequents (i.e., axioms) as well as *logical* and *structural* rules. The latter are divided into *internal* and *external rules*. The internal structural rules deal with formulas within components while the external ones manipulate whole components of a hypersequent. Axioms, logical and internal structural rules are essentially the same as in sequent calculi. The only difference is the presence of a *side hypersequent* G representing a (possibly empty) hypersequent. The standard external structural rules are external weakening (ew) and external contraction (ec)

$$\frac{G}{G \mid \Gamma \Rightarrow A}\ (ew) \qquad\qquad \frac{G \mid \Gamma \Rightarrow A \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A}\ (ec)$$

In hypersequent calculi it is possible to define additional external structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi compared to ordinary sequent calculi.

A *derivation* in a hypersequent calculus is a labeled finite tree with a single root (called the *end hypersequent*), with axioms at the top nodes, and each node-label connected with the label of the (immediate) successor nodes (if any) according to one of the rules.

Definition 2. *The* generic interpretation *of a* sequent $\Gamma \Rightarrow B$, *denoted by* $\mathcal{I}(\Gamma \Rightarrow B)$, *is defined as* $\bigwedge \Gamma \supset B$, *where* $\bigwedge \Gamma$ *stands for the conjunction of the formulas in* Γ ($\top$ *if* Γ *is empty), or* $\neg \bigwedge \Gamma$, *if* B *is empty. The generic interpretation of a* hypersequent $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$ *is defined as* $\mathcal{I}(\Gamma_1 \Rightarrow A_1) \vee \ldots \vee \mathcal{I}(\Gamma_n \Rightarrow A_n)$.

[2] If one prefers sequences over multisets as basic objects of inference then permutation rules have to be added to the calculus.

Let **HIL** be the hypersequent calculus for **IL** whose rules are those of **LJ** augmented with side hypersequents and with in addition (ew) and (ec). More precisely, **HIL** consists of

Axioms

$$A \Rightarrow A$$

Cut Rule

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid A, \Gamma \Rightarrow C}{G \mid \Gamma \Rightarrow C}\ (cut)$$

(*Internal and External*) *Structural Rules*

$$\frac{G \mid \Gamma \Rightarrow C}{G \mid \Gamma, A \Rightarrow C}\ (iw, l) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow C}\ (iw, r)$$

$$\frac{G \mid \Gamma, A, A \Rightarrow C}{G \mid \Gamma, A \Rightarrow C}\ (ic) \qquad (ec)\ (ew)$$

Logical rules

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \supset B}\ (\supset, r) \qquad \frac{G \mid \Gamma \Rightarrow A \quad G \mid B, \Gamma \Rightarrow C}{G \mid \Gamma, A \supset B \Rightarrow C}\ (\supset, l)$$

$$\frac{G \mid \Gamma, A \Rightarrow}{G \mid \Gamma \Rightarrow \neg A}\ (\neg, r) \qquad \frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma, \neg A \Rightarrow}\ (\neg, l)$$

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B}\ (\wedge, r) \qquad \frac{G \mid \Gamma, A_i \Rightarrow C}{G \mid \Gamma, A_1 \wedge A_2 \Rightarrow C}\ (\wedge_i, l)_{i=1,2}$$

$$\frac{G \mid \Gamma \Rightarrow A_i}{G \mid \Gamma \Rightarrow A_1 \vee A_2}\ (\vee_i, r)_{i=1,2} \qquad \frac{G \mid \Gamma, A \Rightarrow C \quad G \mid \Gamma, B \Rightarrow C}{G \mid \Gamma, A \vee B \Rightarrow C}\ (\vee, l)$$

Quantifier rules

$$\frac{G \mid A(t), \Gamma \Rightarrow B}{G \mid (\forall x)A(x), \Gamma \Rightarrow B}\ (\forall, l) \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow (\forall x)A(x)}\ (\forall, r)$$

$$\frac{G \mid A(a), \Gamma \Rightarrow B}{G \mid (\exists x)A(x), \Gamma \Rightarrow B}\ (\exists, l) \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow (\exists x)A(x)}\ (\exists, r)$$

The rules $(\forall, r)$ and $(\exists, l)$ must obey the eigenvariable condition: the free variable a must not occur in the lower *hypersequent*.

Remark 2. In the above two-premise rules, side hypersequents in both premises are the same. Because of the presence of (ew) and (ec), one can derive equivalent versions of these rules where the side hypersequents of both premises are simply joined together in the conclusion.

HGI is defined by adding to **HIL** the following rules for $\Box$:

$$\frac{G \mid \Gamma, A \Rightarrow C}{G \mid \Gamma, \Box A \Rightarrow C}\ (\Box, l) \qquad \frac{G \mid \Box\Gamma \Rightarrow A}{G \mid \Box\Gamma \Rightarrow \Box A}\ (\Box, r)$$

$$\frac{G \mid \Box\Gamma, \Gamma' \Rightarrow A}{G \mid \Box\Gamma \Rightarrow\ \mid \Gamma' \Rightarrow A}\ (cl_\Box)$$

where $\Box\Gamma$ denotes any set of $\Box$-formulas, i.e., formulas prefixed by $\Box$.

Remark 3. The above rules for $\Box$ were introduced in [7] to define a hypersequent calculus for the logic considered in [4], namely, **LC** (i.e., intuitionistic propositional calculus with axiom (Lin) $(A \supset B) \vee (B \supset A)$, see [9]) extended with $\Box$ (called Δ there). This logic coincides with propositional **GIF**.

Note that $(\Box, l)$ and $(\Box, r)$ are the (hypersequent versions of the) usual **S4**-sequent rules for $\Box$, while $(cl_\Box)$ says that $\Box$-formulas behave like boolean formulas (as established by axiom (G5)). Indeed, replacing $\Box\Gamma$ with Γ in $(cl_\Box)$, one obtains the following rule

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow A}{G \mid \Gamma \Rightarrow\ \mid \Gamma' \Rightarrow A}\ (cl)$$

defining a single-conclusioned calculus for classical logic once it is added to **HIL** (see, e.g., [7]).

Lemma 1. *If* $\vdash_{\mathbf{HGI}} \Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$ *then* $\vdash_{\mathbf{scGI}} \Rightarrow \Box\mathcal{I}(\Gamma_1 \Rightarrow A_1), \ldots, \Box\mathcal{I}(\Gamma_n \Rightarrow A_n)$.

Proof. By induction on the length of the proof. The claim is true for axioms since $\vdash_{\mathbf{scGI}} \Rightarrow \Box(A \supset A)$. We have to prove that this holds for each rule of **HGI**. Here below we show some relevant examples.

$(\forall, r)$: By induction hypothesis we have $\vdash_{\mathbf{scGI}} \Rightarrow \Box\mathcal{I}(G), \Box(\bigwedge\Gamma \supset A(a))$, where a does not occur in G, Γ and A. Therefore applying $(\forall, r)'$ we get $\Rightarrow \Box\mathcal{I}(G), \Box(\bigwedge\Gamma \supset A(a)) \vdash_{\mathbf{scGI}} \Rightarrow \Box\mathcal{I}(G), \forall x\Box(\bigwedge\Gamma \supset A(x))$. The claim follows by cut being $\vdash_{\mathbf{scGI}} \forall x\Box(\bigwedge\Gamma \supset A(x)) \Rightarrow \Box(\bigwedge\Gamma \supset \forall x A(x))$.

$(\Box, r)$: By induction hypothesis we have $\vdash_{\mathbf{scGI}} \Rightarrow \Box\mathcal{I}(G), \Box(\bigwedge\Box\Gamma \supset A)$. Since $\vdash_{\mathbf{scGI}} \Box(\bigwedge\Box\Gamma \supset A) \Rightarrow \Box(\bigwedge\Box\Gamma \supset \Box A)$, the claim follows by cut.

$(cl_\Box)$: By induction hypothesis we have $\vdash_{\mathbf{scGI}} \Rightarrow \Box\mathcal{I}(G), \Box(\bigwedge\Box\Gamma \wedge \bigwedge\Gamma' \supset A)$. Since $\vdash_{\mathbf{scGI}} \Box(\bigwedge\Box\Gamma \wedge \bigwedge\Gamma' \supset A), \bigwedge\Box\Gamma \Rightarrow \Box(\bigwedge\Gamma' \supset A)$, by cut we have $\Rightarrow \Box\mathcal{I}(G), \Box(\bigwedge\Box\Gamma \wedge \bigwedge\Gamma' \supset A) \vdash_{\mathbf{scGI}} \bigwedge\Box\Gamma \Rightarrow \Box\mathcal{I}(G), \Box(\bigwedge\Gamma' \supset A)$. The claim follows applying $(\neg, r)'$ and $(\Box, r)'$.

Theorem 1. $\vdash_{\mathbf{HGI}} \Rightarrow P$ *if and only if* $\models_{\mathbf{GI}} P$.

Proof. $(\Longrightarrow)$ If $\vdash_{\mathbf{HGI}} \Rightarrow P$, by Lemma 1, $\vdash_{\mathbf{scGI}} \Rightarrow \Box P$. But $\vdash_{\mathbf{scGI}} \Box P \Rightarrow P$. Hence $\vdash_{\mathbf{scGI}} \Rightarrow P$. The claim follows by Proposition 1.

$(\Longleftarrow)$ Since axioms and rules of **LJ** (including cut) as well as $(G4)$ are contained in **HGI**, it suffices to prove that the latter derives $(G1)-(G3)$ and $(G5)$. The claim then follows by [22]. Here below we display the derivations of $(G2)$, $(G3)$ and $(G5)$ (that of $(G1)$ is straightforward).

$(G2)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(b) \Rightarrow A(b)}{\Box A(b) \Rightarrow A(b)}\,{\scriptstyle(\Box,\mathrm{l})}}{\Rightarrow A(b) \mid \Box A(b) \Rightarrow}\,{\scriptstyle(\mathrm{cl}_\Box)}}{\Rightarrow A(b) \mid \forall x \Box A(x) \Rightarrow}\,{\scriptstyle(\forall,\mathrm{l})}}{\Rightarrow \forall x A(x) \mid \forall x \Box A(x) \Rightarrow}\,{\scriptstyle(\forall,\mathrm{r})}}{\Rightarrow \Box \forall x A(x) \mid \forall x \Box A(x) \Rightarrow}\,{\scriptstyle(\Box,\mathrm{r})}}{\forall x \Box A(x) \Rightarrow \Box \forall x A(x) \mid \forall x \Box A(x) \Rightarrow \Box \forall x A(x)}\,{\scriptstyle(\mathrm{iw,r})+(\mathrm{iw,l})}}{\forall x \Box A(x) \Rightarrow \Box \forall x A(x)}\,{\scriptstyle(\mathrm{ec})}
$$

$(G3)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box A \Rightarrow \Box A \qquad \dfrac{\dfrac{B \Rightarrow B}{\Box B \Rightarrow B}\,{\scriptstyle(\Box,\mathrm{l})}}{\Box B \Rightarrow \ \mid \ \Rightarrow B}\,{\scriptstyle(\mathrm{cl}_\Box)}}{\Box A \supset \Box B, \Box A \Rightarrow \ \mid \ \Rightarrow B}\,{\scriptstyle(\supset,\mathrm{l})}}{\Box A \supset \Box B \Rightarrow \ \mid \Box A \Rightarrow \ \mid \ \Rightarrow B}\,{\scriptstyle(\mathrm{cl}_\Box)}}{\Box A \supset \Box B \Rightarrow \ \mid \Box A \Rightarrow B}\,{\scriptstyle(\mathrm{ec})+(\mathrm{iw,r})+(\mathrm{iw,l})}}{\Box A \supset \Box B \Rightarrow \ \mid \ \Rightarrow \Box A \supset B}\,{\scriptstyle(\supset,\mathrm{r})}}{\Box A \supset \Box B \Rightarrow \ \mid \ \Rightarrow \Box(\Box A \supset B)}\,{\scriptstyle(\Box,\mathrm{r})}}{\Box A \supset \Box B \Rightarrow \Box(\Box A \supset B) \mid \Box A \supset \Box B \Rightarrow \Box(\Box A \supset B)}\,{\scriptstyle(\mathrm{iw,r})+(\mathrm{iw,l})}}{\Box A \supset \Box B \Rightarrow \Box(\Box A \supset B)}\,{\scriptstyle(\mathrm{ec})}
$$

$(G5)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Box A \Rightarrow \Box A}{\Rightarrow \Box A \mid \Box A \Rightarrow}\,{\scriptstyle(\mathrm{cl}_\Box)}}{\Rightarrow \Box A \mid \ \Rightarrow \neg \Box A}\,{\scriptstyle(\neg,\mathrm{r})}}{\Rightarrow \Box A \vee \neg \Box A \mid \ \Rightarrow \Box A \vee \neg \Box A}\,{\scriptstyle(\vee,\mathrm{r})'\mathrm{s}}}{\Rightarrow \Box A \vee \neg \Box A}\,{\scriptstyle(\mathrm{ec})}
$$

Remark 4. In $(\Box, r)$ and $(cl_\Box)$ one can replace $\Box\Gamma$ with $\overline{\Gamma}$, i.e. the corresponding set of $\Box$-closed formulas. The resulting rules are easily seen to be equivalent to the original ones since, on the one hand, each $\Box$-formula is a $\Box$-closed formula, on the other hand $\Box\overline{\Gamma} \vdash_{\mathbf{HGI}} \overline{\Gamma}$ and $\overline{\Gamma} \vdash_{\mathbf{HGI}} \Box\overline{\Gamma}$.

The proof of Theorem 1 relies on the fact that (cut) is one of the rules of **HGI**. In the following section we give a constructive proof that (cut) is in fact eliminable from **HGI**-derivations.

2.3. Cut-elimination and Consequences

Recall that Gentzen's cut-elimination method proceeds by eliminating the *uppermost cut* in a derivation by a double induction on the complexity of the cut formula and on the sum of its left and right ranks, where the right (left) rank of a cut is the number of consecutive (hyper)sequents containing the cut formula, counting upward from the right (left) upper sequent of the cut. In his original proof of the cut-elimination theorem for sequent calculus [11], Gentzen met the following problem: If the cut formula is derived by contraction (here called (ic)), the permutation of cut with (ic) does not necessarily move the cut higher up in the derivation. To solve this problem, he introduced the multi-cut rule – a derivable generalization of cut.

In hypersequent calculus a similar problem arises when one tries to permute cut with (ec), see e.g. [2]. In analogy with Gentzen's solution, a way to overcome this problem is to introduce a suitable (derivable) generalization of Gentzen's multi-cut rule allowing certain cuts to be reduced *in parallel*. E.g. to prove cut-elimination in the hypersequent calculus for **LC** (i.e. propositional **IF**), Avron used the following induction hypothesis [2] ("extended multi-cut rule"):

If $H \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$, where A^{n_i} stands for $A, \ldots, A$, n_i times, are cut-free provable, so is $H \mid \Gamma, \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Gamma, \Sigma_k \Rightarrow B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$.

Nevertheless, this extended multi-cut rule works only for hypersequent calculi in which one can derive the following generalized left rules for $\vee$ and $\exists$:

$$\dfrac{G \mid A, \Gamma_1 \Rightarrow C_1 \mid \ \ldots \ \mid A, \Gamma_n \Rightarrow C_n \quad G \mid B, \Gamma_1 \Rightarrow C_1 \mid \ \ldots \ \mid B, \Gamma_n \Rightarrow C_n}{G \mid A \vee B, \Gamma_1 \Rightarrow C_1 \mid \ \ldots \ \mid A \vee B, \Gamma_n \Rightarrow C_n} \ (\vee, l)^*$$

$$\dfrac{G \mid A(a), \Gamma_1 \Rightarrow C_1 \mid \ \ldots \ \mid A(a), \Gamma_n \Rightarrow C_n}{G \mid \exists x A(x), \Gamma_1 \Rightarrow C_1 \mid \ \ldots \ \mid \exists x A(x), \Gamma_n \Rightarrow C_n} \ (\exists, l)^*$$

($(\vee, l)^*$ and $(\exists, l)^*$ are indeed needed in permuting upward a cut with $(\vee, l)$ and $(\exists, l)$, see [7]). These generalized rules are not derivable in **HGI**. For, assume otherwise that $(\vee, l)^*$ is derivable in **HGI**. Then, in **HGI** one could prove the linearity axiom (Lin) $(A \supset B) \vee (B \supset A)$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow A \vee B}(\vee,r) \quad \dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \mid A \Rightarrow B}(ew) \quad \dfrac{B \Rightarrow B}{B \Rightarrow A \mid B \Rightarrow B}(ew)}{A \vee B \Rightarrow A \mid A \vee B \Rightarrow B}(\vee,l)^*}{B \Rightarrow A \mid A \vee B \Rightarrow B}(iw,l)+(cut) \quad \dfrac{A \Rightarrow A}{A \Rightarrow A \vee B}(\vee,r)}{A \Rightarrow B \mid B \Rightarrow A}(iw,l)+(cut)}{\Rightarrow A \supset B \mid B \supset A}2x(\supset,r)}{\dfrac{\Rightarrow (A \supset B) \vee (B \supset A) \mid (A \supset B) \vee (B \supset A)}{\Rightarrow (A \supset B) \vee (B \supset A)}(ec)}2x(\vee,r)$$

while (Lin) is not valid in **GI** (a counterexample can be easily constructed in any *not* linearly ordered cHa). This would contradict Theorem 1.

A different solution to the problem due to (ec) is to eliminate cuts — starting from a *largest* one (w.r.t. the number of connectives, modalities and quantifiers) — without shifting them upward. This cut-elimination method was introduced by Schütte-Tait [15,16] for sequent calculus and used in [5] for the hypersequent calculus for **IF**. Following [5], to eliminate a cut with non-atomic cut formula here we proceed as follows: we invert one of the two premises of the cut and we use it to replace the cut by smaller ones exactly in the place(s) in which the cut formula (of the remainig premise of the cut) is introduced. This requires us to trace up the occurrences of the cut formula through the derivation ending in a premise of the cut. To this purpose we introduce below the notion of *decoration of a formula in a derivation* d. This essentially amounts in the (marked) derivation obtained by following up and marking in d all occurrences of the considered formula starting from the end hypersequent of d: If at some stage any marked occurrence of the formula is contracted by (ic) or it belongs to a component contracted by (ec), we mark (and trace up) both occurrences of the formula from the premise.

Henceforth we write $d, S_1 \dots S_n \vdash_{\mathbf{HGI}} \mathrm{H}$ if d is a derivation in **HGI** of H from the premises $S_1 \dots S_n$. The introduced formulas in logical and quantifier rules as well as in $(\Box, r)$ and $(\Box, l)$ will be called *principal formulas.*

Definition 3. *Let* $d \vdash_{\mathbf{HGI}} \mathrm{H}$ *and* A *be a formula in* H *that is not the cut formula of any cut in* d*. The* decoration of A in d *is inductively defined as follows: we denote by* A^* *any marked occurrence of* A*. Given a hypersequent* H' *in* d *with some (not necessarily all) marked* A*'s. Let* R *be the rule introducing* H'*. We divide some cases according to* R*.*

1. *R is a logical rule, a quantifier rule,* $(\Box, r)$ *or* $(\Box, l)$*:*
   1.1 A is principal in R, e.g.,

$$\frac{G \mid \Gamma' \Rightarrow C'}{G \mid \Gamma \Rightarrow C}\ (R)$$

   (a) Suppose that $A^* \in \Gamma$. $A^* \in \Gamma'$ if and only if A^* is an occurrence of a formula in Γ which is not the principal formula. Moreover, the marked formulas of G in the premise of R, are as in the conclusion. That is, for each $\{\Sigma \Rightarrow B\} \in G$, $A^* \in \Sigma$ if and only if $A^* \in \Sigma$ of the corresponding component belonging to the conclusion of R.
   (b) Suppose that C is A^*. The marked formulas of G in the premise of R are as in the conclusion.
   1.2 A is not principal in R. The marked formulas of the premise of R are as in the conclusion.
   If R is a two premises rule, the definition is analogous.
2. *R is (ew), (cut) or $(cl_\Box)$. The marked formulas of the premise(s) of R are as in the conclusion.*
3. *R is (ic).*

3.1 A^* *is the contracted formula, then* A^*, A^* *belongs to the premise of* R*. The remaining formulas in the premise of* R *are marked as in the conclusion.*

3.2 A^* *is not the contracted formula. Analogous to case 2.*

4. R *is* (iw, l), (iw, r)*. Analogous to case 1.*
5. R *is* (ec)*. Similar to case 3.*

Definition 4. *The* complexity $|A|$ *of a formula* A *is inductively defined as:*

- $|A| = 0$ *if* A *is atomic*
- $|A \wedge B| = |A \vee B| = |A \supset B| = \max(|A|, |B|) + 1$
- $|\Box A| = |\forall x A(x)| = |\exists x A(x)| = |A| + 1$

The length $|d|$ *of a derivation* d *is the maximal number of inference rules + 1 occurring on any branch of* d*. The* cut-rank $\rho(d)$ *of* d *is the maximal complexity of cut formulas in* $d + 1$*.* ($\rho(d) = 0$ *if* d *is cut-free).*

Let $d({}^s/_x)$ and $H({}^s/_x)$ denote the result of substituting the term s for all free occurrences of x in the derivation $d(x)$ and in the hypersequent $H(x)$, respectively.

Lemma 2 (Substitution). *If* $d(x) \vdash_{\mathbf{HGI}} H(x)$*, then* $d({}^s/_x) \vdash_{\mathbf{HGI}} H({}^s/_x)$*, with* $|d({}^s/_x)| = |d(x)|$ *and* $\rho(d({}^s/_x)) = \rho(d(x))$*, where* s *only contains variables that do not occur in* $d(x)$*.*

Lemma 3 (Inversion).

(i) If $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \vee \mathrm{B} \Rightarrow \mathrm{C}$ *then one can find* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{C}$ *and* $d_2 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{B} \Rightarrow \mathrm{C}$

(iii) If $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{A} \wedge \mathrm{B}$ *then one can find* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{A}$ *and* $d_2 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{B}$

(iv) If $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{A} \supset \mathrm{B}$ *then one can find* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{B}$

(v) If $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \neg\mathrm{A}$ *then one can find* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \Rightarrow$

(vi) If $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \Box\mathrm{A}$ *then one can find* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{A}$

(vii) If $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \exists \mathrm{x} \mathrm{A}(\mathrm{x}) \Rightarrow \mathrm{C}$ *then one can find* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A}(\mathrm{a}) \Rightarrow \mathrm{C}$*, for any* a *that does not occur free in* G, Γ, A *or* C

(viii) If $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \forall \mathrm{x} \mathrm{A}(\mathrm{x})$ *then one can find* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{A}(\mathrm{a})$*, for any* a *that does not occur free in* G, Γ *or* A

such that $\rho(d_i) \leq \rho(d)$ *and* $|d_i| \leq |d|$*, for* $i = 1, 2$*.*

Proof. The proof is by induction on $|d|$. We outline (i) and (vi).
We consider the last inference R in d.

(i) R is a logical rule, a quantifier rule, $(\Box, r)$ or $(\Box, l)$
(a) The indicated occurrence of $A \vee B$ is the principal formula of R. E.g.

$$\frac{\overset{\vdots\, d'}{G \mid \Gamma, B \Rightarrow C} \quad \overset{\vdots\, d''}{G \mid \Gamma, A \Rightarrow C}}{G \mid \Gamma, A \vee B \Rightarrow C}$$

The required derivations d_1 and d_2 are just d'' and d'.
(b) The indicated occurrence of $A \vee B$ is not the principal formula of R. Suppose e.g. that d ends in a rule $(\supset, l)$ whose premises are

$$d' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{Y}, \mathrm{A} \vee \mathrm{B} \Rightarrow \mathrm{C} \text{ and } \mathrm{d}'' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \vee \mathrm{B} \Rightarrow \mathrm{X}$$

and the conclusion is $G \mid \Gamma, X \supset Y, A \vee B \Rightarrow C$. Applying the induction hypothesis to d' and d'' we obtain $d_1' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{Y}, \mathrm{A} \Rightarrow \mathrm{C}$ and $d_2' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{Y}, \mathrm{B} \Rightarrow \mathrm{C}$ as well as $d_1'' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{X}$ and $d_2'' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{B} \Rightarrow \mathrm{X}$, respectively, with $|d_i'|, |d_i''| < |d|$ and $\rho(d_i'), \rho(d_i'') \leq \rho(d)$, for $i = 1, 2$. Then $(\supset, l)$ can be applied to d_1' and d_1'' (resp. d_2' and d_2'') to obtain the derivation d_1 (resp. d_2) with the required properties.
If R is $(cl_\Box)$ the claim trivially follows by induction and subsequent application of $(cl_\Box)$.
R is a(n internal or external) structural rule. If R is (ew) or (ec) and the indicated occurrence of $A \vee B$ is in G, or if R is an internal weakening or (ic) and $A \vee B$ is not the contracted formula, the claim trivially follows by induction hypothesis and subsequent application(s) of R. Assume R is (ec) and the indicated occurrence of $A \vee B$ is in the contracted component. E.g. d ends as follows

$$\dfrac{\begin{matrix} \vdots\, d' \\ G \mid \Gamma, A \vee B \Rightarrow C \mid \Gamma, A \vee B \Rightarrow C \end{matrix}}{G \mid \Gamma, A \vee B \Rightarrow C}$$

By induction hypothesis one obtains the derivations $d_1' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{C} \mid \Gamma, \mathrm{A} \vee \mathrm{B} \Rightarrow \mathrm{C}$ and $d_2' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{B} \Rightarrow \mathrm{C} \mid \Gamma, \mathrm{A} \vee \mathrm{B} \Rightarrow \mathrm{C}$ with $|d_i'| < |d|$ and $\rho(d_i') \leq \rho(d)$, for $i = 1, 2$. The required derivations d_1 and d_2 are obtained by applying again the induction hypothesis to d_1' and d_2' followed by an application of (ec). The remaining case is similar.
If R is cut. Analogous to case i(b).

(vi) We illustrate the case in which R is a logical inference. The remaining cases are handled as in (i).
(a) If the indicated occurrence of $\Box A$ is the principal formula of R. E.g.

$$\dfrac{\begin{matrix} \vdots\, d' \\ G \mid \Box\Gamma \Rightarrow A \end{matrix}}{G \mid \Box\Gamma \Rightarrow \Box A}$$

The required derivation d_1 is just d'.
(b) The indicated occurrence of $\Box A$ is not the principal formula of R. Suppose e.g. that d ends in a rule $(\vee, l)$ whose premises are

$$d' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{X} \Rightarrow \Box\mathrm{A} \text{ and } \mathrm{d}'' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{Y} \Rightarrow \Box\mathrm{A}$$

and the conclusion is $G \mid \Gamma, X \vee Y \Rightarrow \Box A$. Applying the induction hypothesis to d' and d'' we obtain the derivations $d_1' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{X} \Rightarrow$

A and $d_1'' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{Y} \Rightarrow \mathrm{A}$ with $|d_1'|, |d_1''| < |d|$. Then $(\vee, l)$ can be applied to d_1' and d_1'' to obtain the derivation d_1 with the required properties.

In the following $H[^B/_A]$ will indicate the hypersequent H in which we uniformly replace each occurrence of A by B.

Lemma 4. *Let* $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{C}$*, where* A *is an atomic formula that is not the cut formula of any cut in* d*. One can find a derivation* $d', G \mid \Gamma \Rightarrow A \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{C}$ *with* $\rho(d') = \rho(d)$*.*

Proof. Let us consider the decoration of A in d starting from $\Gamma, A^* \Rightarrow C$. We first replace A^* everywhere in d with Γ and we add G to all the hypersequents in d (previously applying the Substitution Lemma, if needed). For each hypersequent $G \mid B \Rightarrow B$, we add an application of (ew) to recover the original axiom $B \Rightarrow B$ of d. Note that A^* can originate

1. in an axiom. Then the axiom is transformed into $G \mid \Gamma \Rightarrow A$,
2. by an internal weakening. The weakening on A^* is replaced by stepwise weakenings of formulas $B \in \Gamma$.
3. by an external weakening. The weakening of the component S is replaced by a weakening on $S[^\Gamma/_{A^*}]$.

One can check that this procedure results in a derivation $d_1, G \mid \Gamma \Rightarrow A \vdash_{\mathbf{HGI}} \mathrm{G} \mid \mathrm{G} \mid \Gamma, \Gamma \Rightarrow \mathrm{C}$ with $\rho(d_1) = \rho(d)$. d' is therefore obtained by applying (ec) and (ic), as necessary.

Lemma 5. *In* **HGI** *non-atomic axioms can be derived from atomic axioms.*

Lemma 6 (Reduction). *Let* $d_0 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{A}$ *and* $d_1 \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A} \Rightarrow \mathrm{C}$ *both with cut-rank* $\rho(d_i) \leq |A|$*. Then we can find a derivation* $d \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{C}$ *with* $\rho(d) \leq |A|$*.*

Of course, one could derive $G \mid \Gamma \Rightarrow C$ by an application of the cut rule, but the resulting derivation would then have cut-rank $|A| + 1$.

Proof. By Lemma 5 we can assume d_0 and d_1 have atomic axioms. If A is atomic, by Lemma 4 one can find a derivation $d', G \mid \Gamma \Rightarrow A \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma \Rightarrow \mathrm{C}$ such that $\rho(d') = \rho(d_1)$. The required derivation d is obtained by concatenating the derivations d_0 and d'.

Suppose that A is not atomic.

- $A = B \supset D$. Let us consider the decoration of A in d_1 starting from $G \mid \Gamma, (B \supset D)^* \Rightarrow C$. We first replace $(B \supset D)^*$ everywhere in d_1 with Γ and we add G to all the hypersequents in d_1 (previously applying the Substitution Lemma, if needed). For each hypersequent $G \mid B \Rightarrow B$, we add an application of (ew) to recover the original axiom $B \Rightarrow B$ of d_1. Note that the resulting tree is not a derivation anymore. We have then to consider the following "correction steps" according to the cases in which the marked occurrence of $B \supset D$ originated:
  (i) as principal formula of a logical inference. We replace $(\supset, r)$ inferences of $B \supset D$ by cut rules as follows: every "inference step" of the kind

$$\dfrac{\begin{array}{c}\vdots\\ G' \mid G \mid \Psi \Rightarrow B\end{array} \quad \begin{array}{c}\vdots\\ G' \mid G \mid \Psi, D \Rightarrow C'\end{array}}{G' \mid G \mid \Psi, \Gamma \Rightarrow C'}\ (\supset,\mathrm{l})$$

is replaced by (let $G'' = G \mid G'$)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ G'' \mid \Psi \Rightarrow B\end{array}}{G'' \mid \Psi, \Gamma \Rightarrow B}\ (\mathrm{iw,l}) \quad \dfrac{\begin{array}{c}\vdots\, d_0'\\ G \mid \Gamma, B \Rightarrow D\end{array}}{G'' \mid \Gamma, \Psi, B \Rightarrow D}\ (\mathrm{ew})+(\mathrm{iw,l})}{G'' \mid \Psi, \Gamma \Rightarrow D}\ (\mathrm{cut}) \quad \dfrac{\begin{array}{c}\vdots\\ G'' \mid \Psi, D \Rightarrow C'\end{array}}{G'' \mid \Psi, \Gamma, D \Rightarrow C'}\ (\mathrm{iw,l})}{G'' \mid \Psi, \Gamma \Rightarrow C'}\ (\mathrm{cut})$$

where the missing premise of the first cut rule (i.e. $G'' \mid \Gamma, \Psi, B \Rightarrow D$) is obtained from d_0 using the Inversion Lemma.

(ii) By an internal weakening: The weakening on $(B \supset D)^*$ is replaced by stepwise weakenings of formulas $X \in \Gamma$.

It is easy to check that this procedure results in a derivation $d' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \mathrm{G} \mid \Gamma, \Gamma \Rightarrow \mathrm{C}$ with $\rho(d') \leq |A|$. Hence, the required derivation is obtained by applying (ec) and (ic), as necessary.

- Cases $A = B \wedge D$, $A = \neg B$ and $A = \forall x A(x)$ are treated analogously.
- $A = \exists x A(x)$. Let us consider the decoration of A in d_0 starting from $G \mid \Gamma \Rightarrow (\exists x A(x))^*$. We first replace in d_0 all the components $\{\Psi \Rightarrow (\exists x A(x))^*\}$ by $\{\Psi, \Gamma \Rightarrow C\}$ and we add G to all the hypersequents in d_0 (previously applying the Substitution Lemma, if needed). For each hypersequent $G \mid B \Rightarrow B$, we add an application of (ew) to recover the original axiom $B \Rightarrow B$ of d_0. This does not result in a correct derivation anymore. Correction step (ii) is handled by stepwise weakenings of formulas $X \in \Gamma$ and C. While if $(\exists x A(x))^*$ originated as the principal formula of $(\exists, r)$, we replace every "inference step" of the kind

$$\dfrac{\begin{array}{c}\vdots\\ G' \mid G \mid \Psi \Rightarrow A(t)\end{array}}{G' \mid G \mid \Psi, \Gamma \Rightarrow C}\ (\exists,\mathrm{r})$$

by

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ G' \mid G \mid \Psi \Rightarrow A(t)\end{array}}{G \mid G' \mid \Psi, \Gamma \Rightarrow A(t)}\ (\mathrm{iw,l}) \quad \dfrac{\begin{array}{c}\vdots\, d_1'(t)\\ G \mid \Gamma, A(t) \Rightarrow C\end{array}}{G \mid G' \mid \Gamma, \Psi, A(t) \Rightarrow C}\ (\mathrm{ew})+(\mathrm{iw,l})}{G' \mid G \mid \Psi, \Gamma \Rightarrow C}\ (\mathrm{cut})$$

where $d_1'(t) \vdash_{\mathbf{HGI}} \mathrm{G} \mid \Gamma, \mathrm{A(t)} \Rightarrow \mathrm{C}$ is obtained by the Inversion Lemma and the Substitution Lemma. Let us call d' the obtained tree. One can check that $d' \vdash_{\mathbf{HGI}} \mathrm{G} \mid \mathrm{G} \mid \Gamma, \Gamma \Rightarrow \mathrm{C}$ with $\rho(d') \leq |A|$. Hence, the required derivation is finally obtained by applying (ec) and (ic), as necessary.
- Case $A = B \vee D$ is treated analogously.

– $A = \Box B$. Let us consider the decoration of A in d_0 and d_1 starting from $G \mid \Gamma \Rightarrow (\Box B)^*$ and $G \mid \Gamma, (\Box B)^* \Rightarrow C$, respectively. If Γ is a set of $\Box$-formulas then the proof proceeds similarly as for $A = B \supset D$. Suppose that Γ is not a set of $\Box$-formulas. We first replace in d_0 all the components $\{\Psi \Rightarrow (\Box B)^*\}$ by $\{\Psi, \Gamma \Rightarrow C\}$ and we add G to all the hypersequents in d_0 (previously applying the Substitution Lemma, if needed). For each hypersequent $G \mid B \Rightarrow B$, we add an application of (ew) to recover the original axiom $B \Rightarrow B$ of d_0. As usual, this does not result in a correct derivation anymore. Correction step (ii) is handled by stepwise weakenings of formulas $X \in \Gamma$ and C. Correction step (i) amounts in applying repeatedly the following procedure: Let us consider a "subderivation" of this tree ending in an uppermost application of $(\Box, r)$ introducing $(\Box B)^*$ in d_0, i.e.,

$$\dfrac{\begin{array}{c}\vdots\, d_0^i \\ G_i \mid G \mid \Box\Gamma_i \Rightarrow B\end{array}}{G_i \mid G \mid \Box\Gamma_i, \Gamma \Rightarrow C}\,{\scriptstyle(\Box,\mathrm{r})}$$

(Note that in the above "subderivation" the only incorrect step is the last one, namely the application of $(\Box, r)$). The whole "subderivation" above is replaced by $d_1^i \vdash_{\mathbf{HGI}} \mathrm{G_i \mid G \mid \Box\Gamma_i, \Gamma \Rightarrow C}$ obtained similarly to the derivation d' in the case $A = B \supset D$. Indeed, we first replace $(\Box B)^*$ everywhere in d_1 with $\Box\Gamma_i$ and we add $G_i \mid G$ to all the hypersequents in d_1, previously applying the Substitution Lemma, if needed, together with suitable applications of internal and external weakening. Applications of $(\Box, l)$ that introduced $(\Box B)^*$ in d_1 are replaced by suitable cuts (on B) in which the missing premise of the cut rule is $d_0^i \vdash_{\mathbf{HGI}} \mathrm{G_i \mid G \mid \Box\Gamma_i \Rightarrow B}$. d_1^i is then obtained by applying (ec) as necessary.

Theorem 2 (Cut-elimination). *If a hypersequent H is derivable in* **HGI** *then H is derivable in* **HGI** *without using the cut rule.*

Proof. Let $d \vdash_{\mathbf{HGI}} \mathrm{H}$ and $\rho(d) > 0$. The proof proceeds by a double induction on $(\rho(d), n\rho(d))$, where $n\rho(d)$ is the number of cuts in d with cut-rank $\rho(d)$. Indeed, let us take in d an uppermost cut with cut-rank $\rho(d)$. By applying the Reduction Lemma to its premises either $\rho(d)$ or $n\rho(d)$ decreases.

As an immediate consequence of the above theorem one has the *subformula property*: every formula P valid in **GI** has a derivation in **HGI** which only contains subformulas of P. Among other things, this ensures that **GI** is a conservative extension of **IL** (i.e., for each formula P not containing $\Box$, $\models_{\mathbf{GI}} P$ if and only of $\models_{\mathbf{IL}} P$). Another important corollary of cut-elimination is (a version of) the *mid-(hyper)sequent theorem*. Indeed, Gentzen showed that a certain separation between propositional and quantifier inferences can be achieved in deriving a prenex formula in the sequent calculus **LK** for classical logic (see, e.g., [18]). In the case of **LJ**, this theorem only holds for the fragment of the calculus without the rule $(\vee, l)$. Here below we prove the mid-(hyper)sequent theorem for a fragment of **HGI** properly including **LJ** without $(\vee, l)$.

Theorem 3. *Any* **HGI**-*derivation d of a hypersequent H s.t.*

1. *H only contains prenex formulas*
2. *in d, the rule $(\vee, l)$ is applied only to formulas in which (at least) one of the disjuncts is a $\Box$-formula*

can be transformed into a derivation of H in which no propositional rule is applied below any application of a quantifier rule.

Proof. Theorem 2 and Lemma 5 provide us with a cut-free derivation d' of H from atomic axioms. Note that the $(\vee, l)$ rule with a principal formula of the kind $\Box A \vee B$ can be simulated, without using cuts, by the following one

$$\frac{G \mid \Box A, \Gamma \Rightarrow C_1 \quad G \mid B, \Gamma \Rightarrow C_2}{G \mid \Box A \vee B, \Gamma \Rightarrow C_1 \mid \Box A \vee B, \Gamma \Rightarrow C_2} \ (\vee^{\Box}, l)$$

This rule can be derived in **HGI** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{G \mid \Box A, \Gamma \Rightarrow C_1}{G \mid \Box A \Rightarrow \ \mid \Gamma \Rightarrow C_1}\,(\mathrm{cl}_{\Box})}{G \mid \Box A, \Gamma \Rightarrow C_2 \mid \Gamma \Rightarrow C_1}\,(\mathrm{iw,l})'\mathrm{s}{+}(\mathrm{iw,r}) \qquad G \mid B, \Gamma \Rightarrow C_2}{G \mid \Box A \vee B, \Gamma \Rightarrow C_2 \mid \Gamma \Rightarrow C_1}\,(\vee,\mathrm{l})}{G \mid \Box A \vee B, \Gamma \Rightarrow C_1 \mid \Box A \vee B, \Gamma \Rightarrow C_2}\,(\mathrm{iw,l})$$

We define the order of a quantifier inference in d' as the number of propositional inferences under it, and the order of d' as the sum of the orders of its quantifiers inferences. The proof proceeds by induction on the order of d' as follows: In d' we replace all the applications of $(\vee, l)$ by applications of $(\vee^{\Box}, l)$. It is easy to check that the rule $(\vee^{\Box}, l)$ can be shifted upward over any quantifier inference. As an example we outline a case that does not work in **LJ**. Thus suppose d' contains a $(\exists, r)$ inference above a $(\vee^{\Box}, l)$ inference and so that all the inferences in between are structural, e.g.,

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\gamma' \\ G' \mid \Gamma' \Rightarrow A(t)\end{array}}{\begin{array}{c}G' \mid \Gamma' \Rightarrow \exists x A(x) \\ \vdots\,\gamma \\ G \mid \Box B, \Gamma \Rightarrow \exists x A(x)\end{array}}\,(\exists,\mathrm{r}) \qquad G \mid A, \Gamma \Rightarrow C}{G \mid A \vee \Box B, \Gamma \Rightarrow \exists x A(x) \mid A \vee \Box B, \Gamma \Rightarrow C}\,(\vee^{\Box},\mathrm{l})$$

where γ contains structural inferences. We reduce the order of d' as follows:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\gamma' \\ G' \mid \Gamma' \Rightarrow A(t) \\ \vdots\,\gamma \\ G \mid \Box B, \Gamma \Rightarrow A(t)\end{array} \qquad G \mid A, \Gamma \Rightarrow C}{G \mid A \vee \Box B, \Gamma \Rightarrow A(t) \mid A \vee \Box B, \Gamma \Rightarrow C}\,(\vee^{\Box},\mathrm{l})}{G \mid A \vee \Box B, \Gamma \Rightarrow \exists x A(x) \mid A \vee \Box B, \Gamma \Rightarrow C}\,(\exists,\mathrm{r})$$

The cases involving the rules for introducing $\Box$ are straightforward.

Corollary 1. *Herbrand's theorem holds for the prenex fragment of* **GI** *in which each disjunctive (sub)formula with negative polarity has at most one disjunct that is not a* $\Box$*-formula.*

3. Global Intuitionistic Fuzzy Logic (GIF)

3.1. Preliminaries

Intuitionistic Fuzzy Logic **IF** was defined by Takeuti and Titani in [19] to be the logic of the complete Heyting algebra $[0, 1]$. **IF** also coincides with first-order Gödel logic [12] based on the truth-values set $[0, 1]$. Takeuti and Titani characterized **IF** by a calculus which extends **LJ** by several axioms as well as the following rule, expressing the density of the ordered set of truth-values:

$$\frac{\Gamma \Rightarrow C \vee (A \supset p) \vee (p \supset B)}{\Gamma \Rightarrow C \vee (A \supset B)} \ (TT)$$

where p is a propositional eigenvariable (i.e., it does not occur in the conclusion).

In the following we use the deductive system for **IF**, defined in [17], consisting of **LJ** with axioms $(Lin) \ \Rightarrow (A \supset B) \vee (B \supset A)$ and $(\vee\forall)$ $\Rightarrow \forall x(A(x) \vee B) \supset (\forall x A(x) \vee B)$, where x does not occur in B. (Note that (Lin) and $(\vee\forall)$ say that Kripke models for **IF** are *linearly ordered* and with *constant domains*, see [10].) In [20] **IF** was extended with the operator $\Box$ of **GI**. The resulting logic was called Global Intuitionistic Fuzzy logic **GIF**. A formula P is *valid* in **GIF** iff for all interpretations v_D on the $[0, 1]$-Heyting algebra with a globalization $\Box$ satisfying $(*)$ (see Section 2.1), $v_D(P) = 1$.

A deductive system for **GIF** was defined in [20] by adding to the one of **IF** the axioms and rules $(G1) - (G4)$ (see Section 2.1) together with the following axioms:

$$(G5') \ \neg\neg\Box A \Rightarrow \Box A, \quad (G6) \ \neg\neg A \Rightarrow \Box\neg\neg A, \quad (G7) \ \Rightarrow \Box(A \supset B) \vee \Box(B \supset A)$$

We refer to this system as **scGIF**.

Proposition 2 ([20]). $\models_{\mathbf{GIF}} P$ *if and only if* $\vdash_{\mathbf{scGIF}} P$.

3.2. An Analytic calculus for **GIF**

Let **HGIF** be the hypersequent calculus obtained by adding to **HGI** Avron's rule:

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow A \quad G \mid \Gamma, \Gamma' \Rightarrow A'}{G \mid \Gamma \Rightarrow A \mid \Gamma' \Rightarrow A'} \ (com)$$

Remark 5. As discussed in [3], using (ic) and (iw, l), (com) is inter-derivable with the *communication* rule

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow A \quad G \mid \Gamma_1, \Gamma_1' \Rightarrow A'}{G \mid \Gamma, \Gamma_1 \Rightarrow A \mid \Gamma', \Gamma_1' \Rightarrow A'} \ (com')$$

introduced in [2] to define a hypersequent calculus for propositional **IF**. Hence **HGIF** coincides with Baaz and Zach's calculus **HIF** for **IF** (see [8]) extended with the rules for $\Box$ introduced in [7] (see Remark 3).

In the following we refer to the fragment of **HGIF** consisting of **HIL**+(*com*) as **HIF**.

Lemma 7. *If* $\vdash_{\mathbf{HGIF}} \Gamma_1 \Rightarrow A_1 \mid \dots \mid \Gamma_n \Rightarrow A_n$ *then* $\vdash_{\mathbf{scGI}+(G7)} \Rightarrow \Box\mathcal{I}(\Gamma_1 \Rightarrow A_1), \dots, \Box\mathcal{I}(\Gamma_n \Rightarrow A_n)$.

Proof. By Lemma 1 we only have to check the case involving the (*com*) rule. By induction hypothesis $\vdash_{\mathbf{scGI}+(G7)} \Rightarrow P_1$ and $\vdash_{\mathbf{scGIF}+(G7)} \Rightarrow P_2$, where P_1 and P_2 abbreviate $\Box\mathcal{I}(G) \vee \Box(\bigwedge \Gamma \wedge \bigwedge \Gamma' \supset A)$ and $\Box\mathcal{I}(G) \vee \Box(\bigwedge \Gamma \wedge \bigwedge \Gamma' \supset A')$, respectively. Since $\Rightarrow P_1, \Rightarrow P_2 \vdash_{\mathbf{scGI}} \Box(\bigwedge \Gamma \supset \bigwedge \Gamma') \vee \Box(\bigwedge \Gamma' \supset \bigwedge \Gamma) \Rightarrow \Box\mathcal{I}(G), \Box(\bigwedge \Gamma \supset A), \Box(\bigwedge \Gamma' \supset A')$ the claim follows by (*cut*) with axiom ($G7$).

Theorem 4. $\vdash_{\mathbf{HGIF}} P$ *if and only if* P *is valid in* **GIF**.

Proof. ($\Longrightarrow$) Similar to the proof of Theorem 1, using Lemma 7.

($\Longleftarrow$) By Proposition 2 we have to show that (*Lin*), ($\vee\forall$) and $(G1)-(G4), (G5')-(G7)$ are derivable in **HGIF**. See the proof of Theorem 1 for the derivations in **HGI** (and therefore in **HGIF**) of $(G1)-(G4)$. Derivations in **HIF** (and therefore in **HGIF**) of (*Lin*) and ($\vee\forall$) can be found in [8,7]. The derivation of $(G5)'$ is straightforward. That of ($G6$) uses (com).

Using **HGIF** one can define an alternative axiomatization for **GIF**. Indeed

Corollary 2. *An alternative deductive system for* **GIF** *is obtained by adding* ($G7$) *to any deductive system for* **GI**.

Proof. Follows by Lemma 7 and Proposition 1.

Note that in the above axiomatization for **GIF**, ($\vee\forall$) turns out to be redundant.

In fact, part ($\Longrightarrow$) of Theorem 4 can be proved in a more direct way. This proof leads to a different deductive system for **GIF** that makes no use of Barcan's axiom ($G2$).

Alternative proof of part ($\Longrightarrow$) of Theorem 4

By induction on the length of the derivation of P. For each rule of **HGIF** we show that ($*$) whenever **scGIF** derives the generic interpretations of its premise(s), **scGIF** derives the generic interpretation of its conclusion too.

The claim then follows by Proposition 2.

For the cases involving axioms, internal structural rules, (*ew*), (*ec*) and logical rules see the analogous proof for the propositional fragment of **HIL** and propositional **IL** outlined, e.g., in [7].

The claim is easily true in the case of the rules $(\forall, l)$, $(\exists, r)$ and $(\Box, l)$.

(*com*): By induction hypothesis $\vdash_{\mathbf{scGIF}} \Rightarrow P_1$ and $\vdash_{\mathbf{scGIF}} \Rightarrow P_2$, where P_1 and P_2 abbreviate $\mathcal{I}(G) \vee (\bigwedge \Gamma \wedge \bigwedge \Gamma' \supset A)$ and $\mathcal{I}(G) \vee (\bigwedge \Gamma \wedge \bigwedge \Gamma' \supset A')$, respectively. Since $\Rightarrow P_1, \Rightarrow P_2 \vdash_{\mathbf{LJ}} (\bigwedge \Gamma \supset \bigwedge \Gamma') \vee (\bigwedge \Gamma' \supset \bigwedge \Gamma) \Rightarrow \mathcal{I}(G) \vee (\bigwedge \Gamma \supset A) \vee (\bigwedge \Gamma' \supset A')$ the claim follows by cut with (Lin).

$(\forall, r)$: By induction hypothesis $\vdash_{\mathbf{scGIF}} \Rightarrow \mathcal{I}(G) \vee (\bigwedge \Gamma \supset A(a))$. We have $\Rightarrow \mathcal{I}(G) \vee (\bigwedge \Gamma \supset A(a)) \vdash_{\mathbf{LJ}} \Rightarrow \forall x(\mathcal{I}(G) \vee (\bigwedge \Gamma \supset A(x)))$. Since a occurs neither in G nor in $\bigwedge \Gamma, A$, we may assume that x occurs neither of them. Therefore $\Rightarrow \mathcal{I}(G) \vee (\bigwedge \Gamma \supset A(a)) \vdash_{\mathbf{LJ}+(\vee\forall)} \Rightarrow \mathcal{I}(G) \vee \forall x(\bigwedge \Gamma \supset A(x))$. The desired result follows since $\forall x(\bigwedge \Gamma \supset A(x)) \supset (\bigwedge \Gamma \supset \forall x A(x))$ is derivable in **LJ**.

$(\exists, l)$: the proof is similar.

$(\Box, r)$: First notice that $\vdash_{\mathbf{LJ}+(G1)+(G4)+(G7)} \Box(A \vee B) \Rightarrow \Box A \vee \Box B$. By induction hypothesis $\vdash_{\mathbf{scGIF}} \Rightarrow \mathcal{I}(G) \vee (\bigwedge \Box\Gamma \supset A)$. Since one has $\Rightarrow \mathcal{I}(G) \vee (\bigwedge \Box\Gamma \supset A) \vdash_{\mathbf{LJ}+(G1)+(G4)+(G7)} \Rightarrow \Box\mathcal{I}(G) \vee \Box(\bigwedge \Box\Gamma \supset A)$ and $\vdash_{\mathbf{LJ}+(G1)+(G4)} \Box(\bigwedge \Box\Gamma \supset A) \Rightarrow \bigwedge \Box\Gamma \supset \Box A$, the claim follows applying (G1), (G4) and the rules of **LJ**.

$(cl_\Box)$: By induction hypothesis $\vdash_{\mathbf{scGIF}} \Rightarrow \mathcal{I}(G) \vee \bigwedge \Box\Gamma \wedge (\bigwedge \Gamma' \supset A)$. Since $\vdash_{\mathbf{LJ}+(G1)+(G4)} \bigwedge \Box\Gamma \wedge (\bigwedge \Gamma' \supset A), \bigwedge \Box\Gamma \vee \neg \bigwedge \Box\Gamma \Rightarrow \neg \bigwedge \Box\Gamma \vee (\bigwedge \Gamma' \supset A)$ and $\vdash_{\mathbf{LJ}+(Lin)+(G1)+(G4)+(G5')} \Rightarrow \bigwedge \Box\Gamma \vee \neg \bigwedge \Box\Gamma$ the claim follows applying the axioms and rules of $\mathbf{LJ} + \{(G1), (G4)\}$.

Remark 6. The claim $(*)$ does not hold for **HGI** (and **GI**), as $(\vee\forall)$ is used essentially for the case $(\forall, r)$.

Corollary 3. *An alternative deductive system for* **GIF** *is obtained by adding* $(G1), (G4), (G5')$ *and* $(G7)$ *to Takano's system for* **IF**

Proof. Follows by carefully inspecting the above proof.

In the deductive system of Corollary 3, axioms (G7) and $(G5')$ can be replaced by $(\Box 4)$ $\Box(A \vee B) \supset \Box A \vee \Box B$ and $(G5)$, respectively. (Note that $\vdash_{\mathbf{LJ}+(G4)+(Lin)+(\Box 4)} (G7)$ and $\vdash_{\mathbf{LJ}+(G1)+(G4)+(G5)} (G5')$). Moreover, the rule (G4) can be substituted by the necessitation rule

$$\frac{A}{\Box A}\ (\Box)$$

together with axioms $(\Box 2)$ $\Box A \supset \Box\Box A$ and $(\Box 3)$ $\Box(A \supset B) \supset (\Box A \supset \Box B)$. This leads to the deductive system for **GIF** of [13], obtained by adding to that of **IF** the axioms and rules used by Baaz in [4] (see Remark 3), namely: (G5) as well as the axioms and rules for $\Box$ in modal logic **S4**, i.e., (G1), $(\Box 2)$, $(\Box 3)$ and $(\Box)$ together with axiom $(\Box 4)$.

Theorem 5 (Cut-elimination). *If a hypersequent H is derivable in* **HGIF** *then H is derivable in* **HGIF** *without using the cut rule.*

Proof. It is enough to check that the (com) rule does not spoil the cut elimination procedure outlined in Section 2.3. We therefore have to check that the logical rules remain invertible in **HGIF** and that the Substitution Lemma holds. See the cut-elimination proof in [5] for the rules of **HIF**. Cases involving the rules for $\Box$ are straightforward.

Lemma 8 [7]. *The generalized rules* $(\vee, l)^*$ *and* $(\exists, l)^*$ *of Section 2.3 can be derived in* **HGIF**.

Proof. $(\vee, l)^*$: For $n = 1$, the claim follows applying $(\vee, l)$. Otherwise, using only (ec), (iw, l) and (com) one can derive $G \mid A, \Gamma_1 \Rightarrow C_1 \mid B, \Gamma_2 \Rightarrow C_2 \mid \ldots \mid B, \Gamma_n \Rightarrow C_n$ from the premises of $(\vee, l)$. Hence by applying $(\vee, l)$ one obtains (a) $G \mid A \vee B, \Gamma_1 \Rightarrow C_1 \mid B, \Gamma_2 \Rightarrow C_2 \mid \ldots \mid B, \Gamma_n \Rightarrow C_n$. Similarly one can derive (b) $G \mid A \vee B, \Gamma_1 \Rightarrow C_1 \mid A, \Gamma_2 \Rightarrow C_2 \mid \ldots \mid A, \Gamma_n \Rightarrow C_n$. The desired result follows by iteratively applying the above argument to (a) and (b). The proof of $(\exists, l)^*$ is similar.

Remark 7. An alternative proof – in Gentzen style – of the cut-elimination theorem for **HGIF** can be formulated using (the following generalization of) Avron's extended multi-cut rule (see Section 2.3): if both the hypersequents $H \mid \Gamma_1 \Rightarrow A \mid \ldots \mid \Gamma_n \Rightarrow A$ and $H \mid \Sigma_1, A^{n_1} \Rightarrow B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \Rightarrow B_k$ are cut-free provable in **HGIF**, so is

- $H \mid \Sigma_1, \Gamma \Rightarrow B_1 \mid \ldots \mid \Sigma_k, \Gamma \Rightarrow B_k$, where Γ is $\Gamma_1, \ldots, \Gamma_n$, if A is not a $\Box$-formula
- $H \mid \Sigma_1 \Rightarrow B_1 \mid \ldots \mid \Sigma_k \Rightarrow B_k \mid \Gamma_1 \Rightarrow \mid \ldots \mid \Gamma_n \Rightarrow$, otherwise.

The above formulation was used in [7] to prove cut elimination in the hypersequent calculus for *propositional* **GIF**. It is easy to check that it works also in presence of quantifiers rule (i.e., for **HGIF**).

3.3. Consequences of cut-elimination

As shown in [8], in analogy with classical logic, the mid(hyper)sequent theorem holds for the prenex fragment[3] of **IF**. This result can be easily extended to **GIF**.

Theorem 6. *Any* **HGIF***-derivation d of a hypersequent H only containing prenex formulas can be transformed into a derivation of H in which no propositional rule is applied below any application of a quantifier rule.*

Proof. Is similar to the same proof for **HIF** (see [8]). First observe that in **HGIF** non-atomic axioms are cut-free derivable from atomic axioms. Next note that $(\vee, l)$ can be simulated using (com) by the following one

$$\frac{G \mid A, \Gamma \Rightarrow C_1 \quad G \mid B, \Gamma \Rightarrow C_2}{G \mid A \vee B, \Gamma \Rightarrow C_1 \mid A \vee B, \Gamma \Rightarrow C_2}\ (\vee', l)$$

This rule can be derived in **GIF** using $(iw, l), (ew), (com)$ and $(\vee, l)$. $(\vee', l)$ can be shifted upward any quantifier rule as in the case of $(\vee^{\Box}, l)$ (see the proof of Theorem 3).

Corollary 4. *Herbrand's theorem holds for the prenex fragment of* **GIF**.

See [6] for a *semantical proof* of the above corollary.

[3] **IF** does not admit equivalent prenex normal forms as in classical logic.

Corollary 5. *Let P be a quantifier-free formula of* **GIF** *and* $\mathsf{Q}_i \in \{\forall, \exists\}$ *for each* $i = 1, \ldots, n$*: If*

$$\models_{\mathbf{GIF}} \Box \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n) \text{ then } \models_{\mathbf{GIF}} \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n \Box P(y_1, \ldots, y_n)$$

Proof. Let d be the derivation of $\Rightarrow \mathsf{Q}_1 y_1 \ldots \mathsf{Q}_n y_n P(y_1, \ldots, y_n)$ obtained by Theorem 6. The required derivation d' is simply constructed by applying, in d, the rule $(\Box, r)$ to mid(hyper)sequent(s) (i.e. the hypersequent(s) in d separating propositional and quantifier inferences). The claim follows by Theorem 4.

Remark 8. If P is not a quantifier-free formula, the above theorem does not hold. Indeed, let P be the formula $(\Box\exists y A(y)) \supset A(x)$. $\exists x \Box P$ is not valid in **GIF**. For, assume otherwise that $\models_{\mathbf{GIF}} \exists x \Box P$. Then $\exists x((\Box\exists y A(y)) \supset \Box A(x))$ would be valid[4] in **GIF** as shown by:

$$\begin{array}{l}
\models_{\mathbf{GIF}} \exists x \Box((\Box\exists y A(y)) \supset A(x)) \\
\models_{\mathbf{GIF}} \exists x((\Box\Box\exists y A(y)) \supset \Box A(x)) \\
\models_{\mathbf{GIF}} \exists x((\Box\exists y A(y)) \supset \Box A(x))
\end{array}$$

while $\models_{\mathbf{GIF}} \Box\exists x P$ as follows by Theorem 4 and the **HGIF**-derivation:

$$\begin{array}{cl}
A(b) \Rightarrow A(b) & \\ \hline
A(b), \Box\exists y A(y) \Rightarrow A(b) & (\mathrm{iw},\mathrm{l}) \\ \hline
A(b) \Rightarrow (\Box\exists y A(y)) \supset A(b)) & (\supset,\mathrm{r}) \\ \hline
A(b) \Rightarrow \exists x((\Box\exists y A(y)) \supset A(x)) & (\exists,\mathrm{r}) \\ \hline
\exists y A(y) \Rightarrow \exists x((\Box\exists y A(y)) \supset A(x)) & (\exists,\mathrm{l}) \\ \hline
\Box\exists y A(y) \Rightarrow \exists x((\Box\exists y A(y)) \supset A(x)) & (\Box,\mathrm{l}) \\ \hline
\Box\exists y A(y) \Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) & (\Box,\mathrm{r}) \\ \hline
\Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) \mid \Box\exists y A(y) \Rightarrow & (\mathrm{cl}_\Box,\mathrm{l}) \\ \hline
\Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) \mid \Box\exists y A(y) \Rightarrow A(a) & (\mathrm{iw},\mathrm{r}) \\ \hline
\Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) \mid\ \Rightarrow (\Box\exists y A(y) \supset A(a)) & (\supset,\mathrm{r}) \\ \hline
\Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) \mid\ \Rightarrow \exists x((\Box\exists y A(y)) \supset A(x)) & (\exists,\mathrm{r}) \\ \hline
\Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) \mid\ \Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) & (\Box,\mathrm{r}) \\ \hline
\Rightarrow \Box\exists x((\Box\exists y A(y)) \supset A(x)) & (\mathrm{ec})
\end{array}$$

As mentioned before, Takeuti and Titani used the density rule (TT) to axiomatize **IF**. Takano [17] has later shown *semantically* that this rule is

[4] a counterexample can be constructed considering an interpretation v_D, in the $[0,1]$-Heyting algebra with $\Box$, s.t. $\bigvee_{p \in D} A(p) = 1$ while for each $p \in D$, $A(p) < 1$.

not needed to characterize **IF** and he posed the question whether a *syntactical* elimination of (TT) is also possible. This question was answered affirmatively in [8] where the following version of (TT)

$$\frac{G \mid \Pi \Rightarrow p \mid p, \Gamma \Rightarrow C}{G \mid \Pi, \Gamma \Rightarrow C}\ (tt) \quad \text{where } p \text{ does not occur in the conclusion}$$

was shown to be eliminable from **HIF**-derivations. The same result holds for **GIF** as stated by the theorem below:

Theorem 7. *Any derivation d of a hypersequent H in* $\mathbf{HGIF} + (tt)$ *can be transformed into a derivation of H not containing applications of* (tt).

This follows by induction on the number of applications of (tt) using the following lemma.

Lemma 9. *If d is a* **HGIF**-*derivation of*

$$G \mid \Phi_1 \Rightarrow p \mid \ \ldots\ \mid \Phi_n \Rightarrow p \mid p^{k_1}, \Psi_1 \Rightarrow A_1 \mid \ \ldots\ \mid p^{k_m}, \Psi_m \Rightarrow A_m$$

where p is a propositional variable not occurring in G, Φ_i, $\Psi_j i$ and A_j (for $i = 1, \ldots, n$ and $j = 1, \ldots, m$), then there is a **HGIF**-*derivation of*

$$G \mid \Phi_1, \ldots, \Phi_n, \Psi_1 \Rightarrow A_1 \mid \ \ldots\ \mid \Phi_1, \ldots, \Phi_n, \Psi_m \Rightarrow A_m$$

Proof. Proceeds by induction on the length of d. By Theorem 5 we can assume that d is cut-free. We divide cases according to the last inference I in d.

(1) I applies to sequents in G. Then the induction hypothesis can be applied to the premise(s) of I and appropriate inferences added below.

(2) I is a logical inference, a quantifier inference, $(\Box, l)$ or $(\Box, r)$.

(2.1) I has only one premise and $I \neq (\exists, l)$. Then the induction hypothesis can be applied to this premise and the desired hypersequent can be derived by applying I. Note that the case $I = (\Box, r)$ can only apply to sequents in G.

(2.2) I has two premises or I is $(\exists, l)$. If I is a right rule (i.e. I is applied to A_j) the case is straightforward. As an example of a left rule, consider the case in which I is $(\supset, l)$ and its premises are, say,

$$G \mid \Phi_1 \Rightarrow p \mid \ldots \mid \Phi_n \Rightarrow p \mid p^{k_1}, \Psi_1 \Rightarrow A_1 \mid \ldots \mid p^{k_m}, \Psi_m \Rightarrow A_m \mid \Gamma_1 \Rightarrow A \quad \text{and}$$

$$G \mid \Phi_1 \Rightarrow p \mid \ldots \mid \Phi_n \Rightarrow p \mid p^{k_1}, \Psi_1 \Rightarrow A_1 \mid \ldots \mid p^{k_m}, \Psi_m \Rightarrow A_m \mid B, \Gamma_2 \Rightarrow p.$$

Let $\Phi = \Phi_1, \ldots, \Phi_n$. The induction hypothesis provides us with

$$G \mid \Phi, \Psi_1 \Rightarrow A_1 \mid \ \ldots\ \mid \Phi, \Psi_m \Rightarrow A_m \mid \Gamma_1 \Rightarrow A \quad \text{and}$$

$$G \mid B, \Gamma_2, \Phi, \Psi_1 \Rightarrow A_1 \mid \ \ldots\ \mid B, \Gamma_2, \Phi, \Psi_m \Rightarrow A_m.$$

The desired hypersequent is obtained by applying $(\supset, l)$ successively m times, together with some (w, l), (ew) and (ec).

I is $(\vee, l)$. Suppose e.g. its premises are

$$G \mid A, \Phi_1 \Rightarrow p \mid \ \ldots \ \mid \Phi_n \Rightarrow p \mid p^{k_1}, \Psi_1 \Rightarrow A_1 \mid \ \ldots \ \mid p^{k_m}, \Psi_m \Rightarrow A_m \quad \text{and}$$

$$G \mid B, \Phi_1 \Rightarrow p \mid \ \ldots \ \mid \Phi_n \Rightarrow p \mid p^{k_1}, \Psi_1 \Rightarrow A_1 \mid \ \ldots \ \mid p^{k_m}, \Psi_m \Rightarrow A_m.$$

Let $\Phi = \Phi_1, \ldots, \Phi_n$. By induction hypothesis we obtain

$$G \mid A, \Phi, \Psi_1 \Rightarrow A_1 \mid \ \ldots \ \mid A, \Phi, \Psi_m \Rightarrow A_m \quad \text{and}$$

$$G \mid B, \Phi, \Psi_1 \Rightarrow A_1 \mid \ \ldots \ \mid B, \Phi, \Psi_m \Rightarrow A_m.$$

The desired hypersequent follows by Lemma 8. Case $I = (\exists, l)$ is treated analogously.

(3) I is a structural inference. If I is other than (com) and $(cl_\Box)$ the proof is straightforward. The case in which I is (com) can be handled as in [8].

I is $(cl_\Box)$. Suppose e.g. its premise is

$$G \mid \Phi_1, \Box\Gamma \Rightarrow p \mid \ \ldots \ \mid \Phi_n \Rightarrow p \mid p^{k_1}, \Psi_1 \Rightarrow A_1 \mid \ \ldots \ \mid p^{k_m}, \Psi_m \Rightarrow A_m$$

Note that in the conclusion of I, $\Box\Gamma$ can only belong to a component $\Box\Gamma \Rightarrow$ in G. Let $\Phi = \Phi_1, \ldots, \Phi_n$. By induction hypothesis we have

$$G \mid \Phi, \Box\Gamma, \Psi_1 \Rightarrow A_1 \mid \ \ldots \ \mid \Phi, \Box\Gamma, \Psi_m \Rightarrow A_m.$$

The desired hypersequent is obtained by applying $(cl_\Box)$ and (ec) m and $m-1$-times respectively.

Remark 9. In the proof of the above lemma for **HIF** outlined in [8], cases $I = (\vee, l)$ and $I = (\exists, l)$ were handled introducing several new cuts. Therefore Baaz and Zach's elimination procedure of the rule (tt) does not work directly for cut-free proofs.

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