Algebraic proof theory for substructural logics: cut-elimination and completions

Agata Ciabattoni[a], Nikolaos Galatos[b], Kazushige Terui[*,c]

[a]*Department of Formal Languages, Vienna University of Technology, Favoritenstrasse 9-11, 1040 Wien, Austria*
[b]*Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208, USA*
[c]*Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*

Abstract

We carry out a unified investigation of two prominent topics in proof theory and order algebra: cut-elimination and completion, in the setting of substructural logics and residuated lattices.

We introduce the *substructural hierarchy* – a new classification of logical axioms (algebraic equations) over full Lambek calculus **FL**, and show that a stronger form of cut-elimination for extensions of **FL** and the MacNeille completion for subvarieties of pointed residuated lattices coincide up to the level $\mathcal{N}_2$ in the hierarchy. Negative results, which indicate limitations of cut-elimination and the MacNeille completion, as well as of the expressive power of structural sequent calculus rules, are also provided.

Our arguments interweave proof theory and algebra, leading to an integrated discipline which we call *algebraic proof theory*.

Key words: Substructural logic, Gentzen system, residuated lattices, cut-elimination, structural rule, MacNeille completions
2000 MSC: 03B47, 06F05, 03G10, 08B15

1. Introduction

The algebraic and proof-theoretic approaches to logic have traditionally developed in parallel, non-intersecting ways. This paper is part of a project to identify the connections between these two areas and apply methods and techniques from each field to the other in the setting of substructural logics. The emerging discipline may be named *algebraic proof theory*. The main contribution of the paper is to reveal the connection between (a stronger form of)

*Corresponding author. Partly supported by JSPS KAKENHI 21700041.
Email addresses: agata@logic.at (Agata Ciabattoni), ngalatos@du.edu (Nikolaos Galatos), terui@kurims.kyoto-u.ac.jp (Kazushige Terui)

cut-elimination for sequent calculi and the MacNeille completion for the corresponding algebraic models, established by interweaving proof theoretic and algebraic arguments.

Sequent calculi have played a central role in proof theory (see, e.g., [43], [9], [35]). *Strongly analytic* sequent calculi – that is calculi in which proofs from atomic assumptions only consist of formulas already contained in the statement to be proved – are useful for establishing various properties. These include consistency, conservativity and interpolation. Analyticity, as well as its strengthened version referring to derivations from atomic assumptions, mainly follow from the fundamental theorem of *cut-elimination* which states the redundancy of the cut rule. Sequent calculi have been proposed for various logics. Here we are interested in *substructural logics* (see, e.g., [20, 37]), i.e., logics which may invalidate some of the structural rules. They encompass among many others classical, intuitionistic, intermediate, fuzzy, linear and relevant logics. In general, a substructural logic is any axiomatic extension of *full Lambek calculus* **FL**, a calculus equivalent to Gentzen's sequent calculus **LJ** for intuitionistic logic without structural rules. In this setting, additional properties are often imposed on **FL** by means of axioms or *structural rules*. As cut-elimination is not preserved in general under the addition of axioms, the following question is of great importance:

> *Given an axiom, is it possible to transform it into a "good" structural rule— i.e. one which preserves strong analyticity when added to* **FL***?*

Substructural logics correspond to subvarieties of (pointed) *residuated lattices* (see, e.g., [26]), via a Tarski-Lindenbaum construction. The strong correspondence between them (known as *algebraization*), together with rich tools from universal algebra, has allowed for a fruitful algebraic study of substructural logics (see [20]). An important technique here is *completion*, that is to embed a given ordered algebraic structure into a complete one. Here we are interested in a particular completion method known as the *(Dedekind-)MacNeille completion*, which generalizes Dedekind's embedding of the rational numbers into the reals [29]. It admits a nice abstract characterization due to [5, 38]. Moreover, it preserves all existing joins and meets, hence is useful for proving completeness of predicate logics with respect to complete algebras, see [34]. Although the MacNeille completion applies to all individual residuated lattices, it may produce a residuated lattice that is not in a given variety, containing the original one. Hence an important question here is:

> *Given a variety of pointed residuated lattices, is it closed under MacNeille completions? Or equivalently, given an equation over residuated lattices, is it preserved by MacNeille completions?*

The two questions, above raised in different contexts, are in fact deeply related. The connection can be naively understood by noticing that both are concerned with some conservativity properties (cf. Lemmas 5.13 and 5.19). However, to establish the exact correspondence between strong analyticity and the

MacNeille completion and to demonstrate their limitations, it seems that it is not enough to merely combine *results* of algebra and proof theory; it is necessary to integrate *techniques* from each discipline in a more intimate and systematic way.

The emerging theory, called algebraic proof theory, consists of two basic ideas:

1. Proof theoretic treatment of algebraic equations,
2. Algebraization of proof theoretic methods.

1. Proof theoretic treatment of algebraic equations. An important idea stemming from proof theory is to classify logical formulas into a hierarchy according to their syntactic complexity, i.e., how difficult they are to deal with. The most prominent example is the arithmetical hierarchy in Peano arithmetic. Inspired by the latter and the notion of *polarity* coming from proof theory of linear logic [1], we introduce a hierarchy $(\mathcal{N}_n, \mathcal{P}_n)$ on equations, called *substructural hierarchy* (Section 3.1).

Another prominent feature of our proof-theoretic approach is a special emphasis on quasiequations. Most of the algebraic contributions to our field have focused on equational classes. However, even when the class of algebraic models is defined by equations, a reformulation of the latter into equivalent quasiequations can be useful. This becomes apparent in view of the connection to proof theory, where a transformation of axioms (equations) into suitable structural rules (quasiequations) is essential for cut-elimination. Remarkably, such a transformation is also a key step when proving preservation under MacNeille completions.

We describe a procedure, which applies to axioms/equations at a low level in the substructural hierarchy (up to the class $\mathcal{N}_2$) and transforms them into equivalent structural rules/quasiequations (Section 3). We also present a procedure for transforming the generated rules/quasiequations into '*analytic*' ones which behave well with respect to both strong analyticity and the MacNeille completion (Section 4). The latter procedure applies to any '*acyclic*' structural rule/quasiequation, or to any structural rule/quasiequation in presence of the weakening rule (integrality). These two procedures together allow the introduction of strongly analytic sequent calculi for all logics semantically characterized by (acyclic) $\mathcal{N}_2$-equations over residuated lattices. These calculi are uniform and their introduction is algorithmic.

2. Algebraization of proof theoretic methods. Syntactic proofs of cut-elimination are often cumbersome and not modular in the sense that each time a new rule is added to a sequent calculus cut-elimination has to be reproved from the outset. More importantly, syntactic proofs are available only for predicative systems, and not for second order logics with the full comprehension axiom. These situations have motivated the investigation of semantic proofs for cut-elimination (e.g., [39, 32, 33, 31, 6, 22, 19]) even though one loses concrete algorithms to eliminate cuts from a given proof, and so the claim should be more precisely called *cut admissibility*.

As observed in [6], the algebraic essence of cut-elimination lies in the construction of a *quasihomomorphism* from an intransitive structure $\mathbf{W}$ (called Gentzen structure) to a complete (and transitive) algebra $\mathbf{W}^+$:

$$\mathbf{W} \quad \overset{\text{quasihom.}}{\longrightarrow} \quad \mathbf{W}^+.$$

The intransitive structure corresponds to a cut-free system, as the cut rule corresponds to transitivity of the algebraic inequation $\leq$. If the original structure $\mathbf{W}$ is already transitive, the construction above is nothing but the MacNeille completion. Thus cut-elimination and completion are of the same nature, and the common essence is well captured in terms of *residuated frames*, which abstract both residuated lattices and sequent calculi for substructural logics [19].

We contribute to the algebraization of proof theory by showing that analytic structural rules/quasiequations are preserved by the above construction. Similar arguments have already appeared in [12, 11], but here the use of residuated frames allows us to give a *unified* proof of the two facts that (i) analytic rules preserve a strong form of cut-elimination (strong analyticity) and (ii) analytic quasiequations are preserved by MacNeille completions (Section 5).

Both strong analyticity and closure under completions imply some conservativity properties with respect to extensions with infinitary formulas. A proof theoretic argument shows that conservativity in turn implies that the involved structural rules/quasiequations are equivalent to analytic ones (Section 6). This leads to the equivalence of statements (1)-(3) below for any set R of $\mathcal{N}_2$-equations/axioms or structural rules/quasiequations:

1. R is equivalent to a set of analytic structural rules which preserve strong analyticity when added to (any infinitary extension of) $\mathbf{FL}$.
2. The class of FL-algebras satisfying R is closed under MacNeille completions.
3. Every infinitary extension of $\mathbf{FL}+R$ is a conservative extension of $\mathbf{FL}+R$.

An example of an equation/axiom in $\mathcal{N}_2$ which does not satisfy any of (1)-(3) is also presented. This indicates the limitations of strong analyticity and MacNeille completions within $\mathcal{N}_2$. Our results also shed light on the expressive power of structural sequent rules, which is discussed in Section7.

Related work. Syntactic and semantic conditions for a sequent calculus to admit (a stronger form of) cut elimination are contained, e.g., in [41, 14, 4, 3]. While these works focus on *calculi*, our current project focuses on *logics* (defined by axioms), and investigates under which conditions they admit a strongly analytic sequent calculus.

Also, the substructural hierarchy and the transformations of axioms into structural rules were introduced in [12] for the commutative case and in [13] for the commutative and involutive case. While these two papers are proof theoretic, [11] makes use of their ideas for purely algebraic purposes. The current paper unifies both directions.

Preservation of equations under completions is an old and mature topic, see e.g. the survey [24]. Among many works, paper [42] investigates MacNeille completions of arbitrary lattice expansions (which include FL-algebras). The methodology in [42] provides a topological perspective on equations preserved by MacNeille completions, that is complementary to our proof theoretic perspective.

Closely related to MacNeille completions are *canonical extensions* [27, 28] (recall a deep result in [17]: preservation under MacNeille completions implies preservation under canonical extensions for arbitrary monotone bounded lattice expansions, which include bounded FL-algebras). Canonical extensions of FL-algebras are studied in [40]. Following some previous works, the paper identifies a class of equations preserved by canonical extensions by means of a tree labeling algorithm, that is complementary to our method. Finally, following [15], [16] contains a (quasi)equation-transformation procedure which is based on the so-called Ackermann's lemma, as in the case of our transformation procedure (cf. Lemma 3.4).

2. Preliminaries

2.1. Full Lambek calculus and substructural logics

We start by recalling our base calculus: the sequent system **FL**. The *formulas* of **FL** are built from propositional variables $p, q, r, \ldots$ and constants 1 (unit) and 0 by using binary logical connectives $\cdot$ (fusion), $\backslash$ (right implication), $/$ (left implication), $\wedge$ (conjunction) and $\vee$ (disjunction). **FL** *sequents* are expressions of the form $\Gamma \Rightarrow \Pi$, where the left-hand-side (LHS) Γ is a finite (possibly empty) sequence of formulas of **FL** and the right-hand-side (RHS) Π is single-conclusion, i.e., it is either a formula or the empty sequence. The sequent calculus rules of **FL** are displayed in Figure 1. Letters α, β stand for formulas, Π stands for either a formula or the empty set, and $\Gamma, \Delta, \ldots$ stand for finite (possibly empty) sequences of formulas. $\neg\alpha$ and $\alpha \leftrightarrow \beta$ will be used as abbreviations for $\alpha\backslash 0$ and $(\alpha\backslash\beta) \wedge (\beta\backslash\alpha)$ respectively, while α^n and $\alpha^{(n)}$ for the formula $\alpha \cdot \ldots \cdot \alpha$ and the sequence $\alpha, \ldots, \alpha$ (n times), respectively.

Roughly speaking, **FL** is obtained by dropping all the structural rules (exchange (e), contraction (c), left weakening (i) and right weakening (o); see Figure 2), from the sequent calculus **LJ** for intuitionistic logic. Also, **FL** (together with $\top$ and $\bot$ below) is the same as noncommutative intuitionistic linear logic without exponentials.

Remark 2.1. Often, the constants $\top$ (true) and $\bot$ (false) and the rules

$$\frac{}{\Gamma \Rightarrow \top}\ \top r \qquad\qquad \frac{}{\Gamma_1, \bot, \Gamma_2 \Rightarrow \Pi}\ \bot l$$

are added to the language and rules of **FL**, respectively; the resulting sequent calculus is denoted by $\mathbf{FL}_\bot$. The results in our paper hold for both **FL** and $\mathbf{FL}_\bot$.

The notion of proof in **FL** (and in the mentioned extensions) is defined as usual. If there is a proof in **FL** of a sequent s from a set of sequents S, we write $S \vdash^{seq}_{\mathbf{FL}} s$. If $\Phi \cup \{\psi\}$ is a set of formulas, we write $\Phi \vdash_{\mathbf{FL}} \psi$, if $\{ \Rightarrow \phi : \phi \in \Phi\} \vdash^{seq}_{\mathbf{FL}} \Rightarrow \psi$. Clearly, $\vdash^{seq}_{\mathbf{FL}}$ and $\vdash_{\mathbf{FL}}$ are consequence relations on the sets of sequents and formulas, respectively. When no confusion arises, we will omit the superscript and write simply $\vdash_{\mathbf{FL}}$ for $\vdash^{seq}_{\mathbf{FL}}$.

The calculus **FL** serves as the main system for defining substructural logics, the latter being simply (the sentential part of) axiomatic extensions of **FL**. A *substructural logic* is simply a set of formulas closed under $\vdash_{\mathbf{FL}}$ and substitution.

2.2. Polarities

Following [1], the logical connectives of $\mathbf{FL}_\perp$ are classified into two groups: connectives $1, \perp, \cdot, \vee$ (resp. $0, \top, \backslash, /, \wedge$), for which the left (resp. right) logical rule is invertible, are said to have *positive* (resp. *negative*) *polarity.* Here a rule is *invertible* if the conclusion implies the premises. E.g., for $(\vee l)$ (cf. Figure 1) we have:

$$\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Pi \dashv\vdash_{\mathbf{FL}_\perp} \{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Pi,\ \ \Gamma_1, \beta, \Gamma_2 \Rightarrow \Pi\}$$

Connectives of the same polarity interact well with each other. Indeed, for positive connectives,

$$\alpha \cdot 1 \leftrightarrow \alpha, \qquad \alpha \vee \perp \leftrightarrow \alpha, \qquad \alpha \cdot \perp \leftrightarrow \perp, \qquad \alpha \cdot (\beta \vee \gamma) \leftrightarrow (\alpha \cdot \beta) \vee (\alpha \cdot \gamma)$$

are provable in $\mathbf{FL}_\perp$, while for negative connectives, we have:

$$\alpha \wedge \top \leftrightarrow \alpha, \qquad (1 \to \alpha) \leftrightarrow \alpha, \qquad (\alpha \to \top) \leftrightarrow \top, \qquad (\perp \to \alpha) \leftrightarrow \top,$$
$$(\alpha \to (\beta \wedge \gamma)) \leftrightarrow (\alpha \to \beta) \wedge (\alpha \to \gamma), \qquad ((\alpha \vee \beta) \to \gamma) \leftrightarrow (\alpha \to \gamma) \wedge (\beta \to \gamma),$$

where $\alpha \to \beta$ stands for either $\alpha\backslash\beta$ and β/α, uniformly in each formula.

We stipulate that polarity is reversed on the left hand side of implications. For instance, the $\vee$ on the left-hand side of $\to$ in the last equivalence is considered negative.

Since connectives $\vee, \wedge, \cdot$ have units $\perp, \top, 1$ respectively, we will adopt a natural convention: $\beta_1 \vee \cdots \vee \beta_m$ (resp. $\beta_1 \wedge \cdots \wedge \beta_m$ and $\beta_1 \cdots \beta_m$) stands for $\perp$ (resp. $\top$ and 1) if $m = 0$.

2.3. Structural rules

Structural rules are described by using three types of *metavariables*:

- metavariables for formulas: $\alpha, \beta, \gamma, \ldots$
- metavariables for sequences of formulas: $\Gamma, \Delta, \Sigma, \ldots$
- metavariables for *stoups* (i.e., for either the empty set or a formula): Π.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \alpha, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \Pi}\ (\mathbf{cut}) \qquad \frac{}{\alpha \Rightarrow \alpha}\ (\mathbf{init}) \qquad \frac{}{\Rightarrow 1}\ (1r)$$

$$\frac{\Gamma_1, \alpha, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \cdot \beta, \Gamma_2 \Rightarrow \Pi}\ (\cdot l) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}\ (\cdot r) \qquad \frac{\Gamma_1, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Pi}\ (1l)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \Pi}{\Delta_1, \Gamma, \alpha \backslash \beta, \Delta_2 \Rightarrow \Pi}\ (\backslash l) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \backslash \beta}\ (\backslash r) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0l)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta_1, \beta, \Delta_2 \Rightarrow \Pi}{\Delta_1, \beta / \alpha, \Gamma, \Delta_2 \Rightarrow \Pi}\ (/l) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta / \alpha}\ (/r) \qquad \frac{}{0 \Rightarrow}\ (0r)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Pi \quad \Gamma_1, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \Rightarrow \Pi}\ (\vee l) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee r_1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\vee r_2)$$

$$\frac{\Gamma_1, \alpha, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Pi}\ (\wedge l_1) \qquad \frac{\Gamma_1, \beta, \Gamma_2 \Rightarrow \Pi}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \Rightarrow \Pi}\ (\wedge l_2) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r)$$

Figure 1: Inference Rules of **FL**

Some examples of structural rules are displayed in Figure 2. An *instance* of the contraction rule (c) is for example

$$\frac{p \wedge q, 0,\ r \vee 1, r \vee 1,\ p/q \Rightarrow}{p \wedge q, 0,\ r \vee 1,\ p/q \Rightarrow}$$

which is obtained by instantiating Γ by the sequence $p \wedge q, 0$ of concrete formulas, α by the concrete formula $r \vee 1$, Δ by p/q, and Π by the empty set. Therefore, (c) represents (or specializes to) many rules, so formally it should be called a metarule. In practice, the distinction between metarules and rules is understood implicitly and both are refereed to as rules.

Note that the following is not an instance of (c)

$$\frac{p \wedge q, 0,\ r \vee 1, s, r \vee 1, s,\ p/q \Rightarrow}{p \wedge q, 0,\ r \vee 1, s,\ p/q \Rightarrow}$$

but is an instance of (*seq-c*) with instantiation of Σ by the concrete sequence $r \vee 1, s$. Hence (c) and (*seq-c*) are different rules, even though they have the same strength. Similar distinctions may be observed on the right hand side of a sequent. It is instructive to think about the differences among

$$\frac{\Gamma \Rightarrow \beta}{\alpha, \Gamma \Rightarrow \beta}\ (w1) \qquad \frac{\Gamma \Rightarrow}{\alpha, \Gamma \Rightarrow}\ (w2) \qquad \frac{\Gamma \Rightarrow \Pi}{\alpha, \Gamma \Rightarrow \Pi}\ (w3)$$

$$\frac{\Gamma, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi}\ (i) \qquad \frac{\Sigma \Rightarrow}{\Sigma \Rightarrow \alpha}\ (o) \qquad \frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \Delta \Rightarrow \Pi}\ (c)$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Pi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \Pi}\ (e) \qquad \frac{\Gamma, \alpha, \Delta \Rightarrow \Pi}{\Gamma, \alpha, \alpha, \Delta \Rightarrow \Pi}\ (exp) \qquad \frac{\overbrace{\alpha, \dots, \alpha}^{m} \Rightarrow \beta}{\underbrace{\alpha, \dots, \alpha}_{n} \Rightarrow \beta}\ (knot^n_m)$$

$$\frac{\Gamma, \Sigma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (seq\text{-}c) \qquad \frac{\Sigma, \Sigma \Rightarrow}{\Sigma \Rightarrow}\ (wc) \qquad \frac{\Gamma, \Sigma_1, \Delta \Rightarrow \Pi \quad \Gamma, \Sigma_2, \Delta \Rightarrow \Pi}{\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi}\ (min)$$

$$\frac{\Sigma \Rightarrow \quad \Gamma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (mix) \qquad \frac{\{\Gamma, \Sigma_{i_1}, \dots, \Sigma_{i_m}, \Delta \Rightarrow \Pi\}_{i_1, \dots, i_m \in \{1, \dots, n\}}}{\Gamma, \Sigma_1, \dots, \Sigma_n, \Delta \Rightarrow \Pi}\ (anl\text{-}knot^n_m)$$

Figure 2: Examples of Structural Rules

The rule $(w1)$ may be applied only when there is a formula on the RHS, while $(w2)$ only when the RHS is empty; $(w3)$ can be applied in both cases.

In general, a *single-conclusion structural rule* (*structural rule* for short) is any rule of the form $(n \geq 0)$

$$\frac{\Upsilon_1 \Rightarrow \Psi_1 \quad \cdots \quad \Upsilon_n \Rightarrow \Psi_n}{\Upsilon_0 \Rightarrow \Psi_0}\ (r)$$

where each Υ_i is a specific sequence of metavariables (allowed to be of both types: metavariables for formulas or for sequences of formulas), and each Ψ_i is either empty, a metavariable for formulas (α), or a metavariable for stoups (Π). $\Upsilon_i \Rightarrow \Psi_i$, with $i = 0, \dots, n$ are called *metasequents*.

Given a set R of structural rules, we denote by $\mathbf{FL}_R$ the system obtained by adding to $\mathbf{FL}$ the rules in R, and by $\vdash^{seq}_{\mathbf{FL}_R}$ the associated consequence relation (often simply written $\vdash_{\mathbf{FL}_R}$).

Two rules (r_0) and (r_1) are *equivalent* (in $\mathbf{FL}$) if the relations $\vdash_{\mathbf{FL}_{(r_0)}}$ and $\vdash_{\mathbf{FL}_{(r_1)}}$ coincide. This holds when the conclusion of (r_0) (and resp. of (r_1)) is derivable from its premises in $\mathbf{FL}_{(r_1)}$ (resp. $\mathbf{FL}_{(r_0)}$). The definition naturally extends to sets of rules.

2.4. Algebraic semantics

The system $\mathbf{FL}$ is algebraizable and its algebraic semantics is the class of pointed residuated lattices, also known as FL-algebras.

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$, such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and for all $a, b, c \in A$,

$$a \cdot b \leq c \text{ iff } b \leq a \backslash c \text{ iff } a \leq c/b.$$

We refer to the last property as *residuation*.

An *FL-algebra* is an expansion of a residuated lattice with an additional constant element 0, namely an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$, such that $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice. In residuated lattices and FL-algebras, we will write $a \leq b$ instead of $a = a \wedge b$ (or equivalently, $a \vee b = b$). Note that $a = b$ is equivalent to $1 \leq a\backslash b \wedge b\backslash a$.

The classes RL and FL of residuated lattices and FL-algebras, respectively, can be defined by equations. Consequently, they are *varieties*, namely classes of algebras closed under subalgebras, homomorphic images and direct products.

Given a class $\mathcal{K}$ of FL-algebras, we say that the equation $s = t$ is a *semantical consequence* of a set of equations E relative to $\mathcal{K}$, in symbols

$$E \models_{\mathcal{K}} s = t,$$

if for every algebra $\mathbf{A} \in \mathcal{K}$ and every valuation f into $\mathbf{A}$, if $f(u) = f(v)$, for all $(u = v) \in E$, then $f(s) = f(t)$. Clearly, $\models_{\mathcal{K}}$ is a consequence relation on the set of equations.

All three relations $\vdash^{seq}_{\mathbf{FL}}$, $\vdash_{\mathbf{FL}}$ and $\models_{\mathsf{FL}}$ are *equivalent*; see [21] and [20]. This is also known as the *algebraization* of $\mathbf{FL}$. Identifying terms of residuated lattices and propositional formulas of $\mathbf{FL}$, we can give translations between sequents, formulas and equations as follows. Given a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \alpha$, the corresponding equation and formula are $\alpha_1 \cdot \ldots \cdot \alpha_n \leq \alpha$ and $(\alpha_1 \cdot \ldots \cdot \alpha_n)\backslash\alpha$; for $\alpha_1, \ldots, \alpha_n \Rightarrow$ we have $\alpha_1 \cdot \ldots \cdot \alpha_n \leq 0$ and $(\alpha_1 \cdot \ldots \cdot \alpha_n)\backslash 0$. To a formula α, we associate $\Rightarrow \alpha$ and $1 \leq \alpha$.

In view of the algebraization, we have that for a set of sequents $S \cup \{s\}$,

$$S \vdash^{seq}_{\mathbf{FL}} s \text{ iff } \varepsilon[S] \models_{\mathsf{FL}} \varepsilon(s)$$

where $\varepsilon(s)$ is the equation corresponding to s.

Bounded FL-algebras are expansions of FL-algebras that happen to be bounded as lattices with two new constants interpreting the bounds ($\bot$, $\top$). The corresponding class $\mathsf{FL}_\bot$ of algebras is the equivalent algebraic semantics of $\mathbf{FL}_\bot$. The existence of bounds excludes interesting algebras, like lattice-ordered groups.

2.5. Interpretation of structural rules

To avoid confusion between the connectives of our language and the connectives of classical logic, we denote the latter by and and $\Longrightarrow$. Recall that a *quasiequation* is a strict universal Horn first-order formula of the form

$$\varepsilon_1 \ \mathsf{and} \ \ldots \ \mathsf{and} \ \varepsilon_n \Longrightarrow \varepsilon_0, \qquad (q)$$

where $\varepsilon_0, \ldots, \varepsilon_n$ are equations. $\varepsilon_1, \ldots, \varepsilon_n$ are the *premises* and ε_0 is the *conclusion.* An FL-algebra $\mathbf{A}$ *satisfies* (q) if $\{\varepsilon_1, \ldots, \varepsilon_n\} \models_{\{\mathbf{A}\}} \varepsilon_0$. Two quasiequations (q_1) and (q_2) are *equivalent* if they are satisfied by the same class of FL-algebras.

We now introduce a class of quasiequations corresponding to structural rules.

Definition 2.2. A quasiequation $\varepsilon_1 \ \mathsf{and} \ \ldots \ \mathsf{and} \ \varepsilon_n \Longrightarrow \varepsilon_0$ is *structural* if each ε_i $(0 \leq i \leq n)$ is an inequation $t \leq u$ where t is a (possibly empty) product of variables and u is either a variable or 0.

Every structural rule can be interpreted by a structural quasiequation as follows. Let Υ be a sequence of metavariables, and Ψ either empty, a metavariable α for formulas, or Π for stoups. Given a fixed bijection between the denumerable sets of variables and metavariables, we define the interpretation $\Upsilon^\bullet$ of Υ as the term in the language of **FL** obtained by replacing the metavariables by their corresponding variables and comma by the connective $\cdot$ (fusion); if Υ is empty, then $\Upsilon^\bullet = 1$. For example, if $\Upsilon = \alpha, \Gamma, \beta, \Gamma$, then $\Upsilon^\bullet = xyzy$. The interpretation $(\Upsilon \Rightarrow \Psi)^\bullet$ of a metasequent $\Upsilon \Rightarrow \Psi$ is defined to be $\Upsilon^\bullet \leq 0$ if Ψ is empty, $\Upsilon^\bullet \leq \alpha^\bullet$, if $\Psi = \alpha$, and $\Upsilon^\bullet \leq \Pi^\bullet$, if $\Psi = \Pi$.

The *interpretation* of a structural rule (let $s, s_1, \ldots, s_n$ be metasequents)

$$\frac{s_1 \quad \cdots \quad s_n}{s} \ (r)$$

is defined to be the structural quasiequation

$$s_1^\bullet \ \mathsf{and} \ \ldots \ \mathsf{and} \ s_n^\bullet \Longrightarrow s^\bullet. \qquad (r^\bullet)$$

For a set R of structural rules, we define $R^\bullet = \{(r^\bullet) : (r) \in R\}$.

Notice that the interpretation disregards the distinction between metavariables for formulas and those for sequences of formulas. Hence there is some freedom when reading back a structural rule from a given structural quasiequation.

Given a set Q of quasiequations, FL_Q will denote the class of all FL-algebras that satisfy Q; clearly FL_Q is a quasi-variety. It follows from the algebraization and from general considerations on the equivalence of consequence relations (see Proposition 7.4 of [36]) that the relations $\vdash^{seq}_{\mathbf{FL}_R}$ and $\models_{\mathsf{FL}_{R^\bullet}}$ are equivalent. In particular, for a set $S \cup \{s\}$ of sequents and a set R of structural rules,

$$S \vdash^{seq}_{\mathbf{FL}_R} s \text{ iff } \varepsilon[S] \models_{\mathsf{FL}_{R^\bullet}} \varepsilon(s)$$

where $\varepsilon(s)$ is the equation corresponding to s.

3. Equations and structural rules

A substructural logic is by definition an extension of **FL** with axioms. However, if one simply adds an axiom to **FL**, one easily loses cut-elimination, the *raison d'être* of proof theory. Hence to apply proof theoretic techniques to substructural logics, one needs to *structuralize* axioms, namely to transform them into suitable structural rules. In algebraic terms, this corresponds to the transformation of equations into structural quasiequations. It is a crucial step when proving that some equations are preserved by MacNeille completions (Def. 5.14).

In this section we investigate which axioms can be structuralized, or equivalently, which equations can be transformed into structural quasiequations.

Class	Equation	Name	Structural rule
$\mathcal{N}_1$	$xx \leq x$	expansion	(exp)
$\mathcal{N}_2$	$xy \leq yx$	exchange	(e)
	$x \leq 1$	left weakening	(i)
	$0 \leq x$	right weakening	(o)
	$x \leq xx$	contraction	(c)
	$x^n \leq x^m$	knotted $(n, m \geq 0)$	$(knot^n_m)$
	$x \wedge \neg x \leq 0$	weak contraction	(wc)
$\mathcal{P}_2$	$1 \leq x \vee \neg x$	excluded middle	none (Prop. 7.1)
	$1 \leq (x\backslash y) \vee (y\backslash x)$	prelinearity	none (Prop. 7.1)
$\mathcal{N}_3$	$x(x\backslash y) = x \wedge y = (y/x)x$	divisibility	none (Prop. 7.1)
	$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$	distributivity	none (Cor. 7.4)
$\mathcal{P}_3$	$1 \leq \neg x \vee \neg\neg x$	weak excluded middle	none (Prop. 7.1)

Figure 3: Some Known Equations

3.1. Substructural hierarchy

To address the problem systematically, we introduce below a classification $(\mathcal{P}_n, \mathcal{N}_n)$ of the terms of $\mathsf{FL}_\perp$ which is analogous to the arithmetical hierarchy (Σ_n, Π_n). Our hierarchy, introduced in [12] for the commutative case, is based on *polarities*; see Section 2.2.

Definition 3.1. For each $n \geq 0$, the sets $\mathcal{P}_n, \mathcal{N}_n$ of terms are defined as follows:

(0) $\mathcal{P}_0 = \mathcal{N}_0 =$ the set of variables.

(P1) $1, \perp$ and all terms of $\mathcal{N}_n$ belong to $\mathcal{P}_{n+1}$.

(P2) If $t, u \in \mathcal{P}_{n+1}$, then $t \vee u, t \cdot u \in \mathcal{P}_{n+1}$.

(N1) $0, \top$ and all terms of $\mathcal{P}_n$ belong to $\mathcal{N}_{n+1}$.

(N2) If $t, u \in \mathcal{N}_{n+1}$, then $t \wedge u \in \mathcal{N}_{n+1}$.

(N3) If $t \in \mathcal{P}_{n+1}$ and $u \in \mathcal{N}_{n+1}$, then $t\backslash u, u/t \in \mathcal{N}_{n+1}$.

Symbolically, we may then write

$$\mathcal{P}_{n+1} = \langle \mathcal{N}_n \rangle_{\vee, \prod} \text{ and } \mathcal{N}_{n+1} = \langle \mathcal{P}_n \cup \{0\} \rangle_{\wedge, \mathcal{P}_{n+1} \rightarrow},$$

namely $\mathcal{P}_{n+1}$ is the set generated from $\mathcal{N}_n$ by means of finite (possibly empty) joins and products, and $\mathcal{N}_{n+1}$ is generated by $\mathcal{P}_n \cup \{0\}$ by means of finite (possibly empty) meets and divisions with denominators from $\mathcal{P}_{n+1}$.

By residuation, any equation ε can be written as $1 \leq t$. We say that ε belongs to $\mathcal{P}_n$ ($\mathcal{N}_n$, resp.) if t does.

Figure 3 classifies some known equations. In terms of logic, they correspond to axioms; for instance, weak contraction and prelinearity correspond to the axioms $\neg(\alpha \wedge \neg\alpha)$ and $(\alpha\backslash\beta) \vee (\beta\backslash\alpha)$, respectively (see Section 2.4).

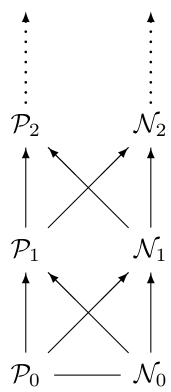

Figure 4: The Substructural Hierarchy

Proposition 3.2.

1. *Every term belongs to some* $\mathcal{P}_n$ *and* $\mathcal{N}_n$.
2. $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$ *and* $\mathcal{N}_n \subseteq \mathcal{N}_{n+1}$ *for every* n.

Hence the classes $\mathcal{P}_n$, $\mathcal{N}_n$ constitute a hierarchy as depicted in Figure 4, which we call the *substructural hierarchy*.

Terms in each class admit the following normal forms.

Lemma 3.3.

(P) If $t \in \mathcal{P}_{n+1}$*, then* t *is equivalent to* $\bot$ *or* $u_1 \vee \cdots \vee u_m$*, where each* u_i *is a product of terms in* $\mathcal{N}_n$.

(N) If $t \in \mathcal{N}_{n+1}$*, then* t *is equivalent to* $\top$ *or* $\bigwedge_{1 \leq i \leq m} l_i \backslash u_i / r_i$*, where each* u_i *is either* 0 *or a term in* $\mathcal{P}_n$*, and each* l_i *and* r_i *are products of terms in* $\mathcal{N}_n$.

Proof. We will prove the lemma by simultaneous induction of the two statements.

Statement (P) is clear for $t = \bot$. The case $t = 1$ is a special case for $m = 1$ and u_1 the empty product. If (P) holds for $t, u \in \mathcal{P}_{n+1}$, then it clearly holds for $t \vee u$. For $t \cdot u$, we use the fact that multiplication distributes over joins.

Statement (N) is clear for $t = \top$. For $t = 0$ we take $m = 1$, $l_1 = r_1 = 1$ and $u_1 = 0$. If (N) holds for $t, u \in \mathcal{N}_{n+1}$, then it clearly holds for $t \wedge u$. If $t \in \mathcal{P}_{n+1}$ and $u \in \mathcal{N}_{n+1}$, we know that $t = t_1 \vee \cdots \vee t_m$, for t_i a product of terms in $\mathcal{N}_n$, where $m = 0$ yields the empty join $t = \bot$. We have $t \backslash u = (t_1 \vee \cdots \vee t_m) \backslash u = (t_1 \backslash u) \wedge \cdots \wedge (t_m \backslash u)$. Moreover, by the induction hypothesis, for all $j \in \{1, \ldots, m\}$, $t_j \backslash u = t_j \backslash (\bigwedge_{1 \leq i \leq k} l_i \backslash u_i / r_i) = \bigwedge_{1 \leq i \leq k} t_j \backslash (l_i \backslash u_i / r_i) = \bigwedge_{1 \leq i \leq k} (l_i t_j) \backslash u_i / r_i$; the empty meet $\top$ is obtained for $k = 0$. $\square$

As a consequence of the above lemma, every equation ε in $\mathcal{N}_2$ is equivalent to a finite set $NF(\varepsilon)$ of equations of the form $t_1 \cdots t_m \leq u$, where $u = 0$ or $u_1 \vee \cdots \vee u_k$ and each u_i is a product of variables. Furthermore, each t_i is of the form $\bigwedge_{1 \leq j \leq n} l_j \backslash v_j / r_j$, where $v_j = 0$ or a variable, and l_j and r_j are products of variables. We call $NF(\varepsilon)$ the *normal form* of ε.

In the sequel, we frequently use the following lemma, corresponding to *Ackermann's Lemma* in [15, 16].

Lemma 3.4. *A quasiequation* $(q)\ \varepsilon_1$ and $\ldots$ and $\varepsilon_n \Longrightarrow t_1 \cdots t_m \leq u$ *is equivalent to either one of*

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_n \text{ and } u \leq x_0 \Longrightarrow t_1 \cdots t_m \leq x_0 \qquad (q')$$

$$\varepsilon_1 \text{ and } \ldots \text{ and } \varepsilon_n \text{ and } x_1 \leq t_1 \text{ and } \ldots \text{ and } x_m \leq t_m \Longrightarrow x_1 \cdots x_m \leq u \qquad (q'')$$

where $x_0, \ldots, x_m$ *are fresh variables.*

Proof. We will prove the equivalence of (q) and (q'). Assume the premises of (q'). Then (q) entails $t_1 \cdots t_m \leq u$. Since $u \leq x_0$ by assumption, we have $t_1 \cdots t_m \leq x_0$. For the converse direction, note that (q') with x_0 instantiated by u entails (q). □

3.2. From $\mathcal{N}_2$-equations to structural quasiequations

We show that the equations in $\mathcal{N}_2$ correspond to structural quasiequations, and hence to structural rules. Our proof is constructive and provides a method to generate those quasiequations (see also the corresponding result in [12] for Hilbert axioms over $\mathbf{FL}_\perp$ with exchange).

Theorem 3.5. *Every equation in $\mathcal{N}_2$ is equivalent to a finite set of structural quasiequations.*

Proof. Let ε be an equation in $\mathcal{N}_2$ and let $t_1 \cdots t_m \leq u \in NF(\varepsilon)$. By Lemma 3.4, ε is equivalent to a quasiequation

$$x_1 \leq t_1 \text{ and } \cdots \text{ and } x_m \leq t_m \Longrightarrow x_1 \cdots x_m \leq u,$$

where $x_1, \ldots, x_m$ are fresh variables. Since each t_i is of the form $\bigwedge_{1 \leq j \leq n} l_j \backslash v_j / r_j$, $x_i \leq t_i$ can be replaced with n premises $l_1 x_i r_1 \leq v_1, \ldots, l_n x_i r_n \leq v_n$. We apply this replacement to all $x_i \leq t_i$. If u is 0, then the resulting quasiequation is already structural. Otherwise, $u = u_1 \vee \cdots \vee u_k$. We replace the conclusion by $x_1 \cdots x_m \leq x_0$ and add k premises $u_1 \leq x_0, \ldots, u_k \leq x_0$ with x_0 a fresh variable. The resulting quasiequation is structural, and is equivalent to the original one by Lemma 3.4. □

Example 3.6. Using the algorithm contained in the proof of the theorem above, the weak contraction axiom $\neg(\alpha \wedge \neg\alpha)$ is turned into an equivalent structural rule. Indeed, it corresponds to the equation $x \wedge \neg x \leq 0$ and is successively transformed as follows:

$$\begin{array}{ll} \longrightarrow & z \leq x \wedge \neg x \Longrightarrow z \leq 0, \\ \longrightarrow & z \leq x \text{ and } z \leq \neg x \Longrightarrow z \leq 0, \\ \longrightarrow & z \leq x \text{ and } xz \leq 0 \Longrightarrow z \leq 0. \end{array}$$

From the last quasiequation, one can read back a structural rule

$$\frac{\beta \Rightarrow \alpha \quad \alpha, \beta \Rightarrow}{\beta \Rightarrow} \ (wc').$$

To obtain the final form (wc) which preserves strong analyticity (see Figure 2), we will apply the transformation in Section 4.2 (analytic completion); see Example 4.10.

3.3. From structural quasiequations to $\mathcal{N}_2$-equations?

Having established that $\mathcal{N}_2$-equations correspond to structural quasiequations, we may ask the converse question. Namely, do all structural quasiequations correspond to $\mathcal{N}_2$-equations? If not, do they correspond to equations at all? The following proposition provides a negative answer to both questions. We also identify a large class of structural quasiequations (*$\mathcal{N}_2$-solvable quasiequations*) which correspond to $\mathcal{N}_2$-equations.

Proposition 3.7. *Not every structural quasiequation is equivalent to an equation.*

Proof. Consider the quasiequation $1 \leq 0 \Rightarrow x^2 \leq 0$. We construct an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ which satisfies the quasiequation and a homomorphic image of $\mathbf{A}$ which does not. Hence the quasiequation cannot be equivalent to an equation.

As A we take the set $\{\bot, a, 1, \top\}$, where $0 = a$ and $\bot < a < 1 < \top$. Now, $\mathbf{A}$ is completely specified by defining multiplication. We define $\bot$ as an absorbing element for A ($\bot x = x\bot = \bot$), $\top$ as an absorbing element for $\{a, 1, \top\}$ and a as an absorbing element for $\{a, 1\}$. It is easy to see that $\mathbf{A}$ is a residuated lattice (which is denoted by $\mathbf{T}_3[\mathbf{2}]$ in [18]) that satisfies the quasiequation vacuously.

We redefine $0 = 1$ in the subalgebra of $\mathbf{A}$ on the set $\{\bot, 1, \top\}$ to obtain $\mathbf{B}$. It is easy to see that the map that sends a to 1 and fixes the other elements is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$, but $\mathbf{B}$ does not satisfy the quasiequation. $\square$

Remark 3.8. The argument above can be repeated for many structural quasiequations with single premise $1 \leq 0$ and a non-valid equation as conclusion.

We now give a sufficient condition for a structural quasiequation to be equivalent to an equation.

Definition 3.9. A structural quasiequation

$$t_1 \leq u_1 \text{ and } \ldots \text{ and } t_n \leq u_n \Longrightarrow t \leq u,$$

is said to be *solvable* if there is a substitution σ, called a *solution*, such that the following holds in all FL-algebras:

(solv1) $\sigma(t_i) \leq \sigma(u_i)$ for all $1 \leq i \leq n$, and

(solv2) $t_i \leq u_i$ for all $1 \leq i \leq n$ implies $x \leq \sigma(x)$ for every x occurring in t, and $\sigma(x) \leq x$ for x occurring in u (and $\sigma(x) = x$ for x occurring in both).

It is called $\mathcal{N}_2$-*solvable* if $\sigma(t) \leq \sigma(u)$ is an $\mathcal{N}_2$-equation.

The structural quasiequation constructed in the proof of Theorem 3.5 is $\mathcal{N}_2$-solvable; indeed, the substitution σ given by $\sigma(x_i) = t_i$ for $1 \leq i \leq m$ and $\sigma(x_0) = u$ provides a solution.

Proposition 3.10. *Every solvable (resp. $\mathcal{N}_2$-solvable) quasiequation is equivalent to an equation (resp. $\mathcal{N}_2$-equation).*

Proof. We will show that a structural quasiequation

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_n \leq u_n \Longrightarrow t \leq u \tag{q}$$

with solution σ is equivalent to the equation

$$\sigma(t) \leq \sigma(u). \tag{e}$$

Assume that (e) holds. Given the premises of (q), we obtain $x \leq \sigma(x)$ when x occurs in t and $\sigma(x) \leq x$ when $u = x$ by condition (solv2). Therefore, (e) yields $t \leq \sigma(t) \leq \sigma(u) \leq u$, the conclusion of (q).

Conversely, if (q) holds, then every substitution instance holds, as well. So we have

$$\sigma(t_1) \leq \sigma(u_1) \text{ and } \dots \text{ and } \sigma(t_n) \leq \sigma(u_n) \Longrightarrow \sigma(t) \leq \sigma(u). \tag{$\sigma(q)$}$$

By condition (solv1), all the premises of $(\sigma(q))$ hold, so we get $\sigma(t) \leq \sigma(u)$. □

We present below two classes of $\mathcal{N}_2$-solvable quasiequations. Let us call a structural quasiequation

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_n \leq u_n \Longrightarrow t \leq u \tag{q}$$

pivotal if one can find a variable x_i (a *pivot*) in each t_i which does not belong to $\{u_1, \dots, u_n\}$.

Proposition 3.11. *Every pivotal quasiequation is $\mathcal{N}_2$-solvable.*

Proof. If (q) is pivotal, it can be written as

$$l_1 x_1 r_1 \leq u_1 \text{ and } \dots \text{ and } l_n x_n r_n \leq u_n \Longrightarrow t \leq u,$$

where $x_1, \dots, x_n$ are not necessarily distinct, and may occur in some l_i, r_i, but not in any u_i. Define a substitution σ by

$$\sigma(x_i) = x_i \wedge \bigwedge l_j \backslash u_j / r_j$$

for $1 \le i \le n$, where the meet $\bigwedge l_j \backslash u_j / r_j$ is built from those premises $l_j x_j r_j \le u_j$ such that $x_j = x_i$. Let $\sigma(z) = z$ for other variables z. We then have $\sigma(y) \le y$ for every variable y and $\sigma(u_k) = u_k$ for every $1 \le k \le n$.

Now σ satisfies condition (solv1), since

$$\sigma(l_k)\sigma(x_k)\sigma(r_k) \le l_k(l_k \backslash u_k / r_k) r_k \le u_k = \sigma(u_k).$$

As to (solv2), the premises of (q) imply $x_i \le \bigwedge l_j \backslash u_j / r_j$ for $1 \le i \le n$. Hence $x_i = \sigma(x_i)$.

Finally, $\sigma(t) \le \sigma(u)$ clearly belongs to $\mathcal{N}_2$ since it is obtained by substituting $\mathcal{N}_1$-terms into the $\mathcal{N}_1$-equation $t \le u$. $\square$

Example 3.12. The quasiequation $xy \le x$ and $x^2 y \le x \Longrightarrow yx \le y$ is pivotal with the choice of pivot y for both premises. It admits a solution $\sigma(y) = y \wedge (x \backslash x) \wedge (x^2 \backslash x)$ and is equivalent to the $\mathcal{N}_2$-equation $\sigma(y)x \le \sigma(y)$.

The notion of pivotality is motivated by the need of excluding premises with inevitable vicious cycles (cf. Definition 4.1) like

$$x\, y \le x \text{ and } y\, x \le y \Longrightarrow y \le x.$$

However, under certain conditions, some structural quasiequations are solvable even with such cycles. We call a structural quasiequation *one-variable* if its premises involve only one variable x and do not contain any of $1 \le x$, $x \le 0$ and $1 \le 0$.

Proposition 3.13. *Every one-variable quasiequation is $\mathcal{N}_2$-solvable.*

Proof. Suppose that the quasiequation is of the form

$$x^{n_1} \le u_1 \text{ and } \dots \text{ and } x^{n_k} \le u_k \Longrightarrow t \le u$$

where each u_i is either x or 0. By definition and since premises of the form $x \le x$ are redundant, we may assume $n_1, \dots, n_k \ge 2$. We claim that the substitution

$$\sigma(x) = x \wedge (u_1 / x^{n_1 - 1}) \wedge \dots \wedge (u_k / x^{n_k - 1})$$

gives rise to a solution.

To check (solv1) we need to verify that $\sigma(x)^{n_i} \le \sigma(u_i)$ for $1 \le i \le k$. If $u_i = 0$, we have

$$\sigma(x)^{n_i} \le (u_i / x^{n_i - 1}) x^{n_i - 1} \le u_i = \sigma(u_i).$$

On the other hand, if $u_i = x$, we need to show that

$$\sigma(x)^{n_i} \le x \wedge (u_1 / x^{n_1 - 1}) \wedge \dots \wedge (u_k / x^{n_k - 1}).$$

We will show that the left hand side is less than or equal to each of the terms on the right hand side.

As before, we have $\sigma(x)^{n_i} \leq (u_i/x^{n_i-1})x^{n_i-1} \leq u_i = x$. Furthermore, for every $1 \leq r \leq k$ we have

$$\sigma(x)^{n_i}x^{n_r-1} \leq (u_r/x^{n_r-1})(x/x^{n_i-1})x^{n_i-2}x^{n_r-1} \leq u_r.$$

So $\sigma(x)^{n_i} \leq u_r/x^{n_r-1}$.

Finally, it is easy to see that condition (solv2) holds. □

To sum up, we have obtained:

Corollary 3.14. *Every $\mathcal{N}_2$-equation is equivalent to a set of $\mathcal{N}_2$-solvable quasiequations. Conversely, every $\mathcal{N}_2$-solvable quasiequation (e.g., pivotal or one-variable ones) is equivalent to an $\mathcal{N}_2$-equation.*

In terms of logic, the first statement means that every $\mathcal{N}_2$-axiom can be structuralized in the single-conclusion sequent calculus. The second statement can also be rephrased accordingly.

In Section 7, we will show that "good" structural quasiequations (acyclic quasiequations that lack $1 \leq 0$ premises) are equivalent to $\mathcal{N}_2$-equations.

4. Analytic Completion

We have described a procedure for transforming $\mathcal{N}_2$-axioms/equations into structural rules/quasiequations. However, this is not the end of the story, since not all structural rules preserve cut admissibility once added to **FL**. For instance, (**cut**) is not redundant in **FL** extended with the contraction rule (c) in Fig. 2, see e.g. [41]. We will see below that, among structural rules, *acyclic* ones can always be transformed into equivalent *analytic* structural rules, which preserve strong analyticity once added to **FL**. The transformation is also important for a purely algebraic purpose: to show preservation of quasiequations under MacNeille completions.

In Section 4.1, we describe a procedure (we refer to it as *analytic completion*) by means of which any acyclic quasiequation is transformed into an analytic one. The procedure also applies to any set of structural quasiequations (without the assumption of acyclicity) in presence of integrality $x \leq 1$ (left weakening). Our current procedure formalizes and extends to the non-commutative case the procedure sketched in [12] (see also Section 6 of [41] for its origin). In Section 4.2, we illustrate what analytic completion amounts to in terms of structural rules.

4.1. Analytic completion of structural quasiequations

Let us begin with defining two classes of structural quasiequations.

Definition 4.1. Given a structural quasiequation (q) we build its *dependency graph* $D(q)$ in the following way:

- The vertices of $D(q)$ are the variables occurring in the premises (we do not distinguish occurrences).

- There is a directed edge $x \longrightarrow y$ in $D(q)$ if and only if there is a premise of the form $lxr \leq y$.

(q) is said to be *acyclic* if the graph $D(q)$ is acyclic (i.e., has no directed cycles or loops).

The terminology naturally extends to structural rules as well. Also, suppose that an $\mathcal{N}_2$-equation ε is transformed into a set Q of structural quasiequations by the procedure described in the proof of Theorem 3.5. We say that ε is *acyclic* if all quasiequations in Q are.

Example 4.2. A structural quasiequation that is not acyclic is $xy \leq x \Longrightarrow yx \leq y$, or the structural quasiequation $xy \leq z$ and $z \leq x \Longrightarrow yxz \leq y$.

Definition 4.3. An *analytic* quasiequation is a structural quasiequation

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_n \leq u_n \Longrightarrow t_0 \leq u_0$$

which satisfies the following conditions:

Linearity t_0 is a (possibly empty) product of distinct variables $x_1, \dots, x_m$.

Separation u_0 is either 0 or a variable x_0 which is distinct from $x_1, \dots, x_m$.

Inclusion Each t_i $(1 \leq i \leq n)$ is a (possibly empty) product of some variables from $\{x_1, \dots, x_m\}$ (here repetition is allowed). Each u_i $(1 \leq i \leq n)$ is either 0 or u_0.

Given an acyclic quasiequation

$$\varepsilon_1 \text{ and } \dots \text{ and } \varepsilon_n \Longrightarrow \varepsilon_0 \tag{q_0}$$

we transform it into an analytic one in two steps.

1. Restructuring. Suppose that ε_0 is $y_1 \cdots y_m \leq u$. Let $x_0, x_1, \dots, x_m$ be fresh variables which are distinct from each other. Depending on whether u is 0 or a variable, we transform (q_0) into either

$$\varepsilon_1, \dots, \varepsilon_n \text{ and } x_1 \leq y_1, \dots, x_m \leq y_m \Longrightarrow x_1 \cdots x_m \leq 0, \tag{q_1}$$

or

$$\varepsilon_1, \dots, \varepsilon_n \text{ and } x_1 \leq y_1, \dots, x_m \leq y_m \text{ and } u \leq x_0 \Longrightarrow x_1 \cdots x_m \leq x_0. \tag{q_2}$$

(q_1) (or (q_2)) is equivalent to (q_0) by Lemma 3.4, is acyclic since $x_0, \dots, x_m$ are fresh, satisfies linearity, separation and

Exclusion none of $x_1, \dots, x_m$ appears on the RHS of a premise, and x_0 does not appear on the LHS of a premise.

2. Cutting. To obtain a quasiequation satisfying the inclusion condition, we have to eliminate *redundant variables* from the premises, i.e., variables other than $x_0, \ldots, x_m$. We describe below how to remove such variables while preserving acyclicity and exclusion.

Let z be any redundant variable. If z appears only in the RHS of premises, we simply remove all such premises $t_1 \leq z, \ldots, t_k \leq z$ from the quasiequation. The resulting quasiequation is not weaker than the original one since it has less premises. To show that it is not stronger either, observe that premises $t_i \leq z$ in the original quasiequation hold with instantiation of z by $\bigvee t_i$, and the instantiation does not affect the other premises and conclusion. Hence the original quasiequations implies the new one.

If z appears only in the LHS of premises, say $l_1 z r_1 \leq u_1, \ldots, l_k z r_k \leq u_k$, we argue similarly, this time instantiating z by $\bigwedge l_i \backslash u_i / r_i$.

Otherwise, z appears both in the RHS and LHS. Let $S_R = \{s_i \leq z : 1 \leq i \leq k\}$ and $S_L = \{t_j(z, \ldots, z) \leq u_j : 1 \leq j \leq l\}$ be sets of premises which involve z on the RHS and LHS, respectively (where all occurrences of z in t_j are displayed). *By acyclicity, S_R and S_L are disjoint.* We replace $S_R \cup S_L$ with

$$S_C = \{t_j(s_{i_1}, \ldots, s_{i_n}) \leq u_j : 1 \leq j \leq l \text{ and } i_1, \ldots, i_n \in \{1, \ldots, k\}\}$$

The resulting quasiequation implies the original one, in view of transitivity. To show the converse, assume the premises of the new one. By instantiating $z = \bigvee s_i$, all premises in S_R hold and all premises in S_L follow from S_C, since $t_j(\bigvee s_i, \ldots, \bigvee s_i) = \bigvee t_j(s_{i_1}, \ldots, s_{i_n}) \leq u_j$. Hence the original quasiequation yields the conclusion.

Note that acyclicity and exclusion are preserved and that the number of redundant variables decreased by one. Repeating this process, we obtain a quasiequation satisfying exclusion which has no redundant variable. Such a quasiequation satisfies also the inclusion condition, and therefore it is analytic.

Remark 4.4. The assumption of acyclicity is redundant in presence of integrality $x \leq 1$ (left weakening). Indeed, acyclicity was essentially used only in the last step where we needed to ensure that S_L and S_R are disjoint. If an equation belongs to both S_L and S_R, then it is of the form $t(z, \ldots, z) \leq z$, which can be safely removed as it follows directly from integrality.

We have thus proved:

Theorem 4.5. *Every acyclic quasiequation is equivalent to an analytic one. The same holds for any structural quasiequation in presence of integrality $x \leq 1$.*

4.2. Analytic completion of structural rules

We apply the procedure in the previous section to acyclic structural rules (or any structural rules in presence of left weakening) in order to transform them into *analytic rules*. The latter will be shown in Section 5.5 to preserve (a stronger form of) cut admissibility once added to **FL**. These results, together with the procedure contained in the proof of Theorem 3.5, allow for the automated definition of strongly analytic sequent calculi for logics semantically characterized by (acyclic) $\mathcal{N}_2$-equations over residuated lattices.

Any acyclic structural rule (r) can be interpreted as an acyclic quasiequation $(r^\bullet)$ (see Section 2.5). By applying to the latter the completion procedure in the previous section we obtain an analytic quasiequation.

In the sequel, we describe a precise way of reading back an analytic rule from the analytic quasiequation.

Definition 4.6. A structural rule (r) is *analytic* if it has one of the forms

$$\frac{\Upsilon_1 \Rightarrow \quad \dots \quad \Upsilon_k \Rightarrow \quad \Gamma, \Upsilon_{k+1}, \Delta \Rightarrow \Pi \quad \dots \quad \Gamma, \Upsilon_n, \Delta \Rightarrow \Pi}{\Gamma, \Upsilon_0, \Delta \Rightarrow \Pi}\ (r_1)$$

$$\frac{\Upsilon_1 \Rightarrow \quad \dots \quad \Upsilon_n \Rightarrow}{\Upsilon_0 \Rightarrow}\ (r_2)$$

and satisfies:

Linearity Υ_0 is a sequence of distinct metavariables $\Sigma_1, \dots, \Sigma_m$ for sequences.

Separation Γ and Δ are distinct metavariables for sequences different from $\Sigma_1, \dots, \Sigma_m$, and Π is a metavariable for stoups.

Inclusion Each Υ_i $(1 \leq i \leq n)$ is a sequence of some metavariables from $\{\Sigma_1, \dots, \Sigma_m\}$ (here repetition is allowed).

Example 4.7. With reference to Figure 2, the rules (*seq-c*), (*wc*), (*min*), (*mix*) and ($anl\text{-}knot^n_m$) are analytic, while the remaining ones are not.

We can associate to each analytic quasiequation

$$\varepsilon_1 \ \mathsf{and} \ \dots \ \mathsf{and} \ \varepsilon_n \Longrightarrow \varepsilon_0 \qquad (q)$$

an analytic structural rule (q°) as follows. Assume that ε_0 is of the form $x_1 \cdots x_m \leq x_0$; the construction below subsumes the case of $x_1 \dots x_m \leq 0$. We associate to each x_i $(1 \leq i \leq m)$ a metavariable Σ_i for sequences, and to x_0 three metavariables Γ, Δ and Π. If ε_j is of the form $x_{i_1} \cdots x_{i_k} \leq 0$ with $i_1, \dots, i_k \in \{1, \dots, m\}$, let ε_j° be the sequent $\Sigma_{i_1}, \dots, \Sigma_{i_k} \Rightarrow$, and if ε_j is of the form $x_{i_1} \cdots x_{i_k} \leq x_0$, let ε_j° be $\Gamma, \Sigma_{i_1}, \dots, \Sigma_{i_k}, \Delta \Rightarrow \Pi$. We thus obtain a structural rule

$$\frac{\varepsilon_1^\circ \quad \cdots \quad \varepsilon_n^\circ}{\varepsilon_0^\circ}\ (q^\circ)$$

which is clearly analytic.

Conversely, it is clear that every analytic structural rule (r) arises from an analytic quasiequation (q) so that $(r) = (q^\circ)$.

Notice that the above procedure associates a *triple* of metavariables Γ, Δ, Π to the RHS variable x_0. This peculiarity, however, does not affect the meaning of the quasiequation.

Lemma 4.8. *If (q) is an analytic quasiequation, then $(q^{\circ\bullet})$ is equivalent to (q).*

Proof. For simplicity, assume that (q) is of the form

$$t_1 \leq 0 \text{ and } t_2 \leq x_0 \Longrightarrow t_0 \leq x_0. \tag{q}$$

Then we obtain

$$t_1 \leq 0 \text{ and } z_l t_2 z_r \leq z_c \Longrightarrow z_l t_0 z_r \leq z_c \tag{$q^{\circ\bullet}$}$$

We easily see that $(q^{\circ\bullet})$ implies (q) by instantiation $z_l = z_r = 1$, $z_c = x_0$, and conversely (q) implies $(q^{\circ\bullet})$ by $x_0 = z_l \backslash z_c / z_r$. □

Theorem 4.9. *Every acyclic rule is equivalent to an analytic rule. The same holds for arbitrary structural rules in presence of the left weakening rule (i.e. (i) in Figure 2).*

Example 4.10. The weak contraction axiom $\neg(\alpha \wedge \neg\alpha)$ is equivalent to the quasiequation $z \leq x$ and $xz \leq 0 \Longrightarrow z \leq 0$ (see Example 3.6), which is acyclic. The analytic completion yields $zz \leq 0 \Longrightarrow z \leq 0$, which corresponds to the structural rule (wc) in Figure 2.

Example 4.11. The expansion axiom $(\alpha \cdot \alpha)\backslash\alpha$, corresponds to the equation $xx \leq x$ (which can also be seen as a structural quasiequation with no premise). The restructuring step of the completion procedure yields

$$y \leq x \text{ and } z \leq x \text{ and } x \leq w \Longrightarrow yz \leq w$$

and the cutting step gives

$$y \leq w \text{ and } z \leq w \Longrightarrow yz \leq w,$$

which corresponds to the mingle rule (min) in Figure 2.

For further examples, the knotted axioms $\alpha^n\backslash\alpha^m$ $(n, m \geq 0)$ in [25] are transformed into the analytic rules $(\mathit{anl}\text{-}\mathit{knot}^n_m)$ in Figure 2; the verification is left to the reader.

5. Cut-Elimination and MacNeille Completion

Having described a way to obtain analytic structural rules/quasiequations, we now turn to showing that these actually preserve admissibility of cut when added to **FL**, and that they are preserved by MacNeille completions. These two facts are to be proved along the same line of argument. The common part is captured in the framework of residuated frames [19]. The primary use of residuated frames is to generate a complete FL-algebra in such a way that certain properties imposed on a frame are transferred to the algebra it generates (called the dual algebra). After giving an introduction to residuated frames (Section 5.1), we prove the crucial fact that analytic quasiequations are always preserved by the dual algebra construction (Section 5.2). This is one common part in the argument for cut-elimination and preservation under MacNeille completions. Another common part is the construction of a (quasi)homomorphism into the dual algebra, which exists when the considered frame satisfies the logical rules of **FL** (Section 5.3). Past this point, the argument branches. We first prove preservation under MacNeille completions in Section 5.4, and then strong analyticity (i.e. a strong form of cut-elimination) in Section 5.5.

5.1. Preliminaries on residuated frames

We introduce a slightly simplified form of residuated frames; they correspond to *ruz*-frames in [19], up to minor differences.

Definition 5.1. A *residuated frame* is a structure of the form $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, where

- W and W' are sets and N is a binary relation from W to W',
- $(W, \circ, \varepsilon)$ is a monoid, $\epsilon \in W'$, and
- for all $x, y \in W$ and $z \in W'$ there exist elements $x \backslash\!\!\backslash z, z /\!\!/ y \in W'$ such that

$$x \circ y \; N \; z \iff y \; N \; x \backslash\!\!\backslash z \iff x \; N \; z /\!\!/ y.$$

We refer to the last property by saying that the relation N is *nuclear*.

Frames abstract both FL-algebras and the sequent calculus **FL**, as we will observe in the following examples.

Example 5.2. If $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is an FL-algebra, then $\mathbf{W_A} = (A, A, \leq, \cdot, 1, 0)$ is a residuated frame. Indeed, for $x \backslash\!\!\backslash z = x \backslash z$ and $z /\!\!/ y = z/y$ we have that N is nuclear by the residuation property.

Example 5.3. Let W be the free monoid over the set Fm of all formulas. The elements of W are exactly the LHSs of **FL** sequents. We denote by $\circ$ (also denoted by comma) the operation of concatenation on W, by ε the empty sequence (the unit element of $\circ$), and by ϵ the empty stoup.

Note that in the left logical rules of **FL** and in analytic structural rules some sequents are of the form $\Gamma, \alpha, \Delta \Rightarrow \Pi$, where Γ, Δ are sequences of formulas. We want to think of $u = \Gamma, \_\,, \Delta$ as a context applied to the formula α in order to yield the sequence $u(\alpha) = \Gamma, \alpha, \Delta$. The element u can be thought of as a unary polynomial over W, such that the variable appears only once (linear polynomial). Such unary, linear polynomials are also known as *sections* over W and we denote the set they form by S_W.

We take $W' = S_W \times (Fm \cup \{\epsilon\})$ and define the relation N by

$$x \; N \; (u, a) \text{ iff } \vdash_{\mathbf{FL}} (u(x) \Rightarrow a).$$

We have

$$x \circ y \; N \; (u, a) \text{ iff } \vdash_{\mathbf{FL}} u(x \circ y) \Rightarrow a \text{ iff } x \; N \; (u(\_ \circ y), a) \text{ iff } y \; N \; (u(x \circ \_), a).$$

Therefore, N is a nuclear relation where the appropriate elements of W' are given by

$$(u,a)/\!\!/x = (u(\_ \circ x), a) \text{ and } x\backslash\!\!\backslash(u,a) = (u(x \circ \_), a).$$

We denote the resulting residuated frame by $\mathbf{W_{FL}}$. We will often identify $(\_\,, a)$ with the element a of $Fm \cup \{\epsilon\}$.

Alternatively, one can define the relation N by

$$x\ N\ (u,a) \text{ iff } u(x) \Rightarrow a \text{ is derivable in } \mathbf{FL} \text{ without using } (\mathbf{cut}).$$

The resulting structure is again a residuated frame, which we denote by $\mathbf{W}^{cf}_{\mathbf{FL}}$.

Given a residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$, $X, Y \subseteq W$ and $Z \subseteq W'$, we write $x\ N\ Z$ for $x\ N\ z$, for all $z \in Z$, and $X\ N\ z$ for $x\ N\ z$, for all $x \in X$. Let

$$\begin{array}{rcl} X \circ Y & = & \{x \circ y : x \in X, y \in Y\}, \\ X^{\rhd} & = & \{y \in W' : X\ N\ y\}, \\ Z^{\lhd} & = & \{y \in W : y\ N\ Z\}. \end{array}$$

For $x \in W$ and $z \in W'$, we also write $x^{\rhd}$ for $\{x\}^{\rhd}$ and $z^{\lhd}$ for $\{z\}^{\lhd}$. The pair $({}^{\rhd}, {}^{\lhd})$ forms a *Galois connection*

$$X \subseteq Z^{\lhd} \quad \Longleftrightarrow \quad X^{\rhd} \supseteq Z,$$

which induces a map $\gamma_N(X) = X^{\rhd\lhd}$ with the following properties:

1. $X \subseteq \gamma_N(X)$.
2. $X \subseteq Y \Longrightarrow \gamma_N(X) \subseteq \gamma_N(Y)$.
3. $\gamma_N(\gamma_N(X)) = \gamma_N(X)$.
4. $\gamma_N(X) \circ \gamma_N(Y) \subseteq \gamma_N(X \circ Y)$.

Namely, γ_N is a *nucleus* on the powerset $\mathcal{P}(W)$ (see [19]). We say that $X \subseteq W$ is *Galois-closed* if $X = \gamma_N(X)$, or equivalently if there is $Z \subseteq W'$ such that $X = Z^{\lhd}$. The set of Galois-closed sets is denoted by $\gamma_N[\mathcal{P}(W)]$. Let

$$\begin{array}{rcl} X \circ_{\gamma_N} Y & = & \gamma_N(X \circ Y), \\ X \cup_{\gamma_N} Y & = & \gamma_N(X \cup Y), \\ X\backslash Y & = & \{z : X \circ \{z\} \subseteq Y\}, \\ Y/X & = & \{z : \{z\} \circ X \subseteq Y\}. \end{array}$$

We define the *dual algebra* of $\mathbf{W}$ by

$$\mathbf{W}^{+} = (\gamma_N[\mathcal{P}(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{\varepsilon\}), \epsilon^{\lhd}).$$

Lemma 5.4 ([19])**.** *If* $\mathbf{W}$ *is a residuated frame, then* $\mathbf{W}^{+}$ *is a complete FL-algebra.*

As Example 5.3 suggests, the basic relation in a residuated frame is

$$x_1 \circ \cdots \circ x_n \; N \; x_0,$$

where $x_1, \ldots, x_n$ range over W and x_0 ranges over W' (this corresponds to asserting a sequent when $\mathbf{W} = \mathbf{W_{FL}}$). On the other hand, the basic relation in the dual algebra $\mathbf{W}^+$ is

$$X_1 \circ_{\gamma_N} \cdots \circ_{\gamma_N} X_n \subseteq X_0,$$

which is easily shown to be equivalent to

$$X_1 \circ \cdots \circ X_n \subseteq X_0,$$

where $X_0, \ldots, X_n$ range over $\gamma_N[\mathcal{P}(W)]$. These two basic relations are linked by the following lemma:

Lemma 5.5. *Let* $\mathbf{W}$ *be a residuated frame.*

1. *For* $x_1, \ldots, x_n \in W$ *and* $x_0 \in W'$, $x_1 \circ \cdots \circ x_n \; N \; x_0$ *iff* $\gamma_N(\{x_1\}) \circ \cdots \circ \gamma_N(\{x_n\}) \subseteq x_0^{\lhd}$.
2. *For* $X_0, \ldots, X_n \in \gamma_N[\mathcal{P}(W)]$, $X_1 \circ \cdots \circ X_n \subseteq X_0$ *iff* $x_1 \circ \cdots \circ x_n \; N \; x_0$ *for every* $x_1 \in X_1, \ldots, x_n \in X_n, x_0 \in X_0^{\rhd}$.
3. *For* $X_1, \ldots, X_n \in \gamma_N[\mathcal{P}(W)]$, $X_1 \circ \cdots \circ X_n \subseteq \epsilon^{\lhd}$ *iff* $x_1 \circ \cdots \circ x_n \; N \; \epsilon$ *for every* $x_1 \in X_1, \ldots, x_n \in X_n$.

Proof. 1. and 2. are derived as follows:

$$\begin{array}{rll} x_1 \circ \cdots \circ x_n \; N \; x_0 & \text{iff} & x_1 \circ \cdots \circ x_n \in x_0^{\lhd} \\ & \text{iff} & \gamma_N(\{x_1 \circ \cdots \circ x_n\}) \subseteq x_0^{\lhd} \\ & \text{iff} & \gamma_N(\{x_1\}) \circ \cdots \circ \gamma_N(\{x_n\}) \subseteq x_0^{\lhd}. \end{array}$$

$$\begin{array}{rll} X_1 \circ \cdots \circ X_n \subseteq X_0 & \text{iff} & x_1 \circ \cdots \circ x_n \in X_0 \text{ for } x_1 \in X_1, \ldots, x_n \in X_n \\ & \text{iff} & x_1 \circ \cdots \circ x_n \in X_0^{\rhd\lhd} \text{ for } x_1 \in X_1, \ldots, x_n \in X_n \\ & \text{iff} & x_1 \circ \cdots \circ x_n \; N \; x_0 \text{ for } x_1 \in X_1, \ldots, x_n \in X_n, x_0 \in X^{\rhd}. \end{array}$$

3. is similar. ∎

5.2. Preservation of analytic quasiequations

Lemma 5.4 provides us with a canonical way of constructing a complete FL-algebra. We now prove that any analytic quasiequation is preserved by the construction of the dual algebra. This is a key step for proving both cut-elimination with structural rules and preservation of quasiequations under MacNeille completions.

Let us begin with an example.

Example 5.6. Recall that the expansion axiom $(\alpha \cdot \alpha)\backslash\alpha$ corresponds to the analytic quasiequation (min) $x_1 \leq x_0$ and $x_2 \leq x_0 \Longrightarrow x_1x_2 \leq x_0$ (Example 4.11). We now show that this is preserved by the dual algebra construction. Namely, if a residuated frame $\mathbf{W}$ satisfies

$$x_1 \; N \; x_0 \text{ and } x_2 \; N \; x_0 \Longrightarrow x_1 \circ x_2 \; N \; x_0 \qquad (min^N)$$

for every $x_1, x_2 \in W$ and $x_0 \in W'$, the dual algebra $\mathbf{W}^+$ satisfies

$$X_1 \subseteq X_0 \text{ and } X_2 \subseteq X_0 \Longrightarrow X_1 \circ X_2 \subseteq X_0 \qquad (min^+)$$

for every $X_0, X_1, X_2 \in \gamma_N[\mathcal{P}(W)]$. Namely, $\mathbf{W}^+ \models (min)$. To show the conclusion of (min^+), let us take $x_1 \in X_1$, $x_2 \in X_2$ and $x_0 \in X_0^{\rhd}$. We then have $x_1 \; N \; x_0$ and $x_2 \; N \; x_0$ by the premises of (min^+). So $x_1 \circ x_2 \; N \; x_0$ by (min^N). Hence we conclude $X_1 \circ X_2 \subseteq X_0$ by Lemma 5.5.

In general, let $\mathbf{W}$ be a residuated frame and (q) an analytic quasiequation

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_n \leq u_n \Longrightarrow t_0 \leq u_0, \qquad (q)$$

where $t_0 = x_1 \cdots x_m$ and u_0 is either x_0 or 0. By the inclusion condition, each term t_i is a product of variables from $\{x_1, \dots, x_m\}$ and each u_i is either x_0 or 0. When $x_1, \dots, x_m$ range over W, we can think of term t_i denoting an element of W. For instance, if $t_i = x_1x_2x_1$, it denotes $x_1 \circ x_2 \circ x_1 \in W$. If $t_i = 1$, then it denotes $\varepsilon \in W$. Likewise, when x_0 ranges over W', the term u_i denotes an element of W'. The case $u_i = x_0$ is obvious. If $u_i = 0$, then it denotes $\epsilon \in W'$.

We say that a residuated frame $\mathbf{W}$ *satisfies* (q) if

$$t_1 \; N \; u_1 \text{ and } \cdots \text{ and } t_n \; N \; u_n \Longrightarrow t_0 \; N \; u_0 \qquad (q^N)$$

always holds when $x_1, \dots, x_n$ range over W and x_0 ranges over W'.

On the other hand, the dual algebra $\mathbf{W}^+$ satisfies (q) just in case

$$T_1 \subseteq U_1 \text{ and } \cdots \text{ and } T_n \subseteq U_n \Longrightarrow T_0 \subseteq U_0 \qquad (q^+)$$

always holds when $X_0, \dots, X_n$ range over $\gamma_N[\mathcal{P}(W)]$. Here, each T_i stands for $X_{i_1} \circ \cdots \circ X_{i_k}$ when $t_i = x_{i_1} \cdots x_{i_k}$. If $t_i = 1$, then $T_i = \gamma_N(\varepsilon)$. Likewise, if $u_i = 0$, then $U_i = \epsilon^{\lhd}$.

Theorem 5.7. *For any analytic quasiequation (q), $\mathbf{W}$ satisfies (q) if and only if $\mathbf{W}^+$ satisfies it.*

Proof. As to the 'only-if' direction, we assume that (q^N) holds in $\mathbf{W}$, that the premises of (q^+) holds in $\mathbf{W}^+$, and show that the conclusion of (q^+) holds in $\mathbf{W}^+$. Let us assume $u_0 = x_0$. Then the conclusion $T_0 \subseteq U_0$ can be written as $X_1 \circ \cdots \circ X_m \subseteq X_0$. To show this, let us take $x_1 \in X_1$, $\dots$, $x_m \in X_m$ and $x_0 \in X_0^{\rhd}$. Recall that, since (q) is analytic, it contains (only) two types of premises: one of the form $x_{i_1} \cdots x_{i_k} \leq x_0$ and the other of the form $x_{i_1} \cdots x_{i_k} \leq 0$ $(i_1, \dots, i_k \in \{1, \dots, m\})$. The former corresponds to $X_{i_1} \circ \cdots X_{i_k} \subseteq X_0$, and the latter to $X_{i_1} \circ \cdots X_{i_k} \subseteq \epsilon^{\lhd}$ in (q^+). Since we assume all premises of (q^+), Lemma 5.5 yields $x_{i_1} \cdots x_{i_k}\ N\ x_0$ for the former and $x_{i_1} \cdots x_{i_k}\ N\ \epsilon$ for the latter. Namely, all premises of (q^N) hold. So we obtain $t_0\ N\ u_0$ by (q^N), namely $x_1 \circ \cdots \circ x_m\ N\ x_0$. Since this holds for every $x_1 \in X_1, \ldots, x_m \in X_m$ and $x_0 \in X_0^{\rhd}$, we conclude that $X_1 \circ \cdots \circ X_m \subseteq X_0$ by Lemma 5.5. The argument is similar and easier when $u_0 = 0$.

As to the 'if' direction, suppose that $x_1, \ldots, x_n$ range over W and x_0 over W' in (q^N). We consider the instantiation $X_1 = \gamma_N(\{x_1\}), \ldots, X_m = \gamma_N(\{x_m\})$ and $X_0 = x_0^{\lhd}$ in (q^+). Under this instantiation, we have $t_i\ N\ u_i$ iff $T_i \subseteq U_i$ by Lemma 5.5. Hence whenever (q^+) holds in $\mathbf{W}^+$, (q^N) holds in $\mathbf{W}$. $\square$

Remark 5.8. The linearity condition for (q) (see Definition 4.3) is essential for the above argument to go through. To see this, consider a non-analytic quasiequation (q) $x_1x_1x_1 \leq x_0 \Longrightarrow x_1x_1 \leq x_0$. Let us try to derive from the condition (q^N) on $\mathbf{W}$

$$x_1 \circ x_1 \circ x_1\ N\ x_0 \Longrightarrow x_1 \circ x_1\ N\ x_0, \tag{q^N}$$

the condition (q^+) in $\mathbf{W}^+$

$$X_1 \circ X_1 \circ X_1 \subseteq X_0 \Longrightarrow X_1 \circ X_1 \subseteq X_0. \tag{q^+}$$

To prove the conclusion $X_1 \circ X_1 \subseteq X_0$, it is natural to take $x_1 \in X_1, x_2 \in X_1$ and $x_0 \in X_0^{\rhd}$ and try to show $x_1 \circ x_2\ N\ x_0$ by using (q^N). However, the latter does not match the conclusion of (q^N), hence the argument breaks down. This is the reason why we impose the linearity condition on analytic quasiequations (see also [41] and [22] for the need of linearity for cut-elimination).

5.3. Gentzen frames

The dual algebra construction produces a complete FL-algebra $\mathbf{W}^+$ from a given residuated frame $\mathbf{W}$ so that analytic quasiequations are transferred. It remains to show that there exists a suitable (quasi)homomorphism f into $\mathbf{W}^+$, provided that $\mathbf{W}$ satisfies the rules of the sequent calculus $\mathbf{FL}$. For 'cut-free' $\mathbf{W}$, this quasihomomorphism is indeed the algebraic essence of cut-elimination. When $\mathbf{W}$ further satisfies 'cut,' f gives rise to an embedding associated to the MacNeille completion.

We begin by making clear what it means for a frame to satisfy the rules of the sequent calculus. We denote by $\mathcal{L}$ the language of $\mathbf{FL}$. An *$\mathcal{L}$-algebra* is simply an algebra over the language $\mathcal{L}$. It does not need to be an FL-algebra; typically, the set Fm of all formulas forms an $\mathcal{L}$-algebra $\mathbf{Fm}$.

Definition 5.9. A *Gentzen frame* is a pair $(\mathbf{W}, \mathbf{A})$ where

- $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ is a residuated frame, $\mathbf{A}$ is an $\mathcal{L}$-algebra,
- there are injections $\iota : A \longrightarrow W$ and $\iota' : A \longrightarrow W'$ (under which we will identify A with a subset of W and a subset of W'),

$$
\frac{x\ N\ a \quad a\ N\ z}{x\ N\ z}\ (\mathrm{CUT}) \qquad \frac{}{a\ N\ a}\ (\mathrm{Id})
$$

$$
\frac{x\ N\ a \quad b\ N\ z}{a\backslash b\ N\ x\backslash\!\backslash z}\ (\backslash\mathrm{L}) \qquad \frac{x\ N\ a\backslash\!\backslash b}{x\ N\ a\backslash b}\ (\backslash\mathrm{R})
$$

$$
\frac{x\ N\ a \quad b\ N\ z}{b/a\ N\ z/\!\!/x}\ (/\mathrm{L}) \qquad \frac{x\ N\ b/\!\!/a}{x\ N\ b/a}\ (/\mathrm{R})
$$

$$
\frac{a \circ b\ N\ z}{a \cdot b\ N\ z}\ (\cdot\mathrm{L}) \qquad \frac{x\ N\ a \quad y\ N\ b}{x \circ y\ N\ a \cdot b}\ (\cdot\mathrm{R})
$$

$$
\frac{a\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}\ell) \qquad \frac{b\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}r) \qquad \frac{x\ N\ a \quad x\ N\ b}{x\ N\ a \wedge b}\ (\wedge\mathrm{R})
$$

$$
\frac{a\ N\ z \quad b\ N\ z}{a \vee b\ N\ z}\ (\vee\mathrm{L}) \qquad \frac{x\ N\ a}{x\ N\ a \vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{x\ N\ b}{x\ N\ a \vee b}\ (\vee\mathrm{R}r)
$$

$$
\frac{\varepsilon\ N\ z}{1\ N\ z}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon\ N\ 1}\ (1\mathrm{R}) \qquad \frac{}{0\ N\ \epsilon}\ (0\mathrm{L}) \qquad \frac{x\ N\ \epsilon}{x\ N\ 0}\ (0\mathrm{R})
$$

Figure 5: Gentzen rules

- N satisfies the *Gentzen rules* (or rather *conditions*) of Figure 5 for all $a, b \in A$, $x, y \in W$ and $z \in W'$.

A *cut-free Gentzen frame* is defined in the same way, but it is not stipulated to satisfy the (CUT) rule.

Example 5.10. If $\mathbf{A}$ is an FL-algebra, then the pair $(\mathbf{W_A}, \mathbf{A})$ is a Gentzen frame (see Example 5.2).

$(\mathbf{W_{FL}}, \mathbf{Fm})$ is also a Gentzen frame, while $(\mathbf{W}^{cf}_{\mathbf{FL}}, \mathbf{Fm})$ is a cut-free Gentzen frame (see Example 5.3). To see this, notice that the conditions $(\backslash\mathrm{L})$ and $(\backslash\mathrm{R})$ can be equivalently expressed by

$$
\frac{x\ N\ a \quad b\ N\ z}{x \circ a\backslash b\ N\ z} \qquad \frac{a \circ x\ N\ b}{x\ N\ a\backslash b}\,.
$$

Now, recall that in $\mathbf{W_{FL}}$ every $x \in W$ is a sequence Σ of formulas and every $z \in W'$ is a pair $((\Gamma, \_\,, \Delta), \Pi)$. Hence the above two rules mean

$$
\frac{\Sigma \Rightarrow \alpha \quad \Gamma, \beta, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \alpha\backslash\beta, \Delta \Rightarrow \Pi} \qquad \frac{\alpha, \Sigma \Rightarrow \beta}{\Sigma \Rightarrow \alpha\backslash\beta},
$$

which precisely correspond to the inference rules for $\backslash$.

Given two $\mathcal{L}$-algebras $\mathbf{A}$ and $\mathbf{B}$, a *quasihomomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a function $F : A \longrightarrow \mathcal{P}(B)$ such that

$$
\begin{array}{rcll}
c_{\mathbf{B}} & \in & F(c_{\mathbf{A}}) & \text{for } c \in \{0, 1\}, \\
F(a) \star_{\mathbf{B}} F(b) & \subseteq & F(a \star_{\mathbf{A}} b) & \text{for } \star \in \{\cdot, \backslash, /, \wedge, \vee\},\ a, b \in A,
\end{array}
$$

where $X \star_{\mathbf{B}} Y = \{x \star_{\mathbf{B}} y | x \in X, y \in Y\}$ for any $X, Y \subseteq B$.

It is equivalent to the standard notion of homomorphism when $F(a)$ is a singleton for every $a \in A$. The theorem below provides us with a suitable (quasi)homomorphism to the dual algebra.

Theorem 5.11 ([19]).

1. *If $(\mathbf{W}, \mathbf{A})$ is a cut-free Gentzen frame, then*

$$F(a) = \{X \in \gamma_N[\mathcal{P}(W)] : a \in X \subseteq a^{\lhd}\}$$

   is a quasihomomorphism from $\mathbf{A}$ to $\mathbf{W}^+$.
2. *If $(\mathbf{W}, \mathbf{A})$ is a Gentzen frame, then $f(a) = a^{\rhd\lhd} = a^{\lhd}$ is a homomorphism from $\mathbf{A}$ to $\mathbf{W}^+$. Moreover, f is an embedding when N is antisymmetric.*

Proof. 1. We verify the conditions on F for $\star \in \{\wedge, \backslash\}$, referring to [19] for the remaining cases. Let $a, b \in A$, $X \in F(a)$ and $Y \in F(b)$, namely,

$$a \in X \subseteq a^{\lhd}, \qquad b \in Y \subseteq b^{\lhd}.$$

(Case $\star = \wedge$) First, we have $X \cap Y \subseteq a^{\lhd} \cap b^{\lhd} \subseteq (a \wedge b)^{\lhd}$, where the the last inclusion is due to the rule ($\wedge$R) of Figure 5. Second, observe that $a \in X$ implies $a \wedge b \in X$. Indeed, if $z \in X^{\rhd}$ we have $a\ N\ z$, so $a \wedge b\ N\ z$ by the rule ($\wedge$Lℓ). This proves $a \wedge b \in X^{\rhd\lhd} = X$. Similarly, $a \wedge b \in Y$. We have thus established

$$a \wedge b \in X \cap Y \subseteq (a \wedge b)^{\lhd},$$

namely $X \wedge_{\mathbf{W}^+} Y \in F(a \wedge_{\mathbf{A}} b)$.
(Case $\star = \backslash$) Let $x \in X \backslash Y$. Since $a \in X$ and $Y \subseteq b^{\lhd}$, we have $a \circ x \in Y \subseteq b^{\lhd}$. So $a \circ x\ N\ b$, which implies $x\ N\ a \backslash b$ by the rule ($\backslash$R), i.e., $x \in (a \backslash b)^{\lhd}$. This proves $X \backslash Y \subseteq (a \backslash b)^{\lhd}$. To show $a \backslash b \in X \backslash Y$, let $x \in X$ and $z \in Y^{\rhd}$. Since $X \subseteq a^{\lhd}$ and $b \in Y$, we have $x\ N\ a$ and $b\ N\ z$. Hence by the rule ($\backslash$L), we have $a \backslash b\ N\ x \backslash\!\!\backslash z$, i.e. $x \circ a \backslash b\ N\ z$. Since this holds for every $x \in X$ and $z \in Y^{\rhd}$, we conclude $X \circ \{a \backslash b\} \subseteq Y^{\rhd\lhd} = Y$. Namely, $a \backslash b \in X \backslash Y$. We have thus established

$$a \backslash b \in X \backslash Y \subseteq (a \backslash b)^{\lhd},$$

namely $X \backslash_{\mathbf{W}^+} Y \in F(a \backslash_{\mathbf{A}} b)$.

2. From the (Id) rule follows $a \in a^{\lhd}$, so $a^{\rhd\lhd} \subseteq a^{\lhd}$. We also have $a^{\lhd} \subseteq a^{\rhd\lhd}$. To show this, let $x \in a^{\lhd}$, so $x\ N\ a$. For every $z \in a^{\rhd}$, we have $a\ N\ z$, so $x\ N\ z$ by (CUT). Namely $x \in a^{\rhd\lhd}$. As a consequence,

$$F(a) = \{X \in \gamma_N[\mathcal{P}(W)] : a^{\rhd\lhd} \subseteq X \subseteq a^{\lhd}\} = \{a^{\rhd\lhd}\},$$

hence F boils down to a homomorphism.

Suppose that N is antisymmetric and $f(a) = f(b)$. We then have $a \in b^{\rhd}$ and $b \in a^{\rhd}$. Namely, $a\ N\ b$ and $b\ N\ a$, so $a = b$. This proves that f is an embedding. $\square$

5.4. Preservation by MacNeille completions

We already have enough facts to conclude that analytic quasiequations are preserved by MacNeille completions. But before that, let us observe a general fact that preservation under completions implies conservativity with respect to infinitary extensions.

More precisely, let κ be a cardinal. We enrich the set of formulas so that both $\bigwedge_{i\in I}\alpha_i$ and $\bigvee_{i\in I}\alpha_i$ are formulas if α_i is a formula for every $i\in I$, where I is an arbitrary index set with $|I|\leq\kappa$. We also add the following inference rules:

$$\frac{\Gamma_1,\alpha_i,\Gamma_2\Rightarrow\Pi\quad\text{for some } i\in I}{\Gamma_1,\bigwedge_{i\in I}\alpha_i,\Gamma_2\Rightarrow\Pi}\ (\textstyle\bigwedge l)\qquad\frac{\Gamma\Rightarrow\alpha_i\quad\text{for all } i\in I}{\Gamma\Rightarrow\bigwedge_{i\in I}\alpha_i}\ (\textstyle\bigwedge r)$$

$$\frac{\Gamma_1,\alpha_i,\Gamma_2\Rightarrow\Pi\quad\text{for all } i\in I}{\Gamma_1,\bigvee_{i\in I}\alpha_i,\Gamma_2\Rightarrow\Pi}\ (\textstyle\bigvee l)\qquad\frac{\Gamma\Rightarrow\alpha_i\quad\text{for some } i\in I}{\Gamma\Rightarrow\bigvee_{i\in I}\alpha_i}\ (\textstyle\bigvee r)$$

The extension of $\mathbf{FL}_R$ with these infinitary connectives is denoted by $\mathbf{FL}_R^\kappa$. Notice that the cardinality restriction on I is necessary, since otherwise the collection of formulas would constitute a proper class.

Definition 5.12. Let R be a set of structural rules and κ a cardinal. We say that $\mathbf{FL}_R^\kappa$ is a *conservative extension* (*atomic conservative extension*, resp.) of $\mathbf{FL}_R$ if $S\vdash_{\mathbf{FL}_R^\kappa} s$ implies $S\vdash_{\mathbf{FL}_R} s$, whenever S is a set of sequents (resp. atomic sequents), and s is a sequent in the language of $\mathbf{FL}$. Here an *atomic sequent* is a sequent that consists of atomic formulas.

Recall that a *completion* of an algebra $\mathbf{A}$ is a complete algebra $\mathbf{B}$ together with an embedding $\iota:\mathbf{A}\longrightarrow\mathbf{B}$. We often identify A with $\iota[A]$ and do not mention the embedding ι explicitly. We say that a class $\mathcal{K}$ of algebras *admits completions* if every $\mathbf{A}\in\mathcal{K}$ has a completion in $\mathcal{K}$. The following is a general fact, although we only state it for $\mathbf{FL}$ with structural rules.

Lemma 5.13. *Let R be a set of structural rules and $R^\bullet$ the set of quasiequations interpreting them (cf. Section 2.5). If $\mathsf{FL}_{R^\bullet}$ admits completions, then $\mathbf{FL}_R^\kappa$ is a conservative extension of $\mathbf{FL}_R$ for every cardinal κ.*

Proof. Assume $S\vdash_{\mathbf{FL}_R^\kappa} s$. In view of the algebraization of $\mathbf{FL}$ (subsection 2.4), it suffices to show that $\varepsilon[S]\models_{\mathbf{A}}\varepsilon(s)$ holds for every algebra $\mathbf{A}\in\mathsf{FL}_{R^\bullet}$. By assumption, $\mathbf{A}$ has a completion $\mathbf{A}'$ in $\mathsf{FL}_{R^\bullet}$. Since all rules of $\mathbf{FL}_R^\kappa$, including the rules for $\bigwedge$ and $\bigvee$, are sound in $\mathbf{A}'$, we have $\varepsilon[S]\models_{\mathbf{A}'}\varepsilon(s)$. Since $\mathbf{A}$ is (isomorphic to) a subalgebra of $\mathbf{A}'$, $\varepsilon[S]\models_{\mathbf{A}}\varepsilon(s)$. □

Completions of a given algebra are not unique in general. Among them, our frame-based construction yields a particularly important one.

Definition 5.14. Given an FL-algebra $\mathbf{A}$, a completion $\iota:\mathbf{A}\longrightarrow\mathbf{B}$ is called a *MacNeille completion* if $\iota[A]$ is both join-dense and meet-dense in $\mathbf{B}$. Namely, for every element $x\in\mathbf{B}$ there exist $P,Q\subseteq\iota[A]$ such that $x=\bigvee P=\bigwedge Q$.

MacNeille completions of **A** are unique up to isomorphisms that fix A (cf. [5, 38]), hence we usually speak of *the* MacNeille completion.

Proposition 5.15. *Given an FL-algebra* $\mathbf{A}$*,* $\mathbf{W}_{\mathbf{A}}^{+}$ *is the MacNeille completion of* $\mathbf{A}$*.*

Proof. $\mathbf{W}_{\mathbf{A}}^{+}$ is a complete FL-algebra by Lemma 5.4. Since $(\mathbf{W}_{\mathbf{A}}, \mathbf{A})$ is a Gentzen frame with N antisymmetric, $f(a) = \gamma_N(a) = a^{\rhd\lhd} = a^{\lhd}$ is an embedding from $\mathbf{A}$ to $\mathbf{W}_{\mathbf{A}}^{+}$ by Theorem 5.11. Recall that every element of $\mathbf{W}_{\mathbf{A}}^{+}$ is a set $X \subseteq A$ such that $X = X^{\rhd\lhd}$. We have

$$\begin{array}{rclcl} X & = & \bigcup_{\gamma_N}\{\gamma_N(a) : a \in X\} & = & \bigvee\{f(a) : a \in X\}, \\ & = & \bigcap\{a^{\lhd} : a \in X^{\rhd}\} & = & \bigwedge\{f(a) : a \in X^{\rhd}\}. \end{array}$$

The first line follows from the properties of nuclei. For the second line, observe

$$\begin{array}{rcl} b \in X & \iff & b \in X^{\rhd\lhd} \\ & \iff & b \; N \; a \text{ for every } a \in X^{\rhd} \\ & \iff & b \in \bigcap\{a^{\lhd} : a \in X^{\rhd}\}. \end{array}$$

This proves join-density and meet-density. ☐

A notable feature of the MacNeille completion is that it preserves all existing joins and meets. Hence it is useful when proving the completeness theorem for predicate substructural logics with respect to the associated classes of complete FL-algebras (see [34]). We refer to [42] for a general study of MacNeille completions for arbitrary lattice expansions.

Notice that an FL-algebra $\mathbf{A}$ satisfies an analytic quasiequation (q) if and only if $\mathbf{W}_{\mathbf{A}}$ satisfies it. Hence a direct consequence of Theorem 5.7 is the following:

Theorem 5.16. *Analytic quasiequations are preserved by MacNeille completions. Namely, if* $\mathbf{A}$ *satisfies an analytic quasiequation (q), then* $\mathbf{W}_{\mathbf{A}}^{+}$ *also satisfies (q).*

Corollary 5.17. *If E is a set of acyclic* $\mathcal{N}_2$*-equations, the variety* FL_E *of FL-algebras satisfying E admits MacNeille completions, and* $\mathbf{FL}_E^{\kappa}$ *is a conservative extension of* $\mathbf{FL}_E$ *for every cardinal* κ*.*

5.5. Strong analyticity

Turning to the proof-theoretic side, we will give an algebraic proof of cut-elimination for **FL** extended with a set R of analytic structural rules. Actually, we prove a stronger form of cut-elimination which we call strong analyticity, and moreover not just for finitary systems, but also for arbitrary infinitary extensions of $\mathbf{FL}_R$. Roughly speaking, strong analyticity refers to a property that cut rules can be eliminated from a given derivation with nonlogical atomic assumptions, and the resulting cut-free derivations satisfy the subformula property, i.e. they consist of formulas already contained in the statements to be proved. Here we need to mention the subformula property explicitly, since a system that admits cut-elimination might not satisfy the subformula property due to some peculiar structural rules.

Informally, a semantic proof of cut-elimination proceeds as follows:

$$\frac{\vdash \varphi \implies \mathbf{A} \models \varphi \qquad \text{and} \qquad \mathbf{A} \models \varphi \implies \vdash^{\text{cut-free}} \varphi}{\vdash \varphi \implies \vdash^{\text{cut-free}} \varphi},$$

where the first premise is the soundness of the semantics and the second premise is the cut-free completeness. Of course, the crucial step of this argument is to build a suitable semantic model $\mathbf{A}$ which is sound with respect to derivability $\vdash$ on the one hand, and is intensionally associated to the cut-free derivability $\vdash^{\text{cut-free}}$ on the other hand. In our setting, this is achieved by the dual algebra construction from a cut-free Gentzen frame (Lemma 5.4) and the quasihomomorphism given by Theorem 5.11.

Let us now proceed to the formal argument.

Definition 5.18. A set S of sequents is said to be *elementary* if S consists of atomic sequents and is closed under cuts: if S contains $\Sigma \Rightarrow p$ and $\Gamma, p, \Delta \Rightarrow \Pi$, it also contains $\Gamma, \Sigma, \Delta \Rightarrow \Pi$.

A sequent calculus is *strongly analytic* if for any elementary set S and a sequent s *in the finitary language*, if s is derivable from S, then s has a cut-free derivation from S in which only subformulas of formulas in s occur.

Strong analyticity subsumes cut admissibility and subformula property in the usual sense (by taking $S = \emptyset$). We also use this concept for infinitary systems $\mathbf{FL}_R^\kappa$, but notice that the conclusion sequent s is restricted to the finitary language, i.e., it does not contain infinitary $\bigwedge$ or $\bigvee$.

A direct consequence of strong analyticity is atomic conservativity with respect to infinitary extensions.

Lemma 5.19. *Let R be a set of structural rules and κ a cardinal. If $\mathbf{FL}_R^\kappa$ is strongly analytic, then $\mathbf{FL}_R^\kappa$ is an atomic conservative extension of $\mathbf{FL}_R$.*

Proof. Let S be a set of atomic sequents, s a sequent in the language of $\mathbf{FL}$ and suppose that $S \vdash_{\mathbf{FL}_R^\kappa} s$. Then we have $S_0 \vdash_{\mathbf{FL}_R^\kappa} s$, where S_0 is the closure of S under cuts; note that S_0 is elementary. By strong analyticity s has a cut-free derivation from S_0 obeying the subformula property. Hence $S_0 \vdash_{\mathbf{FL}_R} s$, since s is in the language of $\mathbf{FL}$. Since all sequents in S_0 are derivable from S, we conclude $S \vdash_{\mathbf{FL}_R} s$. □

We now prove strong analyticity of $\mathbf{FL}_R^\kappa$, where R is a set of analytic rules. The first thing to do is to build a suitable frame, that is analogous to $\mathbf{W}_{\mathbf{FL}}^{cf}$ of Example 5.3.

Given an elementary set S, we define a frame $\mathbf{W}_{R,S} = (W, W', N, \circ, \varepsilon, \epsilon)$ as follows:

- $(W, \circ, \varepsilon)$ is the free monoid generated by Fm,

- $W' = S_W \times (Fm \cup \{\epsilon\})$,
- $\Sigma\ N\ (C, \Pi)$ iff $C = (\Gamma, \_, \Delta)$ and $\Gamma, \Sigma, \Delta \Rightarrow \Pi$ is cut-free derivable from S in $\mathbf{FL}_R$.

For the next lemma, our specific way of reading back a structural rule (q°) from an analytic quasiequation (q) is crucial.

Lemma 5.20. *$(\mathbf{W}_{R,S}, \mathbf{Fm})$ is a cut-free Gentzen frame. Moreover, $\mathbf{W}^+_{R,S}$ satisfies the quasiequations in $R^\bullet$.*

Proof. The first claim is easily verified as in Example 5.10. For the second claim, we have to verify that $\mathbf{W}^+_{R,S}$ satisfies the quasiequation $(r^\bullet)$ for each analytic rule $(r) \in R$. Since the general case is tedious, let us consider one example which is general enough to grasp the idea. Suppose that (r) is

$$\frac{\Sigma_1 \Rightarrow \quad \Gamma, \Sigma_2, \Sigma_2, \Delta \Rightarrow \Pi}{\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi}\ (r).$$

(r) arises from the analytic quasiequation

$$x_1 \leq 0 \text{ and } x_2 x_2 \leq x_0 \Longrightarrow x_1 x_2 \leq x_0 \qquad (q)$$

so that $(r) = (q^\circ)$. We claim that $\mathbf{W}_{R,S}$ satisfies (q), namely

$$x_1\ N\ \epsilon \text{ and } x_2 \circ x_2\ N\ x_0 \Longrightarrow x_1 \circ x_2\ N\ x_0 \qquad (q^N)$$

holds when x_1, x_2 range over W and x_0 over W'. Since $x_i \in W$ is of the form Σ_i for $i = 1, 2$ and $x_0 \in W'$ is of the form $((\Gamma, \_, \Delta), \Pi)$, (q^N) amounts to the following:

- If $\Sigma_1 \Rightarrow$ and $\Gamma, \Sigma_2, \Sigma_2, \Delta \Rightarrow \Pi$ are cut-free derivable from S in $\mathbf{FL}_R$, then so is $\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \Pi$.

This certainly holds as the rule $(r) \in R$.

Therefore, $\mathbf{W}^+_{R,S}$ satisfies (q) by Theorem 5.7. Notice that the quasiequation (q) is equivalent to $(q^{\circ\bullet})$ by Lemma 4.8, which is in turn equivalent to $(r^\bullet)$ by definition. Therefore $\mathbf{W}^+_{R,S}$ satisfies $(r^\bullet)$. □

We next define a valuation into $\mathbf{W}^+_{R,S}$ which makes all sequents in S true, so that the soundness argument goes through. For each propositional variable p, let

$$S(p) = \{\Gamma : \Gamma \Rightarrow p \in S\} \cup \{p\}$$

and define a valuation f by $f(p) = \gamma_N(S(p))$ and homomorphically extending it to all formulas. Given a sequent s, we say that s is *true under* f if $\models_{\mathbf{W}^+_{R,S}, f} \varepsilon(s)$. This holds when $f(\alpha_1) \circ \cdots \circ f(\alpha_m) \subseteq f(\beta)$ if s is of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$, and when $f(\alpha_1) \circ \cdots \circ f(\alpha_m) \subseteq \epsilon^\triangleleft$ if s is of the form $\alpha_1, \ldots, \alpha_m \Rightarrow$.

Lemma 5.21. *For any formula α, $\alpha \in f(\alpha) \subseteq \alpha^\triangleleft$. Moreover, all sequents in S are true under f.*

Proof. For every propositional variable p, we have $p \in S(p) \subseteq p^{\lhd}$, hence $p \in f(p) \subseteq p^{\lhd}$. Since the function $F(\alpha) = \{X \in \gamma_N[\mathcal{P}(W)] : \alpha \in X \subseteq \alpha^{\lhd}\}$ is a quasi-homomorphism from $\mathbf{Fm}$ to $\mathbf{W}^{+}_{R,S}$ by Theorem 5.11, we can inductively show that $\alpha \in f(\alpha) \subseteq \alpha^{\lhd}$ for every formula α.

To verify the second claim for a sequent of the form $p_1, \ldots, p_n \Rightarrow q$ in S, let $\Gamma_1 \in S(p_1), \ldots, \Gamma_n \in S(p_n)$. Since S is closed under cuts, we have $\Gamma_1, \ldots, \Gamma_n \Rightarrow q$ in S. This shows that $S(p_1) \circ \cdots \circ S(p_n) \subseteq S(q)$, and hence $f(p_1) \circ \cdots \circ f(p_n) \subseteq f(q)$.

For a sequent of the form $p_1, \ldots, p_n \Rightarrow$ in S, let $\Gamma_1 \in S(p_1), \ldots, \Gamma_n \in S(p_n)$. Since S is closed under cuts, $\Gamma_1, \ldots, \Gamma_n \Rightarrow$ belongs to S, we have $S(p_1) \circ \cdots \circ S(p_n) \subseteq \epsilon^{\lhd}$, and hence $f(p_1) \circ \cdots \circ f(p_n) \subseteq f(0)$. □

We are now ready to prove:

Theorem 5.22. *If R is a set of analytic structural rules, $\mathbf{FL}^{\kappa}_{R}$ is strongly analytic for every cardinal κ.*

Proof. Suppose that a sequent s of the form $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is derivable from an elementary set S in $\mathbf{FL}^{\kappa}_{R}$ (the case of $\alpha_1, \ldots, \alpha_m \Rightarrow$ is similar). We build a residuated frame $\mathbf{W}_{R,S}$ and a valuation f as described above. Then all sequents in S are true under f by Lemma 5.21 and all inference rules of $\mathbf{FL}^{\kappa}_{R}$ are sound in $\mathbf{W}^{+}_{R,S}$, since $\mathbf{W}^{+}_{R,S}$ is a complete FL-algebra (thus admitting interpretations of $\bigwedge$, $\bigvee$) and satisfies all structural rules in R by Lemma 5.20. Therefore, we have $f(\alpha_1) \circ \cdots \circ f(\alpha_m) \subseteq f(\beta)$. Hence

$$\alpha_1, \ldots, \alpha_m \in f(\alpha_1) \circ \cdots \circ f(\alpha_m) \subseteq f(\beta) \subseteq \beta^{\lhd},$$

which means that s is cut-free derivable from S in $\mathbf{FL}_R$.

The subformula property is obvious, given that all structural rules are analytic, and thus satisfy the inclusion condition. □

Remark 5.23. In defining strong analyticity, the conclusion sequent s was limited to be in the language of $\mathbf{FL}$ (i.e. without $\bigwedge$, $\bigvee$). This restriction, which greatly simplified our proofs, is however inessential, and indeed it can be removed by suitably modifying the definition of cut-free Gentzen frames.

6. Closing the Cycle

Our achievements so far may be illustrated as follows:

$$
\begin{tikzcd}
 & \boxed{\text{Acyclicity}} \arrow[d, leftrightarrow, "\text{Theorem 4.5}"] & \\
 & \boxed{\text{Analyticity}} \arrow[dl, bend right, "\text{Theorem 5.22}"'] \arrow[dr, bend left, "\text{Theorem 5.16}"] & \\
\boxed{\text{Strong analyticity}} \arrow[dr, bend right, "\text{Lemma 5.19}"'] & & \boxed{\text{MacNeille completion}} \arrow[dl, bend left, "\text{Lemma 5.13}"] \\
 & \boxed{\text{Atomic Conservativity}} &
\end{tikzcd}
$$

Here we close the cycle by showing that atomic conservativity (with $\kappa = \omega$) implies analyticity, that is if $\mathbf{FL}_R^\omega$ is an atomic conservative extension of $\mathbf{FL}_R$ then R is equivalent to a set of analytic structural rules. Since the argument below is of a proof-theoretic nature, we first explain the idea in terms of structural *rules*.

Example 6.1. Consider the rule

$$\frac{\alpha, \beta \Rightarrow \beta}{\beta, \alpha \Rightarrow \beta}\ (we).$$

Let R_0 be a set of structural rules and $R = R_0 \cup \{(we)\}$. Assume that $\mathbf{FL}_R^\omega$ is an atomic conservative extension of $\mathbf{FL}_R$. Although (we) is not acyclic, we claim that it is equivalent to an analytic rule in presence of the other rules in R_0.

First of all, note that (we) is equivalent to

$$\frac{\alpha, \beta \Rightarrow \beta \quad \gamma \Rightarrow \beta \quad \beta \Rightarrow \delta}{\gamma, \alpha \Rightarrow \delta}\ (we')$$

by the restructuring step in Section 4.1 (see also Lemma 3.4). Let a, c, d be propositional variables, and $\overline{b}$ the infinitary formula $\bigvee_{0 \leq n} a^n c$. Let S be the set $\{a^{(k)}, c \Rightarrow d : 0 \leq k\}$. Now, observe that we have

$$\vdash_{\mathbf{FL}^\omega} a, \overline{b} \Rightarrow \overline{b}, \qquad \vdash_{\mathbf{FL}^\omega} c \Rightarrow \overline{b}, \qquad \text{and} \qquad S \vdash_{\mathbf{FL}^\omega} \overline{b} \Rightarrow d,$$

corresponding to the three premises of (we'). Hence we have $S \vdash_{\mathbf{FL}_R^\omega} c, a \Rightarrow d$ by (we'). By the assumption of atomic conservativity, $S \vdash_{\mathbf{FL}_R} c, a \Rightarrow d$. Since a derivation in $\mathbf{FL}_R$ is always finite, there must be an n such that $c, a \Rightarrow d$ is derivable from $S_n = \{a^{(k)}, c \Rightarrow d : 0 \leq k \leq n\}$.

Now we claim that R is equivalent to R_0 with the following rule:

$$\frac{\gamma \Rightarrow \delta \quad \alpha, \gamma \Rightarrow \delta \quad \alpha^{(2)}, \gamma \Rightarrow \delta \quad \dots \quad \alpha^{(n)}, \gamma \Rightarrow \delta}{\gamma, \alpha \Rightarrow \delta}\ (we'')$$

It is clear that (we'') implies (we') because the premises of the latter imply all the premises of the former. On the other hand, we have a derivation of the conclusion of (we'') from the premises in $\mathbf{FL}_R$; it can be easily obtained from the derivation of $c, a \Rightarrow d$ from S_n. This means that R implies (we'').

Notice that (we'') is acyclic, hence it can be transformed into an equivalent analytic rule by the procedure described in Section 4.

The above argument can be generalized. Hence we have:

Theorem 6.2. *Let R be a set of structural rules. If $\mathbf{FL}_R^\omega$ is an atomic conservative extension of $\mathbf{FL}_R$, then R is equivalent to a set of analytic structural rules.*

Proof. We argue in terms of algebra. Let Q be a set of structural quasiequations. We prove that Q is equivalent to a set of analytic quasiequations under the assumption of atomic conservativity: $E \models_{\mathsf{FL}_Q^\omega} \varepsilon$ implies $E \models_{\mathsf{FL}_Q} \varepsilon$ whenever $E \cup \{\varepsilon\}$ is a set of equations of the form $y_1 \dots y_m \leq y_0$ or $y_1 \dots y_m \leq 0$. Here, FL_Q^ω consists of algebras in FL_Q in which all countable joins and meets exist.

Given a non-analytic quasiequation in Q, we apply the analytic completion procedure in Section 4.1 with slight modifications. First, we can apply the restructuring step without any problem to obtain a quasiequation (q). As to the cutting step, let z be a redundant variable in (q) and suppose that z occurs both in the RHS and LHS of premises (otherwise the procedure is just as before).

We classify the premises of (q) into four groups:

- $S_R = \{s_i \leq z : 1 \leq i \leq k\}$, which have z only in the RHS.
- $S_L = \{t_j(z, \dots, z) \leq u_j : 1 \leq j \leq l\}$, which have z only in the LHS.
- $S_M = \{v_j(z, \dots, z) \leq z : 1 \leq j \leq m\}$, which have z in both.
- S_O, the others.

Let T be the least set of terms such that

- $s_i \in T$ for $1 \leq i \leq k$,
- if $w_1, \dots, w_n \in T$, then $v_j(w_1, \dots, w_n) \in T$ for $1 \leq j \leq m$.

Let also

$$S'_L = \{t_j(w_1, \dots, w_n) \leq u_j : 1 \leq j \leq l,\ w_1, \dots, w_n \in T\}.$$

We claim that $S'_L \cup S_O \models_{\mathsf{FL}_Q^\omega} \varepsilon$, where ε is the conclusion of (q). To show this, we consider the instantiation $z = \bigvee T$, which makes sense as countable joins and meets exist in all algebras in FL_Q^ω. All equations in S_R hold under this instantiation and those in S_M hold too, because

$$v_j(\bigvee T, \dots, \bigvee T) = \bigvee v_j(w_1, \dots, w_m) \leq \bigvee T,$$

with $w_1, \ldots, w_m \in T$. Moreover, the equations in S_L under the instantiation follow from S'_L. This shows that $S'_L \cup S_O \models_{\mathsf{FL}^\omega_Q} \varepsilon$, being z a redundant variable (i.e. z does not appear in the conclusion). By atomic conservativity $S'_L \cup S_O \models_{\mathsf{FL}_Q} \varepsilon$, and by compactness, there is a finite subset $S''_L \subseteq S'_L$ such that $S''_L \cup S_O \models_{\mathsf{FL}_Q} \varepsilon$. Let (q') be the quasiequation corresponding to the latter consequence relation. So, Q implies (q').

Conversely (q') implies (q) by transitivity. Hence one can replace (q) in Q by (q'). The number of redundant variables is decreased by one. Hence by repeating this process, we obtain an analytic quasiequation equivalent to (q). $\square$

Let us summarize what we have achieved:

Theorem 6.3.

1. *Every $\mathcal{N}_2$-axiom/equation is equivalent to a set of structural rules/quasiequations.*
2. *For any set R of structural rules, the following are equivalent:*
   - *R is equivalent to a set of acyclic structural rules.*
   - *R is equivalent to a set of analytic structural rules.*
   - *$R^\bullet$ is preserved by MacNeille completions.*
   - *$\mathbf{FL}^\kappa_R$ is a conservative extension of $\mathbf{FL}_R$ for every κ.*
   - *R is equivalent to R' such that $\mathbf{FL}^\kappa_{R'}$ is strongly analytic for every κ.*

   If R implies left weakening (i), all the above hold.
3. *For any set E of $\mathcal{N}_2$-equations, the following are equivalent:*
   - *E is equivalent to a set of acyclic quasiequations.*
   - *E is equivalent to a set of analytic quasiequations.*
   - *The variety FL_E admits MacNeille completions.*
   - *FL_E admits completions.*

   If E implies integrality $x \leq 1$, all the above hold.

It follows that strong analyticity for infinitary extensions $\mathbf{FL}^\kappa_R$ is equivalent to admitting completions as far as $\mathcal{N}_2$ axioms/equations and structural rules/quasiequations are concerned (actually strong analyticity of $\mathbf{FL}^\omega_R$ is enough). Also notably, MacNeille completions are optimal for the subvarieties of FL defined by $\mathcal{N}_2$-equations: if such a subvariety admits completions, it necessarily admits MacNeille completions.

We end this section showing the existence of a structural rule/$\mathcal{N}_2$-equation which does not satisfy any of conditions (2) and (3) of the above theorem. Our proof below exhibits a real interplay between proof-theoretic and algebraic arguments.

Proposition 6.4. *Not all $\mathcal{N}_2$-equations are equivalent to acyclic quasiequations.*

Proof. Consider the equation $y/y \leq y\backslash y$ and denote it by ε. ε is easily seen to be equivalent to

$$xy \leq y \Longrightarrow yx \leq y, \tag{$we^\bullet$}$$

which is an interpretation of the rule (we) in Example 6.1. If (we) is equivalent to an acyclic rule, then $\mathbf{FL}^{\omega}_{(we)}$ is conservative over $\mathbf{FL}_{(we)}$ by Theorem 6.3. Hence by the argument in Example 6.1, (we) is equivalent to a rule of the form

$$\frac{\gamma \Rightarrow \delta \quad \alpha, \gamma \Rightarrow \delta \quad \alpha^{(2)}, \gamma \Rightarrow \delta \quad \ldots \quad \alpha^{(n)}, \gamma \Rightarrow \delta}{\gamma, \alpha \Rightarrow \delta}\ (we'')$$

So, we have

$$\{p^n q \leq v : n \in \omega\} \models_{\mathsf{FL}_\varepsilon} qp \leq v.$$

We will show that this is not the case, by exhibiting an algebra $\mathbf{A}$ in FL_ε and elements $a, b, c \in A$ such that $a^n b \leq c$ for all $n \in \omega$, but $ba \nleq c$.

The equation ε is satisfied by all lattice-ordered groups, since $y/y = yy^{-1} = 1 = y^{-1}y = y\backslash y$. We can take as $\mathbf{A}$ the totally ordered ℓ-group based on the free group on two generators, constructed in [7]; it is shown there that $\mathbf{A}$ satisfies the property: if $1 \leq x^m \leq y$, for all $m \in \omega$, then $x^m \leq y^{-1}xy$, for all $m \in \omega$. Since the ℓ-group is based on the free group on two generators, it is not Abelian. Moreover, since it is totally ordered there exist elements $g, h \in A$ with $1 < g, h$ and $g^m < h$, for all $m \in \omega$; otherwise the ℓ-group would be archimedean, and every totally ordered archimedean ℓ-group is abelian. By the property of the constructed ℓ-group, we get $g^m \leq h^{-1}gh$, namely $g^m h^{-1} \leq h^{-1}g$, for all $m \in \omega$. Now, let $a = g^2$, $b = h^{-1}$, and $c = h^{-1}g$. We have $a^n b = g^{2n}h^{-1} \leq h^{-1}g = c$, for all $n \in \omega$; but $c = h^{-1}g < h^{-1}g^2 = ba$, because $1 < g$, so $ba \nleq c$. ☐

Remark 6.5. The same holds for the system $\mathbf{FL}_\bot$. Since ℓ-groups are not in $\mathsf{FL}_\bot$, we have to slightly modify the above argument. We consider the above ℓ-group and we add two new elements $\bot$, below every element, and $\top$, above every element. Multiplication is extended so that $\top$ is an absorbing element for $A \cup \{\top\}$ and $\bot$ is an absorbing element for $A \cup \{\top, \bot\}$. It is shown in [23] that this construction yields an FL-algebra into which $\mathbf{A}$ embeds. Moreover, it is easy to see that it satisfies $y/y \leq y\backslash y$, as $\top/\top = \top\backslash\top = \top = \bot/\bot = \bot\backslash\bot$.

The above proposition shows the limitations of strong analyticity and MacNeille completions within the class $\mathcal{N}_2$.

7. Expressive Power of Structural Rules

Each $\mathcal{N}_2$-equation can be transformed into equivalent structural quasiequations and hence into structural rules (Theorem 3.5). This shows what structural rules *can express.* In this section we address the converse problem, namely identifying which properties (equations over residuated lattices, or equivalently, Hilbert axioms in the language of $\mathbf{FL}_\bot$) *cannot* be expressed by structural rules.

The proposition below, which easily follows from our analytic completion, essentially says that the expressive power of structural rules cannot go beyond intuitionistic logic.

Proposition 7.1. *Any structural rule (r) is either derivable in Gentzen's* **LJ** *or derives in* **LJ** *every formula (i.e.,* $\mathbf{LJ}_{(r)}$ *is contradictory).*

Proof. We first apply our analytic completion procedure to obtain, by Theorem 4.9, an analytic rule (r') equivalent to (r) in **LJ** (this is always possible in presence of the left weakening rule (i)). Two cases can arise. If (r') has no premises, any formula is derivable in **LJ** extended with (r') (and hence with (r)), as the LHS and the RHS of the conclusion of (r') are disjoint. Otherwise, the conclusion of (r') is derivable from any of its premises by weakening, exchange and contraction due to the separation and inclusion conditions of Definition 4.6. □

Hence structural rules added to **LJ** do not define any proper consistent superintuitionistic logic.

Remark 7.2. Our proof theoretic limitation is in accordance with the limit established in [8] for MacNeille completions for the variety HA of Heyting algebras: there are only three subvarieties of HA closed under MacNeille completions, that is the trivial variety, the whole variety HA, and the variety BA of Boolean algebras. The small mismatch on Boolean algebras is due to the fact that we restrict here to single conclusion sequent calculi: there is of course a multiple conclusion sequent calculus that captures BA, that is Gentzen's **LK**. See [13] for a proof-theoretic analysis of the substructural hierarchy, adapted to commutative multiple conclusion (hyper)sequent calculi.

The limitations of structural rules are however stronger. Indeed, as shown below, even among the properties which do hold in intuitionistic logic (Heyting algebras), only *some* can be captured by structural sequent rules.

Proposition 7.3. *Any equation equivalent to a structural rule is preserved by MacNeille completions in presence of integrality.*

Proof. Let (q) be the equivalent structural quasiequation. Theorem 4.9 ensures that, in presence of integrality $x \leq 1$, (q) is equivalent to a set Q of analytic quasiequations. By Theorem 5.16, Q is preserved by MacNeille completions. □

As a particular case we have

Corollary 7.4. *No structural rule is equivalent to the distributivity axiom.*

Proof. We use Proposition 7.3 and the fact that distributivity is not preserved by MacNeille completions, even in presence of integrality. To see this, consider a bounded distributive lattice $\mathbf{L}$ whose MacNeille completion $\overline{\mathbf{L}}$ is not distributive; such a lattice was constructed in [10]. It easy to see that the ordinal sum $\mathbf{L} \oplus \{1\}$ (obtained by adding a new top element 1 to $\mathbf{L}$) supports a residuated lattice structure, by defining multiplication as $xy = \bot$, for $x, y \in L$ and setting 1 as the unit element. The MacNeille completion of the integral distributive residuated lattice $\mathbf{L} \oplus \{1\}$ is clearly the ordinal sum $\overline{\mathbf{L}} \oplus \{1\}$, which also fails to be distributive. □

In contrast to the negative results above, it follows from our analytic completion that all "natural" structural rules can be expressed by $\mathcal{N}_2$-axioms (Corollary 7.6 below).

Proposition 7.5. *Any analytic quasiequation without any premise* $1 \leq 0$ *is equivalent to an* $\mathcal{N}_2$ *equation.*

Proof. Suppose that the conclusion is of the form $x_1 \cdots x_m \leq x_0$ (the case $x_1 \cdots x_m \leq 0$ is similar). Let $t_1 \leq x_0, \ldots, t_n \leq x_0$ be the premises having x_0 in the RHS, and $s_1 \leq 0, \ldots, s_k \leq 0$ the others. By assumption[1] each s_i is not 1, hence one can pick up a 'pivot' x_j for some $1 \leq j \leq m$ (cf. Proposition 3.11) and write $s_i = l_i x_j r_i$. Define a substitution σ by

$$\begin{array}{rcll} \sigma(x_0) & = & t_1 \vee \cdots \vee t_n, & \\ \sigma(x_j) & = & \bigwedge l_i \backslash 0 / r_i, & \text{for } 1 \leq j \leq m, \end{array}$$

where the meet $\bigwedge l_i \backslash 0 / r_i$ is built from those premises $l_i x_j r_i \leq 0$ for which x_j has been chosen as pivot. It is easy to see that σ is a solution. Hence it provides an equivalent equation by Proposition 3.10, that is easily shown to be $\mathcal{N}_2$. $\square$

Corollary 7.6. *Let* (r) *be any analytic structural rule. If* (r) *does not contain any empty premise* $\Rightarrow$, *then* (r) *is equivalent to an* $\mathcal{N}_2$*-axiom.*

Hence we can reasonably claim that the expressive power of structural rules is essentially limited to $\mathcal{N}_2$.

Concluding Remark: Beyond $\mathcal{N}_2$

Our main theorem shows that within the class $\mathcal{N}_2$ an equation is preserved under MacNeille completions if and only if the corresponding sequent calculus structural rule is analytic. This correspondence does not hold anymore outside the class $\mathcal{N}_2$ as witnessed by $\neg\neg x \leq x$ (involutivity), which belongs to the class $\mathcal{N}_3$. The equation is preserved under MacNeille completions, but it does not correspond to any structural rule by Proposition 7.1.

Having explored the level $\mathcal{N}_2$ rather in depth, our next target are $\mathcal{P}_2$ and $\mathcal{P}_3$. Indeed, consider the prelinearity axiom (see Figure 3). By Proposition 7.1 it cannot be expressed by any structural rule, as it is neither derivable in **LJ** nor contradicts **LJ**. Since prelinearity belongs to $\mathcal{P}_2$, we have:

Corollary 7.7. *There is an equation in* $\mathcal{P}_2$ *which is not equivalent to any equation in* $\mathcal{N}_2$.

This implies that the inclusions $\mathcal{N}_2 \subseteq \mathcal{P}_3$ and $\mathcal{N}_2 \subseteq \mathcal{N}_3$ are proper. It is left open whether all inclusions in the substructural hierarchy (see Figure 4) are proper or not.

[1]The presence of a premise $1 \leq 0$ often leads to the non-existence of an equivalent equation, see Remark 3.8.

Notice that prelinearity can instead be expressed as a structural rule in *hypersequent calculus* – a simple generalization of sequent calculus whose additional machinery is basically adding one more disjunction on top of sequents [2]. In [12] we proved that in the commutative case, all axioms in the class $\mathcal{P}_3'$ (a slight modification of $\mathcal{P}_3$) can be expressed as structural rules in hypersequent calculus which preserve cut admissibility. The recent paper [11] also shows that all equations in $\mathcal{P}_3'$ are preserved by MacNeille completions when applied to subdirectly irreducible algebras. In our subsequent work, we consider the general noncommutative case and perform a simultaneous investigation of (strong) analyticity in hypersequent calculi and closure under suitable completions for arbitrary FL-algebras extended by $\mathcal{P}_3$-equations.

References

[1] J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.

[2] A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4: 225–248, 1991.

[3] A. Avron and O. Lahav, Kripke semantics for Basic Sequent Systems, *Proceedings of Tableaux 2011*, LNCS 6793: 43-57, 2011.

[4] A. Avron and A. Zamansky, Non-deterministic Semantics for Logical Systems, forthcoming in the *Handbook of Philosophical Logic*, 2011.

[5] B. Banaschewski. Hüllensysteme und Erweiterungen von Quasi-Ordnungen. *Z. Math. Logik Grund. Math.*, 2: 35-46, 1956.

[6] F. Belardinelli, P. Jipsen and H. Ono. Algebraic aspects of cut-elimination. *Studia Logica*, 77(2):209–240, 2004.

[7] G. Bergman. Specially ordered groups. *Comm. Algebra*, 12(19-20), 2315–2333, 1984.

[8] G. Bezhanishvili and J. Harding: MacNeille completions of Heyting algebras. The Houston J. of Math. 30(4): 937 952, 2004.

[9] S. Buss. An Introduction to Proof Theory. *Handbook of Proof Theory*, Elsevier Science, pp. 1–78, 1998.

[10] N. Funayama. On the completion by cuts of distributive lattices. *Proc. Imp. Acad. Tokyo*, 20:1–2, 1944.

[11] A. Ciabattoni, N. Galatos and K. Terui. MacNeille Completions of FL-algebras, *Algebra Universalis*, to appear.

[12] A. Ciabattoni, N. Galatos and K. Terui. From axioms to analytic rules in nonclassical logics. *Proceedings of LICS 2008*, pp. 229 – 240, 2008.

[13] A. Ciabattoni, L. Strassburger and K. Terui. Expanding the realm of systematic proof theory. *Proceedings of CSL 2009*, LNCS 5771: 163-178, 2009.

[14] A. Ciabattoni and K. Terui. Towards a semantic characterization of cut-elimination. *Studia Logica*, 82(1):95–119, 2006.

[15] W. Conradie, V. Goranko, and D. Vakarelov. Algorithmic correspondence and completeness in modal logic I: The core algorithm SQEMA. *Logical Methods in Computer Science*, 2(1:5), 2006.

[16] W. Conradie and A. Palmigiano. Algorithmic Correspondence and Canonicity for Distributive Modal Logic. *Annals of Pure and Applied Logic*, to appear.

[17] M. Gehrke, J. Harding and Y. Venema. MacNeille completions and canonical extensions. *Transactions of the American Mathematical Society*, 358: 573-590, 2006.

[18] N. Galatos, *Minimal varieties of residuated lattices*, *Algebra Universalis* 52(2) (2005), 215-239.

[19] N. Galatos and P. Jipsen, Residuated frames with applications to decidability, *Transactions of the AMS*, to appear.

[20] N. Galatos, P. Jipsen, T. Kowalski and H. Ono. *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

[21] N. Galatos and H. Ono. Algebraization, parameterized local deduction theorem and interpolation for substructural logics over **FL**. *Studia Logica*, 83:279-308, 2006.

[22] N. Galatos and H. Ono. Cut-elimination and strong separation for substructural logics: an algebraic approach, *Annals of Pure and Applied Logic*, 161 (2010), no. 9, 1097–1133.

[23] N. Galatos and J. Raftery. Adding involution to residuated structures. *Studia Logica*, 77(2): 181-207, 2004.

[24] J. Harding. Completions of ordered algebraic structures: a survey. *Proceedings of the International Workshop on Interval/Probabilistic Uncertainty and Non-classical Logics. Advances in Soft Computing.* Ono et. al. Eds, Springer, 46: 231–244, 2008.

[25] R. Hori, H. Ono and H. Schellinx. Extending intuitionistic linear logic with knotted structural rules. *Notre Dame Journal of Formal Logic*, 35(2): 219–242, 1994.

[26] P. Jipsen and C. Tsinakis. A survey of residuated lattices, *Ordered Algebraic Structures* (J. Martinez, ed.), Kluwer Academic Publishers, Dordrecht, 19–56, 2002.

[27] B. Jónsson and A. Tarski. Boolean algebras with operators I. *Amer. J. of Math.* 73: 891 -939, 1951.

[28] B. Jónsson and A. Tarski. Boolean algebras with operators II. *Amer. J. of Math.* 74: 127-162, 1952.

[29] H. M. MacNeille. Partially ordered sets. *Trans. Amer. Math. Soc.* 42:416-460, 1937.

[30] S. Maehara. Lattice-valued representation of the cut-elimination theorem. *Tsukuba Journal of Mathematics*, 15: 509 – 521, 1991.

[31] M. Okada and K. Terui, *The finite model property for various fragments of intuitionistic linear logic*, J. Symbolic Logic **64**(2) (1999), 790–802.

[32] M. Okada. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science*, 227: 333– 396, 1999.

[33] M. Okada. A uniform semantic proof for cut-elimination and completeness of various first and higher order logics. *Theoretical Computer Science*, 281: 471– 498, 2002.

[34] H. Ono. Semantics for substructural logics. *Substructural logics*, K. Došen and P. Schröder-Heister, editors, Oxford University, pp. 259–291, 1994.

[35] H. Ono. Proof theoretic methods for nonclassical logic — an introduction. *Theories of Types and Proofs (MSJ Memoir 2)*, M. Takahashi, M. Okada and M. Dezani-Ciancaglini, editors, Mathematical Society of Japan, pp. 207 – 254, 1998.

[36] J. G. Raftery, Correspondences between Gentzen and Hilbert systems. *Journal of Symbolic Logic*, 71(3):903-957, 2006.

[37] G. Restall. *An Introduction to Substructural Logics.* Routledge, London, 1999.

[38] J. Schmidt. Zur Kennzeichnung der Dedekind-MacNeilleschen Hülle einer geordneten Menge. *Archiv. d. Math.*, 7: 241-249, 1956.

[39] K. Schütte. Ein System des Verknüpfenden Schliessens. *Archiv. Math. Logic Grundlagenf.*, Vol. 2, pp. 55–67, 1956.

[40] T. Suzuki. Canonicity results of substructural and lattice-based logics. *Review of Symbolic Logic*, 4(1): 1-42, 2011.

[41] K. Terui. Which Structural Rules Admit Cut Elimination? — An Algebraic Criterion. *Journal of Symbolic Logic*, 72(3): 738 – 754, 2007.

[42] M. Theunissen and Y. Venema. Macneille completions of lattice expansions, *Algebra Universalis*, 57: 143-193, 2007.

[43] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory (2nd Edition)*. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2000.