Mīmāṃsā deontic logic: proof theory and applications*

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Abstract. Starting with the deontic principles in Mīmāṃsā texts we introduce a new deontic logic. We use general proof-theoretic methods to obtain a cut-free sequent calculus for this logic, resulting in decidability, complexity results and neighbourhood semantics. The latter is used to analyse a well known example of conflicting obligations from the Vedas.

1 Introduction

We provide a first bridge between formal logic and the Mīmāṃsā school of Indian philosophy. Flourishing between the last centuries BCE and the 20th century, the main focus of this school is the interpretation of the prescriptive part of the Indian Sacred Texts (the *Vedas*). In order to explain "what has to be done" according to the Vedas, Mīmāṃsā authors have proposed a rich body of deontic, hermeneutical and linguistic principles (*metarules*), called *nyāya*s, which were also used to find rational explanations for seemingly contradicting obligations.

Even though the Mīmāṃsā interpretation of the Vedas has pervaded almost every other school of Indian philosophy, theology and law, little research has been done on the *nyāya*s. Moreover, since not many scholars working on Mīmāṃsā are trained in formal logic, and the untranslated texts are inaccessible to logicians, these deontic principles have not yet been studied using methods of formal logic.

In this paper starting from the deontic *nyāya*s we define a new logic – *basic Mīmāṃsā deontic logic* (bMDL for short) – that simulates Mīmāṃsā reasoning. After introducing the logic as an extension of modal logic S4 with axioms obtained by formalising these principles [3] and providing a cut-free sequent calculus and neighbourhood-style semantics for it, we use bMDL to reason about a well known example of seemingly conflicting obligations contained in the Vedas. This example concerns the malefic sacrifice called *Śyena* and proved to be a stumbling block for many Mīmāṃsā scholars. The solution to this controversy provided by the

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[3] While some of the *nyāya*s we consider are listed in the Appendix of [13], we extracted the remaining ones directly from Mīmāṃsā texts, see [6].

semantics of bMDL turns out to coincide with that of Prabhākara, one of the chief Mīmāṃsā authors, which previous approaches failed to make sense of, e.g., [18]. Our formal analysis relies essentially on the cut-free calculus for bMDL introduced with the aid of the general method from [16].

Through the paper we refer to the following Mīmāṃsā texts: the *Pūrva Mīmāṃsā Sūtra* (henceforth PMS, ca. 3rd c. BCE), its commentary, the *Śābarabhāṣya* (ŚBh), the main subcommentary, Kumārila's *Tantravārttika* (TV).

Related work. Logic (mainly classical) has already been successfully used to investigate other schools of Indian thought. In particular for Navya Nyāya formal analyses have contributed to a fruitful exchange of ideas between disciplines [8], however, no deontic modalities were considered. A logical analysis of the deontic aspects of the *Talmud*, another sacred text, is given in [1]. The deontic logic used there is based on intuitionistic logic and contains an external mechanism for resolving conflicts among obligations. Deontic logics similar but not equivalent to bMDL include Minimal Deontic Logic [9] and extensions of monotone modal logic with some versions of the D axiom [12,17]. The latter papers also introduce cut-free sequent calculi, but do not mix alethic and deontic modalities.

2 Extracting a deontic logic from Mīmāṃsā texts

The use of logic to simulate Mīmāṃsā ways of reasoning is motivated by their rigorous theory of inference and attention for possible violations of it. For instance Kumārila, one of the chief Mīmāṃsā authors, emphasises the fact that a text is not epistemically reliable if the whole chain of transmission is reliable, but not its beginning. The classical example is that of "*a chain of truthful blind people transmitting information concerning colours*" (TV on PMS 1.3.27).

At this point, the problem amounts to which logic should be adopted. The simplest logical system for dealing with obligations is *Standard Deontic Logic* SDL, that extends classical logic by a unary operator $\mathcal{O}$ read as "It is obligatory that..." satisfying the axioms of modal logic KD [2,7]. Though simple and well studied, SDL is not suited to deal with conflicting obligations, which are often present in the Vedas and in Mīmāṃsā reasoning. A well known example from the Vedas consists of the following norms concerning the malefic Śyena sacrifice, which is enjoined in case one desires to harm his enemy, since it kills them:

A. "*One should not harm any living being*"
B. "*One should sacrifice bewitching with the Śyena*"

Any reasonable formalisation of the statements A. and B. leads in SDL to a contradiction. Given that the Mīmāṃsā authors embraced the principle of non-contradiction and invested all their efforts in creating a *consistent* deontic system, to provide adequate formalisations of Mīmāṃsā reasoning a different logic is needed. To this aim we introduce *basic Mīmāṃsā deontic logic* (bMDL) by extracting its properties directly from Mīmāṃsā texts.

The language of bMDL extends that of classical logic with the binary modal operator $\mathcal{O}(\cdot/\cdot)$ from dyadic deontic logics and the unary modal operator $\Box$ of S4. While the latter is used to formalise the auxiliary conditions of general deontic principles, the former allows us to impose conditions on obligations describing the situation in which the obligation holds. Hence a formula $\mathcal{O}(\varphi/\psi)$ can be read as "φ it is obligatory given ψ".

The use of the dyadic operator, which is a reasonably standard approach to avoid the problem with conflicting obligations (see, e.g., [11] and [9]), is also suggested in the metarule "*Each action is prescribed in relation to a responsible person who is identified because of her desire*" (cf. PMS 6.1.1–3).

As described in Sec. 2.1 the properties of the deontic operator $\mathcal{O}(\cdot/\cdot)$ of bMDL (definition below) are directly extracted from the *nyāya*s.

Definition 1. *Basic Mīmāṃsā deontic logic* bMDL *extends (any Hilbert system for)* S4 *with the following axioms (taken as schemata):*

(1) $(\Box(\varphi \to \psi) \wedge \mathcal{O}(\varphi/\theta)) \to \mathcal{O}(\psi/\theta)$
(2) $\Box(\psi \to \neg\varphi) \to \neg(\mathcal{O}(\varphi/\theta) \wedge \mathcal{O}(\psi/\theta))$
(3) $(\Box((\psi \to \theta) \wedge (\theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi)) \to \mathcal{O}(\varphi/\theta)$

The choice to use classical logic as base system, in contrast to the use of intuitionistic logic in Gabbay et al.'s deontic logic of the Talmud [1], is due to various metarules by Mīmāṃsā authors implying the legitimacy of the reductio ad absurdum argument RAA; these include the following (contained in Jayanta's book *Nyāyamañjarī*): "*When there is a contradiction (φ and not φ), at the denial of one (alternative), the other is known (to be true)*". Therefore, if we deny $\neg\varphi$ then φ holds, which gives RAA.

2.1 From Mīmāṃsā *nyāya*s to Hilbert axioms

Axiom (1) arises from three different principles, discussed in [6]; among them the following abstraction of the *nyāya*s in the *Tantrarahasya* IV.4.3.3 (see [5])

> If the accomplishment of X presupposes the accomplishment of Y, the obligation to perform X prescribes also Y.

This principle leads to $(\Box(\varphi \to \psi) \wedge \mathcal{O}(\varphi/\theta)) \to \mathcal{O}(\psi/\theta)$, where we represent the accomplishment of X and Y as φ and ψ respectively, and we stipulate that the conditions on the two prescriptions, represented by θ, are the same. Note that we use the operator $\Box$, here as well as in the following axioms, to guarantee that the correlations between formulae are not accidental.

Axiom (2) arises from the so-called *principle of the half-hen*, which is implemented in different Mīmāṃsā contexts (e.g., TV on PMS 1.3.3); an abstract representation of it is:

> Given that purposes Y and Z exclude each other, if one should use item X for the purpose Y, then it cannot be the case that one should use it at the same time for the purpose Z.

This principle stresses the incongruity of enjoining someone to act in contradiction with himself on some object. The corresponding axiom is $\Box(\psi \to \neg\varphi) \to \neg(\mathcal{O}(\varphi/\theta) \wedge \mathcal{O}(\psi/\theta))$ which guarantees that if φ and ψ exclude each other, then they cannot both be obligatory under the same conditions θ. Finally, Axiom (3) arises from a discussion (in ŚBh on PMS 6.1.25) on the eligibility to perform sacrifices (see [6]), which can be abstracted as follows:

> If conditions X and Y are always equivalent, given the duty to perform Z under the condition X, the same duty applies under Y.

We formalise this principle as $(\Box((\psi \to \theta) \wedge (\theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi)) \to \mathcal{O}(\varphi/\theta)$, where the conditions X and Y are represented by ψ and θ respectively, and φ represents that the action Z is performed.

While the properties of $\mathcal{O}(\cdot/\cdot)$ are taken from Mīmāṃsā texts, the same cannot be done for $\Box$ because Mīmāṃsā authors do not conceptualise necessity as separate from epistemic certainty. The established choices for a logic for the alethic necessity operator $\Box$ are $\mathsf{S4}$ and $\mathsf{S5}$. To keep the system as simple as possible, and not having found any principle motivating the additional properties of $\mathsf{S5}$, we have chosen $\mathsf{S4}$.

3 Proof Theory of bMDL

Hilbert systems are convenient ways of defining logics, but are not very useful for proving theorems in and about the logics (e.g., decidability, consistency).

For this purpose we introduce a cut-free sequent calculus $\mathsf{G}_{\mathsf{bMDL}}$ for bMDL and use it to show that, for certain issues, bMDL simulates Mīmāṃsā ways of reasoning. As usual, a *sequent* is a tuple $\Gamma \Rightarrow \Delta$ of multisets of formulae interpreted as $\bigwedge \Gamma \to \bigvee \Delta$. To construct $\mathsf{G}_{\mathsf{bMDL}}$ we use the translation from axioms to rules and the construction of a cut-free calculus from these rules from [15,16]. Since the latter is not fully automatic, we provide some details.

First, by [16, Thm. 26], we automatically obtain from Def. 1(1)-(3) the rules

$$\frac{\varphi, \psi \Rightarrow \chi \qquad \Rightarrow \varphi, \psi \qquad \chi \Rightarrow \varphi \qquad \theta \Rightarrow \xi \qquad \xi \Rightarrow \theta}{\Box\varphi, \mathcal{O}(\psi/\theta) \Rightarrow \mathcal{O}(\chi/\xi)}\ \mathsf{Mon}'$$

$$\frac{\varphi, \theta \Rightarrow \xi \quad \varphi, \xi \Rightarrow \theta \quad \Rightarrow \varphi, \theta, \xi \quad \theta, \xi \Rightarrow \varphi \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\Box\varphi, \mathcal{O}(\psi/\theta) \Rightarrow \mathcal{O}(\chi/\xi)}\ \mathsf{Cg}$$

$$\frac{\varphi, \psi, \chi \Rightarrow \quad \Rightarrow \varphi, \psi \quad \Rightarrow \varphi, \chi \quad \theta \Rightarrow \xi \quad \xi \Rightarrow \theta}{\Box\varphi, \mathcal{O}(\psi/\theta), \mathcal{O}(\chi/\xi) \Rightarrow}\ \mathsf{D}_2'$$

From these rules we construct a new set of rules saturated under cuts from which the rules above are derivable. This step is not automatic and amounts to repeated *cutting between rules* [16, Def. 7]: given any two rules we obtain a new rule whose conclusion is the result of a cut on a formula principal in the conclusions of both rules, and whose premisses contain all possible cuts between the premisses of the original rules on the variables occurring in this formula. We start from the set

$$\frac{\Gamma^{\Box} \Rightarrow \varphi}{\Gamma \Rightarrow \Box\varphi, \Delta}\ \mathsf{4} \qquad \frac{\Gamma, \Box\varphi, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\ \mathsf{T} \qquad \frac{\Gamma^{\Box}, \varphi \Rightarrow \theta \quad \Gamma^{\Box}, \psi \Rightarrow \chi \quad \Gamma^{\Box}, \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \mathcal{O}(\theta/\chi), \Delta}\ \mathsf{Mon}$$

$$\frac{\Gamma^{\Box}, \varphi \Rightarrow}{\Gamma, \mathcal{O}(\varphi/\psi) \Rightarrow \Delta}\ \mathsf{D}_1 \qquad \frac{\Gamma^{\Box}, \varphi, \theta \Rightarrow \quad \Gamma^{\Box}, \psi \Rightarrow \chi \quad \Gamma^{\Box}, \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\varphi/\psi), \mathcal{O}(\theta/\chi) \Rightarrow \Delta}\ \mathsf{D}_2$$

Fig. 1. The modal rules rules of $\mathsf{G_{bMDL}}$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{W} \quad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{Con}_L \quad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ \mathsf{Con}_R \quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Sigma, \varphi \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Cut}$$

Fig. 2. The structural rules

containing the rules above and those of $\mathsf{S4}$ and first cut the rules $\mathsf{4}$ (Fig. 1) with Mon' and $\mathsf{4}$ with Cg on the boxed formula to obtain the rules

$$\frac{\Gamma^{\Box}, \psi \Rightarrow \chi \quad \theta \Rightarrow \xi \quad \xi \Rightarrow \theta}{\Gamma, \mathcal{O}(\psi/\theta) \Rightarrow \mathcal{O}(\chi/\xi), \Delta} \qquad \frac{\Gamma^{\Box}, \theta \Rightarrow \xi \quad \Gamma^{\Box}, \xi \Rightarrow \theta \quad \psi \Rightarrow \chi \quad \chi \Rightarrow \psi}{\Gamma, \mathcal{O}(\psi/\theta) \Rightarrow \mathcal{O}(\chi/\xi), \Delta}$$

where $\Gamma^{\Box}$ is obtained from Γ by deleting every occurrence of a formula not of the form $\Box\varphi$. Now cutting these two rules in either possible way yields the rule Mon (Fig. 1), and cutting this and $\mathsf{4}$ with D_2' yields D_2. We obtain D_1 closing D_2 under contraction, i.e., identifying φ with θ and ψ with χ and contracting conclusion and premiss.

The sequent calculus $\mathsf{G_{bMDL}}$ consists of the rules in Fig. 1 together with the standard propositional $\mathsf{G3}$-rules (with principal formulae copied into the premisses) [14] and the standard left rule for the constant $\bot$. We write $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is derivable using these rules. We denote extensions of $\mathsf{G_{bMDL}}$ with structural rules from Fig. 2 by appending their names, collecting Con_L and Con_R into Con. E.g., $\mathsf{G_{bMDL}ConW}$ is $\mathsf{G_{bMDL}}$ extended with Contraction and Weakening.

By construction [15,16] we have:

Theorem 1. *The rule* Cut *is admissible in* $\mathsf{G_{bMDL}ConW}$.

Proof. Using the structural rules the system $\mathsf{G_{bMDL}ConW}$ is equivalent to the system $\mathsf{G_{bMDL}}'\mathsf{ConW}$ in which the principal formulae of the propositional rules and the rule T are not copied into the premisses. By construction (and straightforward inspection in the non-principal cases) the rules of $\mathsf{G_{bMDL}}'\mathsf{ConW}$ satisfy the general sufficient criteria for cut elimination established in [15,16]. Cut-free derivations in $\mathsf{G_{bMDL}}'\mathsf{ConW}$ are converted into cut-free derivations in $\mathsf{G_{bMDL}ConW}$ using the structural rules. □

The methods in [15,16] now automatically yield also an $\mathsf{EXPTIME}$-complexity result. However, we consider an explicit proof search procedure for $\mathsf{G_{bMDL}}$ which will be used in Sec. 4. First we establish some preliminary results.

Lemma 1. *The Contraction and Weakening rules are admissible in* $\mathsf{G_{bMDL}}$.

Proof. Admissibility of weakening is proved by induction on the depth of the derivation, while that of contraction follows from the general criteria in [16, Thm. 16] resp. [15, Thm. 2.5.5] since the rule set $\mathsf{G_{bMDL}}$ is contraction closed and already contains the modified versions of T and the propositional rules. □

Thus suffices to consider *set-based sequents*, i.e., tuples of sets of formulae instead of multisets. The rules of $\mathsf{G_{bMDL}}$ are adapted to the set-based setting in the standard way. Since boxed formulae are always copied into the premisses of a rule, the proof search procedure needs to include *loop checking* to avoid infinite branches in the search tree. We do this using histories, i.e., lists of (set-based) sequents, where the last element is interpreted as the current sequent:

Definition 2 (Histories). *A* history $\mathcal{H}$ *is a finite list* $[\Gamma_1 \Rightarrow \Delta_1; \dots ; \Gamma_n \Rightarrow \Delta_n]$ *of set-based sequents, where we write* $\mathsf{last}_L(\mathcal{H})$ *(resp.* $\mathsf{last}_R(\mathcal{H})$*) for* Γ_n *(resp.* Δ_n*) and* $\mathsf{last}(\mathcal{H})$ *for* $\mathsf{last}_L(\mathcal{H}) \Rightarrow \mathsf{last}_R(\mathcal{H})$. *Given another history* $\mathcal{H}' = [\Sigma_1 \Rightarrow \Pi_1; \dots ; \Sigma_m \Rightarrow \Pi_m]$ *with* $n \leq m$ *we write* $\mathcal{H} \preccurlyeq \mathcal{H}'$ *if for all* $i \leq n$ *we have* $\Gamma_i = \Sigma_i$ *and* $\Delta_i = \Pi_i$. *Finally, we write* $\mathcal{H} +\!\!+ \mathcal{H}'$ *for the concatenation of the two histories.*

The proof search procedure for $\mathsf{G_{bMDL}}$ is given in Algorithm 1, where following [10] we call the propositional rules together with the rule T the *static* rules, $\mathsf{Mon}, \mathsf{4}, \mathsf{D}_1, \mathsf{D}_2$ are called *transitional* rules. The algorithm saturates the current sequent under backwards applications of the one-premiss static rules, and then checks whether the result is an initial sequent or could have been derived by a two-premiss static rule or a dynamic rule. The histories are used to prevent the procedure from exploring a sequent twice (modulo weakening).

Lemma 2 (Termination). *The proof search procedure terminates.*

Proof. Given a history $\mathcal{H}$, the number N of different set-based sequents which can be constructed from subformulae of the sequent $\mathsf{last}(\mathcal{H})$ is exponential in the size of $\mathsf{last}(\mathcal{H})$. Hence after at most N-many recursive calls of the procedure the subroutine rejects every rule application. Furthermore, for every sequent there are only finitely many possible (backwards) applications of a rule from $\mathsf{G_{bMDL}}$, so the subroutine is executed only a finite number of times. □

Proposition 1. $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$ *iff the procedure accepts* $[\Gamma \Rightarrow \Delta]$.

Proof. If the procedure accepts the input, then we construct a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G_{bMDL}}$ by following the accepting choices of backwards applications of the rules, and labelling the nodes in the derivation with the sequents $\mathsf{last}(\mathcal{H})$ for the histories $\mathcal{H}$ given as input to the recursive calls of the algorithm.

Conversely, if the set-based sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{G_{bMDL}}$, then by admissibility of Weakening there is a *minimal* derivation of it, i.e., a derivation in which no branch contains two set-based sequents $\Sigma \Rightarrow \Pi$ and $\Omega \Rightarrow \Theta$ such that $\Sigma \Rightarrow \Pi$ occurs on the path between $\Omega \Rightarrow \Theta$ and the root, and such that $\Omega \subseteq \Sigma$ and $\Theta \subseteq \Pi$. By induction on the depth of such a minimal derivation it can then be seen that the procedure accepts the input $[\Gamma \Rightarrow \Delta]$. □

Algorithm 1: The proof search procedure for $\mathsf{G}_{\mathsf{bMDL}}$

Input: A history $\mathcal{H}$
Output: Is $\mathsf{last}(\mathcal{H})$ derivable in $\mathsf{G}_{\mathsf{bMDL}}$ given the history $\mathcal{H}$?

Saturate $\mathsf{last}(\mathcal{H})$ under the one-premiss static rules;
if $\mathsf{last}(\mathcal{H})$ *is an initial sequent* **then**
  accept the history
else
  for *every possible application of a two-premiss static rule to* $\mathsf{last}(\mathcal{H})$ **do**
    for *every premiss* $\Sigma \Rightarrow \Pi$ *of this application* **do**
      recursively call the proof search procedure with input $\mathcal{H}{+}{+}[\Sigma \Rightarrow \Pi]$;
    accept the application if each of these calls accepts
  for *every possible application of a transitional rule to* $\mathsf{last}(\mathcal{H})$ **do**
    for *every premiss* $\Sigma \Rightarrow \Pi$ *of this application* **do**
      if *there is an* $\mathcal{H}' \preccurlyeq \mathcal{H}$ *with* $\Sigma \subseteq \mathsf{last}_L(\mathcal{H}')$ *and* $\Pi \subseteq \mathsf{last}_R(\mathcal{H}')$ **then**
        reject the premiss
      else
        call the proof search procedure with input $\mathcal{H}{+}{+}[\Sigma \Rightarrow \Pi]$;
        accept the premiss if this call accepts
    accept the rule application if each of the premisses is accepted
  accept the history if at least one of the possible applications is accepted

3.1 Inner and Outer Consistency

Having extracted a cut-free calculus from the axioms using the method in [15,16], soundness and completeness w.r.t. bMDL follow by construction (Thm. 2). By the subformula property we then obtain the *inner consistency* of the logic bMDL, i.e., the fact that $\bot$ is not a theorem of the logic. This is one of the most basic requirements that our logic should satisfy. But since bMDL was introduced with the purpose of simulating Mīmāṃsā reasoning, it should also be consistent with respect to the examples considered by the Mīmāṃsā authors such as the Śyena sacrifice, i.e., it should not enable us to derive a contradiction from the formalisations of these examples. We capture this in the notion of *outer consistency* or consistency in presence of global assumptions. To make this precise we consider the consequence relation associated with the logic bMDL and the corresponding relation associated with the calculus $\mathsf{G}_{\mathsf{bMDL}}$. Henceforth we denote by $\mathcal{A}$ any set of formulae of bMDL.

Definition 3. *The usual notion of derivability of a formula φ from a set $\mathcal{A}$ of assumptions in* bMDL *is denoted by $\mathcal{A} \vdash_{\mathsf{bMDL}} \varphi$. Similarly, for a set $\mathcal{S}$ of sequents, a sequent $\Gamma \Rightarrow \Delta$ is* derivable from $\mathcal{S}$ in $\mathsf{G}_{\mathsf{bMDL}}\mathsf{Cut}$ *if there is a derivation of $\Gamma \Rightarrow \Delta$ in $\mathsf{G}_{\mathsf{bMDL}}$ with leaves labelled with initial sequents, zero-premiss rules or sequents from $\mathcal{S}$. We then write $\mathcal{A} \vdash_{\mathsf{bMDL}} \varphi$ resp. $\mathcal{S} \vdash_{\mathsf{G}_{\mathsf{bMDL}}\mathsf{Cut}} \Gamma \Rightarrow \Delta$.*

Theorem 2 (Soundness and Completeness). *For all sets $\mathcal{S}$ of sequents and sequents $\Gamma \Rightarrow \Delta$ we have:*

$$\mathcal{S} \vdash_{\mathsf{G_{bMDL}Cut}} \Gamma \Rightarrow \Delta \quad \textit{iff} \quad \{\bigwedge \Sigma \to \bigvee \Pi \mid \Sigma \Rightarrow \Pi \in \mathcal{S}\} \vdash_{\mathsf{bMDL}} \bigwedge \Gamma \to \bigvee \Delta\ .$$

Proof. The corresponding standard results for the propositional calculi transfer readily to the system bMDL and the Gentzen system $\mathsf{G3}$ with the zero-premiss rules $\dfrac{}{\Rightarrow \theta}$ for each modal axiom schema θ of bMDL. The result then follows from interderivability of these rules with the modal rules from $\mathsf{G_{bMDL}}$ [15,16]. As an example, the derivation of the zero-premiss rule for Axiom (2), where α denotes $\Box(\psi \to \neg\varphi) \to \neg(\mathcal{O}(\varphi/\theta) \wedge \mathcal{O}(\psi/\theta))$, is as follows

$$\cfrac{\cfrac{\cfrac{\mathcal{D}_1}{\Box(\psi \to \neg\varphi), \psi, \varphi \Rightarrow} \qquad \cfrac{}{\Box(\psi \to \neg\varphi), \theta \Rightarrow \theta}\ ax. \qquad \cfrac{}{\Box(\psi \to \neg\varphi), \theta \Rightarrow \theta}\ ax.}{\mathcal{O}(\psi/\theta), \mathcal{O}(\varphi/\theta), \mathcal{O}(\varphi/\theta) \wedge \mathcal{O}(\psi/\theta), \Box(\psi \to \neg\varphi) \Rightarrow \alpha, \neg(\mathcal{O}(\varphi/\theta) \wedge \mathcal{O}(\psi/\theta))}\ \mathsf{D_2}}{\Rightarrow \alpha}\ prop.$$

where the double line denotes multiple applications of the propositional rules and the derivation $\mathcal{D}_1$ is

$$\cfrac{\cfrac{\cfrac{}{\psi \to \neg\varphi, \Box(\psi \to \neg\varphi), \psi, \varphi \Rightarrow \psi}\ ax. \qquad \cfrac{\cfrac{}{\neg\varphi, \psi \to \neg\varphi, \Box(\psi \to \neg\varphi), \psi, \varphi \Rightarrow \varphi}\ ax.}{\neg\varphi, \psi \to \neg\varphi, \Box(\psi \to \neg\varphi), \psi, \varphi \Rightarrow}\ \neg\Rightarrow}{\psi \to \neg\varphi, \Box(\psi \to \neg\varphi), \psi, \varphi \Rightarrow}\ \to\Rightarrow}{\Box(\psi \to \neg\varphi), \psi, \varphi \Rightarrow}\ \mathsf{T}$$

□

Corollary 1. *The logic* bMDL *is consistent, i.e.,* $\bot \notin \mathsf{bMDL}$. □

Proof. Follows by Thm. 2.1 and the fact that the rules of $\mathsf{G_{bMDL}}$ satisfy the subformula property. □

Definition 4. bMDL *enjoys* outer consistency *with respect to* $\mathcal{A}$ *if* $\mathcal{A} \nvdash_{\mathsf{bMDL}} \bot$

By Thm. 2 this condition is equivalent to $\{ \Rightarrow \varphi \mid \varphi \in \mathcal{A}\} \nvdash_{\mathsf{G_{bMDL}Cut}} \Rightarrow \bot$. We now show that bMDL allows us to consistently formalise the seemingly conflicting statements of the Śyena sacrifice. The proof uses the proof search procedure given in Algorithm 1 and the following version of the deduction theorem (see Section 5 for a semantic proof).

Theorem 3. *For every sequent $\Gamma \Rightarrow \Delta$ and set $\mathcal{A}$ of formulae the following are equivalent (writing $\Box\mathcal{A}$ for $\{\Box\varphi \mid \varphi \in \mathcal{A}\}$ taken as a multiset):*

1. $\{ \Rightarrow \varphi \mid \varphi \in \mathcal{A}\} \vdash_{\mathsf{G_{bMDL}Cut}} \Gamma \Rightarrow \Delta$
2. $\{ \Rightarrow \Box\varphi \mid \varphi \in \mathcal{A}\} \vdash_{\mathsf{G_{bMDL}Cut}} \Gamma \Rightarrow \Delta$
3. $\vdash_{\mathsf{G_{bMDL}}} \Box\mathcal{A}, \Gamma \Rightarrow \Delta$.

Proof. 1 → 2: Easily follows by using the rules T and Cut.

2 → 3: Since every rule in $\mathsf{G_{bMDL}}$ copies all boxed formulae in the antecedent from conclusion to premisses, the result of adding the formulae $\{\Box\varphi \mid \varphi \in \mathcal{A}\}$ to the antecedents of every sequent occurring in the derivation of $\Gamma \Rightarrow \Delta$ from $\{\Box\varphi \mid \varphi \in \mathcal{A}\}$ is still a derivation. As this turns every assumption $\Rightarrow \Box\varphi$ into the derivable sequent $\Box\mathcal{A} \Rightarrow \Box\varphi$, the result is a derivation without assumptions. Statement 3 now follows using Cut Elimination (Thm. 1).

3 → 1: Easily follows by using the rules 4 and Cut. □

Thus in order to check whether bMDL enjoys outer consistency w.r.t. a set $\mathcal{A}$ of formulae it is sufficient to check that the sequent $\Box\mathcal{A} \Rightarrow \bot$ is not derivable in $\mathsf{G_{bMDL}}$. Before we formalise the Śyena sacrifice, let us remark that while the operator $\mathcal{O}(\cdot/\cdot)$ only captures *conditional obligations*, we would also like to reason about *unconditional obligations*, i.e., obligations which always have to be fulfilled. We formalise such obligations in the standard way by $\mathcal{O}(\cdot/\top)$. A formula $\mathcal{O}(\varphi/\top)$ then can be read as "it is obligatory that φ provided *anything* is the case", and thus models an unconditional obligation. A formalisation of the problematic situation in the Śyena example (sentences A. and B. in Sec. 2) then is:

1. $\mathcal{O}(\neg\texttt{hrm}/\top)$ for "One should not perform violence on any living being"
2. $\mathcal{O}(\texttt{sy}/\texttt{des\_hrm\_en})$ for "If you desire to harm your enemy you should perform the Śyena"
3. $\texttt{hrm\_en} \to \texttt{hrm}$ for "harming the enemy entails harming a living being"
4. $\texttt{sy} \to \texttt{hrm\_en}$ for "performing the Śyena entails harming the enemy".

with the variables hrm for "performing violence on any living being", sy for "performing the Śyena sacrifice", hrm_en for "harming your enemy", and des_hrm_en for "desiring to harm your enemy".

Theorem 4. bMDL *enjoys outer consistency w.r.t. the Śyena sacrifice, i.e.:*

$$\left\{ \texttt{hrm\_en} \to \texttt{hrm},\ \texttt{sy} \to \texttt{hrm\_en},\ \mathcal{O}(\neg\texttt{hrm}/\top),\ \mathcal{O}(\texttt{sy}/\texttt{des\_hrm\_en}) \right\} \nvdash_{\mathsf{bMDL}} \bot\ .$$

Proof. By Thm. 2 and Thm. 3 it is sufficient to show that the sequent

$$\Box(\texttt{hrm\_en} \to \texttt{hrm}), \Box(\texttt{sy} \to \texttt{hrm\_en}), \Box\mathcal{O}(\neg\texttt{hrm}/\top), \Box\mathcal{O}(\texttt{sy}/\texttt{des\_hrm\_en}) \Rightarrow \bot$$

is not derivable in $\mathsf{G_{bMDL}}$. This is done in the standard way by (a bit tediously) performing an exhaustive proof search following the procedure in Algorithm 1. □

4 Semantics of bMDL

The semantics for bMDL is build on the standard semantics for modal logic S4, i.e., Kripke-frames with transitive and reflexive accessibility relation [2]. The additional modality $\mathcal{O}$ is captured using *neighbourhood semantics* [4], which we modify to take into account only accessible worlds. Intuitively, the neighbourhood map singles out a set of deontically acceptable sets of accessible worlds for certain possible situations, i.e., sets of accessible worlds. As usual, if $R \subseteq W \times W$ is a relation and $w \in W$, we write $R[w]$ for $\{v \in W \mid wRv\}$. Also, for a set X we write X^c for the complement of X (relative to an implicitly given set).

Definition 5. *A* Mīmāṃsā-frame *(or briefly:* m-frame*) is a triple* (W, R, η) *consisting of a non-empty set* W *of* worlds *or* states*, an* accessibility relation $R \subseteq W \times W$ *and a map* $\eta : W \to \mathcal{P}(\mathcal{P}(W) \times \mathcal{P}(W))$ *such that:*

1. *R is transitive and reflexive;*
2. *if $(X, Y) \in \eta(w)$, then $X \subseteq R[w]$ and $Y \subseteq R[w]$;*
3. *if $(X, Z) \in \eta(w)$ and $X \subseteq Y \subseteq R[w]$, then also $(Y, Z) \in \eta(w)$;*
4. *$(\emptyset, X) \notin \eta(w)$;*
5. *if $(X, Y) \in \eta(w)$, then $(X^c \cap R[w], Y) \notin \eta(w)$.*

A Mīmāṃsā-model *(or* m-model*) is a m-frame with a* valuation $\sigma : W \to \mathcal{P}(\mathsf{Var})$.

Intuitively, Condition 1 in Def. 5 corresponds to axioms (4) and (T) of S4, Condition 2 ensures that only accessible worlds influence the truth of a formula $\mathcal{O}(\varphi/\psi)$ and comes from the rules (Mon) and (Cg), Condition 3 corresponds to the rule (Mon), while Conditions 4 resp. 5 correspond to (D_1) resp. (D_2).

Definition 6 (Satisfaction, truth set). *Let $\mathfrak{M} = (W, R, \eta), \sigma$ be a m-model. The* truth set $[\![\varphi]\!]_{\mathfrak{M}}$ *of a formula φ in $\mathfrak{M}$ is defined recursively by*

1. $[\![p]\!]_{\mathfrak{M}} := \{w \in W \mid p \in \eta(w)\}$
2. $[\![\Box\varphi]\!]_{\mathfrak{M}} := \{w \in M \mid R[w] \subseteq [\![\varphi]\!]_{\mathfrak{M}}\}$
3. $[\![\mathcal{O}(\varphi/\psi)]\!]_{\mathfrak{M}} := \{w \in W \mid ([\![\varphi]\!]_{\mathfrak{M}} \cap R[w], [\![\psi]\!]_{\mathfrak{M}} \cap R[w]) \in \eta(w)\}$

and the standard clauses for the boolean connectives. We omit the subscript $\mathfrak{M}$ if the m-model is clear from the context, and we write $\mathfrak{M}, w \Vdash \varphi$ for $w \in [\![\varphi]\!]_{\mathfrak{M}}$. A formula φ is valid *in a m-model $\mathfrak{M}$ if for all worlds w of $\mathfrak{M}$ we have $\mathfrak{M}, w \Vdash \varphi$.*

Note that in clause 3 we slightly deviate from the standard treatment in that we restrict the attention to worlds accessible from the current world.

Lemma 3. *For all rules of $\mathsf{G}_{\mathsf{bMDL}}$ we have: if the interpretations of its premisses are valid in all m-models, then so is the interpretation of its conclusion.*

Proof. We show that if the negation of the interpretation of the conclusion is satisfiable in a m-model, then so is the negation of the interpretation of (at least) one of the premisses. For 4, T and the propositional rules this is standard.

For the modal rules we only show the case of D_2, the other cases being similar. Assume that for the m-model $\mathfrak{M} = (W, R, \eta), \sigma$ the negation of the conclusion is satisfied in $w \in W$, i.e., we have $\mathfrak{M}, \sigma \Vdash \bigwedge \Gamma \wedge \mathcal{O}(\varphi/\psi) \wedge \mathcal{O}(\theta/\chi)$. Then we have $([\![\varphi]\!] \cap R[w], [\![\psi]\!] \cap R[w]) \in \eta(w)$ and $([\![\theta]\!] \cap R[w], [\![\chi]\!] \cap R[w]) \in \eta(w)$. By Cond. 5 in Def. 5 we know that $([\![\varphi]\!]^c \cap R[w], [\![\psi]\!] \cap R[w]) \notin \eta(w)$, hence $[\![\theta]\!] \cap R[w] \neq [\![\varphi]\!]^c \cap R[w]$ or $[\![\psi]\!] \cap R[w] \neq [\![\chi]\!] \cap R[w]$. If the latter does not hold, using this and Cond. 3 we have $[\![\varphi]\!]^c \cap R[w] \subsetneq [\![\theta]\!] \cap R[w]$ and hence we find a world $v \in [\![\varphi]\!] \cap [\![\theta]\!] \cap R[w]$. Then with transitivity we obtain $\mathfrak{M}, \sigma, v \Vdash \bigwedge \Gamma^{\Box} \wedge \varphi \wedge \theta$, and thus the negation of the first premiss of the rule is satisfiable. Otherwise we have $[\![\psi]\!] \cap [\![\chi]\!]^c \cap R[w] \neq \emptyset$ or $[\![\chi]\!] \cap [\![\psi]\!]^c \cap R[w] \neq \emptyset$ and again using transitivity we satisfy the negation of the second or the third premiss of the rule. ☐

Corollary 2 (Soundness of $\mathsf{G_{bMDL}}$). *For every sequent $\Gamma \Rightarrow \Delta$ we have: if $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$, then $\bigwedge \Gamma \to \bigvee \Delta$ is valid in all m-models.*

Proof. By induction on the depth of the derivation, using Lem. 3. □

For completeness we show how to construct a countermodel for a given sequent from a failed proof search for it. For this, fix $\Gamma \Rightarrow \Delta$ to be a sequent not derivable in $\mathsf{G_{bMDL}}$. We build a m-model $\mathfrak{M}_{\Gamma \Rightarrow \Delta} = (W, R, \eta), \sigma$ from a rejecting run of Alg. 1 on input $[\Gamma \Rightarrow \Delta]$, such that $\bigwedge \Gamma \wedge \bigwedge \neg \Delta$ is satisfied in a world of $\mathfrak{M}_{\Gamma \Rightarrow \Delta}$. For this, take the set W of worlds to be the set of all histories occurring in the run of the procedure. To define the accessibility relation we first construct the intermediate relation R' by setting $\mathcal{H} R' \mathcal{H}'$ iff (at least) one of the following holds:

1. $\mathcal{H} \preccurlyeq \mathcal{H}'$; or
2. $\mathcal{H}' \preccurlyeq \mathcal{H}$ and there is a transitional rule application with conclusion $\mathsf{last}(\mathcal{H})$ and a premiss $\Sigma \Rightarrow \Pi$ of this rule application such that $\Sigma \subseteq \mathsf{last}_L(\mathcal{H}')$ and $\Pi \subseteq \mathsf{last}_R(\mathcal{H}')$.

Intuitively, in 2. we add the loops which have been detected by the procedure. The relation R then is defined as the reflexive and transitive closure of R'. To define the function η we first introduce a syntactic version of the truth set notation:

$$|\varphi|_W := \{\mathcal{H} \in W \mid \varphi \in \mathsf{last}_L(\mathcal{H})\}$$

Now we define $\eta : W \to \mathcal{P}(\mathcal{P}(W) \times \mathcal{P}(W))$ by setting for every history $\mathcal{H}$ in W:

$$\eta(\mathcal{H}) := \left\{ (X, Y) \in \mathcal{P}(R[\mathcal{H}])^2 \mid \begin{array}{c} \text{for some formula } \mathcal{O}(\varphi/\psi) \in \mathsf{last}_L(\mathcal{H}) : \\ |\varphi|_W \cap R[\mathcal{H}] \subseteq X \text{ and } |\psi|_W \cap R[\mathcal{H}] = Y \end{array} \right\} .$$

Finally, we define the valuation σ by setting for every variable $p \in \mathsf{Var}$:

$$\sigma(p) := |p|_W .$$

Let us write $\mathfrak{M}_{\Gamma \Rightarrow \Delta}$ for the resulting structure (W, R, η). Then we have:

Lemma 4. *The structure $\mathfrak{M}_{\Gamma \Rightarrow \Delta}, \sigma$ is a m-model.*

Proof. By construction σ is a valuation, R is a transitive and reflexive relation on W, and Conditions 2 and 3 of Def. 5 hold for η. To see that Condition 5 holds, we need to show that if $(X, Y) \in \eta(\mathcal{H})$ then $(X^c \cap R[\mathcal{H}], Y) \notin \eta(\mathcal{H})$. For this we show that whenever $(X, Y) \in \eta(\mathcal{H})$ and $(Z, W) \in \eta(\mathcal{H})$, then $Z \neq X^c \cap R[\mathcal{H}]$ or $Y \neq W$. So assume we have such (X, Y) and (Z, W) in $\eta(\mathcal{H})$. By construction of η there must be formulae $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\theta/\chi)$ in $\mathsf{last}_L(\mathcal{H})$ such that

- $|\varphi|_W \cap R[\mathcal{H}] \subseteq X$ and $|\psi|_W \cap R[\mathcal{H}] = Y$; and
- $|\theta|_W \cap R[\mathcal{H}] \subseteq Z$ and $|\chi|_W \cap R[\mathcal{H}] = W$.

Since both $\mathcal{O}(\varphi/\psi)$ and $\mathcal{O}(\theta/\chi)$ are in $\mathsf{last}_L(\mathcal{H})$, the transitional rule D_2 can be applied to $\mathsf{last}(\mathcal{H})$. Thus the proof search procedure either used the premisses

$$\mathsf{last}_L(\mathcal{H})^{\Box}, \varphi, \theta \Rightarrow \qquad \mathsf{last}_L(\mathcal{H})^{\Box}, \psi \Rightarrow \chi \qquad \mathsf{last}_L(\mathcal{H})^{\Box}, \chi \Rightarrow \psi$$

of this rule application to create new histories by appending them to $\mathcal{H}$, or it found a history $\mathcal{H}' \preccurlyeq \mathcal{H}$ whose last sequent subsumes one of the premisses. In either case for at least one premiss $\Sigma \Rightarrow \Pi$ there is a history $\mathcal{H}'$ s.t. $\Sigma \subseteq \mathsf{last}_L(\mathcal{H}')$ and $\Pi \subseteq \mathsf{last}_R(\mathcal{H}')$ and for which proof search fails. Moreover, for this $\mathcal{H}'$ by construction of R we know that $\mathcal{H}R\mathcal{H}'$. Assume that $\Sigma \Rightarrow \Pi$ is the first premiss. Then $\varphi, \theta \in \mathsf{last}_L(\mathcal{H}')$, and hence $\mathcal{H}' \in |\varphi|_W \cap |\theta|_W \cap R[\mathcal{H}]$ and the latter is non-empty. Then in particular $X^c \cap R[\mathcal{H}] \subseteq (|\varphi|_W \cap R[\mathcal{H}])^c \cap R[\mathcal{H}] = (|\varphi|_W)^c \cap R[\mathcal{H}]$ is not equal to $|\theta|_W \cap R[\mathcal{H}] = Z$. Similarly, if $\Sigma \Rightarrow \Pi$ is one of the remaining premisses we obtain $Y \neq W$. Thus whenever $(X, Y) \in \eta(\mathcal{H})$ and $(Z, W) \in \eta(\mathcal{H})$, then $Z \neq X^c \cap R[\mathcal{H}]$ or $Y \neq W$. The reasoning for Cond. 4 is similar. □

Lemma 5 (Truth Lemma). *For every history $\mathcal{H} \in W$: (i) If $\varphi \in \mathsf{last}_L(\mathcal{H})$, then $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, \sigma, \mathcal{H} \Vdash \varphi$ and (ii) if $\psi \in \mathsf{last}_R(\mathcal{H})$, then $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, \sigma, \mathcal{H} \Vdash \neg\psi$.*

Proof. We prove both statements simultaneously by induction on the complexity of φ resp. ψ. The base case and the cases where the main connective of φ resp. ψ is a propositional or $\Box$ are standard (note that Alg. 1 saturates every sequent under the static rules, i.e., the propositional rules and T, and that every transitional rule copies all the boxed formulae in the antecedent into the premisses). If $\varphi = \mathcal{O}(\theta/\chi)$, then by construction of η we have $(|\theta|_W \cap R[\mathcal{H}], |\chi|_W \cap R[\mathcal{H}]) \in \eta(\mathcal{H})$, and thus $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, \sigma, \mathcal{H} \Vdash \mathcal{O}(\theta/\chi)$. Now suppose that $\psi = \mathcal{O}(\xi/\gamma)$. To see that ψ does not hold in $\mathcal{H}$ we show that for no $\mathcal{O}(\delta/\beta) \in \mathsf{last}_L(\mathcal{H})$ we have $|\delta|_W \cap R[\mathcal{H}] \subseteq |\xi|_W \cap R[\mathcal{H}]$ and $|\beta|_W \cap R[\mathcal{H}] = |\gamma|_W \cap R[\mathcal{H}]$. The result then follows by construction of η and the definition of truth set. If $\mathsf{last}_L(\mathcal{H})$ does not contain any formula of the form $\mathcal{O}(\delta/\beta)$, then $\eta(\mathcal{H})$ is empty and we are done. Otherwise, there is such a $\mathcal{O}(\delta/\beta)$ and the rule Mon can be applied backwards to $\mathsf{last}(\mathcal{H})$. But then from the failed proof search for at least one of the premisses

$$\mathsf{last}_L(\mathcal{H})^{\Box}, \delta \Rightarrow \xi \qquad \mathsf{last}_L(\mathcal{H})^{\Box}, \gamma \Rightarrow \beta \qquad \mathsf{last}_L(\mathcal{H})^{\Box}, \beta \Rightarrow \gamma$$

we obtain a history $\mathcal{H}'$ with $\mathcal{H}R\mathcal{H}'$ whose last sequent subsumes this premiss. But then as above either $|\delta|_W \cap R[\mathcal{H}] \not\subseteq |\xi|_W \cap R[\mathcal{H}]$, if it is obtained from the first premiss, or $|\beta|_W \cap R[\mathcal{H}] \neq |\gamma|_W \cap R[\mathcal{H}]$ otherwise. □

Theorem 5 (Completeness). *For every sequent $\Gamma \Rightarrow \Delta$ we have: if $\bigwedge \Gamma \to \bigvee \Delta$ is valid in every m-model, then $\vdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$.*

Proof. If $\nvdash_{\mathsf{G_{bMDL}}} \Gamma \Rightarrow \Delta$, then by Lem. 2 and Prop. 1 the procedure in Alg. 1 terminates and rejects the input $[\Gamma \Rightarrow \Delta]$. Thus by Lem. 4 and 5 we have $\mathfrak{M}_{\Gamma\Rightarrow\Delta}, [\Gamma \Rightarrow \Delta] \Vdash \bigwedge \Gamma \wedge \neg \bigvee \Delta$ and hence $\bigwedge \Gamma \to \bigvee \Delta$ is not m-valid. □

Since only finitely many histories occur in a run of the proof search procedure, the constructed model is finite and by standard methods we immediately obtain:

Corollary 3. *The logic bMDL has the finite model property and is decidable.* □

5 Applications to Indology

We show now that despite being reasonably simple, bMDL is strong enough to derive consequences about topics discussed by Mīmāṃsā authors (Example 1) and to provide useful insights on the reason why the seemingly conflicting statements in the Śyena example are not contradictory.

Example 1. Consider the following excerpt: "*Since the Veda is for the purpose of an action, whatever in it does not aim at an action is meaningless and therefore must be said not to belong to the permanent Veda*" (PMS 1.2.1). In other words: each Vedic prescription should promote an action. Given that no actual action can have a logical contradiction as an effect, a logical contradiction cannot be enjoined by an obligation. This can be translated into the formula $\neg\mathcal{O}(\bot/\theta)$, one of the forms of axiom D, which is derivable in $\mathsf{G_{bMDL}}$ as follows:

$$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow}\ \bot\Rightarrow}{\mathcal{O}(\bot/\theta) \Rightarrow \neg\mathcal{O}(\bot/\theta)}\ \mathsf{D}_1}{\Rightarrow \neg\mathcal{O}(\bot/\theta)}\ \Rightarrow\neg$$

A logical perspective on the Śyena controversy

In Mīmāṃsā literature many explanations of the reasons why the sentences A. and B. in Sec. 2 are not contradictory have been proposed. We show that the bMDL solution matches the one of Prabhākara, one of the chief Mīmāṃsā authors, and makes it formally meaningful.

Consider the sequent in the proof of Thm. 4. Since it is not derivable in $\mathsf{G_{bMDL}}$, using Algorithm 1 we can construct a model for the formula

$$\Box(\mathtt{hrm\_en} \to \mathtt{hrm}) \wedge \Box(\mathtt{sy} \to \mathtt{hrm\_en}) \wedge \Box\mathcal{O}(\neg\mathtt{hrm}/\top) \wedge \Box\mathcal{O}(\mathtt{sy}/\mathtt{des\_hrm\_en}) \quad (1)$$

However, to make the solution clearer, we define below a simpler model $\mathfrak{M}_0 = (W_0, R_0, \eta_0), \sigma_0$ based on Vedic concepts. The domain W_0 is $\{w_i \mid 1 \leq i \leq 8\}$, represented in Fig. 3 by circles. The accessibility relation R_0 is universal, i.e. for any $1 \leq i, j \leq 8$ it holds that $R_0(w_i, w_j)$; it is not represented in the figure for better readability. The map η_0 is such that $\eta_0(w_i) = \{(X, W_0) \mid X \subseteq W_0,\ \{w_1, w_5\} \subseteq X\} \bigcup \{(Y, \{w_5, w_6, w_7, w_8\}) \mid Y \subseteq W_0,\ \{w_4, w_8\} \subseteq Y\}$. The figure represents only the elements of the neighbourhood of w_1 that are relevant to the valuation of our deontic statements. Each element corresponds to a kind of arrow: solid arrows for the statement about Śyena and dashed ones for the obligation not to harm anyone. An element of the neighbourhood is a pair of sets of states, to represent it we draw an arrow from each state belonging to the second element of the pair to each one belonging to the first element of the pair. The function σ_0 is the valuation of the model and it is such that $\sigma_0(w_1) = \emptyset$; $\sigma_0(w_2) = \{\mathtt{hrm}\}$; $\sigma_0(w_3) = \{\mathtt{hrm}, \mathtt{hrm\_en}\}$; $\sigma_0(w_4) = \{\mathtt{hrm}, \mathtt{hrm\_en}, \mathtt{sy}\}$; $\sigma_0(w_5) = \{\mathtt{des\_hrm\_en}\}$; $\sigma_0(w_6) = \{\mathtt{hrm}, \mathtt{des\_hrm\_en}\}$; $\sigma_0(w_7) = \{\mathtt{hrm}, \mathtt{hrm\_en}, \mathtt{des\_hrm\_en}\}$; and $\sigma_0(w_8) = \{\mathtt{hrm}, \mathtt{hrm\_en}, \mathtt{sy}, \mathtt{des\_hrm\_en}\}$. Clearly $\mathfrak{M}_0$ satisfies all the requirements stated in Def. 5.

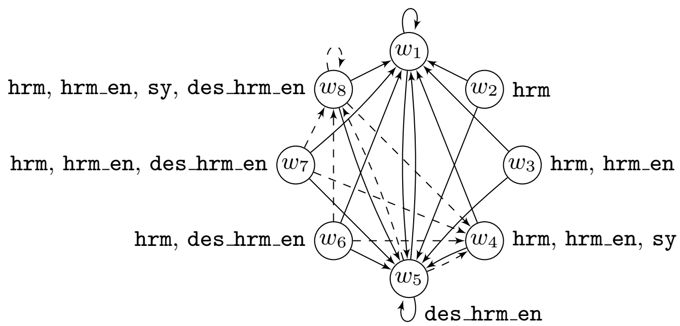

Fig. 3. The model $\mathfrak{M}_0$ for the Śyena controversy

The definition of $\mathfrak{M}_0$ is based on *adhikāra* ([5], pp.147-155), a central concept in Prabhākara's analysis of the Vedas, which identifies the addressee of a prescription through their desire for the results. In the prescription about the Śyena sacrifice, the *adhikāra* corresponds to the desire to harm an enemy; the results correspond to the fact that an enemy is harmed through the performance of Śyena, and, more generally, to the fact that someone is harmed. Some combinations of these facts are impossible if we need to satisfy $\Box(\mathtt{hrm\_en} \to \mathtt{hrm})$ and $\Box(\mathtt{sy} \to \mathtt{hrm\_en})$, thus all the possibilities are the eight states in the model. The accessibility relation accounts for the possible changes of subject's condition. The neighbourhood of a state encodes the obligations holding for that state, and given that these obligations are the same for each state, the neighbourhood is the same too. Thus the arrows show the changes of condition promoted by the obligations.

We show now that the formula (1) is true in the state w_1. First, all its conjuncts without deontic operators are true in all states. Secondly, the formula $\Box\mathcal{O}(\neg\mathtt{hrm}/\top)$ is true in w_1 if $([\![\neg\mathtt{hrm}]\!]_{\mathfrak{M}_0} \cap R_0[s], [\![\top]\!]_{\mathfrak{M}_0} \cap R_0[s]) \in \eta_0(s)$ holds for all s such that $R_0(w_1, s)$. Given that $(\{w_1, w_5\}, W_0)$ belongs to $\eta_0(s)$ for all $s \in W_0$, the formula $\mathcal{O}(\neg\mathtt{hrm}/\top)$ is true in all states. For the formula $\Box\mathcal{O}(\mathtt{sy}/\mathtt{des\_hrm\_en})$ the valuation is similar. Hence $\mathfrak{M}_0$ is a model of (1) and, by Thm. 2 and 3, this provides a semantic proof of Thm. 4.

Among the different solutions for the Śyena controversy, the model $\mathfrak{M}_0$ matches Prabhākara's one which can be summarised in his statement: "*A prescription regards what has to be done. But it does not say that it has to be done*" (*Bṛhatī* I, p. 38, l. 8f). Indeed in state w_1 no conflicting prescriptions are applicable and all obligations are fulfilled. We call this a *Vedic state*. The existence of such a state shows that an agent can find a way not to transgress any prescription, and that the Vedic prescriptions do not imply that the Śyena sacrifice has necessarily to be done. Our model also explains Prabhākara's claim that *the Vedas do not impel one to perform the malevolent sacrifice Śyena, they only say that it is obligatory*, which was wrongly considered meaningless e.g. in [18].

Remark 1. Our analysis highlights that Vedic prescriptions are "instructions to attain desired outcomes" rather than absolute imperatives. A *Vedic state* provides a way not to transgress any obligation, but at the same time there are norms, e.g., the one about Śyena, for those who intend to transgress some obligations, but nonetheless do not want to altogether reject the Vedic principles. This is explicit in another Mīmāṃsā author, Veṅkaṭanātha, who claims that the Śyena is the best way to kill one's enemy if one is determined to transgress the general prescription not to perform violence. This feature suggests a possible use of suitable extensions of bMDL to reason about machine ethics, where indeed choices between actions that should be avoided often arise. Consider a self-driving vehicle that has no choice but to harm some people. There is no perfect solution but, nevertheless, the system should be able to provide instructions that promote imperfect outcomes in order to avoid the worst-case scenario.

6 Conclusions and Future Work

We defined a novel deontic logic justified by principles elaborated by Mīmāṃsā authors over the last 2,500 years, and used its proof theory and semantics to analyse a notoriously challenging example. The fruits of this synergy of Logic and Indology can be gathered from both sides: The vast body of knowledge constituted by Mīmāṃsā texts can provide interesting new stimuli for the logic community, and at the same time our methods can lead to new tools for the analysis of philosophical and sacred texts. Our investigation also raises a number of further research directions, such as (i) a formal analysis of the concept of prohibition as discussed by Mīmāṃsā authors. Moreover, (ii) among the about 200 considered[4] *nyāya*s (50 of which were on deontic principles), some hinted at the need for extending bMDL in various directions: e.g., the principle "*the agent of a duty needs to be the one identified by a given prescription*" (PMS 6.1.1–3) seems to require first-order quantification; some metarules that distinguish between different repetitions of the same action suggest the introduction of *temporal operators*; finally the fact that ŚBh 1.1.1 asserts that the Vedas prevail over other authoritative texts suggests the need of a system to manage conflicts among different authorities, a feature also important for reasoning about ethical machines [3]. Finally, (iii) while the metarules considered for bMDL are common to the Mīmāṃsā school, there are additional principles employed only by specific authors. Their identification and formalisation might shed light on the strength of the different interpretations of various Mīmāṃsā authors and, e.g., help arguing for the conjecture that Kumārila's interpretation is more explicative than Maṇḍana's.

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[4] Not all Mīmāṃsā metarules have been translated from Sanskrit so far, see [6].

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