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These Degrees go to Eleven: Fuzzy Logics and Gradable Predicates

Petr Cintula[a] Berta Grimau[b] Carles Noguera[c] Nicholas J.J. Smith[d]

[a] Institute of Computer Science of the Czech Academy of Sciences
Pod vodárenskou věží 2, 182 07 Prague, Czech Republic
cintula@cs.cas.cz

[b] Institute of Information Theory and Automation of the Czech Academy of Sciences
Pod vodárenskou věží 4, 182 08 Prague, Czech Republic
grimau@utia.cas.cz

[c] Department of Information Engineering and Mathematics, University of Siena
San Niccolò, via Roma 56, 53100 Siena, Italy
carles.noguera@unisi.it

[d] Philosophy, The University of Sydney
Main Quadrangle A14, NSW 2006, Australia
nicholas.smith@sydney.edu.au

Abstract

In the literature on vagueness one finds two very different kinds of degree theory. The dominant kind of account of gradable adjectives in formal semantics and linguistics is built on an underlying framework involving bivalence and classical logic: its degrees are not degrees of truth. On the other hand, fuzzy logic based theories of vagueness—largely absent from the formal semantics literature but playing a significant role in both the philosophical literature on vagueness and in the contemporary logic literature—are logically nonclassical and give a central role to the idea of degrees of truth. Each kind of degree theory has a strength: the classical kind allows for rich and subtle analyses of the comparative form of gradable adjectives and of various types of gradable precise adjectives, while the fuzzy kind yields a compelling solution to the sorites paradox. This paper argues that the fuzzy kind of theory can match the benefits of the classical kind and hence that the burden is on the latter to match the advantages of the former. In particular, we develop a new version of the fuzzy logic approach that—unlike existing fuzzy theories—yields a compelling analysis of the comparative as well as an adequate account of gradable precise predicates, while still retaining the advantage of genuinely solving the sorites paradox.

1 Introduction

This paper looks at two approaches to the semantics of vagueness. Both give a central role to *degrees*—yet they are fundamentally different. On the one hand we have degree-based accounts of the semantics of gradable adjectives. This is the dominant kind of account in the formal semantics literature.[1] On the other hand we have fuzzy logic based theories of

[1]See e.g. Bartsch and Vennemann [1973], Seuren [1973], Cresswell [1976], Stechow [1984], Bierwisch [1989], Kennedy [1997], Kennedy and McNally [2005] and Kennedy [2007].

vagueness. Such theories are largely absent from the formal semantics literature, but play a significant (although not dominant) role in both the philosophical literature on vagueness and in the contemporary logic literature.[2] The latter approach is, from a logical point of view, nonclassical. In particular, it gives a central role to the idea of degrees of *truth*. The former approach, by contrast, is built on an underlying framework involving bivalence and classical logic: it gives a central role to degrees, but they are not degrees of truth.

In this paper we note symmetrical challenges for each kind of approach: there is something that each does well, and the challenge for the other approach is to develop an equally good way of doing it. What the former, classical approach does well is yield an adequate analysis of the comparative as well as account for the meaning of various classes of precise gradable predicates. What the latter, nonclassical approach does well is solve the sorites paradox. In this paper we shall develop a new version of the fuzzy logic approach that yields compelling analyses of the comparative form as well as of various types of gradable precise adjectives, while in turn retaining the advantage of genuinely solving the sorites paradox. Our overall conclusion is that the new version of the fuzzy theory of vagueness proposed here fares considerably better than previous fuzzy accounts and that, when it comes to the sorites, the burden is on the classical approach to show that it can do as well as the fuzzy approach.

Let us clarify the scope of this paper. First, the argument that fuzzy theories of vagueness handle the sorites better than other kinds of theory—including the kinds of classical theory that underly the non-fuzzy degree-based approach to the semantics of vagueness mentioned above—is presented in full elsewhere [Smith, forthcoming]. In this paper, we do not repeat the argument in its entirety and we do not consider what we say in this paper—taken by itself—to constitute a compelling argument that fuzzy theories of vagueness handle the sorites better than other kinds of theory. Here our aim is more limited. We sketch just enough of the fuzzy solution to the sorites presented in Smith [forthcoming] to show that the *new* version of the fuzzy theory of vagueness proposed in this paper can still make use of this solution. That is, in moving to a new version of the fuzzy theory that yields compelling analyses of the comparative form as well as of various types of gradable precise adjectives, we do not break the solution to the sorites that is available to the earlier version of the fuzzy theory of vagueness.

Second, as is well-known, fuzzy logics face some important objections. Of course, solving the sorites and providing semantics for degree constructions found in natural languages are not the only challenges they face. In fact, they have been the target of significant scrutiny and criticism well beyond these particular issues. One of these objections points to its truth-functionality assumption and argues that fuzzy theories do not provide an adequate account of logically complex sentences. This is done mostly by pointing to alleged counterexamples to the consequences of the fuzzy truth functions used to interpret the connectives. Without going into detail, since this topic falls outside the scope of this paper and it has been addressed at length elsewhere (Smith [2008] and Smith [2017]), let us just say that we think there are good replies to this objection. On the one hand, most of these objections are based on the intuitions of their authors, which can often be called into doubt. On the other, they often focus on one particular version of fuzzy logic (what has been called 'philosophers' fuzzy logic'), which need not be endorsed. There are alternatives which lead to more intuitively correct results. This claim, of course, requires further justification, but we leave it aside in this paper. For more detail, see the references above. Another important objection claims that fuzzy logic leads to artificial precision, in the sense that it is artificial or implausible to associate each vague sentence in natural language with a particular real number between 0 and 1 as its degree of truth and thus that any choice of interpretation for the atomic sentences of our language is going to be

[2]See e.g. Goguen [1968–69], Lakoff [1973], Machina [1976] and Smith [2008] on the philosophical side, and Klir and Yuan [1995], Hájek [1998], Novák et al. [1999], Nguyen and Walker [2000] and Cintula et al. [2011 and 2015] on the logical side.

unwarranted. Once again, and without going into details since this, too, has been tackled elsewhere [Smith, 2008], we think there is a good solution to this problem: endorse a form of plurivaluationism. The idea is that the problem of artificial precision does not arise from the fuzzy component of fuzzy theories, but rather because of their assumption that there is only one unique correct model for our language. Drop this assumption and the problem dissipates. What abandoning the assumption of uniqueness amounts to is the acceptance that any discourse has not one but multiple equally correct fuzzy models. These models must all be compatible with our linguistic behaviour (e.g. if all speakers agree that a certain individual is clearly tall in a given context, then any adequate fuzzy model will classify that individual as being definitely tall). In our view, fuzzy plurivaluationism provides an answer to the problem of artificial precision. For further discussion of these matters see the references provided (and cf. also n.26 below).

The paper proceeds as follows. §2 sketches a fuzzy logic based solution to the sorites and points to problems for attempts to solve the sorites that maintain bivalence. §3 presents the technical details behind the fuzzy solution of §2. §4 raises two challenges for this fuzzy approach: on the one hand, that of adequately analysing the comparative form of adjectives; on the other, that of extending the fuzzy account of vagueness to a wider account of gradability. §5 presents the technical details of a kind of fuzzy logic that is somewhat different from the kind presented in §3. §6 shows how the earlier fuzzy solution to the sorites still goes through in the context of the new kind of fuzzy logic, and shows how this new logic provides the basis for a semantics that meets the two challenges above. §7 concludes.

A couple of comments about the structure of the paper are in order before starting. In order to present the version of fuzzy logic that is suitable for our purposes (in §5), uninorm-based logics, we first (in §3) present a closely related class of logics, (left-continuous) t-norm based logics. The point of presenting not only the formalism that we endorse but also some closely related logics is informational. Some of these logics will be much better known by our readers than uninorm-based logics and we think that learning about our proposed formalism by comparing it with those better known alternatives will prove useful. Moreover, note that in §5 we will describe in detail a specific example of a uninorm-based logic — one that does all the jobs we need them to do. This has a twofold aim. First, pedagogical: to illustrate with a particular example what the type of formalism that we require looks like. But it serves another purpose as well: to prove that there is at least one formalism checking all the boxes that we need. However, note that we are not married to this particular logic. Any system that shared with it the same algebraic properties would do.

2 Vagueness and Fuzzy Logic

Vagueness is centrally a property of predicates—for example 'bald', 'tall' and 'heavy'. Predicates that are not vague are said to be precise—for example 'under 1.4m in height', 'weighs at least 500 grams' and 'north of the equator'. Vague predicates are usually identified as those possessing three characteristics: the boundaries of their extensions are blurry; they exhibit borderline cases (things of which we will neither confidently assert nor confidently deny that they fall under the predicate); and they give rise to sorites paradoxes.[3] To explain the latter point: A *sorites series* for a predicate F is a series of objects with the following characteristics:

1. F definitely applies to the first object in the series

2. F definitely does not apply to the last object in the series

3. Each object in the series (except the last one, which has no object after it) is extremely similar to the object after it in all respects relevant to the application of F.

[3]See e.g. Keefe and Smith [1997], Smith [2008].

For example, a series of blocks of concrete ranging in weight from 100kg to 1kg in increments of a gram is a sorites series for 'heavy' and for 'weighs at least 50kg'; a series of pieces of string ranging in length from 100m to 1cm in increments of a millimetre is a sorites series for 'long' and for 'at least 1m in length'; and so on. Given a sorites series, we can generate an associated *sorites argument*:[4]

1. x_1 is F
2. For every x, if x is F then x' is F

   (Or: There is no x such that x is F and x' isn't F. Or all of the following: If x_1 is F then x_2 is F, …, If x_{n-1} is F then x_n is F. Or all of the following: It is not the case both that x_1 is F and x_2 isn't F, …, It is not the case both that x_{n-1} is F and x_n isn't F.)
3. Therefore x_n is F.

Where F is a precise predicate (e.g. 'weighs at least 50 kg'), the sorites argument will be obviously mistaken: there clearly is an x such that x is F and x' is not F. Where F is a vague predicate (e.g. 'is heavy'), the sorites argument will seem genuinely paradoxical: the premises all seem to be true, the reasoning seems to be correct, and yet the conclusion seems to be false. Thus vague predicates, unlike precise ones, give rise to sorites paradoxes.

A requirement on any theory of vagueness is that it solves the sorites paradox. In the current literature on vagueness it is generally agreed that there are two aspects to such a solution. One task is to locate the error in the sorites argument. The second task is to explain why the sorites argument constitutes a paradox as opposed to a simple fallacy.[5] Thus a theory of vagueness must explain why competent speakers find the sorites argument somewhat but not ultimately compelling: why they do not spot the error immediately (as they do when the predicate involved in the sorites argument is precise) and yet are not convinced by the argument to accept the conclusion.

Among existing approaches to vagueness, we believe that the kind best placed to meet this explanatory task is that based on the fundamental guiding idea that truth comes in degrees and formalised using a fuzzy logic.[6] Technical details will be given below, but let us begin with the basic idea, which is that statements may be entirely false—represented by the truth value 0—or completely true—represented by the truth value 1—or have one of infinitely many intermediate degrees of truth—represented by the real numbers greater than 0 and less than 1.[7] Consider for example a series of piles of sand 1 through 10 000, where pile i has i grains of sand and each pile is of a very similar shape to its neighbour(s). The idea will be that claims such as 'Pile 1 is a heap' concerning piles at the end of the series are fully false, claims such as 'Pile 10 000 is a heap' concerning piles at the beginning of the series are fully true, and in between, claims of the form 'Pile n is a heap' have intermediate degrees of truth. Connectives will be associated with functions on the degrees of truth which allow truth values to be assigned to complex formulas in a way analogous to classical logic. For example we might define the truth values of conditional statements as follows (where $||\alpha||$ represents the truth value of α):

$$||\alpha \to \beta|| \quad = \quad \begin{cases} 1 & \text{if } ||\alpha|| \leq ||\beta|| \\ 1 - (||\alpha|| - ||\beta||) & \text{if } ||\alpha|| > ||\beta|| \end{cases}$$

[4] $x_1, \ldots, x_n$ denote the objects in the series from first (x_1) through to last (x_n). x ranges over all objects in the sorites series except the last object and x' is the object immediately after x in the series.

[5] See e.g. Fara [2000].

[6] As mentioned in §1, the argument that fuzzy theories of vagueness handle the sorites better than other kinds of theory is presented elsewhere [Smith, forthcoming] and in this paper we do not repeat the argument in full.

[7] The fundamental idea is that truth comes in degrees. The representation of these degrees of truth by real numbers between 0 and 1 inclusive is convenient but not essential to the very idea of degrees of truth, which could potentially be modelled using other structures. We shall discuss this issue further below.

This is known as the Łukasiewicz conditional. The idea is that its truth value drops below full truth to precisely the extent to which the truth value of the consequent drops below that of the antecedent.

Now consider the following sorites argument:

Pile 10 000 is a heap.

If pile 10 000 is a heap then pile 9 999 is a heap.

If pile 9 999 is a heap then pile 9 998 is a heap.

$\vdots$

If pile 2 is a heap then pile 1 is a heap.

$\therefore$ Pile 1 is a heap.

Let's suppose that 'if... then...' here is read as the Łukasiewicz conditional and that we define validity as follows: on every model on which every premise is true to degree 1, the conclusion is true to degree 1. Then we get the following solution to the sorites. The problem with the argument is that, although it is valid, it is unsound (i.e. it is not the case that every premise is true to degree 1). The first premise is true to degree 1. As for the conditionals, at first both antecedent and consequent are true to degree 1, and so are the conditionals. As we move along the series, we get to a point at which the antecedents are ever so slightly more true than the consequents. In this region, the conditionals are true to a degree ever so slightly less than 1. This continues for a while until both antecedent and consequent are true to degree 0, and hence the conditionals are true to degree 1 again. So why is the argument compelling? Because all the premises are *very nearly* true to degree 1. In normal contexts, we are naturally inclined simply to accept something as true when it is very nearly true—this is a useful approximation. Of course, once we see where the argument leads, we may well reconsider.[8]

Other approaches to vagueness do not offer an equally compelling explanation of why the sorites is a paradox rather than a simple fallacy. Consider for example epistemicism. On this approach, the semantics and logic of vagueness are entirely classical. Vague predicates—like precise ones—have crisp sets as their extensions. Classical reasoning is correct in the presence of vagueness just as it is in the precise realm of mathematics. This is not to say that there is no distinction at all between vague predicates and precise ones: while they are the same from the semantic and logical points of view, there is an epistemological difference between them. Although the extensions of all predicates are crisp sets, with vague predicates we cannot *know* where the borders lie. Thus, according to epistemicism, the blurriness of the boundaries of a vague predicate F is of an epistemic sort: in themselves the boundaries are perfectly sharp but they are hidden behind a veil of ignorance. For all objects x, Fx is true or false; the borderline cases are objects x for which we cannot *know* whether Fx is true or false. The first part of the epistemicist solution to the sorites paradox—saying what is wrong with the argument—is straightforward. According to the epistemicist, there is a sharp cut-off in the sorites series between the last object that is P and the first object that isn't—so the second premise is false. The second part of the solution—saying why we find the argument compelling—is more subtle. According to the epistemicist, we cannot know where the cut-off is—and so we mistakenly think that there is no such cut-off. This is why we are inclined to accept the second premise even though it is in fact false. But this solution faces a serious problem: it does not fit with the usual methodology and rationale of formal semantics of natural language. In formal semantics, the idea is to make explicit the semantic theory that is implicit in the ordinary usage of competent speakers: to codify the semantic facts, implicit grasp of which leads speakers to

[8]Cf. Forbes [1983, pp.243–4], Forbes [1985, pp.171–2], Williamson [1994, pp.123–4], Paoli [2003, p.365] and Paoli [forthcoming].

use the language the way they do. The problem for the epistemicist is that a speaker who internalised the epistemicist's semantic theory would not react to the sorites in the way ordinary speakers do.[9] To put it another way, the epistemicist explanation of speakers' reactions to the sorites argument turns on their being fundamentally mistaken about the semantics of the predicates they are using. For suppose that a speaker did realise that a predicate F had sharp but unknowable boundaries. Then she would not think that the conditional premises of the sorites argument for F are all *true*—even though she would indeed be unable to mark the cut-off point in the sorites series between the F's and the non-F's. For example, suppose we define the predicate 'bearfast' to apply to all and only objects that are moving faster than any polar bear moved on 11th January 1904. We know that some objects are bearfast (e.g. the jet plane flying overhead) and that some are not (e.g. the parked car across the street) but there are many things of which we will neither confidently assert nor confidently deny that they are bearfast—and it seems that we will never be able to gain the information required to classify these cases one way or the other. Yet 'bearfast' does not generate sorites paradoxes. We can set up a sorites series for 'bearfast', beginning with an object moving at great speed and progressing by tiny increments to a stationary object—but the associated sorites argument will be obviously mistaken: there is indeed some object x in the series such that x is bearfast and x' is not. Of course we do not—cannot—know which object it is: but this will not make us think (even for a moment) that the second premise of the sorites argument is *true*. So if ordinary speakers implicitly thought that vague predicates work the way epistemicists say such predicates work, they would not behave as they do with vague predicates. In particular, they would regard sorites arguments for vague predicates as obvious mistakes rather than genuine paradoxes. This sets the epistemicist approach at odds with the usual rules of the game in formal semantics.

Note that the fuzzy approach faces no such problem. In the explanation given above of why the sorites argument is compelling, we do not suppose that speakers mistakenly think that the premises are fully true. Rather, we exploit the fact that someone who takes a statement to be extremely close to fully true would naturally just go along with the statement—at least in normal contexts and until trouble was seen to arise. In everyday contexts we *round* values up or down: not just truth values, but all values. This rounding heuristic—near enough is good enough—is, in general, useful. Sometimes of course it leads to trouble—but in such cases we just refrain from applying it. Indeed the fuzzy theorist can argue that the sorites is just such a case and that this is precisely why, although ordinary speakers are initially willing to go along with the premises, they do not ultimately find the argument fully convincing. The fuzzy theorist can, then, explain speakers' reactions to the sorites in a way compatible with the idea that ordinary speakers internalise the semantic theory that the fuzzy theorist is proposing for vague language. Thus the fuzzy theory runs the gauntlet of presenting a semantic theory according to which the sorites argument is unsound, and yet is such that a speaker who implicitly accepted that semantic theory would not reject the sorites argument as clearly fallacious but would find it compelling. This apparently impossible task is achieved thanks to the general purpose rounding heuristic which initially leads speakers to go along with a premise that they do not take to be fully true.

Other rivals to the fuzzy approach based on two truth values (even views that allow truth value gaps) face the same sort of problem as epistemicism. As mentioned, this has been argued at length in Smith [forthcoming] and the full details will not be repeated here—but let us briefly mention contextualist approaches, because they tend to be favoured by

[9] On terminology: Suppose that some speakers' implicit grasp of certain semantic facts leads them to use language the way they do—and suppose that a certain semantic theory T is a correct codification of those semantic facts. In such a situation, we say that those speakers 'internalise' that semantic theory. This is purely for ease of expression and should not be taken to suggest (for example) that the semantic theory comes first and then speakers learn it. On the contrary, the speakers' usage comes first and the semantic theory seeks to codify those semantic facts, implicit grasp of which leads speakers to use language as they do.

formal semanticists who take a bivalent degree-based approach to vague adjectives. There are many varieties of contextualism but at a certain level of abstraction the core idea is that vague predicates have crisp classical sets as their extensions, but these extensions shift easily and frequently. In particular, they shift in such a way that one cannot truly claim that a is F and b is not F, where a and b are very similar in F-relevant respects (e.g. they are adjacent members of a sorites series for F). While, in a sense, the thought one is trying to express in making such a claim was true before one tried to say it, the act of saying it shifts the context in such a way that it is not true in the context in which one in fact says it.[10] Such accounts are very well placed to explain why speakers cannot identify a boundary between the F's and the non-F's, i.e. an a and b such that a is F and b is not F. However, they are unable to explain why speakers who internalise a contextualist semantics go along with the second premise of the sorites. For this premise is clearly false on every model of the kind that contextualists countenance—and speakers who internalise the contextualist semantics would know this. Contextualists must resort to claiming that ordinary speakers make some sort of *mistake*.

This flouts the rules of the game in formal semantics, where the project is to come up with a semantic theory that codifies what competent speakers internalise, knowledge of which explains why they speak the way they do. For example, Kennedy writes:

> the reason that we are unwilling to reject the second premise is the same as the reason that the positive form is infelicitous in examples involving crisp judgments: doing so would invoke a context (the one that falsifies the universal generalization) in which one of two objects that differ minimally in the property measured by g is claimed to stand out relative to g and the other not. [Kennedy, 2007, 19]

But the semantic theorist can survey the possible interpretations and note that the second premise is false on all of them—indeed that is what Kennedy does earlier in the same paragraph:

> we may assume that in any context, there is in fact a cutoff point between the objects that the positive form is true of and those it is false of based on the degree determined by **s**: the minimal degree needed to 'stand out' relative to g in the context. That is, we maintain bivalence for vague predicates, and assume the inductive premise of the Paradox is false.

When Kennedy says this, he does not "invoke a context (the one that falsifies the universal generalization) in which one of two objects that differ minimally in the property measured by g is claimed to stand out relative to g and the other not". But then an ordinary speaker should be able—indeed should be expected—to do the same thing: to baulk when they hear the inductive premise, because—without invoking any particular context—they can see that it is false in every possible interpretation that could be invoked in any ordinary context.

A formal semantic theory is supposed to make explicit that which competent speakers implicitly grasp and which makes them speak the way they do. If knowledge of the contextualist semantic theory leads Kennedy to reject the inductive premise as being false, then their implicit knowledge of the theory should lead competent speakers to do the same thing.

Again, note that the fuzzy theory faces no problem here. Fuzzy theorists and ordinary speakers *are* expected to do the same thing: to go along with the inductive premise initially (because it is so close to being true, and we generally apply a rounding heuristic as a default, unless or until it is seen to cause problems—that is why extreme pedants stand out as so unusual, if not annoying, in ordinary life) but not to regard the sorites as a sound argument that establishes its conclusion.

[10] On contextualist theories of vagueness see e.g. Kamp [1981], Tappenden [1993], Raffman [1994], Raffman [1996], Soames [1999, ch.7], Fara [2000], Stanley [2003], Shapiro [2006], Kennedy [2007], Åkerman [2009] and Åkerman [2012].

So far so good then for the fuzzy treatment of the sorites—but what about the alternative formulations of the sorites argument: for example, the one that replaces the conditionals 'If pile n' is a heap then pile n is a heap' with the negated conjunctions 'It is not the case both that pile n' is a heap and pile n isn't a heap'? It was on precisely this point that Crispin Wright objected to the fuzzy kind of approach to vagueness and the sorites:

> Can a degree-theoretic account explain the plausibility of the major premises? There is no difficulty, of course, with the usual, quantified conditional form of premise. The explanation will claim that each instance, $Fa \to Fa'$, of $(\forall x)(Fx \to Fx')$ is *almost* true: that its consequent enjoys a degree of truth ever so nearly but not quite as great as that of its antecedent. And this claim will then be followed... by a stipulation that the degree of truth of any universally quantified statement is the minimum of the degrees of truth enjoyed by its instances... But... the major premise doesn't need to be conditional at all. In the case of the Sorites-series of indiscriminable color patches for instance, we could just as well take it in the form
>
> $$(\forall x) - [\mathrm{red}(x) \ \& \ - \mathrm{red}(x')].$$
>
> All the ways of making the conditional form of major premise seem intuitively plausible would be applicable to this conjunctive form... [the degree theorist] needs to explain... with what right such a conjunctive major premise may be regarded as *almost true*; otherwise he cannot explain its plausibility, or duly acknowledge the force of the arguments which seem to sustain it. [Wright, 1987, 251–2]

Wright doubts that the degree theorist can give an account on which such a conjunctive major premise is almost true. Certainly, the conjunctions are not all almost true if one models conjunction as follows—an approach which mirrors the treatment of intersection in Zadeh's [1965] fuzzy set theory and has frequently been assumed in the philosophical literature on fuzzy logic:

$$||\alpha \wedge \beta|| \quad = \quad \min\{||\alpha||, ||\beta||\}$$

However there is a well-studied and well-understood fuzzy logic in which the conditional is treated in the Łukasiewicz fashion mentioned above *and* 'If α then β' is logically equivalent to 'It is not the case that (α and not β)'. Far from being an ad hoc bolt-on addition to avoid Wright's objection, this relationship between the conjunction and the conditional is central to certain t-norm fuzzy logics which we shall introduce in the next section.

3 Mathematical Fuzzy Logic: t-norm-based logics

In this section, we show how the logical tools needed for the treatment of vagueness seen above fit in the very well understood framework of mathematical fuzzy logic. We start from Łukasiewicz predicate logic, as a central example, and then introduce the class of all t-norm-based logics, together with a little historical background for context. Later, in Section 5, we present a broader logical framework necessary for the purposes of the paper. For more details, we refer the reader to the handbook series [Cintula et al., 2011 and 2015] and references therein.

Fuzzy sets were introduced by Lotfi Zadeh [1965] as a mathematical paradigm for handling imprecision and gradual change in engineering applications. They are conceptually very simple: a fuzzy set is nothing more than a classical set X endowed with a function $\mu\colon X \to [0, 1]$ which represents the degree to which an element of the domain X belongs to the fuzzy set. This proposal spawned a whole research area which included mathematical studies, but also technological applications, in particular, to so-called *fuzzy control.* The field came to be known sometimes as *fuzzy logic*, although its mathematical machinery and the concepts investigated were largely unrelated to those typically used and studied in (mathematical) logic.

Mathematical fuzzy logic was developed by Petr Hájek and his collaborators (see Hájek [1998], Gottwald [2001], Mundici [2011], Novák [1990]) as a discipline of mathematical logic aiming at providing solid *logical* foundations for fuzzy set theory by focusing on syntactic formalization of discourse and a notion of logical consequence and thus, among others, creating a viable tool for a logical study of reasoning under vagueness. Over the years the field has grown into a systematic study of fuzzy logics understood as a certain kind of many-valued logics.

Łukasiewicz logic, mentioned in the previous section in relation to the sorites paradox, is an early example of a *genuine* fuzzy logic, that had been studied already for quite some time before the inception of fuzzy sets (see Łukasiewicz [1920], Łukasiewicz and Tarski [1930], Rose and Rosser [1958], Hay [1963]). In the syntactic aspects, this logic is completely classical. A *predicate language* $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{C}, \mathbf{ar} \rangle$, where $\mathbf{P}$ is a non-empty set of *predicate symbols*, $\mathbf{C}$ is a set of *constant symbols* (disjoint from $\mathbf{P}$), and $\mathbf{ar}$ represents the *arity function*, which assigns to each predicate its *arity*.[11] $\mathcal{P}$-terms are just object variables and object constants. Atomic $\mathcal{P}$-formulas are expressions of the form $R(t_1, \ldots, t_n)$, where $R \in \mathbf{P}$ is a predicate of arity n and $t_1, \ldots, t_n$ are $\mathcal{P}$-terms. $\mathcal{P}$-formulas are built from the atomic ones using the implication connective $\to$, the truth constant for *falsum* $\mathtt{f}$ (below we mention other definable connectives), and the quantifiers $\forall$ and $\exists$. As customary, when $\mathcal{P}$ is clear from the context, we will refer to $\mathcal{P}$-terms and $\mathcal{P}$-formulas simply as *terms* and *formulas* and furthermore, formulas without free variables are called *sentences*.

The two non-classical twists come in the semantics:

- Given a predicate language $\mathcal{P} = \langle \mathbf{P}, \mathbf{C}, \mathbf{ar} \rangle$, we define a $\mathcal{P}$-*structure* $\mathbf{M}$ as a tuple

  $$\mathbf{M} = \langle M, \langle R_{\mathbf{M}} \rangle_{R \in \mathbf{P}}, \langle c_{\mathbf{M}} \rangle_{c \in \mathbf{C}} \rangle,$$

  where M is a non-empty domain, $c_{\mathbf{M}}$ is an element of M, and for each n-ary predicate R, $R_{\mathbf{M}}$ is an n-ary $[0,1]$-valued relation, i.e. a function from M^n to $[0,1]$, identified with an element of $[0,1]$ if $n = 0$. Observe that the genuinely non-classical part of this kind of structure lies in the interpretation of predicates, which generalize the notion of fuzzy set. Indeed, if R is a unary predicate, then the interpretation is a function $R_{\mathbf{M}} \colon M \to [0,1]$, i.e. a fuzzy set.

- As truth values of (atomic) predicate formulas are now elements of the interval $[0,1]$, in order to be able to compute the value of an implication between two formulas, we need a function that maps two elements of $[0,1]$ to another element of the interval; we call such function an interpretation of a connective (the implication in this case). Łukasiewicz logic uses the following interpretation of $\to$:

  $$a \to_{Ł} b = \begin{cases} 1 & \text{if } a \leq b, \\ 1 - a + b & \text{otherwise.} \end{cases}$$

  Unsurprisingly, the interpretation of the truth constant $\mathtt{f}$ will be 0. See below for the interpretation of other defined connectives.

We say that a $\mathcal{P}$-structure $\mathbf{M}$ is *crisp* if the codomain of each function $R_{\mathbf{M}}$ is the set $\{0,1\}$. Note that crisp $\mathcal{P}$-structures can be identified with the usual models of classical first-order logic.

We are now ready to define the *truth values* of all $\mathcal{P}$-formulas in a given $\mathcal{P}$-structure $\mathbf{M}$ for an $\mathbf{M}$-valuation v (i.e., a mapping from object variables to the domain $\mathbf{M}$); note

[11] It is relatively straightforward to allow also function symbols in the language and/or consider a special symbol for equality, but we decide to omit them here for simplicity.

that in crisp $\mathcal{P}$-structures the upcoming definition fully coincides with the usual definition of truth in classical logic. First, we extend the evaluation v from object variables to all $\mathcal{P}$-terms by setting $v(c) = c_{\mathbf{M}}$ for each $c \in \mathbf{C}$ and, then, we proceed recursively:

$$\begin{array}{rcll} \|R(t_1,\dots,t_n)\|_{\mathbf{M},v} & = & R_{\mathbf{M}}(v(t_1),\dots,v(t_n)) & \text{for each } n\text{-ary } R \in \mathbf{P} \\ \|\varphi \to \psi\|_{\mathbf{M},v} & = & \|\varphi\|_{\mathbf{M},v} \to_{Ł} \|\psi\|_{\mathbf{M},v} & \\ \|\mathtt{f}\|_{\mathbf{M},v} & = & 0 & \\ \|(\forall x)\varphi\|_{\mathbf{M},v} & = & \inf\{\|\varphi\|_{\mathbf{M},v[x\mapsto d]} \mid d \in M\} & \\ \|(\exists x)\varphi\|_{\mathbf{M},v} & = & \sup\{\|\varphi\|_{\mathbf{M},v[x\mapsto d]} \mid d \in M\}.^{12} & \end{array}$$

Finally, we conclude our definition of semantics by saying that a $\mathcal{P}$-formula φ is a *consequence* of a set of $\mathcal{P}$-formulas Γ, $\Gamma \models_{Ł} \varphi$ in symbols, if for each $\mathcal{P}$-structure $\mathbf{M}$ we have:[13]

if $\|\chi\|_{\mathbf{M},v} = 1$ for each $\chi \in \Gamma$ and each $\mathbf{M}$-evaluation v, then $\|\varphi\|_{\mathbf{M},v} = 1$ for each $\mathbf{M}$-evaluation v.

A formula φ is valid in Łukasiewicz predicate logic, if it is a consequence of the empty set of premises. Note that restricting to the value 1 when defining consequence (and validity) does not diminish the importance of other truth values. Indeed, thanks to these values one can disprove the validity of certain classical tautologies (see below for examples).

Łukasiewicz predicate logic can be given various Hilbert-style presentations, albeit rather complex ones, either with a non-recursively enumerable set of axioms, or by using an infinitary deduction rule (Hay [1963], Ragaz [1981]). However, there is a related weaker finitary logic which has more complex semantics but can be recursively axiomatized (with *modus ponens* and generalization as the only deduction rules) and shares the same propositional fragment with our Łukasiewicz predicate logic (i.e., they coincide when restricted to the language with only nullary predicates); see Hájek [1998] and Section 5 for more details.

Using the given interpretations of the connectives $\to$ and $\mathtt{f}$, we can easily compute the semantics of other connectives definable from $\to$ and $\mathtt{f}$ in the usual (classical) way. Note however that, due to the presence of additional truth values, two classically equivalent definitions of conjunction lead to different functions in this context:[14]

connective	definition	interpretation
$\neg\varphi$	$\varphi \to \mathtt{f}$	$1 - \|\varphi\|$
$\mathtt{t}$	$\neg\mathtt{f}$	1
$\varphi \mathbin{\&} \psi$	$\neg(\varphi \to \neg\psi)$	$\max\{0, \|\varphi\| + \|\psi\| - 1\}$
$\varphi \wedge \psi$	$\neg(\psi \to \neg(\psi \to \varphi))$	$\min\{\|\varphi\|, \|\psi\|\}$
$\varphi \vee \psi$	$(\varphi \to \psi) \to \psi$	$\max\{\|\varphi\|, \|\psi\|\}$

Simple arithmetical manipulations allow one to check the following properties, for each $a, b, c \in [0,1]$,

1. $a \mathbin{\&_{Ł}} b \leq a \wedge_{Ł} b$
2. $a \to_{Ł} b = \neg_{Ł}(a \mathbin{\&_{Ł}} \neg_{Ł} b)$
3. $a \wedge_{Ł} b = \neg_{Ł}(\neg_{Ł} a \vee_{Ł} \neg_{Ł} b)$
4. $a \vee_{Ł} b = \neg_{Ł}(\neg_{Ł} a \wedge_{Ł} \neg_{Ł} b)$
5. $\neg_{Ł}\neg_{Ł} a = a$
6. $a \to_{Ł} b = 1$ iff $a \leq b$
7. $1 \to_{Ł} a = a$
8. $a \mathbin{\&_{Ł}} b \leq c$ iff $a \leq b \to_{Ł} c$.

[12] By $v[x\mapsto d]$ we denote the $\mathbf{M}$-valuation, resulting from v by setting $v[x\mapsto d](x) = d$ and keeping the values of the remaining variables unchanged.

[13] Observe that the notion of consequence we define can be characterized as a *sentential truth-preserving* notion of consequence, i.e., one which is concerned with whether the property of "having value 1 for all valuations" is preserved. Other alternatives, not considered in the paper, include the *non-sentential truth-preserving* consequence, which is concerned with whether the property of "having value 1" is preserved *for each valuation*, or the *degree-preserving consequence*, which is concerned with whether for all valuations the value of conclusion is not less than the value of each of the premises; see Bou et al. [2009].

[14] It is also possible to define a second disjunction, by means of the formula $\neg\varphi \to \psi$, linked with the conjunction & by De Morgan laws. However, since this disjunction connective will not have a general counterpart in the framework introduced later, we will skip it here.

The last property is called the *residuation law* and plays a crucial role in the broader framework of mathematical fuzzy logic we introduce below. The seventh property (together with the sixth one) ensures the validity of the formulas $\varphi \to (\mathtt{t} \to \varphi)$ and $(\mathtt{t} \to \varphi) \to \varphi$, which will be relevant later. But most importantly, let us note that the definitions of both & and $\wedge$ would give conjunction in classical logic. The former is called *strong conjunction* because, by the first and the sixth property, we have $\vDash_{Ł} \varphi \,\&\, \psi \to \varphi \wedge \psi$. The latter is called *lattice conjunction* because of its semantic interpretation (and, for the same reasons, $\vee$ is called *lattice disjunction*). The, perhaps more straightforward syntactic definition of & gives us the validity of the classical laws of non-contradiction and *modus ponens*, which however fail for the lattice conjunction $\wedge$:

$$\vDash_{Ł} \neg(\varphi \,\&\, \neg\varphi) \quad \text{while for } \|\varphi\| = \tfrac{1}{2} \text{ we have} \quad \|\neg(\varphi \wedge \neg\varphi)\| = \tfrac{1}{2}$$
$$\vDash_{Ł} \varphi \,\&\, (\varphi \to \psi) \to \psi \quad \text{while for } \|\varphi\| = \tfrac{1}{2} \text{ and } \|\psi\| = 0 \text{ we have} \quad \|\varphi \wedge (\varphi \to \psi) \to \psi\| = \tfrac{1}{2}$$

As regards *modus ponens*, note that as a *deduction rule* it is valid even when using $\wedge$, i.e., we have:

$$\varphi \wedge (\varphi \to \psi) \vDash_{Ł} \psi.$$

Therefore, in particular, the classical deduction theorem fails in Łukasiewicz logic. However, thanks to the strong conjunction, the logic still retains the following form of deduction theorem: for each set of formulas $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vDash_{Ł} \psi \text{ if and only if there is an } n \geq 0 \text{ such that } \Gamma \vDash_{Ł} \varphi^n \to \psi$$

where φ^n is inductively defined as: $\varphi^0 = \mathtt{t}$, $\varphi^1 = \varphi$, and $\varphi^{i+1} = \varphi^i \,\&\, \varphi$ for $i > 0$. This is a *local* version of deduction theorem, because it does not have the same formulation for all formulas; indeed the number n varies for different choices of the formulas $\Gamma \cup \{\varphi, \psi\}$.[15]

The presence of two (definable) conjunctions is not a particularity of Łukasiewicz logic. Actually, it is rather frequent in non-classical logics to obtain a split of some classical connectives. The high inferential strength of classical logic (indeed, it is well known that as a propositional system it is a maximal consistent logic) conflates on a single connective many properties that in weaker logics are distributed between different connectives. In the case of Łukasiewicz logic (and in the weaker logics we introduce later) we have that

1. the strong conjunction & retains the good relationship between conjunction and implication captured by the residuation law, the law of non-contradiction, and *modus ponens*,

2. the lattice conjunction $\wedge$ keeps the order-related properties arising from its semantic interpretation as the function minimum.

Interestingly enough, classical logic can be obtained as the axiomatic extension of Łukasiewicz logic by adding $\varphi \wedge \psi \to \varphi \,\&\, \psi$, i.e. the other half of the equivalence between both conjunctions.[16]

The presentation of Łukasiewicz predicate logic that we have seen can be easily generalized to accommodate other well-known fuzzy logics. Let us keep most of the syntactic and semantic notions untouched (predicate language, atomic formulas, quantifiers, connectives, structures, valuations etc.). We only (partly) change the definition of truth values of formulas by considering a more general treatment of connectives. First of all, we consider the connectives $\to$, &, $\wedge$, and $\vee$, and the truth constants $\mathtt{f}$ and $\mathtt{t}$ as primitive (despite the fact, irrelevant here, that in the present semantic setting some of them are still interdefinable) and use the additional defined connective $\neg\varphi = \varphi \to \mathtt{f}$. While we continue interpreting $\wedge$, $\vee$, $\mathtt{f}$, and $\mathtt{t}$ as before (i.e., min, max, 0 and 1), now we consider a much wider set of possible interpretations of & and $\to$.

[15] Actually, it can be shown that n can be taken as the minimum number of times one needs to use the premise φ in a formal proof of ψ from $\Gamma \cup \{\varphi\}$ (see Chvalovský and Cintula [2012]).

[16] Alternatively, it can also be obtained by adding the axiom of excluded middle, $\varphi \vee \neg\varphi$, or the axiom of contraction: $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$; see Hájek [1998].

To this end, we define a *t-norm* Klement et al. [2000] as a binary operation on $[0,1]$ which has 1 as the neutral element and is commutative, associative, and monotone. More formally, a t-norm is a function $*\colon [0,1]^2 \to [0,1]$ such that, for each $a, b, c \in [0,1]$, we have:

$$\begin{aligned} a * 1 &= a \\ a * b &= b * a \\ a * (b * c) &= (a * b) * c \\ \text{if } a \leq b, &\ \text{ then } a * c \leq b * c \end{aligned}$$

A t-norm is left-continuous if for each $B \cup \{a\} \subseteq [0,1]$ we have:

$$\sup\{a * b \mid b \in B\} = a * \sup B.$$

A crucial fact is that, for each left-continuous t-norm $*$, there is a unique operation $\Rightarrow_*$ (called the *residuum* of $*$) satisfying the *residuation law* w.r.t. $*$, that is, for each $a, b, c \in [0,1]$, we have:

$$a * b \leq c \ \text{ if, and only if, } \ a \leq b \Rightarrow_* c$$

and it can be easily computed that, for each $a, b \in [0,1]$, we have:

$$a \Rightarrow_* b = \sup\{c \in [0,1] \mid c * a \leq b\}.$$

One may easily check that the functions $a *_{Ł} b = \max\{0, a+b-1\}$ and $a \Rightarrow_{Ł} b = \min\{1, 1-a+b\}$ used above for the interpretation of & and $\to$ in Łukasiewicz logic give an example of a left-continuous (actually, continuous) t-norm and its residuum (we will see other examples soon).

As in the case of the Łukasiewicz t-norm, any left-continuous t-norm $*$ used as interpretation of & will ensure the validity of the law of non-contradiction and *modus ponens*, and moreover, we also have the following for each $a, b \in [0,1]$:

1. $a * b \leq \min\{a, b\}$
2. $a \Rightarrow_* b = 1$ iff $a \leq b$
3. $1 \Rightarrow_* a = a$

Fixing a left-continuous t-norm $*$ and using it and its residuum as interpretations of the strong conjunction & and the implication $\to$, the truth values of formulas in a $\mathcal{P}$-structure $\mathbf{M}$ are recursively defined, for a given $\mathbf{M}$-valuation v, as:

$$\begin{array}{rcll} \|R(t_1, \ldots, t_n)\|^*_{\mathbf{M},v} &=& R_{\mathbf{M}}(v(t_1), \ldots, v(t_n)) & \quad\text{for each } n\text{-ary } R \in \mathbf{P} \\ \|\varphi \wedge \psi\|^*_{\mathbf{M},v} &=& \min\{\|\varphi\|^*_{\mathbf{M},v}, \|\psi\|^*_{\mathbf{M},v}\} & \\ \|\varphi \vee \psi\|^*_{\mathbf{M},v} &=& \max\{\|\varphi\|^*_{\mathbf{M},v}, \|\psi\|^*_{\mathbf{M},v}\} & \\ \|\varphi \,\&\, \psi\|^*_{\mathbf{M},v} &=& \|\varphi\|^*_{\mathbf{M},v} * \|\psi\|^*_{\mathbf{M},v} & \\ \|\varphi \to \psi\|^*_{\mathbf{M},v} &=& \|\varphi\|^*_{\mathbf{M},v} \Rightarrow_* \|\psi\|^*_{\mathbf{M},v} & \\ \|\mathtt{f}\|^*_{\mathbf{M},v} &=& 0 & \\ \|\mathtt{t}\|^*_{\mathbf{M},v} &=& 1 & \\ \|(\forall x)\varphi\|^*_{\mathbf{M},v} &=& \inf\{\|\varphi\|^*_{\mathbf{M},v[x \mapsto d]} \mid d \in M\} & \\ \|(\exists x)\varphi\|^*_{\mathbf{M},v} &=& \sup\{\|\varphi\|^*_{\mathbf{M},v[x \mapsto d]} \mid d \in M\}. & \end{array}$$

Therefore, for each left-continuous t-norm $*$, we can define its corresponding predicate logic by saying that a $\mathcal{P}$-formula φ is a semantic consequence w.r.t. $*$ of a set of $\mathcal{P}$-formulas Γ, $\Gamma \vDash_* \varphi$ in symbols, if for each $\mathcal{P}$-structure $\mathbf{M}$ we have:

$$\text{if } \|\chi\|^*_{\mathbf{M},v} = 1 \text{ for each } \chi \in \Gamma \text{ and each } \mathbf{M}\text{-valuation } v, \text{ then } \|\varphi\|^*_{\mathbf{M},v} = 1 \text{ for each } \mathbf{M}\text{-valuation } v.$$

Observe that, even in this general setting, when restricted to the values $\{0,1\}$, the interpretations of the connectives coincide with the classical connectives (in particular, collapsing & and $\wedge$ into classical conjunction). Therefore, if we use only crisp $\mathcal{P}$-structures, $\vDash_*$ becomes classical predicate logic.

When $*$ is the Łukasiewicz t-norm $*_Ł$, $a *_Ł b = \max\{0, a + b - 1\}$, we retrieve the Łukasiewicz logic defined before: $\models_{*_Ł} = \models_Ł$. Other choices for $*$ give other well-known fuzzy logics (cf. Hájek [1998], Esteva and Godo [2001]). It is worth mentioning the following:[17]

Logic	**T-norm**
Gödel–Dummett logic G	$a *_G b = \min\{a, b\}$
Product logic Π	$a *_\Pi b = ab$ (product of real numbers)
Nilpotent Minimum logic NM	$a *_{NM} b = \begin{cases} \min\{a,b\} & \text{if } a > 1 - b, \\ 0 & \text{otherwise.} \end{cases}$

In the context we have presented here, the weakest logic (called MTL by Esteva and Godo [2001]) is obtained by allowing $*$ to range over *all* left-continuous t-norms, that is,

$$\Gamma \models_{MTL} \varphi \;\text{ iff }\; \Gamma \models_* \varphi \text{ for each left-continuous t-norm } * .$$

The logics MTL, G, and NM, are finitary and recursively axiomatizable (however, Π is neither); see Montagna and Ono [2002], Hájek et al. [2011]. The axiomatic extensions of MTL form a core family of fuzzy logics that has been studied in depth, including all the logics we have mentioned so far. Of course, not all axiomatic extensions of MTL are given by a collection of left-continuous t-norms, e.g., classical logic (extension of MTL by the law of excluded middle) being the principal example.

A particularly important axiomatic extension of MTL is the logic IMTL obtained either by restricting the definition of MTL to t-norms satisfying the equation $(a \Rightarrow_* 0) \Rightarrow_* 0 = a$ or equivalently as the extension of its Hilbert systems by the law of double negation: $\neg\neg\varphi \to \varphi$. It is easy to see that Ł and NM are extensions of IMTL, whereas G and Π are not. In IMTL (and in any of its extensions) we have the definability of implication in terms of negation and conjunction, i.e. $\varphi \to \psi$ is equivalent to $\neg(\varphi \,\&\, \neg\psi)$, which allows the conjunctive formulation of sorites mentioned in the previous section.

4 Two challenges for the fuzzy approach

We have presented a semantics for vague predicates—and more specifically a solution to the sorites paradox, in its many guises (not just the one involving conditionals)—that makes use of a particular fuzzy logic, and we have explained how, far from being an ad hoc assemblage of parts cobbled together to handle vagueness, this logic occupies a central place in the well-studied field of t-norm fuzzy logics. Thus, adopting this logic is not like buying some untested newfangled gadget at a fair from a stall-holder one will likely never see again: it is like buying a mature artefact, the result of a long process of product development, with a rich set of links to related products and services (i.e. a rich set of results about axiomatisability, decidability and all the other things one cares about when choosing a logic).

But suppose we turn from the positive form of vague adjectives such as ‘tall’ and ‘heavy’ to their associated comparative forms: ‘taller’ and ‘heavier’. Once one has degrees of application of the predicate F in the picture it is natural to want to analyse the comparative Fer along the following lines:

(C) a is Fer than b iff a’s degree of Fness is greater than b’s degree of Fness

where ‘greater than’ refers to an ordering of the degrees. Indeed, degree-based accounts are now the dominant approach to gradable adjectives in linguistics, and they generally adopt this sort of analysis of comparatives. Our question now is, could the fuzzy logical

[17] Observe that while $*_G$ and $*_\Pi$ are actually continuous (like $*_Ł$), $*_{NM}$ is only left-continuous.

approach presented above form the basis of a semantic analysis of gradable adjectives along these lines? And the answer seems to be No, for the following reason. Consider the first object in a sorites series for F—an object that is definitely F. For the sake of example, let F be 'tall'. Now instead of going down the series, let's go in the other direction: consider someone b who is even taller than the first person a in the sorites series. So b is taller than a. But a's degree of tallness is already 1 on the approach under consideration, and so there is nowhere to go to make b's degree of tallness greater than a's: there are no degrees above 1; 1 is the top of the ordering of the degrees.[18]

This is not a knockdown objection to the fuzzy approach outlined above—but it is sufficient to motivate the search for a variant fuzzy theory that retains the advantages of the one presented above while also allowing for an analysis of comparatives along the lines just indicated.

The fuzzy approach to vagueness faces a second important challenge: to provide adequate analyses of precise gradable predicates.[19] These are predicates which are gradable, that is, they can be accompanied by degree morphology (e.g. they can appear in comparative form) and which are nonetheless precise, that is, they fail to display some of the surface features of vague predicates (typically, they draw sharp divides between their extensions and their complements, they do not have borderline cases and do not give rise to sorites-style paradoxes). For instance, 'dirty' (at least in some contexts) is precise—it either obtains of an object or it does not—and yet one can certainly say that something is dirtier than something else. (Imagine a head waiter berating a kitchen hand: "This plate is dirty—wash it again!" and then a bit later "Aarrgh, this one is even dirtier!")

One way to check whether a predicate is vague or precise is to observe some of the inferential patterns it validates. For example, the inferences involving antonyms of the following form typically indicate whether a predicate has borderline cases (is vague) or not (is precise):

'This plate is not dirty.' $\vdash$ 'This plate is clean.'

Precise predicates like 'dirty' and 'clean' validate them, while this is not the case for vague predicates:[20]

'Alex is not tall.' $\nvdash$ 'She is short.'

With a fuzzy logic like Łukasiewicz's, we only have the means to account for the meaning of two types of predicates: vague and crisp. The way in which one can distinguish between the two classes in such a setting is by establishing that while vague predicates can map objects to the whole of $[0, 1]$, crisp predicates can only map them to a subset thereof, namely $\{0, 1\}$. In other words, one takes crisp predicates as a limiting case of vague predicates: those which do not make use of the whole algebra of truth values, but only of a very small subset thereof. Because of the simple structure of the fuzzy scales of truth, the fuzzy theory of vagueness is limited in the types of predicates it can analyse. For example, the predicate 'dirty', when used in a precise way, would demand a structure which excludes degrees representing borderline cases, but which nevertheless leaves room for various degrees representing ways of being definitely dirty (not just 1).

As we will see, these two challenges to the fuzzy view can be addressed in one fell swoop. There would seem to be two ways forward: what we might call the Nigel Tufnel approach—extend the scale of degrees above 1—and the Marty DiBergi approach—leave the top of

[18] Paoli [1999, 78], citing Klein [1980] and Williamson [1994], notes an objection along these lines to many-valued and fuzzy approaches to comparatives.

[19] Gradable predicates have received a lot of attention within formal semantics during the last few decades. See, for instance, Kamp [1975], Cresswell [1976], Klein [1980], Stechow [1984], Kennedy [2007], Kennedy and McNally [2005], Burnett [2016].

[20] Note that this pattern also fails for some precise gradable predicates, namely, those whose antonym pairs are contraries (rather than contradictories)—i.e. antonyms A and B such that A is not equivalent to not-B. For instance, it fails for 'empty'/'full', despite these being precise (in some contexts).

the scale at 1 but map a to a lower degree.[21] Of course—and this is part of the point of the joke in the movie (see n.21)—both approaches can be made to lead to what is in effect the same result, just described in two different ways. Nevertheless, there *is* a real and crucially important distinction in the offing, which turns on the notion of 'definitely applies': the notion that we appealed to in the initial characterisation of a sorites series when we said that F definitely applies to the first object in the series. Let's suppose (temporarily) that we retain the idea that degree 1 corresponds to the notion of definite application: that is, for any object a to which F definitely applies, F's degree of application to a will be 1 (or at least 1, should we go the Nigel Tufnel way of adding degrees above 1). Given this stipulation on the meaning or role of the degree 1, the Marty DiBergi approach—leave the top of the scale at 1 but map a to a lower degree—must be rejected. For that approach cannot then capture the initial datum that F definitely applies to the first object in the sorites series for F. So—given the assumption that degree 1 corresponds to the notion of definite application—we need to take the Nigel Tufnel approach, in which the scale of degrees extends above 1, and the first object in the sorites series for F is mapped to 1 or to some degree greater than 1.

Now of course, we could just as well proceed by transferring the role of capturing the idea of definite application from 1 to a lesser degree—say (for the sake of illustration) 0.9—and then we would not need to add degrees above 1. But the point would remain that we would still be countenancing degrees above the degree to which the first object in the sorites series for F is mapped—and this is not something that we countenanced on the original approach to the sorites outlined earlier. This is the real essence of the approach that we wish to explore now. The idea is to have an ordered structure of degrees that includes a particular degree X which captures the idea of definite application—and, unlike the earlier proposal, includes further degrees greater than X. That is, unlike in the earlier proposal, instead of having a single degree (in the earlier presentation, the degree 1) that captures the idea of definite application, we now have a set of them, of which X is the least. For the sake of simplicity and familiarity we shall generally assume that X is 1—which means that we'll be taking the Nigel Tufnel approach of having more degrees above 1. But of course this makes no essential difference: it was always merely convenient, but not essential, to model the degrees of truth as an interval of reals with greatest element 1, and similarly below, it will be convenient but not essential to model them as an interval of reals with greatest element strictly greater than 1.

The idea will be that a statement that is mapped to a degree greater than or equal to 1 (or more generally, X) will be definitely true—just as the idea of F definitely applying to an object a is represented by F's degree of application to a being greater than or equal to 1 (or more generally, X). Accordingly, logical consequence will be modelled in terms of preservation of definite truth: a valid argument will be one such that there is no model in which the premises are all definitely true (have a truth degree greater than or equal to 1, or more generally X) and the conclusion is not (it has a truth degree less than 1, or more generally X).

Note that given the addition of further degrees above 1 (or more generally, X), it will also be natural to say a dual thing about degrees less than 0. In our initial characterisation of a sorites series we said that F definitely does not apply to the last object c in the series. In our earlier approach, degree 0 captured this notion of 'definitely not applying'—but of course there is nothing mandatory about using 0 for this purpose and in general we could have some other set of degrees, as long as some degree Y among them plays the role

[21] These approaches are named after the characters in the following dialogue from the 1984 motion picture *This Is Spin̈al Tap*: Nigel Tufnel: This is a top to a, you know, what we use on stage, but it's very, very special because, if you can see Marty DiBergi: Yeah NT: the numbers all go to eleven. Look, right across the board, eleven, eleven, eleven and MD: Oh, I see. And most amps go up to ten? NT: Exactly. MD: Does that mean it's louder? Is it any louder? NT: Well, it's one louder, isn't it? It's not ten. You see, most, most blokes, you know, be playing at ten, you're on ten here, all the way up, all the way up, all the way up, you're on ten on your guitar, where can you go from there? Where? MD: I don't know. NT: Nowhere. Exactly. What we do is, if we need that extra push over the cliff, you know what we do? MD: Put it up to eleven. NT: Eleven. Exactly. One louder. MD: Why don't you just make ten louder and make ten be the top number and make that a little louder? NT: [long pause] These go to eleven.

of capturing the notion of definite non-application. Now in many sorites series, it will be possible to add a further object d after the last one such that c is Fer than d—for example, an extra person such that the last person in the original sorites series for 'tall' is taller than this extra person. So it will be natural to have the structure of degrees extend *below* 0 (or more generally, Y) and to model the idea of F definitely not applying to an object a by F's degree of application to a being less than or equal to 0 (or more generally, Y). Again, for simplicity and familiarity we shall generally assume that Y is 0—but of course this is not essential.

This kind of approach opens the way to an adequate analysis of comparatives, thus meeting the challenge presented above. Recall the person b who is even taller than the first person a in the sorites series for 'tall'. We can now have it that the degree of application of 'tall' to b is greater than its degree of application to a *and* that the predicate definitely applies to both objects: this will be so if the degree of application to a is at least 1 (or more generally, X) and if the degree of application to b is even higher up the ordering of degrees. Moreover, this is exactly what we need in order to meet the other challenge to the fuzzy account of vagueness: that of analysing precise gradable predicates. Since these predicates are gradable, we need to be able to map objects to more than two degrees (i.e. not just to 0 or 1), but since they are precise, we need to avoid mapping them to the borderline area of the scale. In other words, we need multiple degrees of definite truth (and/or of definite falsity), which is precisely what this approach amounts to.

At this point someone may feel some confusion about how it can be that a predicate F *definitely* or *fully* applies to a, and yet applies *even more* to b. But to make sense of the situation we need only think of a sorites series with further objects added before the first one: the first object is definitely F; and the ones added before it are Fer than the first. If it still seems odd to you to say that a is definitely or fully F and yet b has a higher degree of Fness than a, then you are simply objecting to the idea (which, as we mentioned, is dominant in the literature) of analysing the comparative Fer along the lines of (C) above. That may not be an untenable position, but it is beside the point here: for our aim now is to see whether we can come up with a variant of the fuzzy approach to the sorites presented earlier in this paper that retains its advantages while also allowing for an analysis of the comparative along these lines. In the next section we present the formal details of such an approach. As we shall see, just as the earlier approach found its formal home within the well-understood family of t-norm-based fuzzy logics, the new approach also finds its formal home within a well-understood family of logics: uninorm-based fuzzy logics.

5 Mathematical Fuzzy Logic: uninorm-based logics

In this section, we generalize the logical framework presented in section 3 to a broader family of fuzzy logics amenable to deal with comparatives in the way outlined in the previous section and to provide an account of gradability in general, not just of vagueness. That is, we want logics with a semantics that, not only has many intermediate degrees of truth for statements about borderline cases, but also a multiplicity of degrees for definitely true and for definitely false statements. In particular, we want to present a natural generalization of the semantics of Łukasiewicz logic, which will be very convenient for the analysis proposed in the next section. Indeed, we construct a chain defined over the set of *all* real numbers extended with two limit points ($+\infty$ and $-\infty$),[22] in which

- the values in the open interval $(0, 1)$ are still intended as intermediate degrees of truth,
- 1 is intended as the least degree for definitively true statements and will interpret the truth constant $\mathtt{t}$,

[22] The two limit points give a largest and a least truth-value whose presence is necessary for our purposes of covering a wide class of graded predicates. Of course, these extremal truth values are not actually used in the modelization of all kinds of graded predicates (see Section 6).

- values above 1 are also intended for definitively true statements and $+\infty$, the largest of these values, will interpret the truth constant $\top$
- 0 is intended as the largest degree for definitively false statements and will interpret the truth constant `f` (and thus be used to define the negation connective)
- values below 0 are also intended for definitively false statements and $-\infty$, the least of these values, will interpret the truth constant $\bot$
- the connectives $\to$ and & are defined as in Łukasiewicz logic (but in an untruncated form) as $1-x+y$ and $x+y-1$ (for the bounds we use suitable conventions; see details below).[23]

In order to justify that such a chain (and similar semantical models) is indeed viable and it fits in a reasonable logical framework, we will show the rather simple steps of generalization that we need to take starting from the setting of t-norm-based logics seen above.

We have advocated for models with additional degrees of truth bigger than 1 for definitively true statements (and degrees smaller than 0 for definitely false statements). However, in order to be able to start our technical exposition from t-norms, it is convenient to begin by keeping the real unit interval $[0,1]$ as the set of truth values and interpreting the truth constant `t` and `f` inside this interval (rather than on 1 and 0). As a first and crucial step, we remove one requirement from the semantics of t-norms: we will not require anymore the top of the interval to be the neutral element of $*$. This can be achieved by considering the notion of *conjunctive uninorm*:[24] a binary operation $*$ on $[0,1]$ which is associative, commutative, monotone, 0 is its annihilator ($0*a=0$), and it has a neutral element t ($t*a=a$), which, as we will see, will be the interpretation of `t`. This allows us to distinguish clearly between t-norms and uninorms: T-norms are, naturally, conjunctive uninorms in which the neutral element is $t=1$ (the fact that 0 is an annihilator follows from the monotony and the fact that 1 is the neutral). Let us give two examples of conjunctive uninorms with neutral element $\frac{1}{2}$ (which therefore are not t-norms since there can be only one neutral element):

$$a \circ b = \begin{cases} \min\{a,b\}, & \text{if } b \leq 1-a, \\ \max\{a,b\}, & \text{if } b > 1-a. \end{cases} \qquad a \circ_{CR} b = \begin{cases} \dfrac{ab}{ab+(1-a)(1-b)}, & \text{if } \{a,b\} \neq \{0,1\}, \\ 0, & \text{otherwise.} \end{cases}$$

The second example, called *cross ratio uninorm*, will allow us to obtain the desired extension of the Łukasiewicz t-norm via a suitable transformation.

As in the case of t-norms, the left-continuity of a conjunctive uninorm ensures the existence of a unique function satisfying the residuation law; e.g. for the previous examples we obtain:

$$a \Rightarrow_{\circ} b = \begin{cases} \max\{1-a,b\}, & \text{if } a \leq b, \\ \min\{1-a,b\}, & \text{if } a > b. \end{cases}$$

$$a \Rightarrow_{CR} b = \begin{cases} \dfrac{b(1-a)}{b(1-a)+a(1-b)}, & \text{if } a \neq 0 \text{ and } b \neq 1, \\ 1, & \text{otherwise.} \end{cases}$$

[23] Let us note that we could work instead with the optically more manageable interval $[-0.1, 1.1]$ (yet with the same cardinality), which is perhaps more consonant with our up-to-eleven spirit. Or we could, as well, use the unit internal $[0,1]$ to stress the similarity with t-norms. Any suitable monotone bijection of the extended reals with any chosen closed interval of reals would yield an isomorphic semantics (more details later); however, we would pay the price of losing the natural and simple arithmetic interpretations of conjunction and implication that we have on the extended reals. The choice of a particular closed interval of real numbers as the set of truth values for our algebra is inessential from the point of view of the proposed modeling of the sorites paradox in the next section, so we can pick one just out of aesthetic considerations like those just mentioned.

[24] For details on the theory of uninorms and on the particular examples that we will mention see Cintula et al. [2011 and 2015], Gabbay and Metcalfe [2007] and references thereof.

We can use a left-continuous conjunctive uninorm and its residuum to interpret, respectively, the strong conjunction & and the implication $\to$, while we keep the interpretation of $\wedge$ and $\vee$ respectively as minimum and maximum. To obtain a system with minimal differences with that based on t-norms, the truth constant $\mathtt{t}$ should not be interpreted as 1 but rather as the neutral element t of $*$ (so we preserve the validity of the simple logical law $\mathtt{t} \,\&\, \varphi \to \varphi$ or equivalently $\varphi \to (\mathtt{t} \to \varphi)$). The interpretation of the truth constant $\mathtt{f}$ is given by an element $f \in [0,1]$, sometimes determined by the expected properties of negation (defined, as usual, as $\varphi \to \mathtt{f}$). Finally, we introduce two new truth constants $\top$ and $\bot$ to be interpreted respectively as the top and the bottom elements of the interval, i.e. 1 and 0.

Given a left-continuous conjunctive uninorm $*$ and an element $f \in [0,1]$, we can compute the truth values of all formulas. However, before we give the corresponding definition of semantical consequence, it is important to discuss the role of $\mathtt{t}$ and observe that it conforms with the intuitions given in the previous section. Note e.g. that in the previous example $\circ$, we have $t \Rightarrow_\circ t = \frac{1}{2} \Rightarrow_\circ \frac{1}{2} = \frac{1}{2}$ but $\frac{1}{4} \Rightarrow_\circ \frac{1}{4} = \frac{3}{4}$ (not 1!). This is just a symptom of a more general behavior; indeed, recall that, for any t-norm, we have:

$$a \Rightarrow_* b = 1 \text{ iff } a \leq b,$$

but, for a uninorm with neutral element t, we only have

$$a \Rightarrow_* b \geq t \text{ iff } a \leq b.$$

Therefore, not only t but also some bigger degrees turn out to be values of the law of identity $\varphi \to \varphi$. Taking this into account and, most importantly for our purposes, the desideratum of allowing many values for definitely true statements, we define the set of *designated truth values* $D = \{a \in [0,1] \mid a \geq t\}$. This choice makes t the least designated value, which allows to preserve as well the validity of the logical laws $\varphi \to (\mathtt{t} \to \varphi)$ and $(\mathtt{t} \to \varphi) \to \varphi$.

According to this choice, semantical consequence is defined as: $\Gamma \models_{*,f} \varphi$ if, for each $\mathcal{P}$-structure $\mathbf{M}$ we have:

if $\|\chi\|^*_{\mathbf{M},v} \geq t$ for each $\chi \in \Gamma$ and each $\mathbf{M}$-valuation v, then $\|\varphi\|^*_{\mathbf{M},v} \geq t$ for each $\mathbf{M}$-valuation v.

As in the case of MTL, the intersection of all such logics, called UL, is finitary and has been given a natural Hilbert-style axiomatization (see Gabbay and Metcalfe [2007], Metcalfe and Montagna [2007]). Also, analogously to IMTL, we define IUL as the extension of UL obtained by adding the law of double negation: $\neg\neg\varphi \to \varphi$, which keeps the definability of implication in terms of negation and conjunction.

There is nothing mandatory about using the interval $[0,1]$ for the set of truth values. As said before, we may want to have values bigger than 1 for definitely true statements, or smaller than 0 for definitely false, or we may want to use a different set altogether, maybe not even linearly ordered. In general, we can take an arbitrary set A equipped with the necessary operations to interpret the connectives &, $\to$, $\wedge$, $\vee$, $\mathtt{f}$, $\mathtt{t}$, $\bot$, and $\top$. We say that a tuple $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \mathtt{f}^{\boldsymbol{A}}, \mathtt{t}^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a UL-*algebra* if A is a non-empty domain, $\wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}$ are binary operations on A, and $\mathtt{f}^{\boldsymbol{A}}$, $\mathtt{t}^{\boldsymbol{A}}$, $\bot^{\boldsymbol{A}}$, $\top^{\boldsymbol{A}}$ are elements of A, and

(1) $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a bounded lattice, i.e., the relation $\leq$ defined as: $a \leq b$ iff $a \wedge^{\boldsymbol{A}} b = a$ is a partial order with least element $\bot^{\boldsymbol{A}}$, largest element $\top^{\boldsymbol{A}}$, and $\wedge^{\boldsymbol{A}}$ (resp. $\vee^{\boldsymbol{A}}$) is the infimum (resp. supremum) operation with respect to $\leq$.

(2) $\langle A, \&^{\boldsymbol{A}}, \mathtt{t}^{\boldsymbol{A}} \rangle$ is a commutative monoid, i.e. $\&^{\boldsymbol{A}}$ is associative, commutative, and has $\mathtt{t}^{\boldsymbol{A}}$ as neutral element

(3) For each $x, y, z \in A$, we have: $z \leq x \to^{\boldsymbol{A}} y$ iff $x \,\&^{\boldsymbol{A}}\, z \leq y$

(4) For each $x, y \in A$, we have: $((x \to^{\boldsymbol{A}} y) \wedge^{\boldsymbol{A}} \mathtt{t}^{\boldsymbol{A}}) \vee^{\boldsymbol{A}} ((y \to^{\boldsymbol{A}} x) \wedge^{\boldsymbol{A}} \mathtt{t}^{\boldsymbol{A}}) \geq \mathtt{t}^{\boldsymbol{A}}$.

Note that the third condition is a generalization of the residuation property we have seen in Section 3; the last condition is known as prelinearity.

Let us now introduce three particular classes of UL-algebras: those with involutive negation and those corresponding to MTL and Łukasiewicz logic.[25] We say that a UL-algebra $\boldsymbol{A}$ is

- an IUL-*algebra* if for each $x \in A$, we have: $(x \to^{\boldsymbol{A}} \mathtt{f}^{\boldsymbol{A}}) \to^{\boldsymbol{A}} \mathtt{f}^{\boldsymbol{A}} = x$
- an MTL-*algebra* if $\mathtt{t}^{\boldsymbol{A}} = \top^{\boldsymbol{A}}$ and $\mathtt{f}^{\boldsymbol{A}} = \bot^{\boldsymbol{A}}$
- an MV-*algebra* if it is an MTL-algebra and, moreover, for each $x, y \in A$, we have: $(x \to^{\boldsymbol{A}} y) \to^{\boldsymbol{A}} y = (y \to^{\boldsymbol{A}} x) \to^{\boldsymbol{A}} x$.

Whenever the order $\leq$ is linear we speak about *-chains* instead of *-algebras* (e.g., IUL-chains) and we use prefix *standard* if, moreover, A is the real unit interval $[0,1]$ and $\leq$ is the usual order on reals.

For example, the operations $\circ$ and $\Rightarrow_\circ$ (or $\circ_{CR}$ and $\Rightarrow_{CR}$ respectively) defined above and any element $f \in [0,1]$ give rise to a standard UL-chain. Moreover, if f is taken as $\frac{1}{2}$ in the first case, or for any $f \in (0,1)$ in the second case, the resulting algebras are actually standard IUL-chains. Let us note that the operations $*_{\text{Ł}}$ and $\Rightarrow_{\text{Ł}}$ give rise to a standard MV-chain (the unique standard MV-chain, modulo isomorphism). Finally, let us note that it can be shown that $\boldsymbol{A}$ is a *standard* UL-chain iff $\&^{\boldsymbol{A}}$ is a left-continuous conjunctive uninorm with neutral element $\mathtt{t}^{\boldsymbol{A}}$ and that $\boldsymbol{A}$ is a *standard* MTL-chain iff $\&^{\boldsymbol{A}}$ is a left-continuous t-norm; see Metcalfe and Montagna [2007], Klement et al. [2000].

Now we are ready to formally introduce the semantics we have sketched at the beginning of the section and note that it is indeed based on an IUL-chain isomorphic to the standard IUL-chain defined by $\circ_{CR}$ and an arbitrary $f \in (0, \frac{1}{2})$. We call this chain $\boldsymbol{C}$ and define it in the following way:

- the domain C of $\boldsymbol{C}$ consists of *all* real numbers (denoted as $\mathbb{R}$) extended by the elements $-\infty$ and $+\infty$
- the lattice operations $\wedge^{\boldsymbol{C}}$ and $\vee^{\boldsymbol{C}}$ are defined as minimum and maximum with respect to the natural order of C
- constants $\bot^{\boldsymbol{C}}$, $\mathtt{f}^{\boldsymbol{C}}$, $\mathtt{t}^{\boldsymbol{C}}$, and $\top^{\boldsymbol{C}}$ are defined as $-\infty$, 0, 1, and $+\infty$
- the connectives $\&^{\boldsymbol{C}}$ and $\to^{\boldsymbol{C}}$ are defined schematically as:

$x \&^{\boldsymbol{C}} y$	$y = -\infty$	$y \in \mathbb{R}$	$y = +\infty$
$x = -\infty$	$-\infty$	$-\infty$	$-\infty$
$x \in \mathbb{R}$	$-\infty$	$x + y - 1$	$+\infty$
$x = +\infty$	$-\infty$	$+\infty$	$+\infty$

$x \to^{\boldsymbol{C}} y$	$y = -\infty$	$y \in \mathbb{R}$	$y = +\infty$
$x = -\infty$	$+\infty$	$+\infty$	$+\infty$
$x \in \mathbb{R}$	$-\infty$	$1 - x + y$	$+\infty$
$x = +\infty$	$-\infty$	$-\infty$	$+\infty$

It is easy to see that all the conditions in the definition of IUL-algebra are satisfied and the negation $\neg^{\boldsymbol{C}} x = x \to^{\boldsymbol{C}} \mathtt{f}^{\boldsymbol{C}}$ is the involutive function

$$\neg^{\boldsymbol{C}} x = \begin{cases} 1 - x & \text{for } x \in \mathbb{R} \\ -\infty & \text{for } x = +\infty \\ +\infty & \text{for } x = -\infty \end{cases}$$

Observe that the simple arithmetical expressions used to define $\&^{\boldsymbol{C}}$ and $\to^{\boldsymbol{C}}$ when both arguments are real numbers coincide with those of Łukasiewicz's operations for the strong conjunction and the implication, respectively, with the only difference that here they are not truncated to fit in the interval $[0,1]$.

[25] This correspondence obeys a precise technical sense, which we will not explain here; see right after the definition and the end of the section for some hints.

As mentioned above, it can be shown that $\boldsymbol{C}$ is isomorphic to any standard IUL-chain defined by $\circ_{CR}$ and $f \in (0, \frac{1}{2})$; e.g. for $f = \frac{1}{3}$ we use the following mapping (for other fs just use a different suitable basis of the logarithm):

$$h\colon [0,1] \to \mathbb{R} \cup \{-\infty, +\infty\} \qquad \text{defined as} \qquad h(x) = \begin{cases} 1 + \log_2(\frac{x}{1-x}) & \text{if } x \in (0,1) \\ -\infty & \text{if } x = 0 \\ +\infty & \text{if } x = 1 \end{cases}$$

Therefore using any suitable strictly increasing bijection $h_2\colon [0,1] \to [-0.1, 1.1]$ sending $\frac{1}{3}$ to 0 and $\frac{1}{2}$ to 1 we obtain the isomorphic copy of $\boldsymbol{C}$ mentioned in footnote 23, i.e., the copy defined on the interval $[-0.1, 1.1]$ which, while having the same cardinality, is optically more manageable interval and perhaps more consonant with our up-to-eleven spirit.

The notions of $\mathcal{P}$-structure, truth value of a formula and semantical consequence have to be naturally changed to accommodate a wider range of possible sets of truth values. Let us fix an arbitrary UL-algebra $\boldsymbol{A}$ and define the following notions:

- An $\boldsymbol{A}$-based $\mathcal{P}$-structure $\mathbf{M}$ is a tuple

$$\mathbf{M} = \langle M, \langle R_{\mathbf{M}} \rangle_{R \in \mathbf{P}}, \langle c_{\mathbf{M}} \rangle_{c \in \mathbf{C}} \rangle,$$

  where M is a non-empty domain, each $c_{\mathbf{M}}$ is an element of M, and for each n-ary predicate R, $R_{\mathbf{M}}$ is an n-ary A-valued relation, i.e. a function from M^n to A, identified with an element of A if $n = 0$.

- The truth values of formulas in an $\boldsymbol{A}$-based $\mathcal{P}$-structure $\mathbf{M}$ are recursively defined, for a given $\mathbf{M}$-valuation v, as:

$$\begin{array}{rcll} \|R(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{M},v} & = & R_{\mathbf{M}}(v(t_1), \ldots, v(t_n)) & \text{for each } n\text{-ary } R \in \mathbf{P} \\ \|\varphi \circ \psi\|^{\boldsymbol{A}}_{\mathbf{M},v} & = & \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} \circ^{\boldsymbol{A}} \|\psi\|^{\boldsymbol{A}}_{\mathbf{M},v} & \text{for each } \circ \in \{\&, \to, \wedge, \vee\} \\ \|\circ\|^{\boldsymbol{A}}_{\mathbf{M},v} & = & \circ^{\boldsymbol{A}} & \text{for each } \circ \in \{\mathtt{f}, \mathtt{t}, \bot, \top\} \\ \|(\forall x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} & = & \inf\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x \mapsto d]} \mid d \in M\} & \\ \|(\exists x)\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} & = & \sup\{\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v[x \mapsto d]} \mid d \in M\}. & \end{array}$$

  Note that, since we do not assume $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ to be a *complete* lattice, some of the suprema and infima need not exist. In that case, we take the value of the corresponding formula as undefined. If values of all formulas are defined (for all evaluations) we say that $\mathbf{M}$ is a *safe* $\mathcal{P}$-structure.

- The designated truth values in the algebra $\boldsymbol{A}$ are defined as $D = \{a \in A \mid a \geq \mathtt{t}^{\boldsymbol{A}}\}$ and, therefore, the semantical consequence w.r.t. $\boldsymbol{A}$ is given in the following way: $\Gamma \models_{\boldsymbol{A}} \varphi$ if, for each safe $\boldsymbol{A}$-based $\mathcal{P}$-structure $\mathbf{M}$, we have:

  if $\|\chi\|^{\boldsymbol{A}}_{\mathbf{M},v} \geq \mathtt{t}^{\boldsymbol{A}}$ for each $\chi \in \Gamma$ and each $\mathbf{M}$-evaluation v, then $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} \geq \mathtt{t}^{\boldsymbol{A}}$ for each $\mathbf{M}$-evaluation v.

For any given class $\mathbb{K}$ of UL-algebras, we define the following consequence relation:

$$\Gamma \models_{\mathbb{K}} \varphi \text{ iff } \Gamma \models_{\boldsymbol{A}} \varphi \text{ for each } \boldsymbol{A} \in \mathbb{K}.$$

Particular choices of $\mathbb{K}$ give interesting logics:

- the classes $\mathbb{UL}$ and $\mathbb{UL}^\ell$ of all UL-algebras/UL-chains yield two different recursively axiomatizable logics. It can be shown that $\models_{\mathbb{UL}^\ell}$ coincides with the logic $\models_{\mathrm{UL}}$ of left-continuous conjunctive uninorms (i.e., the standard UL-chains) and it can be axiomatized as the extension of $\models_{\mathbb{UL}}$ by the following form of the so-called *law of constant domains*: $\forall x(\varphi \vee \psi) \to \forall x\varphi \vee \psi$ (where x is not free in ψ). Interestingly enough, when restricted to propositional languages both logics coincide (see Metcalfe and Montagna [2007]).

- the analogous claim holds for the classes $\mathbb{MTL}$ and $\mathbb{MTL}^{\ell}$ of all MTL-algebras/MTL-chains; the only difference is that the logic $\models_{\mathbb{MTL}^{\ell}}$ now coincides with the logic $\models_{\mathrm{MTL}}$ of left-continuous t-norms (see Esteva and Godo [2001]).
- interestingly the classes of all MV-algebras/MV-chains yield the same recursively axiomatizable logic (i.e., not two different logics as before), which is in general *weaker* than the Łukasiewicz logic (which can be obtained as its extension by a simple but infinitary deduction rule) but both logics coincide when restricted to propositional languages (see Hájek [1998], Hay [1963]).

Finally, we introduce a notion dual to that of designated truth values which, although not used for the definition of consequence, plays an important role in the next section. Given a UL-algebra $\boldsymbol{A}$, the set of its *anti-designated truth values* is $D' = \{x \in A \mid x \leq \mathtt{f}^{\boldsymbol{A}}\}$. It can be shown, by easy algebraic manipulations, that negation (recall that it is defined as $\neg^{\boldsymbol{A}} a = a \rightarrow^{\boldsymbol{A}} \mathtt{f}^{\boldsymbol{A}}$) turns designated truth values into anti-designated, and vice versa. Here's a graphical depiction of the regions in a UL-chain, such as the chain $\boldsymbol{C}$ and its isomorphic copy over $[-0.1, 1.1]$ defined above, that will be helpful later:

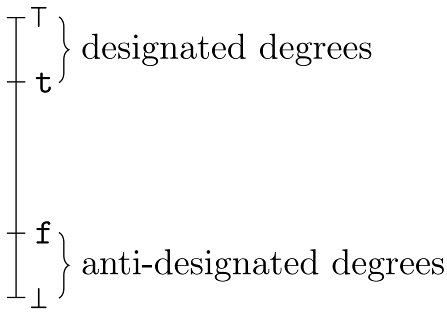


Figure 1: UL-chain

6 The fuzzy approach revisited

In this final section, we show how the fuzzy account of vagueness can be extended so that it provides an adequate analysis of comparative statements and of the meaning of gradable predicates in general—including but not limited to vague predicates. As mentioned above, both issues are addressed by means of the same strategy: adopting a UL-algebra as our structure of truth values.

But, first of all, we must show that our newly adopted uninorm-based fuzzy logic retains the advantages of t-norm-based logics, such as Łukasiewicz's. In particular, we must show that there is an adequate UL-chain-based logic on which we can recover the solution to sorites described in Section 2. As it happens, the bounded IUL-chain $\boldsymbol{C}$, described in the previous section, is a suitable candidate. Here we show how we can use it to this end, but first, let us clarify what our algebra of truth values looks like. Recall that, according to the definition of $\&^{\boldsymbol{C}}$ on $C = \mathbb{R} \cup \{-\infty, +\infty\}$, 1 is the unit element ($\mathtt{t}^{\boldsymbol{C}} = 1$) and thus, as explained above, it follows that 1 is also the minimum designated value (since 1 is not the top of the algebra, we have multiple designated values, namely, $D = [1, +\infty]$). On the other hand, the set of anti-designated values is not determined by $\&^{\boldsymbol{C}}$, but we have stipulated it taking $\mathtt{f}^{\boldsymbol{C}} = 0$, and hence $D' = [-\infty, 0]$). Given our aims (recall: correctly accounting for the sorites while at the same time modelling comparative statements and gradable predications) we needed $\mathtt{f}^{\boldsymbol{C}}$ to obey a couple of constraints: $\mathtt{f}^{\boldsymbol{C}} \neq -\infty$ and $\mathtt{f}^{\boldsymbol{C}} < \mathtt{t}^{\boldsymbol{C}}$ (ensuring that D' contains multiple elements, that it cannot overlap with D and that there must be values that are neither in D nor in D'—in other words, that the resulting structure is of the form depicted in Figure 1).

Now, consider the following sorites argument, in which each person n has exactly n euros (so each person forming the sorites series has one euro less than the previous person):

Person 102 000 is rich.

If person 102 000 is rich, then person 101 999 is rich.

If person 101 999 is rich, then person 101 998 is rich.

$\vdots$

If person 2 is rich, then person 1 is rich.

$\therefore$ Person 1 is rich.

First, we need to formalize the argument in a predicate language. Take a language $\mathcal{P}$ with object constants $p_1, \ldots, p_{102\,000}$ and a unary predicate R (for 'rich').

The argument can now be formalized as:

$R(p_{102\,000})$

$R(p_{102\,000}) \to R(p_{101\,999})$

$R(p_{101\,999}) \to R(p_{101\,998})$

$\vdots$

$R(p_2) \to R(p_1)$

$\therefore\ R(p_1)$

Now, take a $\mathcal{P}$-structure $\mathbf{M}$, with a domain M containing exactly the 102 000 people under consideration and such that, for each $1 \leq i \leq 102\,000$, $(p_i)_{\mathbf{M}}$ is person i. It only remains to define $H_{\mathbf{M}}\colon M \to C$ as an adequate truth assignment for the atomic statements that appear in the argument (i.e. the truth value of $R(p_i)$ will be $H_{\mathbf{M}}(i)$). Since we now have more than one designated and anti-designated values, we are going to describe a truth assignment that takes advantage of this fact (although this is not compulsory in general, of course), for the sake of illustration.

So suppose that any person who has 101 001 euros or more is considered, by most speakers, to be clearly rich. Similarly, suppose that any person who has 1 000 euros or less is taken as a clear-cut case of non-rich by most speakers. Finally, suppose that speakers hesitate about whether anyone who has between 1 001 and 101 000 euros is rich. In such scenario, the following would be a suitable assignment.[26]

Persons from 1 to 1 000 should get mapped into $[-\infty, 0]$, those from 1 001 to 101 000 into $(0, 1)$, and those from 101 001 to 102 000 into $[1, +\infty]$. If we choose the truth degrees for the borderline cases to be evenly distributed throughout the interval $(0, 1)$, they will

[26]However, note that this is not the only suitable assignment. For the sake of simplicity, we shall not make this explicit in our formulations, but we advocate a view—called 'fuzzy plurivaluationism'—which leaves room for the acceptance of multiple equally-suitable models. On this view, instead of each vague discourse being associated with a unique intended fuzzy model, it is associated with multiple acceptable fuzzy models. The acceptable models are all those that our usage and usage dispositions do not rule out as being incorrect interpretations of our language. These models must be adequate, in the sense that they must respect the meaning-determining facts (i.e. what speakers say, and the circumstances in which they say these things; what speakers are disposed to say in possible circumstances; etc). For example, they must all assign the adjective 'rich' a model such that any individual who all speakers agree is paradigmatically rich is assigned a designated value. Moreover, for any two people such that one is clearly richer than the other, all adequate models must assign 'rich' a model that maps the former to a higher degree than the latter. The idea is that, even though the meaning-determining facts do not suffice to pin down a single adequate fuzzy model, this does not necessarily lead to abandoning the fuzzy route. We just take all adequate models to be on a par. For more details, see Smith [2008].

be separated by a distance of $\frac{1}{100\,000}$, which naturally we could also use as the distance between the truth values of the people mapped into the designated and the anti-designated area. More precisely, we will have $\|R(p_n)\|^{\boldsymbol{C}}_{\mathbf{M}} = \frac{-1}{100} + \frac{n}{100\,000}$, for each $n \in \{1, \dots, 102\,000\}$.

With such an assignment, we have available an interpretation of the sorites argument that is analogous to the one presented in Section 2. Once again, the problem with the argument is that, although it is valid, it is unsound. The first premise, $R(p_{102\,000})$ is true to degree $\frac{101}{100}$, so it is definitely true. As for the conditionals, since we did not assign any two atomic statements the same value (we could have, but we assumed that this was not the case), the antecedents are always ever so slightly more true than the consequents. Thus all the conditional premises are true to a constant degree slightly less than 1. Indeed, for each $n \in \{1, \dots, 102\,000\}$, $\|R(p_n) \to R(p_{n-1})\|^{\boldsymbol{C}}_{\mathbf{M}} = 1 - \|R(p_n)\|^{\boldsymbol{C}}_{\mathbf{M}} + \|R(p_{n-1})\|^{\boldsymbol{C}}_{\mathbf{M}} = 1 - (\frac{-1}{100} + \frac{n}{100\,000}) + \frac{-1}{100} + \frac{n-1}{100\,000} = 1 - \frac{1}{100\,000}$.

Therefore, the argument appears compelling because all the premises are definitely true or very nearly definitely true and, in ordinary contexts, we are naturally inclined to accept something as true when it is very nearly true. However, the argument is unsound, because not all premises are definitely true.

Hence by using $\to^{\boldsymbol{C}}$ as the interpretation of the conditional, we can recover what we argued to be a suitable treatment of sorites in Section 2. Note, however, that our particular choice of algebra, the IUL-chain, is not essential to the proposal. There is a range of options to explore and it is possible that future research will lead to other suitable candidates. All we need is that our algebra has the same structural properties as $\boldsymbol{C}$. In particular, one may argue that introducing an algebra of truth values over the whole real line (even extended with bounds) is something of an overkill, especially because ultimately our truth assignment has only taken values in the interval $[0, 1]$ plus some values slightly bigger than 1 and some values slightly smaller than 0. This may be the case in this particular example, but, as illustrated in the rest of this section, we wanted to offer an algebra of truth values capable of supporting models for a big variety of gradable predicates for which it may be natural to employ more values, including the bounds. Moreover, $\boldsymbol{C}$ has the advantage that its operations are extremely simple, as we have just demonstrated. In any case, readers uncomfortable with the choice of $\boldsymbol{C}$ have an easy way out: pick instead an isomorphic copy on any smaller real interval of your liking, such as the traditional $[0, 1]$ itself or any slightly expanded interval like $[-0.1, 1.1]$. The latter has the advantage of being quite intuitive, as it captures more directly the idea behind the up-to-eleven reference. However, as mentioned, there is a trade-off: we gain an apparently more manageable scale, but we lose the simplicity our logical operations enjoyed before.

To finish, observe that, given our interpretation of the conditional as coinciding with that of the Łukasiewicz conditional, we retain the advantage of being able to account for the formulation of the sorites argument which replaces the conditionals 'If person n is rich then person $n-1$ is rich' with the negated conjunctions 'It is not the case both that person n is rich and person $n-1$ isn't rich'. As mentioned in Section 2, this follows from the fact that the Łukasiewicz interpretation of 'If α then β' deems it equivalent with 'It is not the case that (α and not β)'.

We conclude that nothing is lost by the adoption of a logic based on IUL-chains: we can still account for the sorites paradox as is customarily done in fuzzy theories of vagueness. Next we show that not only nothing is lost, but in fact there are some things to be gained.

6.1 Non-borderline comparatives

In this section, we show how a semantic analysis based on an IUL-chain such as $\boldsymbol{C}$ can accommodate a proper treatment of comparatives.

Recall that with a fuzzy theory of vagueness of the sort described in Section 2, we could not account for true non-borderline comparatives, that is, true comparatives between objects which are definite instances of a certain predicate. In the new setting, we can. In order to present our analysis in detail, let us first describe the compositional interpretation we propose for comparatives (it is as one would expect, but it is still worth making it explicit). Our analysis of comparatives depends on a prior analysis of the positive unmarked position, e.g. 'Alex is tall', 'The cup is full', 'The stick is straight'. For us, a positive unmarked statement will denote a function from the domain of individuals U to the algebra of truth values $\boldsymbol{C}$ (type $\langle e,t\rangle$):[27]

$$[\![\text{'tall'}]\!] = \lambda x.\mathbf{tall}(x)$$

Now, building on this basic form, the comparative morpheme is analyzed as a function from a property to a binary relation between individuals (type $\langle\langle e,t\rangle,\langle\langle e,e\rangle,t\rangle\rangle$):

$$[\![\text{'-er/more than'}]\!] = \lambda\mathbf{f}.\lambda x.\mathbf{f}(x) > \lambda y.\mathbf{f}(y)$$[28]

Thus, comparative statements are analyzed as binary relations between individuals (type $\langle\langle e,e\rangle,t\rangle$). For instance, the comparative with respect to tallness is analysed as:

$$[\![\text{'taller than'}]\!] = \lambda x.\mathbf{tall}(x) > \lambda y.\mathbf{tall}(y)$$

We can finally go back to the issue of non-borderline comparatives. Let us recall what the problem was with another example. Take Jeff Bezos and Bill Gates, two definitely rich people. In many contexts, the following would be taken as true:

'Bezos is richer than Gates.'

Let's take r as the characteristic function of the extension of 'rich' and b and g as picking out Bezos and Gates, respectively. According to the compositional analysis just described, this comparative would be true iff $r(b) > r(g)$ is true. Since both $r(b)$ and $r(g)$ are definitely true, both are evaluated inside D. The problem was that in a fuzzy logic such as Łukasiewicz's, $D = \{1\}$, and hence $r(b) = r(g)$. The advantage of working on the IUL-chain $\boldsymbol{C}$ is that we now have available a plurality of truth values corresponding to full truth and full falsity and hence have room for assignments that map definite instances (respectively, definite counterexamples) of a predicate to different values. Regarding the example we just saw, r can now be defined so that it maps b and g to different values $d, d' \in D$, respectively, such that $d > d'$, as desired.

6.2 Gradable precise predicates

Finally, we show how a uninorm-based approach opens the door to a semantic analysis of gradability in general, not just vagueness. As explained in §4, with a fuzzy logic like Łukasiewicz's, we could only account for the meaning of two types of predicates. This was because there was only one structurally salient way of restricting the range of our interpretation functions: restricting them to the top and bottom of the real unit interval. This left us with only two types of scales, one for vague predicates (i.e. the unrestricted $[0,1]$) and one for crisp predicates (i.e. $\{0,1\}$). In our new setting, based on the bounded IUL-chain $\boldsymbol{C}$, there are more ways in which we can restrict the range of our interpretation functions, since there is a larger variety of structurally different subsets of the domain of $\boldsymbol{C}$. For example, not only can we define interpretation functions that map objects to a single designated degree, but we can now also define an interpretation function that maps objects to multiple designated degrees.

[27] Although this may seem very obvious in this setting, other proposals are not able to offer such a simple analysis. For instance, Kennedy [2007] needs to pose a null morpheme transforming so-called 'bare adjectives', which denote measure functions, into proper predicates, understood as classical crisp predicates.

[28] This particular choice of analysis forbids comparatives involving different predicates (e.g. 'Alex is taller than he is smart'), but we could easily modify it so as to leave room for these (by analysing the comparative as a function from a pair of properties to a binary relation). Nevertheless, since this topic falls outside the scope of this article, we will stick to the analysis above.

In this section, we show that by following the same procedure as before (i.e. restriction of the interpretation functions' co-domains) and working on $\boldsymbol{C}$ instead of $[0, 1]$, we can distinguish more types of predicates than just vague and crisp. The new kind of predicates we are now able to analyze are gradable precise, that is, predicates which can occur in degree constructions while at the same time not having borderline cases, not having blurry boundaries and not giving rise to sorites paradoxes. Amongst gradable precise predicates, we can distinguish various salient types of predicates, but, before describing them, let us make a few caveats about the predicates we are prepared to analyze at this point.

First of all, our semantics will cover only linear predicates, that is, predicates all of whose applications are comparable. More precisely, F is linear iff for any objects a and b, exactly one of the following is assertible: 'a is more F than b', 'b is more F than a' or 'a is exactly as F as b'. Secondly, for the sake of simplicity, we limit our analysis to monadic predicates—the treatment of polyadic ones does not pose any additional difficulties. Moreover, we focus exclusively on adjectival predicates (i.e. predicates consisting of a copula and an adjective), since an account of other kinds of predicates would take us too far afield. Finally, note that strictly speaking our semantics does not analyze predicates simpliciter, but uses of predicates. Most predicates can be used in ways that differ with respect to their form of gradability in different contexts. For instance, 'empty' can be used in two slightly different ways. In one way, which we label 'strict', 'empty' means 'completely devoid of content'. This is the sense in which you would use the predicate, say, in a lab setting. By contrast, one can use it in a more relaxed way, to mean 'practically devoid of content'. For instance, you would use it in this way to refer to a theater with only two people attending a play. We label this other use 'loose'.[29] Thus, when giving examples, sometimes we will have to clarify the use to which we are referring.[30] Nevertheless, we have and will continue to speak as if we are analyzing predicates simpliciter, for the sake of conciseness—just keep in mind that we must always be understood as speaking of uses of predicates.

We are finally ready to describe the taxonomy of gradable precise predicates our account is equipped to deal with.[31] As one would expect, some predicates clearly fall in a unique category and can be taken as paradigmatic examples thereof, while others are not easily classifiable. We will always try to go for prototypical cases in our presentation.

First of all, we can distinguish two general kinds of gradable precise predicates. To begin with, we find, as a limiting case, crisp or bigraded predicates, that is, predicates which have only two degrees of applicability (e.g. 'even' (for numbers) or 'Canadian' (in the legal sense, as applied to people)). This is only a degenerate case of gradable predicate, since in fact these are predicates that cannot appear in degree constructions (at least not under their usual understanding). The rest of precise predicates are multigraded and thus, in a sense, properly gradable. Among these, we find three salient subclasses: highest-standard, lowest-standard and intermediate-standard predicates.

Highest-standard predicates demand that the object reaches a maximum amount of a certain quality for their applicability (e.g. 'full', 'flat', 'clean', 'straight'). By contrast, lowest-standard predicates demand that the object surpasses a minimum such amount (e.g. 'dirty', 'bent', 'bumpy', 'impure').[32] Note that highest and lowest-standard predicates are not exactly analogous: the former demand that a certain degree be reached while the latter demand that a certain degree be surpassed.

[29] For more details on this distinction, see Burnett [2016, Ch. 3].

[30] This is one of the ways in which context interacts with the semantic analysis of gradable predicates, but it is not the only one. For instance, another point where context must be taken into account is to determine the exact form of the extension of (a use of) a predicate (and, possibly, anti-extension, depending on the kind of analysis). This is the sense in which 'tall' has different extensions depending on whether we are talking, say, about basketball players or about 10 year-old children. The latter collections of objects are usually referred to as 'the class of comparison' (see Kennedy [2007], Burnett [2016]) and are usually provided by the context.

[31] This taxonomy is based, in part, on those in Paoli [1999] and Kennedy and McNally [2005].

[32] In all these cases, we are focusing on the precise use of the predicates in question.

One way to classify predicates into these two classes is to observe the distribution of certain patterns of inference from comparative statements.[33]

For highest-standard predicates, the following type of inference is intuitively valid:

'Your glass is fuller than my glass.' $\vdash$ 'My glass is not full.'

But not this pattern:

'Your glass is fuller than my glass.' $\nvdash$ 'Your glass is full.'

The opposite occurs with lowest-standard ones:

'This stick is more bent than that stick.' $\vdash$ 'This stick is bent.'

'This stick is more bent than that stick.' $\nvdash$ 'That stick is not bent.'

Finally, we have intermediate-standard predicates, which demand that an intermediate standard be surpassed for their applicability (e.g. 'acute' (for angles), 'basic' (in the chemical sense, applied to solutions)). For instance, a solution is basic if it has a pH higher than 7, which is intermediate within the pH scale. For these predicates the inferences from comparatives just described do not go through, just as it occurs with vague predicates (the first two examples concern a vague predicate and the last two, an intermediate-standard predicate):

'Jo is taller than Alex.' $\nvdash$ 'Jo is tall.'

'Jo is taller than Alex.' $\nvdash$ 'Alex is not tall.'

'Vinegar is more basic than hydrochloric acid.' $\nvdash$ 'Vinegar is basic.'

'Bleach is more basic than ammonia.' $\nvdash$ 'Ammonia is not basic.'

However, intermediate-standard predicates can be distinguished from vague ones because, as we said above, being precise, they do not display the surface characteristics of vagueness. For example, they fail to have borderline cases: e.g. there are no aqueous solutions for which speakers (with the adequate knowledge) would hesitate to assert or deny that they are basic, neutral or acidic. Consequently, the inference pattern involving antonyms that we saw above is valid for intermediate-standard adjectives (unlike for vague ones). In the following example, suppose that Alex is a baby and that 'healthy'/'underweight' are being used in the precise medical sense of meeting (respectively, not meeting) a certain weight guideline for babies:

'Alex is not underweight.' $\vdash$ 'Alex is healthy.'

To sum up, our analysis will cover only linear predicates, including vague and precise ones. Amongst the latter we find crisp and multigraded predicates, amongst which in turn we find highest, lowest and intermediate-standard predicates.

We said that we would distinguish between different kinds of predicates by restricting, for each kind, the part of the IUL-chain $\boldsymbol{C}$ their interpretations can map individuals to. Thus, gradable predicates do not in general use the whole of C, but subsets thereof. In fact, we take vague predicates to be the only ones that are completely unrestricted from the point of view of the algebra: they can map individuals to the whole of C. By contrast, precise predicates map objects to a proper subset of C, in particular, to a subset of $D \cup D'$ (recall that D is the set of designated truth values and D', the set of anti-designated truth values).

[33]These distributions are described in Kennedy [2007].

Having the whole algebra available for the interpretation of vague predicates allows us to capture the fact that for a very small change with respect to the applicability of the underlying vague property, there can only be a very small change with respect of truth. This is only possible if they are associated with a set of truth values that has no gaps in it. And it is precisely this feature what opens the door to the corresponding sorites paradox (under our analysis).

Naturally, we want adequate models that respect the meaning-determining facts. For us, this means that, given a vague predicate F, for each individual a that gives a determinately true (resp. false) instance of F, the sentence Fa must be given a designated (resp. anti-designated) truth value. Moreover, the mapping must be such that if an individual a is Fer than an individual b, then Fa must be given a value higher than Fb. This monotonicity condition makes sense of some of the so-called 'penumbral connections' of vague predicates (see Fine [1975]), i.e. the fact that if Alex is considered bald, and Jim has less hair than Alex, then Jim should also be considered bald.

Now, how do we distinguish between the various types of precise predicates? Each type is associated with a particular restriction on $D \cup D'$ (see Figure 2 for some illustrative diagrams of the restrictions on the codomains associated with each kind of predicate). First, bigraded predicates map individuals to $\{-\infty, +\infty\}$, i.e. objects mapped to $+\infty$ by the function corresponding to 'even number' are even numbers and those mapped to $-\infty$ are odd numbers.

Second, highest-standard predicates map individuals onto $D' \cup \{+\infty\}$.[34] In other words, they can have as image the whole set of anti-designated truth values and one designated truth value. It is easy to see why: the anti-designated truth values represent all the ways in which an object may fail to have the corresponding property (e.g. all the ways in which a cup may fail to be empty) and the unique designated truth value denotes the single way in which an object may have the property (e.g. the only way in which a cup may be empty, that is, completely devoid of content). This range restriction suffices to capture the fact that highest-standard predicates demand that objects hit the maximal standard in order to be instances of the predicate.

Third, the opposite occurs with lowest-standard predicates. The relevant subset of C is $D \cup \{-\infty\}$.[35] The only anti-designated truth value represents the only way of not having the property in question (e.g. the only way for a stick not to be bent: being completely straight), whereas the designated truth values can be used to capture the variety of ways in which an object may have the property (e.g. all the ways in which a stick may be bent). Once again, this restriction captures the fact that lowest-standard predicates demand that objects surpass the minimal standard in order to be instances of the predicate.

Finally, intermediate-standard adjectives can map objects to the whole set $D \cup D'$. This is because for this kind of predicates there are many ways in which they may hold true or false of an object (e.g. there are various ways for a solution to be basic, all mapped to designated truth values, and also to not be basic, all mapped to anti-designated truth values).

Obviously, besides the restrictions on the kind of subalgebra employed in each case, in modelling gradable precise predicates we have to impose again the same kind of monotonicity condition that we introduced for vague predicates. No other restrictions are required, leaving a lot of freedom for particular assignments of truth values.

This finalizes the presentation of our semantics of acceptable models for gradable predicates, that is, those that give a reasonable semantic analysis of the gradable predicates in question. Note that this approach changes the notion of consequence in our first-order language setting. Let us elaborate on this point. For the purposes of this explanation, we label as $\vdash_{log}$ the logical consequence given by all $\mathcal{P}$-structures $\langle \boldsymbol{C}, \mathbf{M} \rangle$ where $\boldsymbol{C}$ is our

[34] We could have chosen any other designated value. For present purposes, it does not need to be the top of the chain. However, we choose the top for simplicity and to capture the intuition that bigraded predicates are a limiting case of any other type of gradable precise predicates.

[35] Once again, we could have chosen any other anti-designated value.

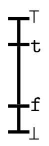
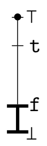
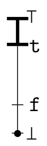
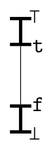
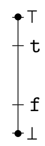
Vague	Highest-st.	Lowest-st.	Intermediate-st.	Bigraded
				

Figure 2: The scales associated with predicates with different types of gradability, where the bold parts mark the degrees to which the predicates of that type can map individuals.

chosen IUL-chain and the predicates of the language are interpreted without any restrictions. Moreover, we label as $\vdash_{sem}$ the consequence relation given by the same kind of $\mathcal{P}$-structures, but in which the language is required to be interpreted according to the semantic analysis above. More precisely, given a sentence ψ and a finite set of sentences Φ involving a set of predicates $\{P^1, \ldots, P^n\}$, we define:

- $\Phi \vdash_{log} \psi$ iff for every safe $\mathcal{P}$-structure $\langle \boldsymbol{C}, \mathbf{M} \rangle$, if for each $\varphi \in \Phi$ $||\varphi||^{\boldsymbol{C}}_{\mathbf{M}} \in D$, then $||\psi||^{\boldsymbol{C}}_{\mathbf{M}} \in D$,
- $\Phi \vdash_{sem} \psi$ iff for every safe $\mathcal{P}$-structure $\langle \boldsymbol{C}, \mathbf{M} \rangle$ such that each $P^i_{\mathbf{M}}$ is a mapping of the kind corresponding to P^i according to the stipulations above, if for each $\varphi \in \Phi$ $||\varphi||^{\boldsymbol{C}}_{\mathbf{M}} \in D$, then $||\psi||^{\boldsymbol{C}}_{\mathbf{M}} \in D$.

Clearly, $\Phi \vdash_{log} \psi$ implies $\Phi \vdash_{sem} \psi$, but not the other way around. Let us illustrate it with some examples. Above we claimed that, for instance, the following inference is valid:

'Your glass is fuller than my glass.' $\vdash$ 'My glass is not full.'

while the following one is not:

'Your glass is fuller than my glass.' $\nvdash$ 'Your glass is full.'

Let us show how our logical/semantic apparatus handles this situation. To formalize these sentences of natural language, let us take a predicate language with two object constants c, d, a unary predicate F, and a binary predicate MF. With these symbols, the premise of both inferences would be formalized as $MF(c,d)$, the first conclusion would be $\neg F(d)$, and the second would be $F(c)$.[36] Now, the situation is the following:

1. $MF(c,d) \nvdash_{log} \neg F(d)$,
2. $MF(c,d) \nvdash_{log} F(c)$,
3. $MF(c,d) \vdash_{sem} \neg F(d)$,
4. $MF(c,d) \nvdash_{sem} F(c)$,

That is, from the logical point of view (i.e. just in virtue of the logical form of the involved sentences) both inferences are invalid; by contrast, the former is semantically valid, while the latter is not. The failure of the first two inferences needs no justification, since this

[36]Formulas of the form $MF(c,d)$ correspond to our old $\mathbf{f}(c) > \mathbf{f}(d)$ (i.e. the result of multiple β-reductions on $\lambda\mathbf{g}\lambda\mathbf{f}.\lambda x.\mathbf{g}(x) > \lambda y.\mathbf{f}(y)$). These two representations belong to two different levels of linguistic analysis. The lambda calculus expressions make explicit the compositional steps by which we get to the semantic analysis of complex statements, in this case, comparatives. By contrast, the first-order notation employed here simply expresses the resulting propositional content of those statements. In a sense, we can see the lambda expressions as providing instructions to arrive at the semantics of a binary predicate MF, given a corresponding monadic predicate F.

is already the situation in classical logic and here we are dealing with a weaker logical system. Let us consider now the third inference. Assume that $\langle \boldsymbol{C}, \mathbf{M}\rangle$ is a safe $\mathcal{P}$-structure such that $\|MF(c,d)\|^{\boldsymbol{C}}_{\mathbf{M}} \in D$ and it complies with the semantic stipulations corresponding to the involved predicates. By the analysis of comparatives, we have that $\|F(c)\|^{\boldsymbol{C}}_{\mathbf{M}} > \|F(d)\|^{\boldsymbol{C}}_{\mathbf{M}}$. Moreover, since 'full' is a highest-standard predicate, we have imposed that $F_{\mathbf{M}}$ maps individuals onto $D' \cup \{+\infty\}$. Therefore, it is clear that $\|F(d)\|^{\boldsymbol{C}}_{\mathbf{M}} \in D'$ and hence $\|\neg F(d)\|^{\boldsymbol{C}}_{\mathbf{M}} = \|F(d) \to \overline{0}\|^{\boldsymbol{C}}_{\mathbf{M}} \in D$. The validity of these kind of inferences shows the usefulness of the consequence relation $\vdash_{sem}$: we know that in any state of things in which $M(c,d)$ is true, because of logical and linguistic reasons, $\neg F(d)$ must be true as well, which we could not know without involving non-logical facts. Finally, to show that $MF(c,d) \nvdash_{sem} F(c)$ we need to produce a counterexample in the class of acceptable models: it suffices to require that $\|F(d)\|^{\boldsymbol{C}}_{\mathbf{M}} < \|F(c)\|^{\boldsymbol{C}}_{\mathbf{M}} < \mathtt{t}$.

There is a sense in which the semantic notion of consequence can be reduced to the logical one. Indeed, the classes of acceptable models could be determined (up to isomorphism) by sentences of the first-order language involving predicates for the corresponding gradable properties and, for each gradable predicate F, a binary predicate MF used (as above) to represent comparatives.[37] Roughly speaking, we would need to write down axioms to force MF to behave like a strict order, force predicates to satisfy the required monotonicity conditions, force each type of gradable predicate to behave as described above, etc.

7 Conclusion

In the formal semantics literature, the dominant approach to vague adjectives involves a semantics employing degrees, built on an underlying framework involving bivalence and classical logic. These degrees are not degrees of truth and the approach is very different from fuzzy logic based theories of vagueness. In this paper we have posed a challenge to such approaches: come up with a solution to the sorites that is as good as the fuzzy solution outlined above. In particular, it must conform to the accepted modus operandi of formal semantics, whereby the semantic theory is an explicit formulation of something that competent speakers implicitly grasp, that guides and explains their linguistic behaviour. We have suggested that the kinds of epistemicist and contextualist solutions typically favoured by proponents of such formal semantic theories fail in this regard. This is not to say that no solution is possible: but there is a real challenge here for proponents of the bivalent approach—one that has not yet been met. On the other side of the coin, fuzzy theories of vagueness face challenges of their own: analyse the comparative along the lines of (C), or else in some other equally compelling manner—and provide analyses of predicates which are both gradable and precise. We have responded to these challenges by presenting a version of the fuzzy logic approach that makes use of a new structure of truth degrees, the bounded IUL-chain $\boldsymbol{C}$. The fact that this scale has a more general structure than $[0,1]$ makes it suitable to accommodate (C) and to adequately analyse the main types of gradable precise predicates, while still allowing us to retain the usual fuzzy solution to sorites.

Our conclusion is twofold. On the one hand, we claim that the burden is on the degree-based approach to show that it can match the advantages of the fuzzy solution to the sorites. On the other hand, we believe the fuzzy account developed in this article carries a significant improvement over prior fuzzy theories, providing not only a satisfactory account of the sorites, but also an elegant and unified analysis for a variety of natural language phenomena. Of course, the linguistic constructions relevant for a complete account of gradability are numerous and much work in providing a fuzzy account for them remains to be done. However, in this article we have set the foundations for such an account, thereby taking the first steps towards vindicating the fuzzy theory of vagueness and gradability as a serious contender in the formal semantics arena.[38]

[37] Nevertheless, the question of the axiomatizability of the logic of $\boldsymbol{C}$ is left for future research.

[38] We are grateful to the anonymous referees for helpful comments. The research leading to these results

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