Journal of Applied Non-Classical Logics
Vol. 00, No. 00, Month 201X, 1–31

RESEARCH ARTICLE

A modal theorem-preserving translation of a class of three-valued logics of incomplete information

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(*v1.0 released September 2012*)

There are several three-valued logical systems that form a scattered landscape, even if all reasonable connectives in three-valued logics can be derived from a few of them. Most papers on this subject neglect the issue of the relevance of such logics in relation with the intended meaning of the third truth value. Here, we focus on the case where the third truth-value means *unknown*, as suggested by Kleene. Under such an understanding, we show that any truth-qualified formula in a large range of three-valued logics can be translated into KD as a modal formula of depth 1, with modalities in front of literals only, while preserving all tautologies and inference rules of the original three-valued logic. This simple information logic is a two-tiered classical propositional logic with simple semantics in terms of epistemic states understood as subsets of classical interpretations. We study in particular the translations of Kleene, Gödel, Łukasiewicz and Nelson logics. We show that Priest logic of paradox, closely connected to Kleene's, can also be translated into our modal setting, just exchanging the modalities *possible* and *necessary*. Our work enables the precise expressive power of three-valued logics to be laid bare for the purpose of uncertainty management.

Keywords: Three-valued logics; modal logic; uncertainty; incomplete information.

1. Introduction

Classical Boolean logic has a remarkable advantage over many others: the definition of its basic connectives is clear and consensual, even if the truth values *true* (1) and *false* (0) can be interpreted in practice in different ways. Moreover, there is a complete agreement on its model-based semantics. Its formal setting seems to ideally capture the "targeted reality", that of propositions being true or false in each possible world. The situation is quite different with many-valued logics, where we replace the two truth values by an ordered set with more than two truth values. The simplest case is three-valued logic where we add a single intermediate value, here denoted by $\frac{1}{2}$. Naively, we might think that three-valued logic should be as basic as Boolean logic: the set $\{0, \frac{1}{2}, 1\}$ is the simplest example of a bipolar scale (Dubois and Prade, 2008), isomorphic to the set of signs $\{-, 0, +\}$. However, there are quite a number of three-valued logics, since the extension to three values of the Boolean connectives is not unique. Worse, there is

This is an Author's Accepted Manuscript of an article published in Journal of Applied Non-Classical Logic, 23(4), 321-352, available online at: http://www.tandfonline.com/10.1080/11663081.2013.863491.
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no agreement on the intuitive interpretation of this third truth value in the literature. Several interpretations have been proposed. Here is a (probably not exhaustive) list:

(1) *Possible*: the oldest interpretation due to Łukasiewicz (Borowski, 1970). Unfortunately, it seems to have introduced some confusion between modalities and truth values, that is still looming in some parts of the many-valued logic literature; see the discussions in (Font and Hájek, 2002).
(2) *Half-true*: it is the natural understanding in formal fuzzy logic (Hájek, 1998): if it is true that a man whose height is 1.80 m. is tall and it is false that a man with height 1.60 m. is tall, we can think that it is half-true that a man whose height is 1.70 m. is tall. In this view, truth becomes a matter of degree (Zadeh, 1975). Then $\frac{1}{2}$ captures the idea of *borderline*.
(3) *Undefined*: this vision is typical of the studies on recursive functions modelled by logical formulas and it can be found in the book of Kleene (1952). A formula is not defined if some of its arguments are out of its domain. So, in this case the third truth value has a contaminating effect through recursion.
(4) *Unknown*: in the same book, Kleene suggests this alternative interpretation of the intermediate value. It is the most usual point of view outside the fuzzy set community. Unfortunately, it suffers from the confusion between truth value and epistemic state, which generates paradoxes (Dubois, 2008; Dubois and Prade, 2001; Urquhart, 1986), just like the Łukasiewicz proposal, if truth-functionality is assumed.
(5) *Inconsistent*: in some sense, it is the dual of "unknown". Several paraconsistent logics try to tame the notion of contradiction by means of a truth-functional logic (da Costa and Alves, 1981; Priest, 1979), for instance, while Belnap (1977) considers both *unknown* and *inconsistent* as additional truth-values. This standpoint has been criticized as also generating paradoxes (Dubois, 2008; Fox, 1990).
(6) *Irrelevant*: this point of view is similar to "undefined" but with the opposite effect: abstention. If a component of a formula has $\frac{1}{2}$ as truth value, the truth value of the whole formula is determined by the remaining components. This is at work in the logic of Sobociński (1952), and the logic of conditional events (Dubois and Prade, 1994).

In the present work,[1] we are interested in the fourth interpretation *unknown* of the third truth value $\frac{1}{2}$ popularized by Kleene (this includes the Łukasiewicz view). Kleene logic has been used in logic programming (Fitting, 1985), formal concept analysis (Burmeister and Holzer, 2005) and databases (Codd, 1979; Grant, 1980) to model such notions as null-values.

However, the use of a truth-functional logic such as Kleene or Łukasiewicz logic accounting for the idea of *unknown* has always been controversial (see discussions in (Urquhart, 1986), more recently the second author (Dubois, 2008)). In a nutshell, the loss of properties such as the law of excluded middle, when moving from two to three truth values including *unknown* sounds questionable. Indeed, in Kleene logic the negation operation applied to $\frac{1}{2}$ yields $\frac{1}{2}$: so if a proposition α is assigned $\frac{1}{2}$, its negation $\neg\alpha$ is also assigned $\frac{1}{2}$, and so are the disjunction $\alpha \sqcup \neg\alpha$ and the conjunction $\alpha \sqcap \neg\alpha$ in Kleene logic. Typically, assigning $\frac{1}{2}$ to α may mean that the available recursive computation method cannot decide whether α is true or false, hence not for its negation $\neg\alpha$ and so, not for $\alpha \sqcup \neg\alpha$, $\alpha \sqcap \neg\alpha$ either. However, if the actual truth-value of α is 0 or 1, any expression of the form $\alpha \sqcup \neg\alpha$ cannot be but assigned 1, and likewise 0 to $\alpha \sqcap \neg\alpha$, even if the procedure cannot find it recursively. It is easy to let the computer detect these

[1] It is is an extended and completely revised version of a conference paper (Ciucci and Dubois, 2012).

patterns and avoid assigning $\frac{1}{2}$ to such ontic tautologies or contradictions.

As a matter of fact, if the third truth-value means *unknown*, it suggests that the corresponding three-valued logic aims at capturing epistemic notions, and so does Łukasiewicz view of *possible* as a third truth value. Clearly, *unknown* means that true and false are *possible*[1]. So it is natural to bridge the gap between such three-valued logics and modal epistemic logics. Already in 1921, Tarski had the idea of translating the modalities *possible* and *necessary* into Łukasiewicz three-valued logic. The modal *Possible* is defined on $\{0, \frac{1}{2}, 1\}$ as $\Diamond x = \neg x \to_L x = \min\{2x, 1\}$ with Łukasiewicz negation and implication. In this translation, *possible* thus means that the truth value is at least $\frac{1}{2}$. So the question is: which of the two is the most suitable language for handling partial ignorance? the one of modal logic or the one of three-valued logic? This paper addresses this issue for the class of three-valued logics with monotonic conjunctions and implications that extend Boolean connectives, by translating them into a very elementary modal logic, less expressive even than S5.

This point of view is opposite to Tarski's: rather than trying to translate modal logic into a three-valued one (which is provably hopeless (Beziau, 2011)), it seems more feasible and fruitful to do the converse. We propose a theorem-preserving translation of three-valued logics in a modal setting. According to the epistemic nature of the interpretation of $\frac{1}{2}$ here chosen, the framework of some epistemic logic looks like a natural choice for a target language. Unsurprisingly, as shown in the following, modal logic is more expressive than all the three-valued logics of *unknown*. Note that the idea of using modal logic as a general target language for explicating logics with more concise languages is in fact not new. The oldest similar attempt is that of Gödel who provided a theorem-preserving translation of intuitionistic logic into the modal logic S4 (Gödel, 1933), a translation studied in more details by McKinsey and Tarski (1948). Translations of three-valued logics into modal logic are not new either. For instance, Duffy (1979), and very recently Kooi and Tamminga (2013) use S5 as a target language. Minari (2002) applies the above Tarski expression of the modal *possible* to Wajsberg axioms of Łukasiewicz logic, and studies the resulting modal system. More generally, Demri (2000) has proposed an embedding of finite many-valued logics into von Wright's logic of elsewhere. We can also cite the modal translation of the 5-valued equilibrium logic into a bimodal logic with only two possible worlds, by Fariñas del Cerro and Herzig (2011). In many cases, the semantics on the modal side relies on Kripke-style relations.

The main contribution of the paper is to point out that we do not need the full language of S5 in order to capture three-valued logics exactly in a modal setting, let alone full-fledged accessibility relations for the semantics. A very simple two-tiered propositional logic called MEL having a very simple and intuitive semantics, is enough to capture Łukasiewicz logics, hence all other three-valued logics in the class we consider here. It is an elementary variant of epistemic logic, sufficient for declaring a Boolean proposition to be unknown at the syntactic level. Its language is a fragment of the KD language with modal formulas of depth 1 and modalities in front of literals only. The motivation of this translation is to better understand the meaning of 3-valued connectives and formulas in the scope of handling incomplete information. Moreover, the above cited translations into S5, like (Kooi and Tamminga, 2013), focus on the separation between valid, invalid and contingent formulas only (as expected with S5). In contrast, we here deal with the issue of inference of a formula from a set of formulas in three-valued logics, and show it translates into inference from a knowledge base in MEL.

[1]Actually,Łukasiewicz proposed this idea for the study of contingent futures: it is possible that the battle will be won and it is possible that the battle will be lost.

The paper develops as follows: first, we recall the Minimal Epistemic Logic (Banerjee and Dubois, 2009) MEL,[2] where we can express only Boolean propositional formulas prefixed by a modality and Boolean combinations thereof. It has a simple semantics in term of non-empty subsets of interpretations. In Section 3, we review truth-tables for basic connectives of three-valued logics under minimal requirements of monotonicity and coincidence with Boolean truth-tables, and recall that only very few connectives are needed to generate all the other ones (we essentially need the minimum and its residuated implication, plus an involutive negation). Some three-valued logics like Łukasiewicz's can express all the others. In section 4, we show how it is possible to express semantic constraints on the truth-value of three-valued propositions by means of Boolean modal formulas, and we describe the one-to-one correspondence between three-valued valuations and partial classical models. In the remaining sections, we provide theorem-preserving translations of several three-valued logics into MEL. We lay bare in each case the proper fragment of the language of MEL that can encode the translation of these three-valued logics. Section 5 deals with 3-valued Łukasiewicz logic $Ł_3$ and shows that it exactly corresponds to the fragment of the MEL language where modalities are placed only in front of literals. We also show that reasoning from a set of formulas in $Ł_3$ can be achieved in MEL by classical inference from its translation. We also translate Nelson logic (also LPF in (Avron, 1991)), which is known to be equivalent to $Ł_3$. Section 6 considers the translation into MEL of other logics that are less expressive than $Ł_3$ (Kleene and Gödel-Heyting three-valued logics), plus a semantic variant of Kleene logic (the Logic of Paradox) which is paraconsistent. Section 7 wraps up the results obtained so far, comparing the modal translations of all fourteen truth-qualified three-valued conjunctions and implications laid bare in Section 3. Perspectives toward translations of other multivalued logics, having different intuitions, into the modal setting are outlined.

2. A simple information logic

The usual truth values *true* (1) and *false* (0) are of ontological nature (which means that they are part of the definition of what we call *proposition*[1]), whereas *unknown* sounds epistemic: it reveals a knowledge state according to which the truth value of a proposition (in the usual Boolean sense) in a given situation is out of reach (for instance one cannot compute it, either by lack of computing power, or due to a sheer lack of information). It corresponds to the epistemic state of an agent that can neither assert the truth of a Boolean proposition nor its falsity.

Admitting that the concept of "unknown" refers to a knowledge state rather than to an ontic truth value, we may, instead of adding a specific truth-value, augment the syntax of Boolean propositional logic (BPL) with the capability of stating that we ignore the truth value (1 or 0) of propositions. The natural framework to syntactically encode knowledge or belief regarding Boolean propositions is modal logic, and in particular, the logic KD. Nevertheless, only a very limited fragment of this language is needed here: the language MEL (Banerjee and Dubois, 2009).

Consider a set of propositional variables $\mathcal{V} = \{a, b, c, \dots, p, \dots\}$ and a standard propositional language $\mathcal{L}$ built on these symbols along with the Boolean connectives of conjunction and negation $(\wedge, ')$. As usual, disjunction $\alpha \vee \beta$ stands for $(\alpha' \wedge \beta')'$, implication $\alpha \Rightarrow \beta$ stands for $\alpha' \vee \beta$, and tautology $\top$ for $\alpha \vee \alpha'$. Let us build another propositional language $\mathcal{L}_\Box$ whose set of propositional variables is of the form $\mathcal{V}_\Box = \{\Box\alpha : \alpha \in \mathcal{L}\}$

[2]In that paper, the acronym stands for Meta-Epistemic Logic, excluding the case of an agent reasoning on its own beliefs.
[1]and not that they represent Platonist ideals.

to which the classical connectives can be applied. It is endowed with a modality operator $\Box$ expressing certainty, that encapsulates formulas in $\mathcal{L}$. We denote by $\alpha, \beta, \ldots$ the propositional formulas of $\mathcal{L}$, and $\phi, \psi, \ldots$ the modal formulas of $\mathcal{L}_\Box$. In other words:

$$\mathcal{L}_\Box = \Box\alpha : \alpha \in \mathcal{L}|\phi'|\phi \wedge \psi|\phi \vee \psi|\phi \Rightarrow \psi.$$

The logic MEL uses the language $\mathcal{L}_\Box$ with the following axioms:

(1) $\phi \Rightarrow (\psi \Rightarrow \phi)$
(2) $(\psi \Rightarrow (\phi \Rightarrow \mu)) \Rightarrow ((\psi \Rightarrow \phi) \Rightarrow (\psi \Rightarrow \mu))$
(3) $(\phi' \Rightarrow \psi') \Rightarrow (\psi \Rightarrow \phi)$
(RM) : $\Box\alpha \Rightarrow \Box\beta$ if $\vdash \alpha \Rightarrow \beta$ in BPL.
(M) : $\Box(\alpha \wedge \beta) \Rightarrow (\Box\alpha \wedge \Box\beta)$
(C) : $(\Box\alpha \wedge \Box\beta) \Rightarrow \Box(\alpha \wedge \beta)$
(N) : $\Box\top$
(D) : $\Box\alpha \Rightarrow \Diamond\alpha$

and the inference rule is modus ponens. As usual, the modality *possible* ($\Diamond$) is defined as $\Diamond\alpha \equiv (\Box\alpha')'$. The first three axioms are those of BPL and the other ones are those of modal logic KD. In this setting, (M) and (C) can be replaced by axiom (K):

$$(\mathrm{K}) : \Box(\alpha \Rightarrow \beta) \Rightarrow (\Box\alpha \Rightarrow \Box\beta).$$

It points out the fact that MEL language is the "subjective" fragment of the language of S5 (i.e., the one without "objective" Boolean formulas α combined or not with modal ones). We can justify the minimality property of the modal language $\mathcal{L}_\Box$ for reasoning about incomplete information: in $\mathcal{L}_\Box$, we can only express at the syntactic level that a proposition in BPL is certainly true, certainly false or unknown as well as all the logical combinations of these assertions.

The MEL semantics is very simple but it stands in contrast with usual modal semantics in terms of accessibility relations, that are not needed here as we do not nest modalities. Let Ω be the set of $\mathcal{L}$-interpretations: $\{\omega : \mathcal{V} \to \{0,1\}\}$. The set of models of α is $[\alpha] = \{\omega : \omega \models \alpha\}$. A (meta)-interpretation of $\mathcal{L}_\Box$ is a *non-empty* set $E \subseteq \Omega$ of interpretations of $\mathcal{L}$ understood as an epistemic state[1]. We define satisfiability as follows:

- $E \models \Box\alpha$ if $E \subseteq [\alpha]$ (α is certainly true in the epistemic state E)
- $E \models \phi \wedge \psi$ if $E \models \phi$ and $E \models \psi$;
- $E \models \phi'$ if $E \models \phi$ is false.

MEL is sound and complete with respect to this semantics (Banerjee and Dubois, 2009); see (Banerjee *et al.*, 2013) for a direct proof.

Some comments help positioning our simple information logic with respect to the standard way of envisaging modal epistemic logics as well as uncertainty theories:

- Unlike epistemic logics, MEL is not a flat extension of propositional logic enriched with modal symbols. It is a two-tiered logic, where both layers are propositional. Its language $\mathcal{L}_\Box$ is disjoint from $\mathcal{L}$, contrary to the language of S5. Moreover the deduction theorem holds in MEL, contrary to usual modal logics.
- In standard modal logic, the set of models of $\Box\alpha$ is a subset of Ω, just as BPL propositions α (all the interpretations whose images via the accessibility relation are included in the set of models of α), while here, the set of models of $\Box\alpha$ is a subset of the power set of Ω.

[1]The non-emptiness of E is enforced by axiom (D).

- We can debate whether MEL is an epistemic or a doxastic logic. Our formalism does not take side, since axiom (D) is valid in both S5 and KD45 and the axiom (T) of knowledge ($\Box\alpha \Rightarrow \alpha$) is not expressible in MEL. We kept the term "epistemic" in reference to the idea of an information state, whether it is consistent with reality or not. Moreover, MEL is not concerned with introspection, and only deals with reasoning about the beliefs of an external agent as revealed by the latter.
- We remark that in this framework, uncertainty modeling is Boolean and can be described in possibility theory (Dubois and Prade, 2001). The satisfiability $E \models \Diamond\alpha$ means $E \cap [\alpha] \neq \emptyset$. By definition, it can be written as $\Pi([\alpha]) = 1$ in the sense of a possibility measure Π computed with the possibility distribution given by the characteristic function of the non-empty set E. Intuitively, $E \models \Diamond\alpha$ then means that the agent has not enough information for discarding α as being false, or in other words, that α does not contradict the agent 's epistemic state. Likewise, the satisfiability $E \models \Box\alpha$ can be written as $N([\alpha]) = 1 - \Pi([\alpha]) = 1$ in the sense of a necessity measure. It expresses the certainty that α is true. Axioms (M) and (C) lay bare the connection with possibility theory, as they state the equivalence between $\Box\alpha \wedge \Box\beta$ and $\Box(\alpha \wedge \beta)$ (which also writes $N(\alpha \wedge \beta) = \min(N(\alpha), N(\beta))$). In probabilistic terms, $\Diamond\alpha$ stands for the probability of α being positive, while $\Box\alpha$ expresses that the probability of α is 1, provided that E is the support of the distribution.

3. Connectives in three-valued logics

The idea that *unknown* can be a truth value seems to originate from a common usage in natural language, creating a confusion between *true* and *certainly true* (or yet provable), *false* and *certainly false*. Indeed, in the spoken language, saying "it is true that..." is often short for "I know it is true that...". We mix up, in this way, the idea of truth *per se* with the assertion of truth. The latter reveals something about the information possessed by the speaker (its epistemic state), namely that he or she knows that a proposition is true. The value *unknown* attached to a proposition α ($\Diamond\alpha \wedge \Diamond\alpha'$ in MEL) is thus in conflict with *certainly true* ($\Box\alpha$) and *certainly false* ($\Box\alpha'$), not with the ontological truth values *true* and *false*. In this context, it sounds strange to add *unknown* to the usual truth-set as a full-fledged truth-value.

Accordingly, we shall not use the same symbols for Boolean truth values and the ones of the three-valued logic as long as $\frac{1}{2}$ means *unknown*. For the sake of clarity, we will use 0 and 1 for ontic truth-values in the Boolean case, $\mathbf{0}$ and $\mathbf{1}$ for their epistemic counterparts in the three-valued case. The truth set $\mathbf{3} = \{\mathbf{0}, \frac{1}{2}, \mathbf{1}\}$ contains epistemic values, as opposed to 0 and 1. Moreover, we equip $\mathbf{3}$ with a total order $\leq$: $\mathbf{0} < \frac{1}{2} < \mathbf{1}$, often referred to as the truth ordering (Belnap, 1977).

Three-valued logics assume connectives are compositional. Conjunction, implication and negation on the set of values $\mathbf{3}$ can be defined by minimal intuitive properties.

Definition 1. *A conjunction on* $(\mathbf{3}, \leq)$ *is a binary mapping* $*$ *from* $\mathbf{3} \times \mathbf{3}$ *to* $\mathbf{3}$ *such that*

(C1) If $x \leq y$ *then* $x * z \leq y * z$;
(C2) If $x \leq y$ *then* $z * x \leq z * y$;
(C3) $\mathbf{0} * \mathbf{0} = \mathbf{0} * \mathbf{1} = \mathbf{1} * \mathbf{0} = \mathbf{0}$ *and* $\mathbf{1} * \mathbf{1} = \mathbf{1}$.

We note that (C3) requires that $*$ be an extension of the connective AND in Boolean logic. Then, the monotonicity properties (C1-C2) imply $\frac{1}{2} * \mathbf{0} = \mathbf{0} * \frac{1}{2} = \mathbf{0}$. If we consider all the possible cases, there are 14 conjunctions satisfying Definition 1. Among them, only six are commutative and only five associative. These five conjunctions are already known in literature and precisely, they have been studied in the following logics: Sette (1973), Sobociński (1952), Łukasiewicz (Borowski, 1970), Kleene (1952), Bochvar (1981). In Table 1, we list all the 14 conjunctions. The idempotent and commutative Kleene

$*$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{0}$		
$\mathbf{1}$	$\mathbf{0}$		$\mathbf{1}$

n.	$\frac{1}{2} * \frac{1}{2}$	$\mathbf{1} * \frac{1}{2}$	$\frac{1}{2} * \mathbf{1}$	name / inventor
1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	Sette
2	$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{1}$	quasi conjunction/Sobociński
3	$\frac{1}{2}$	$\mathbf{1}$	$\frac{1}{2}$	
4	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{1}$	
5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	min/interval conjunction/Kleene
6	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	
7	$\mathbf{0}$	$\mathbf{0}$	$\frac{1}{2}$	
8	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	Bochvar external
9	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{0}$	
10	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$	
11	$\mathbf{0}$	$\frac{1}{2}$	$\frac{1}{2}$	Łukasiewicz
12	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$	
13	$\mathbf{0}$	$\mathbf{1}$	$\frac{1}{2}$	
14	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	

Table 1. All conjunctions on **3** according to Definition 1

conjunction and disjunction (the minimum, denoted by $\sqcap$ and the maximum denoted by $\sqcup$) are present in **3** due the total order assumption ($x \sqcap y = y \sqcap x = x$ if and only if $x \leq y$.)

In the case of implication, we can give a general definition, which extends Boolean logic and supposes monotonicity (decreasing in the first argument, increasing in the second).

Definition 2. *An implication on* $(\mathbf{3}, \leq)$ *is a binary mapping* $\rightarrow$ *from* $\mathbf{3} \times \mathbf{3}$ *to* $\mathbf{3}$ *such that*

(I1) If $x \leq y$ *then* $y \Rightarrow z \leq x \rightarrow z$*;*
(I2) If $x \leq y$ *then* $z \Rightarrow x \leq z \rightarrow y$*;*
(I3) $\mathbf{0} \rightarrow \mathbf{0} = \mathbf{1} \rightarrow \mathbf{1} = \mathbf{1}$ *and* $\mathbf{1} \rightarrow \mathbf{0} = \mathbf{0}$.

From the above definition we derive the identities $x \rightarrow \mathbf{1} = \mathbf{1}$, $\mathbf{0} \rightarrow x = \mathbf{1}$ and the inequality $\frac{1}{2} \rightarrow \frac{1}{2} \geq \max(\mathbf{1} \rightarrow \frac{1}{2}, \frac{1}{2} \rightarrow \mathbf{0})$. There are 14 implications satisfying this definition. Nine of them are known in the literature and have been studied. Besides those implications named after the five logics mentioned above, there are also those named after Jaśkowski (1969), Gödel (1932), Nelson (1949), Gaines-Rescher (Gaines, 1976). The complete list is given in Table 2.

Gödel implication is present in the lattice $(\mathbf{3}, \leq)$ using the residuation:

$$x \sqcap y \leq z \text{ if and only if } x \leq y \rightarrow_G z,$$

such that $y \rightarrow_G z = \mathbf{1}$ if $y \leq z$ and z otherwise. Then $(\mathbf{3}, \leq)$ is called a Heyting chain.

Finally, there are only three possible negations that extend the Boolean negation, that is, preserve $\mathbf{0}' = \mathbf{1}$ and $\mathbf{1}' = \mathbf{0}$:

(1) $\sim\frac{1}{2} = \mathbf{0}$. We call it an intuitionistic negation (as it satisfies the law of contradiction, not the excluded middle law).
(2) $\neg\frac{1}{2} = \frac{1}{2}$. It is an involutive negation.

$\rightarrow$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{1}{2}$			$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$		$\mathbf{1}$

n.	$\frac{1}{2} \rightarrow \frac{1}{2}$	$\mathbf{1} \rightarrow \frac{1}{2}$	$\frac{1}{2} \rightarrow \mathbf{0}$	name / inventor
1	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	
2	$\frac{1}{2}$	$\mathbf{0}$	$\mathbf{0}$	Sobociński
3	$\frac{1}{2}$	$\mathbf{0}$	$\frac{1}{2}$	
4	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{0}$	Jaśkowski
5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	(strong) Kleene
6	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	Sette
7	$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	
8	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	
9	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{1}$	Nelson
10	$\mathbf{1}$	$\frac{1}{2}$	$\mathbf{0}$	Gödel
11	$\mathbf{1}$	$\frac{1}{2}$	$\frac{1}{2}$	Łukasiewicz
12	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	Bochvar external
13	$\mathbf{1}$	$\mathbf{0}$	$\frac{1}{2}$	
14	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	Gaines–Rescher

Table 2. All implications according to Definition 2.

(3) $-\frac{1}{2} = \mathbf{1}$. We call it a paraconsistent negation (as it satisfies the law of excluded middle, not the one of contradiction).

The intuitionistic negation is definable by means of Gödel implication and the truth-constant $\mathbf{0}$ as $\sim x = x \rightarrow_G \mathbf{0}$, and the paraconsistent one using Nelson implication instead, as $-x = x \rightarrow_N \mathbf{0}$.

Finally, despite the existence of several known systems of three-valued logics, we can use, in the above setting, only one encompassing three-valued structure to express all connectives. That is, all the connectives satisfying the above definitions, can be obtained from a structure equipped with few primitive ones (Ciucci and Dubois, 2013b). In the following, we denote by $\mathbf{3}$ the set of three elements without any structure and by $\overline{3}$ the same set equipped with the usual order $\mathbf{0} < \frac{1}{2} < \mathbf{1}$ or equivalently, $\overline{3} = (\mathbf{3}, \sqcap, \rightarrow_G)$.

Proposition 1 (Ciucci and Dubois (2013b)). *All 14 conjunctions and implications can be expressed in any of the following systems:*

- $(\overline{3}, \neg) = (\mathbf{3}, \sqcap, \rightarrow_G, \neg)$;
- $(\overline{3}, \rightarrow_K)$ *where* $\rightarrow_K$ *is Kleene implication* $(x \rightarrow_5 y = \neg x \sqcup y)$;
- $(\mathbf{3}, \rightarrow_L, \mathbf{0})$ *where* $\rightarrow_L$ *is Łukasiewicz implication* $(x \rightarrow_{11} y = \min(1, 1 - x + y))$;
- $(\mathbf{3}, \rightarrow_K, \sim, \mathbf{0})$ *where* $\rightarrow_K$ *is Kleene implication and* $\sim$ *the intuitionistic negation.*

So, in the first two cases, we assume a Heyting chain, whereas in the other two, we can derive it from the other connectives. We remark also that the intuitionistic negation can be replaced by the paraconsistent negation in the last item. The above result differs from functional completeness, since Proposition 1 only deals with three-valued functions that coincide with Boolean connectives on $\{\mathbf{0}, \mathbf{1}\}$.

4. The principles of the translation

Let T be a truth set and $S \subseteq T$ a non-empty subset of truth values. A truth-qualified statement is of the form: *the truth-value of* α *lies in* S, where α is a formula in some language. It means that only the truth values in S are possible for α in the considered knowledge state of an agent (the values outside S are impossible).

In the case of Boolean logic, we consider statements $t(\alpha) \in S \subseteq \{0, 1\}$ where t is a

Boolean valuation. It is a possibly incomplete description of the agent knowledge about the truth state of α in the current state of the world. We can then model epistemic terms *certainly true*, *certainly false* and *unknown* by the respective subsets of Boolean truth values $S = \{1\}$, $\{0\}$ and $\{0,1\}$[1]. For instance, the truth-qualified statement $t(\alpha) \in \{1\}$ encodes *certainly true* since the only possible truth value is 1 (*true*). Mixing up the ontological *true* and the epistemic *certainly true* is the same as confusing an element with a singleton.

In the following we consider a three-valued logic based on propositional variables $\mathcal{V} = \{a, b, c, \ldots, p, \ldots\}$. Stricto sensu, we should not use the same notation for three-valued propositional variables and Boolean ones. However, we will do it for the sake of simplicity. If v is a three-valued valuation, the assertion $v(a) \in S \subseteq \mathbf{3}$ is a partial description of the knowledge state of an agent concerning an atomic Boolean proposition a. Here, we identify $\{1\}$ with $\mathbf{1}$, $\{0\}$ with $\mathbf{0}$, and $\{0,1\}$ with $\frac{1}{2}$, and consider $\mathbf{3}$ as a set of epistemic truth-values. For instance, $v(a) \in \{\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}}\}$ means that we know the agent either is certain that a is false, or ignores if a is true or not. In the following this is the kind of statement we shall translate into MEL.

4.1 From three-valued truth-qualified statements to MEL

Let $\mathcal{L}_3$ denote a language supporting the three-valued connectives introduced in the previous section. If we interpret the three epistemic truth-values $\mathbf{0}, \mathbf{1}, \frac{1}{2}$ as *certainly true*, *certainly false* and *unknown* respectively, we can translate into MEL the assignment of one or more of such truth-values to a proposition $\alpha \in \mathcal{L}_3$. Let $\mathbb{V}$ be the set of three-valued valuations on the set of variables $\mathcal{V}$. We denote by $\mathcal{T}(v(\alpha) \in S)$ the translation into MEL of the set $\{v : v(\alpha) \in S\}$ corresponding to the statement $v(\alpha) \in S$. Formally, it is a function $\mathcal{T} : 2^{\mathbb{V}} \to \mathcal{L}_\Box$ from subsets of ternary valuations to the modal language $\mathcal{L}_\Box : \{v : v(\alpha) \in S\} \mapsto \phi = \mathcal{T}(v(\alpha) \in S)$. In the special case of atomic propositions, we define it as follows, in agreement with the intended meaning of the epistemic truth-values:

$$\mathcal{T}(v(a) = \mathbf{1}) = \Box a \qquad \mathcal{T}(v(a) = \mathbf{0}) = \Box a'$$

from which it follows:

$$\mathcal{T}(v(a) \geq \tfrac{1}{2}) = \Diamond a; \qquad \mathcal{T}(v(a) \leq \tfrac{1}{2}) = \Diamond a';$$
$$\mathcal{T}(v(a) = \tfrac{1}{2}) = \Diamond a \wedge \Diamond a'; \qquad \mathcal{T}(v(a) \in \{\mathbf{0}, \mathbf{1}\}) = \Box a \vee \Box a'.$$

These definitions shed light on the acceptability or not of the excluded middle law and the contradiction principle in the presence of the value *unknown*: a is always ontologically true or false, but $\Box a \vee \Box a'$ is not a tautology nor is $\Diamond a \wedge \Diamond a'$ a contradiction in MEL. The latter means that it is known the agent knows the truth-value of a but the agent did not reveal it.

Given this translation method it becomes clear that the assignment of "truth-values" to any formula in a three-valued logic can be translated into a formula in MEL obtained by combining atomic formulas of the form $\Box a, \Box a'$ for variables $a \in \mathcal{V}$. Indeed, each expression in a three-valued logic is the combination of subformulas by some of primitive unary or binary connective defined by a truth-table. Assigning a truth-value to the formula (e.g., $\mathbf{1}$) leads to constraints on the truth-values of the subformulas, which in

[1]Belnap (1977) follows another convention where $\{0,1\}$ represents a conjunction of truth values and encodes the contradiction while the empty set represents *unknown*.

turn determines constraints on the truth-values of subsubformulas, and so on, until reaching constraints on the truth-value of elementary variables of the language, all of which can be translated into MEL as per the above translation rules. It is clear that the original formula will be translated into a logical formula in MEL where involved connectives express Boolean dependencies between constraints on the truth-values of 3-valued variables. Under the above translation principles it is clear that any translated truth-qualified three-valued formula will belong to a fragment of the MEL language where we can put modalities only in front of literals, that is, $\mathcal{L}^{\ell}_{\Box} \subset \mathcal{L}_{\Box}$ defined by

$$\mathcal{L}^{\ell}_{\Box} = \Box a | \Box a' | \phi' | \phi \wedge \psi | \phi \vee \psi.$$

4.2 From three-valued semantics to epistemic semantics

At the semantic level, we shall map 3-valued valuations to special epistemic states that serve as interpretations of the sublanguage $\mathcal{L}^{\ell}_{\Box}$ of MEL. Given a 3-valued valuation v, a partial Boolean model, denoted by E_v, is naturally defined by $t(a) = 1$ if and only if $v(a) = \mathbf{1}$ and $t(a) = 0$ if and only if $v(a) = \mathbf{0}$. Such an epistemic state E_v has a particular (rectangular) form that makes it a partial model: it is the set of Boolean models of a non-contradictory conjunction of literals $\wedge_{v(a)=\mathbf{1}} a \bigwedge \wedge_{v(a)=\mathbf{0}} a'$. So, the consequence of interpreting the third truth-value as *unknown* is that we must interpret three-valued valuations as partial models, which are special cases of MEL interpretations.

Conversely, to any MEL interpretation E (a disjunction of propositional interpretations) we can assign a single 3-valued interpretation v_E defined as follows:

$$\forall a, v_E(a) = \begin{cases} \mathbf{1} & E \vDash \Box a \\ \mathbf{0} & E \vDash \Box a' \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

The map $E \mapsto v_E$ is not bijective. It defines an equivalence relation on epistemic states. Namely, $\{E : v_E = v\}$ is the set of epistemic states that are indistinguishable by the three-valued valuation v. Define the rectangular closure of a set E of propositional valuations as the set of models of $\wedge_{E \subseteq [a]} a \bigwedge \wedge_{E \subseteq [a']} a'$ (the conjunctions of literals known as true in the epistemic state E). Clearly, $E_v = \cup\{E : v_E = v\}$ is the unique partial Boolean model induced by v, and is the rectangular closure $r(E)$ of any epistemic state $E \in \{E : v_E = v\}$. Note that $\forall v \in \mathbb{V}, E_v \neq \emptyset$.

We can show that the MEL logic restricted to the language $\mathcal{L}^{\ell}_{\Box}$ is sound and complete with respect to the set of partial models of the propositional language $\mathcal{L}$.

Lemma 2. *$\forall \phi \in \mathcal{L}^{\ell}_{\Box}, \forall E \in 2^{\Omega} \setminus \{\emptyset\}, E \models \phi$ if and only if $r(E) \models \phi$.*

Proof. We proceed by induction.

For a literal x of BPL, if $E \models \Box x$, then E is the set of models of a formula of the form $x \wedge \alpha$, where α does not contain the variable associated with x. It is then clear that $r(E) = x \wedge r([\alpha])$, hence $r(E) \subseteq [x]$. The converse is obvious.

Suppose $E \models (\Box x)'$, that is $E \not\models \Box x$. Hence $r(E) \not\models \Box x$ either, since $E \subseteq r(E)$. Conversely, we know that if $E \models \Box x$ then $r(E) \models \Box x$ from the previous lines.

For conjunction, since $\Box x \wedge \Box y$ is equivalent to $\Box(x \wedge y)$ for two literals x and y, then, if $E \models \Box(x \wedge y)$, E is the set of models of a formula of the form $x \wedge y \wedge \beta$; the same technique as for literals can be used to conclude the equivalence with $r(E) \subseteq [x \wedge y]$. More generally, in formulas $\Box\alpha \in \mathcal{L}^{\ell}_{\Box}$, the BPL formula α corresponds to a conjunction of literals.

For disjunctions, $E \models \Box\alpha \vee \Box\beta$ is equivalent to $E \models \Box\alpha$ or $E \models \Box\beta$, which (inductive assumption) is equivalent to $r(E) \models \Box\alpha$ or $r(E) \models \Box\beta$, which in turn is equivalent to $r(E) \models \Box\alpha \vee \Box\beta$. □

Proposition 3. *Let ϕ be a formula and Γ a set of formulas in the language of $\mathcal{L}^{\ell}_{\Box}$. Then, $\Gamma \vdash \phi$ if and only if $\forall v \in \mathbb{V}, E_v \models \Gamma$ implies $E_v \models \phi$.*

This is a direct consequence of Lemma 2. This result leads us to the completeness of MEL restricted to the language $\mathcal{L}^{\ell}_{\Box}$ with respect to a three-valued semantics defined by $v \models \phi \in \mathcal{L}^{\ell}_{\Box}$ if and only if $E_v \models \phi$, due to the bijection between three-valued valuations v and partial Boolean models E_v. Given a three-valued logic system, our translation methodology consists in showing that the following statements are equivalent:

- For a given set B of three-valued formulas and a three-valued logic formula α, $B \vdash \alpha$ (using axioms and inference rules of the three-valued logic).
- $\{\mathcal{T}(v(\beta) \in D), \beta \in B\} \vdash \mathcal{T}(v(\alpha) \in D)$ in MEL, where D is the set of designated truth-values in the three-valued logic (that is, **1**, unless otherwise specified).

In the following, we consider four known three-valued logics (Kleene, Gödel three-valued intuitionistic, Łukasiewicz and Nelson-LPF logics) and show that, insofar as the third truth-value means *unknown*, they can be expressed in MEL, in the above sense. The two first ones can be expressed in, and are less expressive than, the latter. Especially, we show that MEL restricted to the language $\mathcal{L}^{\ell}_{\Box}$ exactly captures any of Łukasiewicz and Nelson logics as we will see in the next sections. Additionally, we also consider Priest Logic of Paradox.

5. From Łukasiewicz and Nelson three-valued logics to MEL and back

Łukasiewicz three-valued logic $Ł_3$ possesses a language based on $(\mathcal{V}, \rightarrow_L, \neg)$, powerful enough to express all connectives laid bare in section 3. It has been axiomatized by Wajsberg (1931), using the following axioms and the modus ponens rule:

(W1) $(\alpha \rightarrow_L \beta) \rightarrow_L ((\beta \rightarrow_L \gamma) \rightarrow_L (\alpha \rightarrow_L \gamma))$
(W2) $\alpha \rightarrow_L (\beta \rightarrow_L \alpha)$
(W3) $(\neg\beta \rightarrow_L \neg\alpha) \rightarrow_L (\alpha \rightarrow_L \beta))$
(W4) $(((\alpha \rightarrow_L \neg\alpha) \rightarrow_L \alpha) \rightarrow_L \alpha)$

The truth-table of the implication $\rightarrow_L$ is given in Table 3. It corresponds to the arithmetic expression $\min(1, 1 - x + y)$. The involutive negation of Kleene logic is recovered as $\neg a := a \rightarrow_L \mathbf{0}$. The formulas $\alpha \rightarrow_L \alpha$ and $\neg(\alpha \rightarrow_L \alpha)$ correspond to the tautology and the contradiction, and have truth-values **1** and **0**, respectively.

$\rightarrow_L$	**0**	$\frac{1}{2}$	**1**
0	**1**	**1**	**1**
$\frac{1}{2}$	$\frac{1}{2}$	**1**	**1**
1	**0**	$\frac{1}{2}$	1

$\odot$	**0**	$\frac{1}{2}$	**1**
0	**0**	**0**	**0**
$\frac{1}{2}$	**0**	**0**	$\frac{1}{2}$
1	**0**	$\frac{1}{2}$	**1**

$\oplus$	**0**	$\frac{1}{2}$	**1**
0	**0**	$\frac{1}{2}$	**1**
$\frac{1}{2}$	$\frac{1}{2}$	**1**	**1**
1	**1**	**1**	**1**

Table 3. Łukasiewicz implication, conjunction and disjunction truth tables.

We can also define two pairs of conjunction and disjunction connectives denoted by $(\sqcap, \sqcup)$ and $(\odot, \oplus)$. The first pair is Kleene's, recovered as $a \sqcup b = (a \rightarrow_L b) \rightarrow_L b$ and $a \sqcap b = \neg(\neg a \sqcup \neg b)$. Numerically, they correspond to well-known idempotent conorms and t-norms (Klement *et al.*, 2000): $\max(a, b)$ and $\min(a, b)$, respectively. The other pair is $a \oplus b := \neg a \rightarrow_L b$ and $a \odot b := \neg(\neg a \oplus \neg b)$ explicitly described in Table 3.

Numerically, they correspond to well-known nilpotent conorms and t-norms: $\min(1, a+b)$ and $\max(0, a+b-1)$, respectively. Then the contradiction $\mathbf{0}$ is also expressed as $a \odot \neg a$.

5.1 Translating the basic connectives in Ł3

Łukasiewicz implication is translated into MEL as follows. First, consider the translation of $v(\alpha \to_L \beta) = \mathbf{1}$. It is important, as inference in Ł$_3$ is based on the propagation of the designated truth-value $\mathbf{1}$ across deduction steps. It is clear from the truth-table that $v(\alpha \to_L \beta) = \mathbf{1}$ if and only if the two Boolean conditions are satisfied:

- if $v(\alpha) = \mathbf{1}$ then $v(\beta) = \mathbf{1}$
- if $v(\alpha) \geq \frac{1}{2}$ then $v(\beta) \geq \frac{1}{2}$.

It thus yields the translation, using Boolean conjunction and implication:

$$\mathcal{T}(v(\alpha \to_L \beta) = \mathbf{1}) = [\mathcal{T}(v(\alpha) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\beta) = \mathbf{1})] \wedge [\mathcal{T}(v(\alpha) \geq \tfrac{1}{2}) \Rightarrow \mathcal{T}(v(\beta) \geq \tfrac{1}{2})]$$

The translation of $\mathcal{T}(v(\alpha \to_L \beta) = \mathbf{1})$ is the same for all the 3-valued residuated implications. Likewise $v(\alpha \to_L \beta) \geq \frac{1}{2}$ only requires that $v(\beta) \geq \frac{1}{2}$ whenever $v(\alpha) = \mathbf{1}$. The translation is thus:

$$\mathcal{T}(v(\alpha \to_L \beta) \geq \tfrac{1}{2}) = \mathcal{T}(v(\alpha) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\beta) \geq \tfrac{1}{2})$$

In the case of atoms, we can use the modal translations of $v(a) = \mathbf{1}$, etc., and get:

$$\mathcal{T}(v(a \to_L b) = \mathbf{1}) = (\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)$$

and

$$\mathcal{T}(v(a \to_L b) \geq \tfrac{1}{2}) = \Box a \Rightarrow \Diamond b.$$

Under the epistemic stance, $v(a \to_L b) = \mathbf{1}$ thus means: if a is certain then so is b and if a is possible then so is b. This interpretation was not at all obvious to guess in the language of Ł$_3$.

The translation of Kleene conjunction and disjunction can be achieved likewise, although in a simpler way as $v(\alpha \sqcup \beta) = \mathbf{1}$ if and only if $v(\alpha) = \mathbf{1}$ or $v(\beta) = \mathbf{1}$, and $v(\alpha \sqcap \beta) = \mathbf{1}$ if and only if $v(\alpha) = \mathbf{1}$ and $v(\beta) = \mathbf{1}$, etc. It is then easy to check that

$$\begin{aligned}
\mathcal{T}(v(\alpha \sqcap \beta) \geq i) &= \mathcal{T}(v(\alpha) \geq i) \wedge \mathcal{T}(v(\beta) \geq i), i \geq \tfrac{1}{2} \\
\mathcal{T}(v(\alpha \sqcup \beta) \geq i) &= \mathcal{T}(v(\alpha) \geq i) \vee \mathcal{T}(v(\beta) \geq i), i \geq \tfrac{1}{2} \\
\mathcal{T}(v(\alpha \sqcap \beta) \leq i) &= \mathcal{T}(v(\alpha) \leq i) \vee \mathcal{T}(v(\beta) \leq i), i \leq \tfrac{1}{2} \\
\mathcal{T}(v(\alpha \sqcup \beta) \leq i) &= \mathcal{T}(v(\alpha) \leq i) \wedge \mathcal{T}(v(\beta) \leq i), i \leq \tfrac{1}{2}
\end{aligned}$$

In the case of atoms, it is clear that

$$\mathcal{T}(v(a \sqcap b) = \mathbf{1}) = \Box a \wedge \Box b \text{ and } \mathcal{T}(v(a \sqcup b) = \mathbf{1}) = \Box a \vee \Box b.$$

The translation of the connectives $\odot$ and $\oplus$ is:

$$\mathcal{T}(v(\alpha \oplus \beta) = \mathbf{1}) = \mathcal{T}(v(\alpha) = \mathbf{1}) \vee \mathcal{T}(v(\beta) = \mathbf{1}) \vee (\mathcal{T}(v(\alpha) \geq \tfrac{1}{2}) \wedge \mathcal{T}(v(\beta) \geq \tfrac{1}{2}))$$
$$\mathcal{T}(v(\alpha \oplus \beta) \geq \tfrac{1}{2}) = \mathcal{T}(v(\alpha) \geq \tfrac{1}{2}) \vee \mathcal{T}(v(\beta) \geq \tfrac{1}{2})$$
$$\mathcal{T}(v(\alpha \odot \beta) = \mathbf{1}) = \mathcal{T}(v(\alpha) = \mathbf{1}) \wedge \mathcal{T}(v(\beta) = \mathbf{1})$$
$$\mathcal{T}(v(\alpha \odot \beta) \geq \tfrac{1}{2}) = [\mathcal{T}(v(\alpha) \geq \tfrac{1}{2}) \wedge \mathcal{T}(v(\beta) = \mathbf{1})] \vee [\mathcal{T}(v(\alpha) = \mathbf{1}) \wedge \mathcal{T}(v(\beta) \geq \tfrac{1}{2})]$$

For atoms, we see that:

$$\mathcal{T}(v(a \oplus b) = \mathbf{1}) = \Box a \vee \Box b \vee (\Diamond a \wedge \Diamond b)$$

and

$$\mathcal{T}(v(\alpha \odot \beta) = \mathbf{1}) = \Box a \wedge \Box b.$$

Note that while the truth of Kleene disjunction $a \sqcup b$ corresponds to the requirement that one of a and b be certain, $a \oplus b$ corresponds to a very loose view of the disjunction of two atoms, which remains valid if both conjuncts are unknown. Besides, asserting the truth of a conjunction in Ł$_3$ leads to the same translation for the two conjunctions (but asserting falsity would lead to different translations).

The negation $\neg\alpha$ in Ł$_3$ is the involutive one, and its translation clearly yields:

$$\mathcal{T}(v(\neg\alpha) = \mathbf{1}) = \mathcal{T}(v(\alpha) = \mathbf{0}) = (\mathcal{T}(v(\alpha) \geq \tfrac{1}{2}))'$$
$$\mathcal{T}(v(\neg\alpha) \geq \tfrac{1}{2}) = \mathcal{T}(v(\alpha) \leq \tfrac{1}{2}) = (\mathcal{T}(v(\alpha) = \mathbf{1}))'$$

For atoms, $\mathcal{T}(v(\neg a) = \mathbf{1}) = \Box a'$, and $\mathcal{T}(v(\neg a) = \frac{1}{2}) = \mathcal{T}(v(a) = \frac{1}{2}) = \Diamond a \wedge \Diamond a'$.

Note that in Ł$_3$ the top and bottom element in $\mathbf{3}$ are translated (computing respectively $\mathcal{T}(v(a \to_L a) = \mathbf{1})$ and $\mathcal{T}(v(a \odot \neg a) = \mathbf{1})$), into $((\Box a)' \vee \Box a) \wedge ((\Diamond a)' \vee \Diamond a)$ and $\Box a \wedge \Box a'$, respectively, which are indeed tautologies and contradictions in MEL, respectively, hence semantically equivalent to $\Box\top$ and $\Box\bot$, respectively.

Example 3. *Let us translate axiom (W2) applied to atoms:*

$$\mathcal{T}(v(a \to_L (b \to_L a)) = \mathbf{1}) =$$
$$[\mathcal{T}(v(a) = \mathbf{1}) \Rightarrow \mathcal{T}(v(b \to_L a)) = \mathbf{1})] \wedge [\mathcal{T}(v(a) \geq \tfrac{1}{2}) \Rightarrow \mathcal{T}(v(b \to_L a) \geq \tfrac{1}{2})] =$$
$$[\Box a \Rightarrow ((\Box b \Rightarrow \Box a) \wedge (\Diamond b \Rightarrow \Diamond a))] \wedge [\Diamond a \Rightarrow (\Box b \Rightarrow \Diamond a)]$$

This Łukasiewicz axiom is translated into a MEL tautology: indeed it is the conjunction of two tautologies. This result can be generalized to all axioms of Ł$_3$, as we will see in Proposition 6. On the other hand, we started from a formula containing two literals and we ended with a MEL formula with 4 literals. That is, during the translation we gain in interpretability but we lose in terms of complexity of the formula. In the worst case, we may have an exponential growth in the terms of literals (see Proposition 10).

Let $\mathcal{L}_\Box^{\text{Ł}}$ be the syntactic fragment of the MEL language obtained by translating truth-qualified Ł$_3$ formulas into MEL. From the above considerations, it is formed of formulas of MEL where modalities appear only in front of literals. It is clear that $\mathcal{L}_\Box^{\text{Ł}} \subseteq \mathcal{L}_\Box^{\ell}$, the MEL language fragment $\mathcal{L}_\Box^{\ell}$ made of *all* formulas where modalities are just in front of literals. From $\mathcal{L}_\Box^{\ell}$ to Ł$_3$, we can actually prove the converse translation is possible:

Proposition 4. *For any formula in $\phi \in \mathcal{L}^{\ell}_{\Box}$, there exists a formula α in Ł$_3$ such that ϕ is logically equivalent to $\mathcal{T}(v(\alpha) = \mathbf{1})$ in MEL.*

Proof. Formulas in the language $\mathcal{L}^{\ell}_{\Box}$ can be equivalently expressed using $\Box a|\Box a'|\Diamond a|\Diamond a'|\phi \vee \psi|\phi \wedge \psi$ without the explicit use of an outer negation ϕ'. So, the translation θ from $\mathcal{L}^{\ell}_{\Box}$ to Łukasiewicz logic is recursively defined as (we write $\theta(\phi)$ as short for $\theta(t(\phi) = 1)$): $\theta(\Box a) = a$, $\theta(\Box a') = \neg a$, $\theta(\Diamond a') = a \to_L \neg a$, $\theta(\Diamond a) = \neg a \to_L a$, $\theta(\phi \wedge \psi) = \theta(\phi) \sqcap \theta(\psi)$, $\theta(\phi \vee \psi) = \theta(\phi) \sqcup \theta(\psi)$. □

In particular, Tarski's translation from $\Diamond\alpha$ into $\neg\alpha \to_L \alpha$ is thus recovered, however only if α is a literal.

To sum up, the image of the language Ł$_3$ via the translation mapping $\mathcal{T}$ in the MEL language $\mathcal{L}_{\Box}$ is exactly $\mathcal{L}^{\ell}_{\Box}$, i.e., its fragment with modalities in front of literals only.

5.2 Using MEL to reason in Ł$_3$

We are now in a position to compare the logic Ł$_3$ and the restriction of MEL to the sublanguage $\mathcal{L}^{\ell}_{\Box}$. Syntactic inference in Ł$_3$ uses Wajsberg axioms and the modus ponens rule. At the semantic level, if B_L is a set of formulas in Ł$_3$ (understood as a knowledge base), then $B_L \models \alpha$ means that whenever $v(\beta) = \mathbf{1}, \forall\beta \in B_L$, we do have that $v(\alpha) = \mathbf{1}$. Ł$_3$ is sound and complete with respect to this semantics (Gottwald, 2001). This semantic inference can be expressed in MEL by:

$$\wedge_{\beta \in B_L} \mathcal{T}(v(\beta) = \mathbf{1}) \vdash \mathcal{T}(v(\alpha) = \mathbf{1}).$$

So the question to be addressed in this subsection is whether this inference in the restriction of MEL to $\mathcal{L}^{\ell}_{\Box}$ is equivalent to the inference in Ł$_3$, in other words whether this "sublogic" of MEL captures the logic Ł$_3$ exactly.

To simplify notation, we may in the following occasionally (especially in proofs) write $\mathcal{T}_1(\alpha)$ in place of $\mathcal{T}(v(\alpha) = \mathbf{1})$, and $\mathcal{T}^{\geq}_{1/2}(\alpha)$ in place of $\mathcal{T}(v(\alpha) \geq \frac{1}{2})$.

First we can generalize the result on the correspondence between tautologies in both logics:

Lemma 5. *If α is a formula in Ł$_3$, then $\mathcal{T}(v(\alpha) \geq \frac{1}{2}) \vee \mathcal{T}(v(\alpha) \leq \frac{1}{2})$ is a tautology in MEL.*

The proof is by induction on the structure of α.

- $\alpha = a$. We have $\mathcal{T}(v(a) \geq \frac{1}{2}) \vee \mathcal{T}(v(a) \leq \frac{1}{2}) = \Diamond a \vee \Diamond a' = \Box a \Rightarrow \Diamond a$, that is axiom (D).
- $\alpha = \neg\beta$. $\mathcal{T}(v(\neg\beta) \geq \frac{1}{2}) \vee \mathcal{T}(v(\neg\beta) \leq \frac{1}{2}) = \mathcal{T}(v(\beta) \leq \frac{1}{2}) \vee \mathcal{T}(v(\beta) \geq \frac{1}{2})$ and then, it is sufficient to use the induction.
- $\alpha = \alpha_1 \to_L \alpha_2$. So, $\mathcal{T}(v(\alpha_1 \to_L \alpha_2) \geq \frac{1}{2}) \vee \mathcal{T}(v(\alpha_1 \to_L \alpha_2) \leq \frac{1}{2})$ is translated into $[\mathcal{T}(v(\alpha_1) \geq \frac{1}{2}) \Rightarrow \mathcal{T}(v(\alpha_2) \geq \frac{1}{2})] \vee [\mathcal{T}(v(\alpha_1) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\alpha_1) = \mathbf{1}]' \vee [\mathcal{T}(v(\alpha_1) \geq \frac{1}{2}) \Rightarrow \mathcal{T}(v(\alpha_2) \geq \frac{1}{2})]'$ which is a tautology since the first and the last terms together are in the form $\phi \vee \phi'$.

We could prove the same result for other disjunctions of translated truth-assignment of three-valued formulas, like, e.g., $\mathcal{T}(v(\alpha) \geq \frac{1}{2}) \vee \mathcal{T}(v(\alpha) = \mathbf{0})$, or yet $\mathcal{T}(v(\alpha) = \mathbf{0}) \vee \mathcal{T}(v(\alpha) = \frac{1}{2}) \vee \mathcal{T}(v(\alpha) = \mathbf{1})$. As all three-valued connectives considered in this paper can be expressed in the language of Ł$_3$, the above results are valid for any 3-valued formula written with the connectives in Tables 1 and 2. Lemma 5 is useful to prove the following result:

Proposition 6. *If α is an axiom in Ł$_3$, then $\mathcal{T}(v(\alpha) = \mathbf{1})$ is a tautology in MEL.*

Proof. See Appendix A. □

The other direction, from MEL to Ł$_3$, would be more problematic. Indeed, in the sublanguage $\mathcal{L}^{\ell}_{\Box}$, some of the MEL axioms then become uninteresting or cannot be expressed. Axiom (D) can be translated back when restricted to literals. On atoms, this axiom reads $\Box a \Rightarrow \Diamond a$ whose translation is $(a \to_L \neg a) \vee (\neg a \to_L a)$ which is a tautology since in Łukasiewicz logic any formula of the kind $(\alpha \to_L \beta) \vee (\beta \to_L \alpha)$ is a tautology. $\Box\top$ can be translated by any Łukasiewicz tautology, say for instance $a \to_L a$.

Axioms of Propositional Logic applied to MEL literals can be translated back and it is possible to check whether they become tautologies in Ł$_3$ (this is left to the reader). For instance, consider Axiom 1 using atomic formula $\Box a$ and with any $\mathcal{L}^{\ell}_{\Box}$-formula ϕ, we have: $\theta([\Box a \Rightarrow (\phi \Rightarrow \Box a)]) = [(a \to_L \neg a) \vee \theta(\neg\phi) \vee a]$ and $[(a \to_L \neg a) \vee a]$ is a tautology in Ł$_3$.

In contrast, axioms (M) and (C) cannot be expressed in $\mathcal{L}^{\ell}_{\Box}$ since $\Box(a \wedge b)$ is not a formula of this language (even if in MEL, $\Box(a \wedge b)$ and $\Box a \wedge \Box b$ are equivalent). Axiom RM on BPL literals becomes uninteresting, since $a \Rightarrow b$ is never a BPL tautology for distinct atoms, etc.

We note that the issue of translating MEL axioms to Ł$_3$ is not a real concern for our purpose. Indeed, here, we are only trying to simulate Ł$_3$ inside MEL. So, we need to

- translate truth-qualified formulas of Ł$_3$ into the language $\mathcal{L}_{\Box}$;
- use MEL inference rule to simulate Ł$_3$ modus ponens.

We have seen that the first item is feasible. About the second one, we have to show that from $T_1(\alpha)$ and $T_1(\alpha \to_L \beta)$ we can deduce $T_1(\beta)$. Now, the translation of $T_1(\alpha \to_L \beta)$ is by definition $[T_1(\alpha) \Rightarrow T_1(\beta)] \wedge [T^{\geq}_{1/2}(\alpha) \Rightarrow T^{\geq}_{1/2}(\beta)]$. This means that $[T_1(\alpha) \Rightarrow T_1(\beta)]$ is valid and by modus ponens in BPL we get $T_1(\beta)$.

The following proposition is crucial to ensure the equivalence between the models of true formulas in Ł$_3$ and the epistemic models of their translation into MEL.

Proposition 7. *Let α be a formula in Ł$_3$. For each model v of α, the epistemic state E_v is a model (in the sense of MEL) of $\mathcal{T}(v(\alpha) = \mathbf{1})$. Conversely, for each model in the sense of MEL (epistemic state) E of $\mathcal{T}(v(\alpha) = \mathbf{1})$ the 3-valued interpretation v_E is a model of α in the sense that $v_E(\alpha) = \mathbf{1}$.*

Proof. The proposition can be proved by induction on the structure of the formula α.

At first let us prove that from a model v we get a model E_v, that is from $v(\alpha) \in S$ we get E_v is a model of $\mathcal{T}(v(\alpha) \in S)$, where by "$\in S$" we mean $= \mathbf{0}| = \mathbf{1}| \geq \frac{1}{2}| \leq \frac{1}{2}$. If α is a literal, $\alpha = a|\neg a$, then the proof immediately follows by definition of E_v.

Otherwise, for a general formula, we make the inductive hypothesis: if $v(\alpha) \in S$ then $E_v \vDash \mathcal{T}(v(\alpha) \in S)$. Then, we distinguish the two cases of

- negation $\neg\alpha$. Let us suppose that $v(\neg\alpha) = \mathbf{1}$ (the case $v(\neg\alpha) = \mathbf{0}$ is handled dually). Then, we get $v(\alpha) = \mathbf{0}$ and by inductive hypothesis: $E_v \vDash \mathcal{T}(v(\alpha) = \mathbf{0}) = \mathcal{T}(v(\neg\alpha) = \mathbf{1})$, the last equality being valid by definition of $\mathcal{T}$.

  Finally, if $v(\neg\alpha) \geq \frac{1}{2}$ (similarly for $v(\neg\alpha) \leq \frac{1}{2}$) it means that $v(\alpha) \leq \frac{1}{2}$. By inductive hypothesis, $E_v \vDash \mathcal{T}(v(\alpha) \leq \frac{1}{2}) = \mathcal{T}(v(\neg\alpha) \geq \frac{1}{2})$.
- implication $\alpha \to_L \beta$. At first, let us suppose that $v(\alpha \to_L \beta) = 1$. By definition of $\to_L$ this is true when ($v(\alpha) \leq \frac{1}{2}$ or $v(\beta) = \mathbf{1}$) and ($v(\alpha) = \mathbf{0}$ or $v(\beta) \leq \frac{1}{2}$)). By inductive hypothesis, the fact that $\mathcal{T}(v(\alpha) \leq \frac{1}{2}) = \mathcal{T}(v(\alpha) = \mathbf{1})'$ and definition of Boolean implication, we easily get the thesis. The other cases are handled similarly.

Conversely, if we show that

$$v_E(\alpha) = \begin{cases} \mathbf{1} & \text{if } E \vDash \mathcal{T}(v(\alpha) = \mathbf{1}) \\ \mathbf{0} & \text{if } E \vDash \mathcal{T}(v(\alpha) = \mathbf{0}) \\ \frac{1}{2} & \text{otherwise} \end{cases} \tag{1}$$

then the thesis immediately follows. The case where α is an atom is a simple translation of the definition of v_E. Let us make the inductive hypothesis that equation (1) holds for generic α, β and prove that it holds also for $\neg\alpha$ and $\alpha \rightarrow_L \beta$.

- the case of negation. If $E \vDash \mathcal{T}(v(\neg\alpha) = \mathbf{1})$ then $E \vDash \mathcal{T}(v(\alpha) = \mathbf{0})$ and by induction we get $v_E(\alpha) = \mathbf{0}$ and so $v_E(\neg\alpha) = \mathbf{1}$ Similarly, for $E \vDash \mathcal{T}(v(\neg\alpha) = \mathbf{0})$.
- the case of implication. If $E \vDash \mathcal{T}(v(\alpha \rightarrow_L \beta) = \mathbf{1})$ then by definition $E \vDash [\mathcal{T}(v(\alpha) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\beta) = \mathbf{1})] \wedge [\mathcal{T}(v(\alpha) \geq \frac{1}{2}) \Rightarrow \mathcal{T}(v(\beta) \geq \frac{1}{2})]$. This means that $(E \vDash \mathcal{T}(v(\alpha) = \mathbf{1})'$ or $E \vDash \mathcal{T}(v(\beta) = \mathbf{1})$) and ($E \vDash \mathcal{T}(v(\alpha) \geq \frac{1}{2})'$ or $E \vDash \mathcal{T}(v(\beta) \geq \frac{1}{2})$). By induction we have: $(v(\alpha) = \mathbf{0}$ or $v(\beta) = \mathbf{1})$ and $(v(\alpha) \leq \frac{1}{2}$ or $v(\beta) \geq \frac{1}{2})$, from which we get the thesis $v(\alpha \rightarrow_L \beta) = \mathbf{1}$ by definition of Łukasiewicz implication.
  The case $E \vDash \mathcal{T}(v(\alpha \rightarrow_L \beta) = \mathbf{0})$ is handled similarly.

□

Moreover, since the sublanguage $\mathcal{L}^\ell_\Box$ is exactly the Łukasiewicz fragment of the MEL language, putting together Propositions 3, 6 and 7, we obtain the equivalence between inference in Ł$_3$ and inference in the corresponding linguistic restriction of MEL.

First, from the above results we get the following:

Lemma 8. *Let ϕ be a formula in $\mathcal{L}^\ell_\Box$ and $\theta(\phi)$ its translation in Łukasiewicz logic. If $E \vDash_{MEL} \phi$, then $v_E \vDash_{\mathcal{L}} \theta(\phi)$, where v_E is the unique three-valued valuation associated to the partial model $r(E)$.*

Proof. We proceed by induction.

- $\phi = \Box a$, then $\theta(\phi) = a$ and $v_E(a) = 1$. So, $v_E(\theta(\phi)) = 1$.
- $\phi = \Box a'$, then $\theta(\phi) = \neg a$ and $v_E(a) = 0$. So, $v_E(\theta(\phi)) = 1$.
- $\phi = \Diamond a$, then $\theta(\phi) = \neg a \rightarrow_L a$ and $v_E(a) \geq \frac{1}{2}$. So $v_E(\theta(\phi)) = 1$.
- $\phi = \Diamond a'$. Same as the previous case.
- $\phi = \phi_1 \wedge \phi_2$. Then, we know by induction that $v_E(\theta(\phi_1)) = v_E(\theta(\phi_2)) = 1$ and from $\theta(\phi) = \theta(\phi_1) \sqcap \theta(\phi_2)$ the thesis follows.
- $\phi = \phi_1 \vee \phi_2$. Same as the $\wedge$ case.

□

Finally, we reach the main equivalence result of this section, showing that in so far as the third truth-value refers to the idea of *unknown*, Łukasiewicz logic is exactly captured by a sublogic of modal logic.

Proposition 9. *Let α be a formula in Łukasiewicz logic Ł$_3$ and B_L a set of formulas in this logic. Then, $B_L \vdash \alpha$ in Ł$_3$ iff $\mathcal{T}(B_L) \vdash \mathcal{T}(v(\alpha) = \mathbf{1})$ in MEL .*

Proof. Both MEL and Łukasiewicz logic are sound and complete. So, it is enough to show that $B_L \vdash \alpha$ iff $\mathcal{T}(B_L) \vDash_{MEL} \mathcal{T}(v(\alpha) = \mathbf{1})$. One direction is the application of Proposition 7 to the present case and the other is given by Lemma 8. □

Another issue to consider is the complexity of the MEL formulas obtained by the translation from Ł$_3$. Indeed, we can see that the resulting formula is more complex in the number of literals compared to the original Ł$_3$ formula, with an exponential growth. Already when translating the Łukasiewicz implication, we see that $\mathcal{T}(v(\alpha \to_L \beta = \mathbf{1}))$ yields a significantly larger MEL formula. We can quantify this growth in the size of translated formulas more precisely:

Proposition 10. *Let n be the number of literals appearing in an Ł$_3$ formula α and $\#\ell_1(n)$ be the number of (modal) literals in the translation $\mathcal{T}(v(\alpha) = \mathbf{1})$. Then,*

$$\#\ell_1(n) \leq c_1\Big(\frac{1-\sqrt{5}}{2}\Big)^n + c_2\Big(\frac{1+\sqrt{5}}{2}\Big)^n - 3 \tag{2}$$

with c_1, c_2 constants.

Proof. The worst case is when α is of the following form $(((a \to_L b) \to_L c) \to_L d \cdots$. For $n = 1, 2$, it is clear that $\#\ell_1(1) = 1, \#\ell_1(2) = 4$ (by checking $\mathcal{T}_1(a \to_L b)$). Let $\#\ell_{1/2}(n)$ be the number of literals appearing in the translation of $v(\alpha) \geq \frac{1}{2}$ if α contains n literals. It is clear that $\#\ell_{1/2}(1) = 1, \#\ell_{1/2}(2) = 2$ (using $\mathcal{T}_{1/2}(a \to_L b)$). Now consider $\alpha = (a \to_L b) \to_L c$:

- $\mathcal{T}_1(\alpha) = (\mathcal{T}_1(a \to_L b) \Rightarrow \Box c) \wedge (\mathcal{T}_{1/2}(a \to_L b) \Rightarrow \Diamond c)$, so that $\mathcal{T}_1(\alpha) = ((\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)) \Rightarrow \Box c) \wedge ((\Box a \Rightarrow \Diamond b) \Rightarrow \Diamond c)$ and $\#\ell_1(3) = 8$.
- $\mathcal{T}_{1/2}(\alpha) = (\mathcal{T}_1(a \to_L b) \Rightarrow \Diamond c = ((\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)) \Rightarrow \Diamond c$ so that $\#\ell_{1/2}(3) = 5$

More generally consider the formula $\alpha \to_L b$:

$$\begin{aligned} \mathcal{T}_1(\alpha \to_L b) &= (T_1(\alpha) \Rightarrow \Box b) \wedge (T_{1/2}(\alpha) \Rightarrow \Diamond b) \\ \mathcal{T}_{1/2}(\alpha \Rightarrow b) &= \qquad\qquad (T_1(\alpha) \Rightarrow \Diamond b) \end{aligned}$$

It yields the following recursions, assuming the number of literals in α is $n-1$:

$$\begin{aligned} \#\ell_1(n) &= \#\ell_1(n-1) + \#\ell_{1/2}(n-1) + 2 \\ \#\ell_{1/2}(n) &= \qquad\quad \#\ell_1(n-1) + 1 \end{aligned}$$

Injecting the second equation into the first leads to the recursive equation

$$\#\ell_1(n) = \#\ell_1(n-1) + \#\ell_1(n-2) + 3$$

One can check it holds for the case $n = 3$. It can be seen that, up to constants, this is Fibonacci series, whose solution can be computed by difference equation techniques (Elaydi, 1995) yielding expression (2). The constants c_1, c_2 can be computed by substituting the case $n = 2$ and $n = 3$ (whose solution is known) in equation (2). $\Box$

We can claim, however that this loss in concision is counterbalanced by a gain in interpretability, as for instance, the meaning of Łukasiewicz connectives in the setting of incomplete information handling is laid bare by the translation. Indeed, we see that declaring $a \to_L b$ as true in Ł$_3$ means (after its translation into MEL): if a is certain, then so is b, and if a is not impossible, then so is b. Note that if the truth of some atomic propositions is known and encoded in MEL, such rules can be triggered, and can derive the certainty of other atomic proposition, in a style very similar to logic programming. One may conjecture that the behavior of a rule "$a \leftarrow b_1, \ldots, b_n$" in logic programming, can be captured by means of the formula $(\Box b_1 \wedge \cdots \wedge \Box b_2) \Rightarrow \Box a$ in MEL, expressing facts as $\Box a$.

5.3 Nelson logic

The three-valued Nelson logic N_3 (Vakarelov, 1977), also known as classical logic with a strong negation, uses the language built on $(\mathcal{V}, \sqcap, \sqcup, \rightarrow_N, \neg, -)$. It also corresponds to the LPF logic in (Avron, 1991). The part of N_3 based on connectives $(\sqcap, \sqcup, \rightarrow_N, -)$ satisfies the axioms of propositional Boolean logic:

(B1) $\alpha \rightarrow_N (\beta \rightarrow_N \alpha)$
(B2) $(\alpha \rightarrow_N (\beta \rightarrow_N c)) \rightarrow_N ((\alpha \rightarrow_N \beta) \rightarrow_N (\alpha \rightarrow_N c))$
(B3) $(-\alpha \rightarrow_N -\beta) \rightarrow_N (\beta \rightarrow_N \alpha)$

and the other negation $\neg$ satisfies the additional six axioms:

(V1) $\neg\alpha \rightarrow_N (\alpha \rightarrow_N \beta)$
(V2) $\neg(\alpha \rightarrow_N \beta) \leftrightarrow_N (\alpha \sqcap \neg\beta)$
(V3) $\neg(\alpha \sqcap \beta) \leftrightarrow_N \neg\alpha \sqcup \neg\beta$
(V4) $\neg(\alpha \sqcup \beta) \leftrightarrow_N \neg\alpha \sqcap \neg\beta$
(V5) $\neg - \alpha \leftrightarrow_N \alpha$
(V6) $\neg\neg\alpha \leftrightarrow_N \alpha$

The semantics is given by Nelson algebras (Cignoli, 1986), that is Kleene algebras where a further implication $a \rightarrow_N b = a \rightarrow_G (\neg a \sqcup b)$ always exists for any a, b and it satisfies $(a \wedge b) \rightarrow_N c = a \rightarrow_N (b \rightarrow_N c)$. This implication is not equal to its contraposition $\neg b \rightarrow_N \neg a$. An elementary example is the three-valued Kleene algebra $(\{\mathbf{0}, \frac{1}{2}, \mathbf{1}\}, \sqcap, \sqcup, \neg, \mathbf{0}, \mathbf{1})$ equipped with Nelson implication $\rightarrow_N$, given in Table 4 (left), also $=\rightarrow_9$ in Table 2. Apart from Kleene implication, it is the only other one such that $(a \rightarrow b) \rightarrow a = a$. The designated truth-value is $\mathbf{1}$. The negation $-$, defined as $-a := a \rightarrow_N \mathbf{0}$, is the one we called paraconsistent, such that $-\frac{1}{2} = \mathbf{1} = -\mathbf{0}$.

$\rightarrow_N$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$

$\leftrightarrow_N$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$
$\frac{1}{2}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$
$\mathbf{1}$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$

Table 4. Nelson implication and equivalence on three-values.

Nelson equivalence (Table 4 on the right) is not much demanding and confuses the values $\frac{1}{2}$ and $\mathbf{0}$. In fact, if we merge these two truth-values, we are left with Boolean logic and the two negations will coincide. Besides we can notice that the deduction theorem holds in the form $v(\alpha \rightarrow_N \beta) = \mathbf{1}$ if and only if $v(\alpha) = \mathbf{1}$ implies $v(\beta) = \mathbf{1}$, which is false with Łukasiewicz implication, and contrasts with its counterpart in G_3. Nelson logic also exhibits a constructivist flavor for the notion of falsity, in the sense that $v(-(A \sqcap B)) = \mathbf{1}$ if and only if $v(-A) = 1$ or $v(-B) = 1$, while in G_3, we have that $v(\sim (A \sqcup B)) = \mathbf{0}$ if and only if $v(\sim A) = 0$ or $v(\sim B) = 0$.

In order to translate all formulas of Nelson logic into MEL, it is sufficient to give the translation of the implication and the associated negation, the other connectives being the same as the ones in Kleene logic encountered in the previous subsections via Łukasiewicz logic.

$$\mathcal{T}(v(-\alpha) = \mathbf{1}) = \mathcal{T}(v(\alpha) \leq \tfrac{1}{2})$$
$$\mathcal{T}(v(-\alpha) = \mathbf{0}) = \mathcal{T}(v(\alpha) = \mathbf{1})$$
$$\mathcal{T}(v(\alpha \rightarrow_N \beta) = \mathbf{1}) = \mathcal{T}(v(\alpha) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\beta) = \mathbf{1})$$
$$\mathcal{T}(v(\alpha \rightarrow_N \beta) \geq \tfrac{1}{2}) = \mathcal{T}(v(\alpha) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\beta) \geq \tfrac{1}{2})$$

For atoms, it holds

$$\mathcal{T}(v(a \to_N b) = \mathbf{1}) = \Box a \Rightarrow \Box b$$

$$\mathcal{T}(v(a \to_N b) \geq \tfrac{1}{2}) = \Box a \Rightarrow \vee\Diamond b$$

The first identity gives the meaning of Nelson implication in the epistemic approach, namely if α is certain then β is certain. This implication may look more natural in MEL than residuated ones or Kleene's.

It turns out that Nelson implication can be defined by means of Łukasiewicz implication as:

$$a \to_N b := a \to_L (a \to_L b),$$

and conversely Łukasiewicz implication can be defined as by contrapositive symmetrisation of Nelson implication (Avron, 1991):

$$a \to_L b := (a \to_N b) \sqcap (\neg b \to_N \neg a).$$

Actually all the results pertaining to Łukasiewicz logic also apply to the three-valued Nelson logic $N_3 = (\mathcal{V}, \sqcap, \sqcup, \to_N, \neg, -)$ due to the equivalence of the two logics (Vakarelov, 1977). The expressive power of N_3 is thus the same as $Ł_3$, and their translation into MEL can be carried out in the same fragment $\mathcal{L}^{\ell}_{\Box}$ of the MEL language. Conversely, for the translation from $\mathcal{L}^{\ell}_{\Box}$ into Nelson logic, we must use $\theta(\Diamond a') = -a$ and $\theta(\Diamond a) = -\neg a$.

At the semantic level, an interpretation v in Nelson logic corresponds again to a partial model E_v of propositional logic and Proposition 7 relating valuations satisfying $Ł_3$ formulas and MEL- models of their translations still holds for Nelson logic. In particular, Proposition 6 holds for N_3 axioms, just using their translations into $Ł_3$:

Proposition 11. *If α is an axiom in Nelson logic, then $\mathcal{T}(v(\alpha) = \mathbf{1})$ is a tautology in MEL.*

Proof. Axioms (B1)-(B3) are Boolean axioms, thus they easily follow. We can give the direct proof for (V1), the other axioms being proved similarly. $\mathcal{T}(v(\neg\alpha \to_N (\alpha \to_N \beta)) = \mathbf{1}) = \mathcal{T}(v(\alpha) = \mathbf{0}) \Rightarrow (\mathcal{T}(v(\alpha) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\beta) = \mathbf{1})) = \mathcal{T}(v(\alpha) \geq \frac{1}{2}) \vee \mathcal{T}(v(\alpha) \leq \frac{1}{2}) \vee \mathcal{T}(v(\beta) = \mathbf{1})$ which is a tautology in MEL. □

Finally, using again the equivalence between $Ł_3$ and N_3 the counterpart of Proposition 9 is valid for Nelson logic, namely that if a formula in Nelson logic is a consequence of a knowledge base, it can be proven in MEL using their translations.

6. Special cases

In this section, we consider Kleene and Gödel three-valued logics, that are well-known in the literature and that are expressible in $Ł_3$, but are less expressive. We try to figure out which fragment of the language $\mathcal{L}^{\ell}_{\Box}$ can carry such logics, bearing in mind the third truth-value means *unknown*. Moreover we consider a variant of Kleene logic that has been proposed as a paraconsistent logic, by changing the designated truth-value. Interestingly, even if its aim is to capture the notion of conflict rather than partial ignorance, this logic can also be captured in MEL.

6.1 Kleene logic in MEL

The best known and often used logic to represent uncertainty due to incomplete information is Kleene logic. The connectives are simply the min $\sqcap$, the max $\sqcup$, the involutive negation $\neg$. A material implication $a \to_K b := \neg a \sqcup b$ is then derived. The involutive negation preserves the De Morgan laws between $\sqcap$ and $\sqcup$. As all these connectives can be defined in Ł$_3$, its language can be considered as a fragment of the latter. However the syntax of Kleene logic is the same as the one of propositional logic (replacing $\wedge, \vee, '$ by $\sqcap, \sqcup, \neg$), since only one pair of (idempotent) conjunctions and disjunctions and only one negation is used.

The translation of the basic connectives into MEL was given in the previous section, including Kleene implication. We can also define the latter directly as follows using standard material implication $\Rightarrow$.

$$\mathcal{T}(v(\alpha \to_K \beta) = \mathbf{1}) = \mathcal{T}(v(\alpha) \geq \tfrac{1}{2}) \Rightarrow \mathcal{T}(v(\beta) = \mathbf{1})$$
$$\mathcal{T}(v(\alpha \to_K \beta) \geq \tfrac{1}{2}) = \mathcal{T}(v(\alpha) = \mathbf{1}) \Rightarrow \mathcal{T}(v(\beta) \geq \tfrac{1}{2})$$

If $\alpha = a$, $\beta = b$ are atoms, we obtain $\Box\neg a \vee \Box b$ and $\Diamond\neg a \vee \Diamond b$ respectively. The translation into MEL lays bare the meaning of Kleene implication: $a \to_K b$ is "true" means that b is certain if a is possible (which may sound like a bold, debatable implication).

A knowledge base B_K in Kleene logic K_3 is a conjunction of formulas supposed to have designated truth value $\mathbf{1}$. We can always transform this base in conjunctive normal form (CNF), that is, a conjunction of disjunction of literals (without simplifying terms of the form $a \sqcup \neg a$):

$$\sqcap_{i=1,\ldots,k} \sqcup_{j=1,\ldots,m_i} \ell_j(a_j),$$

where $\ell_j(a_j) = a_j$ or $\neg a_j$ is a three-valued literal. Its translation into MEL clearly consists of the same set of clauses, where we put the modality $\Box$ in front of each literal, namely

$$\mathcal{T}_1(\sqcap_{i=1,\ldots,k} \sqcup_{j=1,\ldots,m_i} \ell_j(a_j)) = \wedge_{i=1,\ldots,k} \vee_{j=1,\ldots,m_i} \Box\ell_j(a_j),$$

where, in the right-hand side, $\ell_j(a_j)$ is now a Boolean literal a_j or a'_j in propositional logic.

Example 4. *Consider the formula* $\alpha = \neg(a \sqcap (\neg(b \sqcup \neg c)))$. *Then,* $\mathcal{T}(v(\alpha) = \mathbf{1}) = \mathcal{T}(v(a \sqcap (\neg(b \sqcup \neg c))) = \mathbf{0})$. *So, we get* $\mathcal{T}(v(a) = \mathbf{0}) \vee \mathcal{T}(v(\neg(b \sqcup \neg c)) = \mathbf{0}) = \Box a' \vee \mathcal{T}(v(b \sqcup \neg c) = \mathbf{1})$ *and finally,* $\Box a' \vee \mathcal{T}(v(b) = \mathbf{1}) \vee \mathcal{T}(v(\neg c) = \mathbf{1}) = \Box a' \vee \Box b \vee \Box c'$. *Note that we could more simply have first put* α *in conjunctive normal form as* $\neg a \vee b \vee \neg c$, *and then put* $\Box$ *in front of each literal.*

As a consequence the fragment of the MEL language that exactly captures the language of Kleene logic contains only conjunctions and disjunctions of MEL atoms of the form $\Box a$ or $\Box a'$:

$$\mathcal{L}^K_\Box = \Box a | \Box a' | \phi \vee \psi | \phi \wedge \psi \subset \mathcal{L}^\ell_\Box.$$

It is clear that this fragment of $\mathcal{L}^\ell_\Box$ forbids negation in front of $\Box$ as well as material implication $\Rightarrow$ between modal atoms. It follows that no axiom of MEL can be expressed in this fragment. The BPL axioms, (RM), (D), require implication and or negation, and syntactically $\Box\top$ is not part of $\mathcal{L}^\ell_\Box$. The latter point reflects the fact that Kleene logic does not have any tautology (there is no formula α in K_3 such that for all $v, v(\alpha) =$

1). So, the translation of any K_3 formula having the form of a Boolean propositional tautology will no longer be a tautology in MEL. For instance, take the BPL axiom 1 (also MEL axiom 1) in Kleene style, i.e., $\alpha \rightarrow_K (\beta \rightarrow_K \alpha)$:

$$\begin{aligned} \mathcal{T}_1(\alpha \rightarrow_K (\beta \rightarrow_K \alpha)) &= \mathcal{T}^{\geq}_{1/2}(\alpha) \Rightarrow \mathcal{T}_1(\beta \rightarrow_K \alpha) \\ &= \mathcal{T}^{\geq}_{1/2}(\alpha) \Rightarrow (\mathcal{T}^{\geq}_{1/2}(\beta) \Rightarrow \mathcal{T}_1(\alpha)) \\ &= (\mathcal{T}^{\geq}_{1/2}(\alpha))' \vee (\mathcal{T}^{\geq}_{1/2}(\beta))' \vee \mathcal{T}_1(\alpha)) \end{aligned}$$

which is not a tautology, as $(\mathcal{T}^{\geq}_{1/2}(\alpha))' \vee \mathcal{T}_1(\alpha)$ excludes the case where $v(\alpha) = 1/2$.

At the semantic level we can use Proposition 7 and apply it to Kleene logic as it is expressible in Ł$_3$.

Corollary 12. *Let α be a formula in Kleene logic. For each model v of α, the epistemic state E_v is a model (in the sense of MEL) of $\mathcal{T}(v(\alpha) = \mathbf{1})$. Conversely, for each model in the sense of MEL (epistemic state) E of $\mathcal{T}(v(\alpha) = \mathbf{1})$ the 3-valued interpretation v_E is a model of α in the sense that $v_E(\alpha) = \mathbf{1}$.*

We can also use the completeness of the restriction of MEL to the language $\mathcal{L}^{\ell}_{\Box}$ with respect to partial models of the form E_v (Proposition 3) and specialise it to the Kleene sublanguage of MEL $\mathcal{L}^{K}_{\Box}$: if $\mathcal{T}(B_K)$ is the MEL translation of a set of Kleene formulas (so $\mathcal{T}(B_K) \subset \mathcal{L}^{K}_{\Box}$, it holds that

$$\begin{gathered} \mathcal{T}(B_K) \vdash \mathcal{T}(v(\alpha) = \mathbf{1}) \text{ in MEL} \\ \text{if and only if for all } v, E_v \vDash \mathcal{T}(B_K) \text{ implies } E_v \vDash \mathcal{T}(v(\alpha) = \mathbf{1}) \\ \text{if and only if for all } v \in \mathbb{V}, v(\beta) = \mathbf{1}, \forall \beta \in B_K \text{ implies } v(\alpha) = \mathbf{1} \text{ in } K_3. \end{gathered}$$

In other words, we can use the MEL inference rules applied to the sublanguage $\mathcal{L}^{K}_{\Box}$ to reason in Kleene logic. We note that the following inference rules that apply to $\mathcal{L}^{K}_{\Box}$ hold in MEL (Banerjee and Dubois, 2009):

- From $\Box a$ and $\Box a' \vee \Box b$,derive $\Box b$ (a special form of modus ponens)
- From $\Box a \vee \Box b$ and $\Box a' \vee \Box c$,derive $\Box b \vee \Box c$ (a counterpart to the resolution principle)

It is then clear that Kleene logic is a propositional logic without tautologies but with such standard rules of inference.

The above result is to be compared with the fact that we can also capture propositional logic in MEL. Consider the following fragment of the language of MEL $\mathcal{L}^{BPL}_{\Box} = \{\Box\alpha, \alpha \in BPL\}$: then as shown in (Banerjee and Dubois, 2009; Dubois, Hajek, and Prade, 2000), $\{\Box\alpha_1, \dots \Box\alpha_k\} \vdash \Box\alpha$ in MEL if and only if $\{\alpha_1, \dots \alpha_k\} \vdash \alpha$ in BPL.

6.2 From three-valued Gödel logic to MEL

Another three-valued logic, known as the here-and-there logic of Heyting (1930), and also the three-valued Gödel (1932) logic is based on the language built from the 4-tuple $(\mathcal{V}, \rightarrow_G, \sqcap, \sim)$, and the axioms are recalled by Pearce (2006). We call it G_3:

(I1) $\alpha \rightarrow_G (\beta \rightarrow_G \alpha)$
(I2) $(\alpha \rightarrow_G (\beta \rightarrow_G \gamma)) \rightarrow_G ((\alpha \rightarrow_G \beta) \rightarrow_G (\alpha \rightarrow_G \gamma)$
(I3) $(\alpha \sqcap \beta) \rightarrow_G \alpha$
(I4) $(\alpha \sqcap \beta) \rightarrow_G \beta$
(I5) $\alpha \rightarrow_G (\beta \rightarrow_G (\alpha \sqcap \beta))$

(I6) $\alpha \rightarrow_G (\alpha \sqcup \beta)$
(I7) $\beta \rightarrow_G (\alpha \sqcup \beta)$
(I8) $(\alpha \rightarrow_G \beta) \rightarrow_G ((\gamma \rightarrow_G \beta) \rightarrow_G (\alpha \sqcup \gamma \rightarrow_G \beta))$
(I9) $(\alpha \rightarrow_G \beta) \rightarrow_G ((\alpha \rightarrow_G \sim\beta) \rightarrow_G (\sim\alpha))$
(I10) $\sim\alpha \rightarrow_G (\alpha \rightarrow_G \beta)$
(I11) $\alpha \sqcup (\sim \beta \sqcup (\alpha \rightarrow_G \beta))$

where $\rightarrow_G$ is the residuum of Kleene conjunction $\sqcap$, $\sim$is the intuitionistic negation, and the Kleene disjunction $\sqcup$ is retrieved as $\alpha \sqcup \beta := [(\alpha \rightarrow_G \beta) \rightarrow_G \beta] \sqcap [(\beta \rightarrow_G \alpha) \rightarrow_G \alpha]$. The truth tables of the implication and negation are given by Table 5. The 10 first

$\rightarrow_G$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\frac{1}{2}$	$\mathbf{1}$

	$\sim$
$\mathbf{0}$	$\mathbf{1}$
$\frac{1}{2}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{0}$

Table 5. Truth table of Gödel implication and negation.

axioms are those of intuitionistic logic. Axiom (I11), due to Hosoi (1996), ensures three-valuedness. To see it note the following result:

Proposition 13. *Consider valuations that attach values in a lattice L to propositions in G_3. Then, $\alpha \sqcup (\sim \beta \sqcup (\alpha \rightarrow_G \beta))$ is a tautology if and only if $L = \overline{3}$.*

Proof. Using the truth-tables, we have that $v(\alpha \sqcup (\sim \beta \sqcup (\alpha \rightarrow_G \beta)) = \max(v(\alpha), v(\sim (\beta)), v(\alpha \rightarrow_G \beta))$. It takes value $\mathbf{1}$ whenever $v(\alpha) = \mathbf{1}$ or $v(\beta) = \mathbf{0}$ or $v(\alpha) \leq v(\beta)$. In order to make all these conditions false, we must assume $\mathbf{0} < v(\beta) < v(\alpha) < \mathbf{1}$. This needs at least 4 distinct totally ordered truth-values. Using three values, Hosoi axiom always holds with truth-value $\mathbf{1}$. □

G_3 is again expressible in Ł$_3$ as Gödel implication $\alpha \rightarrow_G \beta$ is logically equivalent to $-((\alpha \rightarrow_L \beta) \rightarrow_L \beta)$ (Ciucci and Dubois, 2013b), where the paraconsistent negation is defined by $-\alpha = \alpha \rightarrow_L \neg\alpha$ in Ł$_3$. The intuitionistic negation is then $\sim \alpha = \alpha \rightarrow_L (\alpha \odot \neg\alpha)$ in Ł$_3$. The logic G_3 can also be obtained by replacing the 10 first axioms by those of the continuous t-norm logic BL of Hájek (1998) (based on connectives $\rightarrow_G, \sqcap$, and constant $\mathbf{0}$), adding to it axiom $\alpha \rightarrow_G \alpha \sqcap \alpha$ (ensuring idempotence of $\sqcap$), and Hosoi axiom.

The translation $\mathcal{T}(v(\alpha \rightarrow_G \beta) = \mathbf{1})$ is the same as for Łukasiewicz implication. However,

$$\mathcal{T}(v(\sim \alpha) = \mathbf{0}) = \mathcal{T}(v(\alpha) \geq \tfrac{1}{2})$$
$$\mathcal{T}(v(\alpha \rightarrow_G \beta) \geq \tfrac{1}{2}) = \mathcal{T}(v(\alpha) \geq \tfrac{1}{2}) \Rightarrow \mathcal{T}(v(\beta) \geq \tfrac{1}{2})$$

In the case of atoms, $\mathcal{T}(v(a \rightarrow_G b) \geq \frac{1}{2}) = (\Diamond a)' \vee \Diamond b = \Box a' \vee \Diamond b = \Diamond a \Rightarrow \Diamond b$.

The translation $\mathcal{T}(v(\sim \alpha) = \mathbf{1})$ in MEL of Gödel negation is the same as the translation of Kleene negation. We note that the top element $\mathbf{1} = \alpha \rightarrow_G \alpha$ and the bottom element $\mathbf{0} =\sim (\alpha \rightarrow_G \alpha)$ in Gödel logic translate into a tautology and to a contradiction in MEL. Their translation is the same as for Łukasiewicz logic Ł$_3$. In fact, since Gödel logic G_3 is expressible in Ł$_3$, its axioms, after translation into the language of Ł$_3$, become tautologies of Ł$_3$. So, applying our translation and Proposition 6 yield:

Corollary 14. *If α is an axiom of the three-valued Gödel logic, then $\mathcal{T}(v(\alpha) = \mathbf{1})$ is a tautology in MEL.*

Proposition 7 is obviously valid for Gödel logic:

Corollary 15. *Let α be a formula in G_3. For each model v of α, the epistemic state E_v is a model (in the sense of MEL) of $\mathcal{T}(v(\alpha) = \mathbf{1})$. Conversely, for each model in the sense of MEL (epistemic state) E of $\mathcal{T}(v(\alpha) = \mathbf{1})$ the 3-valued interpretation v_E is a model of α in the sense that $v_E(\alpha) = \mathbf{1}$.*

Finding the fragment $\mathcal{G}^{\ell}_{\Box}$ of the MEL language (or of KD) that is necessary and sufficient to exactly capture this three-valued logic is an open problem. Clearly, $\mathcal{G}^{\ell}_{\Box}$ is contained in $\mathcal{L}^{\ell}_{\Box}$ and includes the formulas $\{\Box a, \Box a', a \in \mathcal{V}\}$ (for the negation) and $(\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)$ (to translate the truth of Gödel implication), and their combinations via conjunction and disjunction. The difference between the translations of Ł$_3$ and G_3 into MEL only appears with more complex formulas. There is only a tiny difference between the two translations:

- $\mathcal{T}_1((a \to_G b) \to_G c) = ((\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)) \Rightarrow \Box c)) \wedge ((\Diamond a \Rightarrow \Diamond b) \Rightarrow \Diamond c)$;
- $\mathcal{T}_1((a \to_L b) \to_L c) = ((\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)) \Rightarrow \Box c)) \wedge ((\Box a \Rightarrow \Diamond b) \Rightarrow \Diamond c)$.

Regarding inference, note that in G_3 (contrary to Ł$_3$), the deduction theorem holds, that is $\alpha \vdash \beta$ if and only if $\alpha \to_G \beta$ is a tautology (Hájek, 1998). To prove in G_3 that a formula β is a consequence of a knowledge base $B_G = \{\alpha_1, \ldots, \alpha_n\}$, one may equivalently try to prove that the assertion $\gamma = (\sqcap_{i=1,\ldots,n}\alpha_i) \to_G \beta$ is a G_3-tautology. As a consequence of Proposition 9, we can do the same after translating the inference problem into MEL, since the deduction theorem holds in MEL:

Corollary 16. *Let β be a formula in Gödel logic G_3 and $B_G = \{\alpha_1, \ldots, \alpha_n\}$ a knowledge base in this logic. Then, $B_L \vdash \beta$ in G_3 iff the modal formula $\mathcal{T}_1((\sqcap_{i=1,\ldots,n}\alpha_i) \to_G \beta)$ is a Boolean tautology can be proved from the MEL axioms.*

Proof. As G_3 formulas are expressible in Łukasiewicz logic, tautologies of the former become tautologies of the latter. If $(\sqcap_{i=1,\ldots,n}\alpha_i) \to_G \beta$ is a tautology in G_3 it can be expressed also as a tautology in Ł$_3$. So, we can apply Proposition 9 to the present case: it says that the translation into MEL of any tautology in Ł$_3$ is derivable from the MEL axioms (i.e., is a tautology in MEL). $\Box$

Clearly, in MEL, proving that $(\mathcal{T}_1(\sqcap_{i=1,\ldots,n}\alpha_i) \to_G \beta)$ is a tautology is not easier than proving $\mathcal{T}_1(\beta)$ from $(\mathcal{T}_1(\sqcap_{i=1,\ldots,n}\alpha_i)$. This is left for further research.

6.3 An Example of Paraconsistent Logic: Priest

Priest (1979) Logic of Paradox (PLP) is supposed to tolerate contradictions. In order to do this, it uses the three truth values and the connectives of Kleene logic. The difference lies in the designated truth values, which are $\mathbf{1}$ and $\frac{1}{2}$ in Priest logic. Thus, asserting a formula α means $v(\alpha) \geq \frac{1}{2}$ in Priest logic, hence can be translated as $\Diamond a$ in MEL when α is atom a. More precisely, the translation into MEL of propositional variables and formulas of Priest logic having truth-degree at least $\frac{1}{2}$ is similar to the translation of true formulas of Kleene logic, where we replace $\Box$ with $\Diamond$. More precisely, the translation $\mathcal{T}(v(\alpha) \leq \frac{1}{2})$ into MEL of formulas asserted in PLP follows the rules:

- $\mathcal{T}^{\geq}_{1/2}(a) = \Diamond a$; $\mathcal{T}^{\geq}_{1/2}(\neg a) = \Diamond a'$;
- $\mathcal{T}^{\geq}_{1/2}(\alpha \sqcup \beta) = \mathcal{T}^{\geq}_{1/2}(\alpha) \vee \mathcal{T}^{\geq}_{1/2}(\beta)$;
- $\mathcal{T}^{\geq}_{1/2}(\alpha \sqcap \beta) = \mathcal{T}^{\geq}_{1/2}(\alpha) \wedge \mathcal{T}^{\geq}_{1/2}(\beta)$

- (Kleene implication) $\mathcal{T}^{\geq}_{1/2}(\alpha \to_K \beta) = \mathcal{T}_1(\alpha) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\beta)$, which is $\Box a \Rightarrow \Diamond b$ (or $\Diamond a' \vee \Diamond b$) in the case of atoms. This is a weak implication as the certainty of a only implies the possibility of b.

Any formula α in Priest logic can be rewritten in conjunctive normal form as

$$\sqcup_{i=1,\ldots,k} \sqcap_{j=1,\ldots,m_i} \ell_j(a_j),$$

where $\ell_j(a_j) = a_j$ or $\neg a_j$ is a literal, without simplifying terms of the form $a \sqcap \neg a$, in such a way that $v(\alpha) \geq \frac{1}{2}$ if and only if $v(\sqcup_{i=1,\ldots,k} \sqcap_{j=1,\ldots,m_i} \ell_j(a_j)) \geq \frac{1}{2}$. Its translation into MEL consists of the same set of clauses, where we put the modality $\Diamond$ in front of each literal, namely

$$\vee_{i=1,\ldots,k} \wedge_{j=1,\ldots,m_i} \Diamond \ell_j(a_j),$$

where $\ell_j(a_j)$ is now a literal a_j or a'_j in propositional logic. A knowledge base B in PLP is a conjunction of Kleene logic formulas supposed to have truth-values at least $\frac{1}{2}$. We can always put this knowledge base in disjunctive normal form, which ensures its direct translation into MEL, as a disjunction of conjunctions of literals, each literal prefixed by $\Diamond$.

In particular, if α has the form of a Boolean tautology then its translation (following the above recipe) will also be a tautology in MEL and it is also a tautology in Priest logic ($\vDash_{PLP} \alpha$). In fact Priest logic has the same tautologies as Boolean logic.

As a consequence the fragment of the language of MEL that can exactly encode Priest logic contains elementary formulas of the form $\Diamond a$ or $\Diamond a'$ and is

$$\mathcal{L}^P_\Diamond = \Diamond a | \Diamond a' | \phi \vee \psi | \phi \wedge \psi \subset \mathcal{L}^\ell_\Box.$$

This language is the image of $\mathcal{L}^K_\Box$ replacing necessity modalities by possibility, and is another fragment of $\mathcal{L}^\ell_\Box$. Moreover, we can put any formula in $\mathcal{L}^P_\Diamond$ back in the form of a conjunction of formulas of the form $\Diamond(\vee_{i=1,\ldots,k} \ell_i(a_j))$ due to MEL axioms.

The notion of consequence is defined in PLP as:

Definition 5. *If B is a set of propositions in the language of Kleene logic, then $B \vDash_{PLP} \alpha$ if and only if there does not exist an interpretation v such that $v(\alpha) = \mathbf{0}$ and for all $\beta \in B, v(\beta) \in \{\mathbf{1}, \frac{1}{2}\}$. In other words, if $v(\beta) \geq \frac{1}{2}$, for all $\beta \in B$ then $v(\alpha) \geq \frac{1}{2}$.*

Priest logic is paraconsistent: we do not have $\alpha \sqcap \neg\alpha \models_P \beta$, which is not surprising when translated into MEL, where $\Diamond a \wedge \Diamond a'$ is not a contradiction. The use of Kleene strong connectives in this approach to paraconsistency thus imposes the choice of the modality $\Diamond$ in the translation of atomic assertions in order to capture the behavior of the logic PLP. In a recent paper (Ciucci and Dubois, 2013a), we have shown that at the semantic level, asserting $v(a) \geq \frac{1}{2}$, that is $E_v \models \Diamond a$, must be understood as follows in the scope of paraconsistent logic: each classical interpretation w in E_v should be viewed as a fully informed agent that considers that w is the actual world. So $v(a) \geq \frac{1}{2}$ means that at least one agent thinks a is true, and $v(a) = \frac{1}{2}$ clearly means that there is one agent that thinks a is true and another one that thinks a is false, which explains why in this case, $\frac{1}{2}$ can express the idea of contradiction.

Modus ponens does not hold in Priest logic, since from $\models_P a$ and $\models_P a \to_K b$ we cannot derive that $\models_P b$; in MEL it is easy to see that, likewise, $\Diamond a$, and $\Diamond a' \vee \Diamond b$ do not imply $\Diamond b$. Likewise, the transitivity of implication is lost in Priest logic. In MEL this is because from $\models \Diamond a' \vee \Diamond b$ and $\models \Diamond b' \vee \Diamond c$, one cannot infer $\models \Diamond a' \vee \Diamond c$. In fact the disjunctive syllogism fails in Priest Logic, and indeed, from $\Diamond a'$ and $\Diamond a \vee \Diamond b$ one cannot conclude $\Diamond b$. However all inference rules in Priest logic yield valid inference rules in MEL. To cite a few:

- $\ell(a) \vdash_P \ell(a) \sqcup \ell(b)$; $\{\ell(a), \ell(b)\} \vdash_P \ell(a) \sqcap \ell(b)$
- $a \rightarrow_K (b \rightarrow_K c) \vdash_P b \rightarrow_K (a \rightarrow_K c)$ (both are $\neg a \sqcup \neg b \sqcup c$)
- If $\{a_1, \ldots a_n\} \vdash_P b$ then $\{a_1, \ldots a_{n-1}\} \vdash_P a_n \rightarrow_K b$.

In MEL the latter reads If $\Diamond a_1 \wedge \cdots \wedge \Diamond a_n \vdash \Diamond b$ then $\Diamond a_1 \wedge \cdots \wedge \Diamond a_{n-1} \vdash_P \Diamond a'_n \vee \Diamond b$, which is obvious. So, Priest logic is a propositional logic with all Boolean tautologies but without the usual inference rules, and it is expressible in a fragment of the MEL language made of the elementary formulas of the form $\Diamond a$ or $\Diamond a'$ as well as their conjunctions and disjunctions.

At the semantic level, the epistemic truth-value $\mathbf{0}$ plays in PLP a role similar to the one of the epistemic truth-value $\mathbf{1}$, in Kleene logic. Basically, β is a PLP- consequence of α if $v(\beta) = \mathbf{0}$ implies $v(\alpha) = \mathbf{0}$ for all valuations. It is clear that for any Kleene formula β, $\mathcal{T}(v(\beta) = \mathbf{0})$ ‘ can be expressed in the Kleene fragment $\mathcal{L}^K_\Box$ of $\mathcal{L}_\Box$. Indeed:

- $\mathcal{T}(v(a) = \mathbf{0}) = \Box a'$.
- $\mathcal{T}(v(\neg a) = \mathbf{0}) = \Box a$
- $\mathcal{T}(v(a \sqcup b) = \mathbf{0}) = \Box a' \wedge \Box b'$
- $\mathcal{T}(v(a \sqcap b) = \Box a' \vee \Box b'$

So, inference in Priest Logic can rely on inference in MEL inside the target language $\mathcal{L}^K_\Box$ in the form $\alpha \vDash_{PLP} \beta$ if and only if $\mathcal{T}(v(\beta) = \mathbf{0}) \vdash \mathcal{T}(v(\alpha) = \mathbf{0})$. We can thus capture inference in Priest logic by propagating falsity instead of truth, using inference rules in MEL.

7. The modal translation of all connectives

We have seen in section 3 that 14 conjunctions and implications can be defined on three-values according to some reasonable property given in Definitions 1 and 2. Here, we give the translations of all these connectives (in the case of atomic formulas), when the corresponding formulas have truth-value $\mathbf{1}$. In Table 6 we can see the translation of all the conjunctions and in Table 7 of all the implications.

Conjunction	Translation $\mathcal{T}_1(a * b)$
1 (Sette)	$\Diamond a \wedge \Diamond b$
2,14 (Sobocinski)	$(\Diamond a \wedge \Box b) \wedge (\Box a \wedge \Diamond b)$
3,12,13	$\Box a \wedge \Diamond b$
4,6,10	$\Diamond a \wedge \Box b$
5,7,8,9,11 (Kleene, Bochvar, Łukasiewicz)	$\Box a \wedge \Box b$

Table 6. Translation of all the conjunctions

Implication	Translation $\mathcal{T}_1(a \rightarrow b)$
1–5 (Sobocinski, Jaskowski,Kleene)	$\Diamond a \Rightarrow \Box b$
6,7 (Sette)	$\Diamond a \Rightarrow \Diamond b$
8	$\Box a \Rightarrow \Diamond b$
9,12 (Nelson, Bochvar)	$\Box a \Rightarrow \Box b$
10,11,13,14 (Gödel, Łukasiewicz, Gaines-Rescher)	$(\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)$

Table 7. Translation of all the implications

So, we are able to translate all such logics into a unique one, namely MEL, restricting its language to $\mathcal{L}_{\Box}^{\ell}$, where $\Box$ only appears in front of literals[1]. We, indeed, recall that due to the result in Proposition 1, they either coincide with Łukasiewicz logic or can be expressed in it. So, their translation yields a fragment of $\mathcal{L}_{\Box}^{\ell}$. Now, the translation of three-valued logics in MEL highlights an epistemic semantics for them, and enables a comparison between them. We can see, for instance, that

- the non-commutative behaviour of some conjunctions, translates in a different choice of modalities in front of literals. That is, we have the translations $\Diamond a \wedge \Box b$ or $\Box a \wedge \Diamond b$ on lines 3 and 4 of Table 6;
- the translation of Sette logic reveals the paraconsistent nature of this logic. Indeed, we can see that true formulas consists in the ones where we have a possibility $\Diamond$ in front of atoms;
- on the other hand, Nelson and Bochvar logics are the only two logics such that both conjunction and implication involve only the $\Box$ modality.

We have seen that conversely any formula in $\mathcal{L}_{\Box}^{\ell}$ can be expressed as a formula in Ł$_3$. Interestingly the part of the MEL language that cannot be mapped to any three-valued formula include all formulas where the $\Box$ modality is put in front of a disjunction of literals. Note that any MEL formula can be expressed as (for instance) a disjunction of conjunctions each term of which is a clause prefixed by $\Box$ or the negation thereof.

Typically, $\Box(a \vee b)$ cannot be expressed in Ł$_3$ nor in any other three-valued logic. This is because in such logic it is impossible to know $a \vee b$ without knowing either a or b (only $\Box a \vee \Box b$ can be expressed in three-valued logics). It shed lights on the paradoxes of such truth-functional logics, when it comes to justifying $v(a \sqcap b)$ or $v(a \to_K b)$ as a function of $v(a)$ and $v(b)$ when these truth-values are $\frac{1}{2}$, interpreted as *unknown*. Neither Kleene truth-tables not Łukasiewicz ones sound satisfactory (Urquhart, 1986). However under our translation the fact that $v(a \sqcup b) = \frac{1}{2}$ is clear in that case because $a \sqcup b$ means $\Box a \vee \Box b$ which is indeed false if none of $\Box a$ and $\Box b$ is true. Truth-functionality in Ł$_3$ reduces to something trivial in MEL. Likewise, $v(a \to_L b) = \mathbf{1}$ if $v(a) = v(b) = \frac{1}{2}$ in Ł$_3$ because in those cases, all of $\Diamond a, \Diamond a', \Diamond b, \Diamond b'$ are true, which makes $\mathcal{T}_1(a \to_L b) = (\Box a \Rightarrow \Box b) \wedge (\Diamond a \Rightarrow \Diamond b)$ true as well. However, $v(a \to_K b) = \frac{1}{2}$ in Kleene logic, because it means $\Box a' \vee \Box b$ whose truth we ignore in that same situation.

This limited expressiveness of three-valued logics of incomplete information is related to the fact that the only epistemic states that can be captured by $\mathcal{L}_{\Box}^{\ell}$ are partial model. The full-fledged MEL logic, even if a tiny part of a general modal logic, allows for any kind of epistemic state. Note that restricting to partial models for incomplete information is similar to restricting to probability distributions on Boolean languages made of product of marginal probabilities on variables. So our work makes the expressive power of three-valued logic very clear under an epistemic view of truth-values.

8. Conclusion

This work suggests that the multiplicity of three-valued logics is only apparent. If the third value means *unknown*, the elementary modal logic MEL, restricting its language to the case of modalities appearing only in front of literals, is a natural choice to encode a large class three-valued logics that extend Boolean logic. In the framework of a given

[1]Interestingly, even if MEL has a semantics which can be described in terms of possibility theory (Banerjee and Dubois, 2009), possibilistic logic (Dubois and Prade, 2004) cannot encode such rules as appear on Table 7. Indeed, viewed in the scope of MEL, possibilistic logic uses graded $\Box$ modalities (weights that express the strength of belief), but such formulas can only be combined by conjunctions. Translations of rules such as $\Diamond a \Rightarrow \Box b$, $\Diamond a \Rightarrow \Diamond b$, $\Box a \Rightarrow \Diamond b$, $\Box a \Rightarrow \Box b$ can be captured in generalized possibilistic logic (Dubois *et al.*, 2012a).

application, some connectives make sense, others do not and we can choose the proper logic. The interest in our translation, which is both modular and faithful, is double:

(1) Once translated into modal logic, the meaning of a formula becomes clear since its epistemic dimension is encoded in the syntax, even if in the worst case, the size of a translated formula may grow exponentially in the number of occurrences of the input variables.
(2) We can better measure the expressive power of each three-valued system. In particular it shows that the truth-functionality of three-valued logic is paid by a severe restriction of representation capabilities: we can express knowledge about literals only, which results in a very restrictive use of disjunction.

This work can be extended to more than 3 "epistemic" truth values. However, the target language is then a more expressive modal logic with several necessity modalities of various strength, such as generalized possibilistic logic (Banerjee *et al.*, 2013; Dubois and Prade, 2011) (where the epistemic states are possibility distributions). It is a weigthed extension of MEL as well. For instance, the 5-valued equilibrium logic (Pearce, 2006) (which can encode "answer-set" programming) has been translated into generalized possibilistic logic with weak and strong necessity operators in front of literals, the epistemic states being pairs of nested partial models (Dubois *et al.*, 2012b). In particular, we can thus capture answer-set programming in this generalized MEL logic, by means of rules of the form $(\Box a \wedge \Diamond b') \Rightarrow \Box c$. However, we need more than MEL to properly account for negative literals in the body of the rule ($\Diamond b'$ here)[1].

The idea of expressing a many-valued logic in a two-level Boolean language (one encapsulating the other) put here at work can be adapted to other understandings of the third truth value (such as *contradictory, irrelevant*, etc.) by changing the target language. We have seen the case of Priest logic here. However it is very closely related to Kleene logic and MEL can still be used as a target logic for the translation, by just replacing necessities by possibilities. Recent results (Ciucci and Dubois, 2013a) suggest that this technique applied to other three-valued logics can recover some other paraconsistent logics. When both incomplete information and conflicting information must be handled conjointly, preliminary works related to Belnap logic (Dubois, 2012) indicate that a possible target logic could be a non-regular modal logic such as EMN (Chellas, 1980), restricted to the language of MEL.

Finally, based on our results, one can conjecture that only in the case where the third truth value possesses an ontic nature (that is, when it means *half-true*, admitting that truth is a matter of degree) can a straightforward meaning be given to formulas in propositional languages that use the syntax of Gödel, Łukasiewicz, etc. logics and can their violation of the Boolean axioms such as excluded middle or contradiction laws, be intuitively explained, as in the case of formal fuzzy logics (Hájek, 1998).

Appendix A. Proof of proposition 6

Proposition 6. *If α is an axiom in Ł$_3$, then $\mathcal{T}(v(\alpha) = \mathbf{1})$ is a tautology in MEL.*

Proof. From Ł$_3$ axioms to MEL.
(W1). $\mathcal{T}_1((\alpha \rightarrow_L \beta) \rightarrow_L ((\beta \rightarrow_L \gamma) \rightarrow_L (\alpha \rightarrow_L \gamma)))$ is the conjunction of two MEL

[1] Indeed the behavior of this negation is not properly captured if $\Diamond b' = (\Box b)'$: $\Diamond b'$ must dually correspond to a weaker $\Box$ modality, as explained in (Dubois *et al.*, 2012a,b).

formulas, namely

$$\mathcal{T}_1(\alpha \to_L \beta) \Rightarrow \mathcal{T}_1((\beta \to_L \gamma) \to_L (\alpha \to_L \gamma)) \tag{A1}$$

and

$$\mathcal{T}^{\geq}_{1/2}(\alpha \to_L \beta) \Rightarrow \mathcal{T}^{\geq}_{1/2}((\beta \to_L \gamma) \to_L (\alpha \to_L \gamma)), \tag{A2}$$

which are two tautologies as we are going to show. The first formula (A1) is of the form

$$\phi \Rightarrow (\psi \wedge \chi) = (\phi' \vee \psi) \wedge (\phi' \vee \chi)$$

where

$$\begin{aligned}
\phi' &= (\mathcal{T}_1(\alpha \to_L \beta))' = (\mathcal{T}_1(\alpha) \wedge \mathcal{T}_1(\beta)') \vee (\mathcal{T}^{\geq}_{1/2}(\alpha) \wedge \mathcal{T}^{\geq}_{1/2}(\beta)') \\
\psi &= \mathcal{T}_1(\beta \to_L \gamma) \Rightarrow \mathcal{T}_1(\alpha \to_L \gamma) \\
&= [(\mathcal{T}_1(\beta) \wedge \mathcal{T}_1(\gamma)') \vee (\mathcal{T}^{\geq}_{1/2}(\beta) \wedge \mathcal{T}^{\geq}_{1/2}(\gamma)')] \vee [(\mathcal{T}_1(\alpha)' \vee \mathcal{T}_1(\gamma)) \wedge (\mathcal{T}^{\geq}_{1/2}(\alpha)' \vee \mathcal{T}^{\geq}_{1/2}(\gamma))] \\
\chi &= \mathcal{T}^{\geq}_{1/2}(\beta \to_L \gamma) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\alpha \to_L \gamma) = [\mathcal{T}_1(\beta) \wedge \mathcal{T}^{\geq}_{1/2}(\gamma)'] \vee \mathcal{T}_1(\alpha)' \vee \mathcal{T}^{\geq}_{1/2}(\gamma) \\
&= \mathcal{T}_1(\beta) \vee \mathcal{T}^{\geq}_{1/2}(\gamma) \vee \mathcal{T}_1(\alpha)'
\end{aligned}$$

We show that both $(\phi' \vee \psi)$ and $(\phi' \vee \chi)$ are tautologies.

- $(\phi' \vee \psi)$. From $\mathcal{T}_1(\alpha) \wedge \mathcal{T}_1(\beta)') \vee (\mathcal{T}_1(\beta) \wedge \mathcal{T}_1(\gamma)')$ we can get $(\mathcal{T}_1(\alpha) \wedge \mathcal{T}_1(\beta)') \vee (\mathcal{T}_1(\beta) \wedge \mathcal{T}_1(\gamma)') \vee (\mathcal{T}_1(\alpha) \wedge \mathcal{T}_1(\gamma)')$. We also obtain a dual expression from the terms where $\mathcal{T}_1$ is substituted by $\mathcal{T}^{\geq}_{1/2}$. So, putting everything together, we have $[...] \vee \underline{\underline{(\mathcal{T}_1(\alpha) \wedge \mathcal{T}_1(\gamma)')}} \vee \underline{(\mathcal{T}^{\geq}_{1/2}(\alpha) \wedge \mathcal{T}^{\geq}_{1/2}(\gamma)')} \vee [\underline{\underline{(\mathcal{T}_1(\alpha)' \vee \mathcal{T}_1(\gamma))}} \wedge \underline{(\mathcal{T}^{\geq}_{1/2}(\alpha)' \vee \mathcal{T}^{\geq}_{1/2}(\gamma))}]$ which can easily be verified to be a tautology: underlined terms are the negations of each other;
- $(\phi' \vee \chi)$ is equal, by just changing the order of the terms, to $(\mathcal{T}^{\geq}_{1/2}(\alpha) \wedge \mathcal{T}^{\geq}_{1/2}(\beta)') \vee \mathcal{T}^{\geq}_{1/2}(\gamma) \vee (\mathcal{T}_1(\alpha) \wedge \mathcal{T}_1(\beta)') \vee \mathcal{T}_1(\alpha)' \vee \mathcal{T}_1(\beta)$.
  By distributivity, we have a tautology from $(\mathcal{T}_1(\alpha) \wedge \mathcal{T}_1(\beta)') \vee \mathcal{T}_1(\alpha)' \vee \mathcal{T}_1(\beta)$.

The second formula (equation A2) is of the form:
$(\mathcal{T}_1(\alpha) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\beta)) \Rightarrow \{[(\mathcal{T}_1(\beta) \Rightarrow \mathcal{T}_1(\gamma)) \wedge (\mathcal{T}^{\geq}_{1/2}(\beta) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\gamma))] \Rightarrow (\mathcal{T}_1(\alpha) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\gamma))\}$
$= (\mathcal{T}_1(\alpha) \wedge \mathcal{T}^{\geq}_{1/2}(\beta)') \vee (\mathcal{T}_1(\beta) \wedge (\mathcal{T}_1(\gamma))') \vee (\mathcal{T}^{\geq}_{1/2}(\beta) \wedge (\mathcal{T}^{\geq}_{1/2}(\gamma)') \vee (\mathcal{T}_1(\alpha)' \vee \mathcal{T}^{\geq}_{1/2}(\gamma))$. By distributivity, we obtain the tautology
$\mathcal{T}^{\geq}_{1/2}(\beta)' \vee \mathcal{T}_1(\alpha)' \vee (\mathcal{T}_1(\beta) \wedge \mathcal{T}_1(\gamma)') \vee \mathcal{T}^{\geq}_{1/2}(\beta) \vee \mathcal{T}^{\geq}_{1/2}(\gamma)$.
(W2) The translation of this axiom is the conjunction of the two formulas:

$$[T_1(\alpha) \Rightarrow ((T_1(\beta) \Rightarrow T_1(\alpha)) \wedge (T^{\geq}_{1/2}(\beta) \Rightarrow T^{\geq}_{1/2}(\alpha)))]$$

and

$$T^{\geq}_{1/2}(\alpha) \Rightarrow [T^{\geq}_{1/2}(\beta) \Rightarrow T^{\geq}_{1/2}(\alpha)]$$

The second one is a tautology since $x \Rightarrow (y \Rightarrow x)$ is a tautology in BPL for any formula

x, y. The first one can be developed as the conjunction of:

$$T_1(\alpha) \Rightarrow (T_1(\beta) \Rightarrow T_1(\alpha))$$

and

$$T_1(\alpha) \to (T^{\geq}_{1/2}(\beta) \Rightarrow T^{\geq}_{1/2}(\alpha))$$

Again, the first one is a tautology in BPL, and the second one is a tautology due to Lemma 5 and the fact that $T_1(\alpha)' = T^{\leq}_{1/2}(\alpha)$. As a result, we showed that the translation of (W2) is a conjunction of tautologies, hence a tautology.
(W3). $\mathcal{T}_1(\neg\beta \to_L \neg\alpha) \to_L (\alpha \to_L \beta))$ is translated into a conjunction of two tautologies. The former is: $\mathcal{T}_1(\neg\beta \to_L \neg\alpha) \Rightarrow \mathcal{T}_1(\alpha \to_L \beta)) = \{\underline{\underline{[\mathcal{T}_1(\neg\beta) \Rightarrow \mathcal{T}_1(\neg\alpha)]}} \wedge \underline{[\mathcal{T}^{\geq}_{1/2}(\neg\beta) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\neg\alpha)]}\} \Rightarrow \{\underline{\underline{[\mathcal{T}_1(\alpha)) \Rightarrow \mathcal{T}_1(\beta)]}} \wedge \underline{[\mathcal{T}^{\geq}_{1/2}(\alpha) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\beta)]}\}$, which leads to a formula $\phi \Rightarrow \phi$ in MEL since $\mathcal{T}_1(\neg\beta) \Rightarrow \mathcal{T}_1(\neg\alpha) = \mathcal{T}^{\geq}_{1/2}(\beta)' \Rightarrow \mathcal{T}^{\geq}_{1/2}(\alpha)' = \mathcal{T}^{\geq}_{1/2}(\alpha) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\beta)$ and similarly for the other terms.
The second tautology is: $\mathcal{T}^{\geq}_{1/2}(\neg\beta \to_L \neg\alpha)) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\alpha \to_L \beta)) = [\mathcal{T}_1(\neg\beta) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\neg\alpha)] \Rightarrow [\mathcal{T}_1(\alpha) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\beta)] = [\mathcal{T}^{\geq}_{1/2}(\beta)' \Rightarrow \mathcal{T}_1(\alpha)'] \Rightarrow [\mathcal{T}_1(\alpha) \Rightarrow \mathcal{T}^{\geq}_{1/2}(\beta)]$ which is valid by contraposition of classical implication.
(W4). By a partial translation of the axiom we get the conjunction of two formulas:

$$[(T_1(\alpha \to_L \neg\alpha) \Rightarrow T_1(\alpha)) \wedge (T^{\geq}_{1/2}(\alpha \to_L \neg\alpha) \Rightarrow T^{\geq}_{1/2}(\alpha))] \Rightarrow T_1(\alpha)$$

and

$$[(T_1(\alpha \to_L \neg\alpha)) \Rightarrow T^{\geq}_{1/2}(\alpha)] \Rightarrow T^{\geq}_{1/2}(\alpha)$$

The first one is of the form $((y \Rightarrow x) \wedge z) \Rightarrow x$ which can be verified to be a tautology in BPL. Also the second formula is a tautology of the form $(x \Rightarrow y) \Rightarrow y$. □

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