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Flawless Disagreement in Mathematics

A disagrees with B with respect to a proposition, p, *flawlessly* just in case A believes p and B believes not-p, or vice versa, though neither A nor B is guilty of a cognitive shortcoming – i.e. roughly, neither A nor B is being irrational, lacking evidence relevant to p, conceptually incompetent, insufficiently imaginative, etc.

Potential examples: disagreements over *aesthetic propositions*, such as that Picasso is better painter than Matisse, *epistemic modal propositions*, such as that President Obama can't be in the White House now, and *moral propositions*, such as that killing is wrong.

Define *realism* with respect to propositions of a given sort, F, as the view that (atomic) F-sentences are truth-apt, true or false relevantly independent of people's beliefs, and sometimes true, interpreted at face-value.

A potential upshot of the possibility of flawless disagreement with respect to F-propositions is that *F-realism is false* (i.e. that non-cognitivism, relativism, error-theory, or some other brand of antirealism with respect to F-propositions, is true).

Question: Does "insensitivity to the F-facts", where F-facts would be a priori knowable, count as a cognitive shortcoming (over and above conceptual incompetence, etc.)?

Widely Assumed Answer: No.

Question: Why Not?

Obvious Answer: If it did, then one couldn't argue from flawless disagreement over the likes of aesthetic, epistemic modal, and moral claims to antirealism with respect to corresponding discourses without begging the question against corresponding realists.

Claim: If "insensitivity to the F-facts", where F-facts would be a priori knowable, does not, in general, count as a cognitive shortcoming (over and above conceptual incompetence, insufficient evidence, etc.), then flawless disagreement is possible with respect to mathematical propositions, including "core" ones.

Dilemma: allow that flawless disagreement is possible with respect to mathematical claims, and risk giving up on mathematical realism, or don't allow this, and risk giving up on standard arguments against aesthetic, epistemic modal, and moral realism.

Clarification: I am interested in whether there can be flawless disagreement over mathematical claims *that does amount to a disagreement over logic* – i.e. in whether there can be flawless disagreement over *axioms* of our mathematical theories.

Mathematical Knowledge and Scientific Knowledge

In virtue of what are we justified in believing the axioms of our mathematical theories?

A natural answer is that we are justified in believing them because they are intuitively evident. Familiar axioms such as Extensionality and Pairs seem to be intuitively evident.

But this answer does not seem to be true in general. Consider the Axiom of Replacement. An easy consequence of this is that there is an ordinal greater than all f(x), where f(0) = Aleph_0 and f(x+1) = Aleph_f(x) for all natural numbers, x. This is not intuitively evident (look at how fast it grows!).

A better answer is that we are justified in believing axioms in a way that is analogous to the way in which we are justified in believing empirical scientific hypotheses.

Consider F = ma. This isn't observationally evident. We are justified in believing it because it figures into the best overall systematization of propositions that are so evident.

In a similar way, though not intuitively evident, Replacement plausibly affords the best systematization of propositions which are – such as that there are ordinals as great as or greater than omega + omega.

Intuition vs. Observation

There is not perfect agreement as to what is observationally evident. Some people have impaired perceptual capabilities. Also, observation is "theory-laden".

But it does not seem that disagreements that arise over scientific hypotheses are primarily attributable to disagreements over what is observationally evident. There does not seem to be serious disagreement over *what the data to be systematized* are in science.

The same is not true in mathematics. There seems to be serious disagreement, not only over what theories are true, but over *what the data to be systematized* are in mathematics.

Disagreement in Set Theory

Consider disagreement over the Axiom of Foundation. This axiom is one of the backbones to a central conception of the set-theoretic universe, according to which every set is built out of basic elements at some stage in a transfinite "generation" process.

Advocates and detractors of Foundation are equally aware that Foundation makes no difference for the vast majority of mathematics. What advocates and detractors seem to dispute is the intuitive evidentness of Foundation's consequences.

Advocates of Foundation point out that Foundation disproves the existence of sets that contain themselves, and sets with infinitely-descending chains of membership – calling such sets "pathological" to register the intuitiveness of the view that they don't exist.

Detractors of Foundation cite the same consequences in arguing that Foundation is false – calling the view that they don't exist "unnaturally restrictive" to register the intuitiveness of the view that they *do* exist.

This seems to be a straightforward case of mathematicians appealing to mutually inconsistent propositions as intuitively evident in order to establish their positions – the epistemic analog of empirical scientists appealing to mutually inconsistent propositions as observationally evident in order to establish their positions.

<u>Disagreement in "Core" Mathematics</u>

Disagreements over what mathematical propositions are intuitively evident gives rise to disagreements over "core" mathematical propositions as well.

Consider the proposition, p, that all sets of real numbers are measurable. Some seem to regard p as intuitively evident, while others seem regard it as the opposite.

The Axiom of Choice disproves p. This axiom also implies fundamental results in the "core" areas of analysis, algebra, and topology. For example, it implies the Banach-Tarski theorem in analysis, and that every vector space has a basis in algebra.

Those who seem to regard p as intuitively evident often embrace some alternative to Choice, such as the Axiom of Determinacy. But Determinacy implies that the Banach-Tarski theorem is false, and that it is not the case that every vector space has a basis.

There are even more dramatic examples than this of disagreements over what is intuitively evident that give rise to disagreements over "core" mathematical propositions.

Intuitionists do not seem to regard the fundamental principle of the calculus, that every set of real numbers with an upper bound has a least upper bound, or various core consequences of it as intuitively evident, and reject the heart of classical analysis.

Radical predicativists do not even seem to regard such elementarily number-theoretic principles as the principle of mathematical induction or the principle that every natural number has a successor as intuitively evident, and reject the core of standard arithmetic.

<u>Actual Disagreement to Possible Flawless Disagreement</u>

It is conceivable that mathematical disagreements do not result from conflicting senses of what is intuitively evident, as they seem to, but rather result from cognitive shortcomings.

Perhaps advocates or detractors of Foundation, for example, are systematically incompetent with the sole non-logical concept of their discipline, that of membership.

Or, perhaps they are exceptionally prone to bias, unable to admit to trivialities like that no (some) sets contain themselves out of emotional investment in the opposite view.

Finally, perhaps they are insufficiently imaginative, attentive, or guilty of an error in reasoning. They are certainly not guilty of a lack of relevant evidence.

But this is extremely unlikely. If there were ever disagreements that one should *not* try to explain primarily with reference to cognitive shortcomings, it is the aforementioned ones.

Given that variations in mathematical intuitions serve generally to explain mathematical disagreements, and do not reflect cognitive shortcomings, flawless disagreement is possible with respect to a wide array of mathematical propositions, including "core" ones.

<u>Interlude: Possible Flawless Disagreement to Possible Pervasive Flawless Disagreement</u>

Are there any mathematical propositions over which there could not be flawless dispute?

If such claims as that the set-theoretic universe is hierarchical, that there are non-measurable sets of real numbers, and that every natural number has a successor, can be flawlessly denied, then I can see no reason to think that mathematical claims quite generally could not be.

No mathematical claim is a (first-order) logical truth. Even the favorite example, 2 + 2 = 4, is an intuitively synthetic hypotheses about the relation among some objects, numbers, not to be confused with the logical truth that if there are, say, exactly two apples on the table, and there are exactly two apples on the floor, and no apple on the table is an apple on the floor, then there are exactly four apples on the table or the floor (where the numerical quantifiers here are definable in terms of ordinary quantifiers plus identity).

If an agent's beliefs are coherent in an appropriate sense, then I can see no reason to think that she *must* be guilty of a cognitive shortcoming in denying that 2 + 2 = 4. What is the relevant difference between this and the claim that every natural number has a successor?

<u>Mathematical vs. Philosophical Disagreement</u>

One might worry that what I have shown is that flawless disagreement over *philosophical* propositions that have mathematical consequences is possible. If the kind of disagreement that exists between advocates and detractors of Choice is the kind that exists between fictionalists and typical number theorists, then this is not unsettling.

But, while "philosophical" in a natural sense of that term, disagreements over mathematical axioms are not philosophical in a way that undercuts the strength of arguments from flawless disagreement. In particular, they are not philosophical in a way that disagreements over aesthetic, epistemic modal, or moral matters, are not.