Vagueness and Frege

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Abstract: A constant of Frege's writing is his rejection of indeterminate predicates as found in natural language. This paper follows Frege's remarks on vagueness from the early "Begriffsschrift" to his mature works, drawing brief parallels with the main contemporary theories of vagueness. I critically examine Frege's arguments for the inconsistency of natural language and argue that the inability to accommodate vagueness in his mature ontology is mainly due to heuristic rules of thumb which Frege took as essential, not to a deep problem in his fundamental apparatus.

***Keywords*:** Frege, vagueness, indeterminateness, precision, theory of definition

Introduction

This study[2] grew from two questions. First, where does indeterminateness stand in the context of Gottlob Frege's philosophy and how does he justify his constant rejection of natural language on account of

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[2] The history of this paper predates (Călborean 2020) which contains a very condensed variant of it in Chapter 11.

it? Secondly, can Frege's constant doubts be assuaged by recent theories of vagueness? These two questions can only receive interlocking answers, as the justification Frege provides for his rejection might need to be compared with what we learned from the debate on vagueness started during the 1960s and 1970s, incidentally by some of Frege's rediscoverers, such as Michael Dummett.

As Frege never gave a positive theory of vagueness, there is a danger of introducing too many distinctions he would not have recognized. To avoid anachronism, the method of the paper is to follow Frege's early work in roughly chronological order, up to the Frege of after 1891, where I switch to discussing Frege's stance thematically, in relation to his mature ontology and semantics. Frege's fragments relevant to vagueness are often intermingled with fragments bearing on other topics, reason for which I try to follow his remarks closely and compare them, from place to place, with the main strands of the post-1960 philosophy of vagueness.

Upon analysis, both Frege's fundamental apparatus and common vague predicates survive, his rejection of natural language being unmotivated.

1. "Begriffsschrift" and the Sorites

The aim of Gottlob Frege's work "Begriffsschrift" (1997a) is to provide a core[3] symbolic language for laws of thought, which language is also called `*begriffsschrift*` or `*conceptual script*`. In the preface of the work,

[3] `Core` in the sense that it could later be applied to all sciences by way of special signs so as to become a "single formula language" partially realizing Leibniz's project of *calculus ratiocinator* (Frege 1997a, 50).

Frege speaks of begriffsschrift as being a formula language adequate to express those proofs which can be given by logical means alone. He arguably lists two conditions: first, the language should be able to express a complete chain of inference, so that nothing from intuition can matter to proof and, secondly, it should conserve the utmost precision of inferences and relations (1997a, 48).

Frege speaks of `*begrifflicher Inhalt*` (conceptual content) as consisting of those kind of entities between which such proofs arise (1997a, 49–53) and of which begriffsschrift would therefore make use. Against the tradition of Aristotelian logic, the conceptual content of such phrases as "S defeated P" and "P was defeated by S" is held to be the same. Frege also introduces a `*judgement*` as being the assertion of truth about a conceptual content and a `*function*` as the invariant part of a unitary sub-expression[4] replaceable by some other symbol in its places. After introducing letters and logical symbols into the language, the latter including quantifiers and truth-functional operators expressed as graphical (most being vertical) connections between them, Frege gives nine axioms. Frege's system is a second-order predicate calculus, including what we now call propositional logic and first-order predicate logic. In the final part of "Begriffsschrift", he puts the system to work, proving some theorems of mathematical induction.

While not all philosophers would agree that there are such things as purely logical proofs, conceptual contents common in various linguistic expressions or functions separable "*in thought*"[5], Frege's distinctions seem to make possible, by the end of "Begriffsschrift", a rigorous[6] analysis of mathematical induction, containing proofs which are general and important.

[4] Frege speaks of 'simple or complex symbol' (1997a, 67).

[5] The distinction object-function is fundamental to Frege's project and modern logic (Heck and May 2013, 835).

[6] Rigor is one of the main motivations of Frege's project, comprising the two conditions already noted of nothing coming into a proof unnoticed and of conserving truth, i.e., the possible syntactic verification of correct derivation but also a theory of definition (Frege 1960, XXI).

While Frege's formulation of them makes use of unrestricted second-order quantification, it is recognized that what is now known as `*classical logic*` springs from Frege's "Begriffsschrift" and its unprecedented success in formalizing this kind of proofs[7].

The topic of vagueness appears in this final part of "Begriffsschrift". Frege defines consecutively the notions of a property being hereditary in a sequence[8] (2002, 55), then the notion of an object following another in a sequence[9] and then he arrives at the base proposition of mathematical induction[10]. Frege expresses it in words and adds an aside:

> "We can translate (81) thus:
>
> *If x has a property F that is hereditary in the f-sequence, and if y follows x in the f-sequence, then y has the property F*[11]
>
> For example, let F be the property of being a heap of beans; let f be the procedure of removing one bean from a heap of beans; so that, f(a,b) means the circumstance that b contains all beans of a heap a except one and does not contain anything else. Then by means of our proposition we would arrive at the result that a single bean, or even none at all, is a heap of beans if the property of being a heap of beans is hereditary in the f-sequence. This is not the case in general, however since there are certain z for which F(z) cannot become a judgement on account of the indeterminateness of the notion `heap`"

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[7] For both points see Jean van Heijenoort's introduction to "Begriffsschrift" (Frege 2002, 1).

[8] Sequences, or in Beaney's translation,"f-series" (Frege 1997a, 75) are sets which satisfy $\forall x (Fx \supset \forall y(f(xy) \supset Fy)$. I remark that there is an obvious parallel with the principle of tolerance for vagueness (Călborean 2020, 22).

[9] Also known in Quine's terminology as "proper ancestral" (Frege 2002, 59).

[10] Frege writes in a footnote "Bernoulli's induction rests upon this" (2002, 62). Michael Beaney calls it "the key point of mathematical induction" (Frege 1997a, 77).

[11] Here Frege inserts his footnote containing the rest of the quote (2002, 62).

This is the Sorites paradox. We see that the property of being a heap of beans seems hereditary in the sequence of one-bean subtraction, i.e., that the property of being a heap is not lost by removing one bean. But that's not "the case in general" as Frege puts it, because the property of `*being a heap*` is in some way problematic.

Many philosophers start to discuss vagueness by assuming the existence of borderline cases, those where it is unclear whether the property applies or not[12]. There is a parallel with Frege's certain "z" above: Frege says that since the notion `*heap*` is indeterminate, there are certain *z* where "*F(z)*" cannot become a judgement. The notion of unjudgeable contents, i.e., conceptual contents which cannot be asserted, is once more discussed in "Begriffsschrift", namely when Frege states at #3 that contents such as "house" belong to it (1997a, 53), the heap of beans above being the second such example. But the proposition "Eleven beans are a heap of beans" is quite different from "house". It seems like there is an easy way of saying why the latter cannot become a judgement, namely, it is not predicative, that is, capable (if turned into a judgement) of becoming true or false. But this is precisely Frege's point: the grammatical form of a truth-carrying expression does not guarantee that the expression is also logically truth-asserting. As van Heijenoort puts it "With these few remarks, Frege puts vague predicates outside logic" (1986, 32).

Let us give a common form of the Sorites paradox, covering Dummett's Wang's paradox (Dummett 1996, 99) too:

> *IB (Induction basis): An object corresponding to a number x under measurement m has property P.*

[12] This formulation is very close to Rosanna Keefe and Peter Smith's (1996a, 2).

IS (Inductive step)[13]*: If an object corresponding to a number under measurement m has property P, so does an object corresponding to the next / previous natural number under measurement m.*

C (Conclusion): Objects corresponding to any number under measurement m have property P.

We can take `*group of beans*` as measurement and `*being a heap*` as P, so that we get:

IB: A group of two hundred beans is a heap of beans.

IS: If a is a heap of beans, a group only one bean short of a will be a heap of beans.

C: A group of zero beans is a heap of beans.

Epistemicists such as Timothy Williamson deny *IS*: there is a number in the measurement where the property *P* does not apply, and that number is next to one for which the property does apply (Williamson 1996, 279). Other philosophers, fuzzy theorists among them, deny that the repeated application of the induction step conserves truth (Machina 1996, 200). In comparison, Frege chooses to generally deny the general applicability of the inductive step: he denies that such a predicate can always even be asserted. This means going further than needed. Frege could have gone epistemicist *avant la lettre* and deny that the property is hereditary in the sequence of

[13] Mathematical induction is not necessary for the paradox, *IS* can be replaced with a finite series of modus ponens or conjunctive syllogism (Williamson 1994, 24).

bean-subtraction, saying that there is such a number y smaller by one than x so that x beans is a heap of beans and y beans is not. He could have thought `*heap of beans*` parallel to a sharp-boundary property such as `*natural number in the second dozen*`. The fact that he does not do so raises the question of whether and how he allows some numbers of beans to go through and others not.

Timothy Williamson writes that what Frege has in mind here is that while the notion `*heap*` fails to refer, some of its predications may still be judgements, because those sentences would employ the problematic words as *idioms*, that is, shortcuts or revelatory images based on context or previous experience, that secure truth or falsity to the proposition. He writes: "it is not a cartographer's job to explain why travelers with bad maps or none at all sometimes reach their destinations" (1994, 44). This assumes that we need to read in the early Frege of "Begriffsschrift" his later distinction of sense and reference although it is precisely in Frege's eponymous article for that distinction that he repudiates some main points of "Begriffsschrift"[14]. That is, Frege's first work did not mention conceptual expressions referring at all (Heck 2012, 21–22). He did not deny conceptual content to those predications of certain "*z*", even though the later Frege would deny reference to vague concept-words and, under some interpretations, sense as well (1997b, 178)[15].

Let us take a step back and ask whether the inability of sentences containing vague predicates to become judgements is, for Frege, solely a matter of them not becoming true or false. He does not affirm this. Thus, an alternative is to remember that the analysis of mathematical induction rests on quantifying over properties. And this assumes that there is a common logical form of predicates, so that a symbol can represent them. But indeterminate predicates could be interpreted as exceptions to Frege's theory of sequences: they seem to be hereditary in a sequence, yet they also indicate as absurd the predication of their corresponding C in the soritical

[14] Namely that claiming identity is a relation between names (Frege 2002, 20).
[15] See below at section 4.

series above. Of course, this means that there is a range of "*z*" where there will be trouble, this trouble zone still manifesting itself as lack of truth value for associated judgements. But the deeper problem is that the natural language term does not conform to Frege's expectations of logic. Frege hoped that by removing all particular content irrelevant to validity of proof (2002, 7), a consistent kernel of thought would be revealed, but vague predicates belie it, by embedding *prima facie* logical relations which can be turned into a contradiction by the laws of the system. Therefore, Frege's aside on the Sorites paradox is an illustration of what he expects of logic. Indeterminateness fails the minimal rigor necessary for a formula language based on distinguishing functions and arguments in natural language, without any supplementary semantic or ontological characterization[16]. And vague predicates are not adequately captured in a formula language expressing pure thought, because the pure thoughts they embed lead to contradictions.

Therefore, my first conclusion is that Frege takes indeterminateness as going against the assumptions of his project: that logical relations can be extracted from language without contradiction. Of course, this raises the question of how to get a grip on what logic can be, for Frege, in relation to natural language and what I called prima facie logical relations embedded therein.

Before turning to that issue, note that `*Heap*` could be understood as per the epistemicists, in having a precise border in centimeters[17]. Or be understood statistically, as I argued elsewhere (Călborean 2020), as applying both truthfully and falsely to separate groups of the same number of beans. In both cases, Frege's project would stay the same, the single difference

[16] In "Begriffsschrift" Frege speaks of functions as "expressions", a point Philip Jourdain was to call a "trace of formalism" (Heck and May 2013, 831–32).

[17] On how one can understand Frege as epistemicist, see Stephen Puryear (2013, 123–27).

being that what he treats as embedded logical relationships of `*Heap*` should be nuanced[18].

2. The relation between logic and natural language

There are times when Frege takes linguistic form as determinative of logical distinctions. The grammatical articles make the best example. Frege insists that the definite article marks the difference between objects and concepts up to the point of hypostatizing enigmatic objects corresponding to expressions of the form `*the concept "man"*`[19]. He will introduce a special function "$\backslash\xi$" to play the role of definite article, that of turning a concept into an object when appropriate, by way of his Axiom VI of his "Basic Laws of Arithmetic"[20]. He will also see the indefinite article as determinative of concepts[21] and the German subjunctive mood as determinative of indirect reference (1997i, 162), among many other.

On the other hand, Frege's main achievement is taken to be the revealing of a single logical form underlying various forms of natural

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[18] Contrast van Heijenoort: "Ordinary language is somehow too weak to stand the stress of bivalence and should not be asked to bear up against the requirements of logical rigidity." (1986, 41).

[19] Functions are unsaturated, therefore Frege doesn't mix concepts with objects, making concepts non-referable. Because he cannot accept a definite-article language expression not being a name, or a name not having a referent, he insists that there are such objects standing for expressions of the form "the concept 'x'" (1997b, 174–77). Frege will later review passingly a suggestion that these objects could be somehow identified with the extensions of concepts, but will make nothing of it (1997h, 187).

[20] "Here, then, we have a substitute for the definite article of language, which serves to form proper names out of concept-words" (Frege 2016, 19).

[21] "As soon as a word is used with the indefinite article or in the plural without any article, it is a concept-word" (Frege 1960, 64).

language and distinct from them. As already noted, he shows that the subject-predicate distinction does not belong to logic. He also only uses truth-functional operators, ignoring shading, that is, performative aspects of language. He famously denies that pairs like "Men are mortal" and "Cicero is mortal" are of the same logical form, i.e., he distinguishes subordination of concepts from falling of an object under a concept (1997f, 81). He also distinguishes conceptual marks, under which a concept is subordinated, and which are properties for the objects falling under the concept, from second-level properties, characterizing concepts: "The number of planets is 7" does not mean that 7 is a property of planets and a conceptual mark of `*number of planets*`, but a second-level property of `*number of planets*` (1960, 64).

Frege also defends his appeal to linguistic distinctions in "On Concept and Object" thus:

> "… my own way of [basing logical rules on linguistic distinctions] is something that nobody can avoid who lays down such rules at all, for we cannot understand one another without language, and so in the end we must always rely on other people's understanding words, inflexions, and sentence-construction in essentially the same way as ourselves." (1997h, 184)

Frege then affirms that he's not trying to give a linguistic definition to logical concepts, but only hints, appealing for that purpose to "the general feeling for the German language" (1997h, 184).

Does this mean that once apprehended, the linguistic priors of logical distinctions can be discarded as eliminable from the system? There are commentators that see Frege's mature semantics as applying only to perfect formal languages (Dummett 1996, 109), so for them the answer would be affirmative. But Frege's insistence on some linguistic devices, especially articles, is simply too strong to conform to this interpretation. Frege's constant point of equilibrium was that, for a successful logical system, the conceptual distinctions should lead to successful treatment of logical argument, i.e., results justify the distinctions made. This may be seen as akin to the Rawlsian reflective equilibrium in which logical principles and treatment of particular language contexts are balanced so that maximal explanatory output is achieved. In "Begriffsschrift", Frege rejected the judgeability of vague contents, but he did not deny that `*being a heap of beans*` is indeed a property, since it met his only criterion available: being separable in thought. As Frege develops a semantic theory and a strict theory of definition, ordinary language will come increasingly into attack, and he will constantly reject vague predicates. But Frege will also constantly employ and exemplify his arguments with ordinary-language examples, the latter never being outside his philosophical project.

After "Begriffsschrift", Frege formulates the aim of defining the concept of number and the foundations of arithmetic logically. In his first rejection of the Kantian synthetic nature of arithmetic judgements, he affirms that the realm of arithmetic is the enumerable, and the enumerable comprises anything, including:

> "… inner mental processes and events and even concepts, that stand neither in temporal nor in spatial but only in logical relations to one another. The only barrier to enumerability is to be found in the imperfection of concepts. Bald people for example cannot be enumerated as long as the concept of baldness is not defined so precisely that for any individual there can be no doubt whether he falls under it. Thus the area of the enumerable is as wide as that of conceptual thought." (1997f, 80)

Frege writes that vague predicates such as `*bald*` are not enumerable, thus being imperfect concepts. Enumerability here means at least that there should exist such a number as the number of all individuals falling under the concept. But Frege's argument seems misleading, as he accepts in *#54* of his later *Foundations of Arithmetic* that there are concepts which cannot be counted, those known as non-sortal concepts:

> "We can, for example, divide up something falling under the concept "red" into parts in a variety of ways, without the parts thereby ceasing to fall under the same concept "red". To a concept of this kind no finite number will belong." (1960, 66)

One way to preserve Frege's argument against \`bald\` is to take him speaking instead of the concept *\`bald people\`* having the appearance of a so-called sortal concept but being in fact uncountable. This is similar to how we have read Frege's soritical discussion in the "Begriffsschrift": natural-language terms embed opposite logical intuitions. Without further elaboration, it is unclear why this false appearance cannot be circumscribed. For example, why could the proposition "*A* is a bald person or *A* is not a bald person" not be true, as the supervaluationists hold, without committing oneself to the truth of any of the disjuncts (Fine 1975)? Completely excluding vague concepts from the realm of conceptual thought, as the quote above does, seems unmotivated. That being said, this kind of formulation becomes common in Frege's later work.

3. Frege's ontology and semantics

Since the characterizations of *\`function\`*, *\`argument\`*, *\`concept\`* and *\`predicate\`* are missing or incomplete in "Begriffsschrift", Frege's later works clarify them, taking functions as primary. The function will be defined, on the model of mathematical functions, as a mapping[22] of objects (first-level functions) or functions (for second-level functions) as arguments to objects as values of the function. Functions can be either one-place (monadic) or two-place (dyadic). Monadic functions that map their argument only to the truth-values (the True and the False) are concepts. Dyadic functions that map their arguments to the truth values are relations. The function (or concept or relation) is never an object, it is *unsaturated*. That is why a predicate letter is always written with at least one letter in parentheses, so as to indicate the empty places of the function. By saturation, that is, the coming together of a concept and object as argument, i.e., predication, a proposition is obtained (Frege 1997c, 130–48; 1997h, 181–93).

Frege also introduces a special kind of object standing in one-to-one correspondence with functions: the value-ranges. The reason for their introduction is Frege's Platonism, insisting that numbers are objects, leading to them being defined in terms of objects[23]. The "Foundations of Arithmetic" had introduced *'extensions'* with that role, assuming in a footnote "that it is known what the extension of a concept is" (1960, 79). This means approximately the set of all objects falling under the concept, but Frege generalizes the idea in his "Basic Laws of Arithmetic". It is tempting to see value-ranges as sets of ordered pairs containing every object in the domain and the value of the function at that object, but Frege defines ordered pairs in terms of value-ranges (Heck 2012, 10), which are introduced, controversially, by contextual definition[24].

[22] Frege cannot be said to offer this as a definition for `*function*`. His only definition is negative: that which is not object (1997c, 140).

[23] The so-called "Caesar problem", in the sense that an identity of objects should say what it is for two numbers to be identical (Frege 1960, 79).

[24] The "Basic Laws of Arithmetic" introduce value-ranges at #3 then re-examine them in #10 and #29-#32 (Frege 2016, 7). See Heck for a critical analysis (2012, 129–34).

Twelve years after the publication of "Begriffsschrift" and twelve years before the second volume of his "Basic Laws of Arithmetic", Frege splits "Begriffsschrift"'s conceptual content into *'Sinn'* (sense) and *`Bedeutung`* (reference) in his article "Function and Concept" (1997c)[25]. The reference of an expression is defined as the object or concept represented through the sense of the expression and that can have identity (for objects) or identity-like (for concepts) relations with expressions differing from it only in sense. The second approach to defining the reference is to identify it with the scientific objects or concepts underlying the expression[26]. The sense, as said, is defined as the mode of presentation[27] of the reference. This table results:

Type of expression:	*Reference:*	*Sense:*
Propositions (in direct speech)	Truth-value object: the True or the False	The thought (e.g.: what is common in different-language translations)
Names (definite descriptions and proper names)	The bearer (corresponding real object)	Hidden description (debated)
Concept-words (general expressions)	The concept (unsaturated function)	Sense of the concept-word (debated)

Frege's commitment to objectivity leads him to hypostatize the objects of the True and the False as real. It also requires expressions to find a *reference*. If they cannot do that, it means that they only have senses, they

[25] He will clarify the distinction in further works (Frege 1997i; 1997b).

[26] Based on such Fregean quotes as "The *Bedeutung* is thus shown at every point to be the essential thing for science"(1997b, 178), and "A concept-word must have a sense too and if it is to have a use in science, a *Bedeutung*." (1997b, 180).

[27] Or, for some commentators *the mode of determination* (expression found in a similar context in "Begriffsschrift"), i.e., the way by which the true meaning (reference) is to be reached (Beaney 1997, 23).

are *bedeutungslos*. Fictional names such as "Pegasus" and "Nausicäa" have senses, but no references. So do all propositions containing such names.

Therefore, Frege's ontology contains unsaturated functions and objects, the latter comprising physical objects, truth-values, numbers, and value-ranges. Arguably, for the later Frege, *thoughts* and *senses* more generally may be accepted, as he affirms their objectivity (1997j, 325–45).

Frege's semantics works towards two seemingly opposite directions. First, the context principle, postulated as the second fundamental principle of his "Foundations of Arithmetic"[28] holds that a term has meaning only inside a proposition. But Frege also argues that we can only learn language by deriving the composite meaning from the meaning of the parts[29] and he gives, in his posthumous "Notes for Ludwig Darmstaedter" a fragmentary statement of the so-called *building principle* for both sense and reference. That is, he arguably says that the sense of a complex expression is built from the senses of its parts and that the reference of a complex expression is built from the references of its parts (Frege 1997g, 364–65). Without deciding the matter, we'll note that compositionality will be one reason for Frege's rejection of vague predicates.

How is compositionality supposed to work? Take the proposition "The Earth is round". On the side of sense, one would say that the immutable thought expressed by the sentence has contributions from the senses of its parts (Frege 1997g, 364) and, also, that its sense is that the conditions under which the proposition has the Truth as reference are fulfilled (Frege 2016, 50). Those are, in a truth-conditional reading, the scientific propositions which should be true for the Earth to be round. As for the side of reference, the predicate concept-words *'is round`* refer to a

[28] "Never to ask for the meaning of a word in isolation, but only in the context of a proposition"(Frege 1960, XXII).

[29] "The possibility of our understanding propositions which we have never heard before rests evidently on this, that we construct the sense of a proposition out of parts that correspond to the words" (Frege 1997e, 320).

concept, namely a mapping from any object to the True or the False. The name "Earth" refers to one of those objects, namely the Earth. Hence the application of the reference of the predicate to the reference of its argument results beautifully in the reference of the entire proposition, namely the Truth. Obviously, \`Earth\` and \`is round\` are common natural language terms, yet we seem to have precise scientific understandings of both. Can compositionality work when applied to natural language predicates without such an understanding?

4. Frege's main objections

Having stated the distinctions above, we can now discuss the mature Frege's objections against indeterminate predicates. The best-known such fragment is in "Basic Laws of Arithmetic", #56, we can call it 'the completeness fragment'. Under the heading "Principle of completeness", Frege writes:

> "A definition of a concept (a possible predicate) must be complete; it has to determine unambiguously for every object whether it falls under the concept or not (whether the predicate can be applied to it truly). Thus, there must be no object for which, after the definition, it remains doubtful whether it falls under the concept, even though it may not always be possible, for us humans, with our deficient knowledge, to decide the question. Figuratively, we can also express it like this: a concept must have sharp boundaries. If one pictures a concept with respect to its extension as a region in a plane, then this is, of course, merely an analogy and must be treated with care, though it can be of service here. A concept without sharp boundaries would correspond to a region that would not have a sharp borderline everywhere but would, in places, be completely blurred, merging with its surroundings. This would not really be a region at all; and, correspondingly, a concept without sharp definition is

> wrongly called a concept. Logic cannot recognize such concept-like constructions as concepts; it is impossible to formulate exact laws concerning them. The law of excluded middle is in fact just the requirement, in another form, that concepts have sharp boundaries. Any object Δ either falls under the concept Φ or it does not fall under it: *tertium non datur*. Would, for example, the proposition "Every square root of 9 is odd" have any graspable sense if *square root of 9* were a concept without sharp boundaries? Does the question, "Are we still Christians", indeed have a sense if it is not determined to whom the predicate *Christian* can be truly applied and from whom it must be withheld?" (2016, 70).

Care should be taken after the first read. Frege's main purpose is not to discuss natural language reasoning, as often thought, although that is certainly important (Fine 1975, 279). It is to press the importance of complete definition in mathematics. Thus, the completeness fragment is followed by a detailed critique of *piecemeal definitions* given by Frege's contemporary mathematicians. What Frege understands by "piecemeal" is the habit of introducing and modifying new terms as one likes. He writes it

> "… consists in providing a definition for a special case – for example, for the positive whole numbers – and putting it to use and then, after various theorems, following it up with a second explanation for a different case – for example for the negative whole numbers and for Zero – at which point, all too often, the mistake is committed of once again making determinations for the case already dealt with" (2016, 70).

a) Definitions may be implicit

Therefore, the completeness fragment is a condensation of Frege's position on conceptual definition. It is followed by examples of ambiguous or misleading definitions in mathematics, exemplified with ordinary-language predicates, as an introduction to Frege's discussion of definitions given by Cantor and other mathematicians. How is this to be applied to our current use of language?

For Frege, the reason for which piecemeal definition is unacceptable in mathematics seems to be that one can define and redefine anything. But natural language may resist unprincipled redefinitions, if one assumes there are such things as linguistic norms which stops any one speaker from stipulating *'tall'* to mean whatever they want. Thus, we may suppose that speakers have some, possibly implicit, definitions of common terms, for Frege's argument to be relevant. They may acquire them on learning the language, to the same effect as the explicit – even if piecemeal – definitions of mathematical concepts. Then Frege is justified in drawing a parallel and saying that, if *`Christian'* neither applies nor does not apply to – say – a member of the Church of Jesus Christ of Latter-day Saints because of complex theological debate, it would not fit his definition of a concept. That is, mapping any object to the True or the False.

Therefore, the term *`Christian`* had been defined implicitly, yet not correctly, so it did not turn into a concept. Taken to the extreme, this implies something like Peter's Unger nihilism (2017)[30] in the vagueness debate: natural-language terms simply have no meaning (especially reference) because they do not have a correct (and consistent) definition, at least until Frege's begriffsschrift gives them such a rigorous definition (Weiner 2010).

This extreme interpretation is hard to square with Frege himself relying on natural language and, suggesting, as at the end of the fragment, that such predicates as *`Christian`* may under some circumstances be already acceptable. Implicit definitions may work differently from explicit ones. Therefore, we need to see *why* Frege insists on each (first level) concept being

[30] For a discussion on whether Frege can be read as nihilist, see also Puryear (2013, 136–37)

defined for every object as argument, and this also for natural language predicates.

b) Securing referents and compositionality

Since his distinction between sense and reference, Frege argues that science primarily needs to secure referents, to avoid blind alleys:

> "It seems to be demanded by scientific rigour that we ensure than an expression never becomes *bedeutungslos*; we must see to it that we never perform calculations with empty signs in the belief that we are dealing with objects. People have in the past carried out invalid procedures with divergent infinite series. […] What rules we lay down is a matter of comparative indifference." (1997c, 141).

The plain read of this fragment is a purely heuristic rule, to the effect that time saving in scientific work is preferable. What Frege has in mind is that a definition can introduce (or recognize) mathematical objects into being, but only if it is unambiguous: As he remarks in "Foundations of Arithmetic", there is no problem with the concept of Infinite, as long as is non-ambiguous:

> "Any name or symbol that has been introduced in a logically unexceptionable manner can be used in our enquiries without hesitation, and here our Number ∞_1 [31] is as sound as 2 or 3'" (1960, 97).

[31] This is the way Frege writes aleph-null, the cardinality of natural numbers.

And he accepts contradictory definitions as well (Frege 1960, 87).

In the case of conceptual expressions, their possible referents are concepts, that is, functions mapping objects to truth-values. And, in the completeness fragment above, Frege held that any concept-words lacking values at any object of the domain, i.e., even for ⊙ (The Sun), do not correspond to such a concept. That is, first, because Frege insists that all well-formed formulas of begriffsschrift should have exactly one referent, and that is only assured if for any Δ[32] the logical operators and functions with which it forms more complex expressions define what kind of *referent* results for the combined expression. Compositionality is thus stronger on the side of reference. Such a semantic structure should exist that indicates how the meaning (reference) of complex expressions is built from simpler forms, parallel to the syntactic structure created by the application of rules of inference. And in this semantic structure, the only contribution of parts is to the truth of the complex expression (Williamson 1994, 38). Any contextualism is incompatible with what Frege sees as scientific, namely a single domain of all objects of which any thought is immutable.

A related argument is that any lack of reference can propagate itself through the system, affecting a large number of cases. Frege points out that for any x if "x+1" is *bedeutungslos*, "x+1 = 10" will have no solution, so it will not refer to either the True or the False. It will be *bedeutungslos* as well, illustrating how concepts and functions move together (Frege 1997c, 141). If we accept the sentence "My 49-year-old uncle is a bald person" in our language, without it being true or false, then the meaning of "All bald persons are over 50 years old" is lost, because of the logical relations embedded by language. We can say that "*A* is a bald person or *A* is not a

[32] Δ is any possible object. Heck writes "The term Δ is not supposed to be a name in begriffsschrift at all: It is a formal device […] subject only to the condition that it should refer to some object in the domain" (2012, 58).

bald person" could not be true and it coulyd not be false if *`bald`* was indeterminate of *A*. That's because neither disjunct would stand for a truth-value, and *'or'* is truth-functional. Indeterminateness is extended in all directions by the rules of begriffsschrift. Only by not accepting indeterminate propositions, the law of excluded middle seems to apply; since "C⊙" is a well-formed propositional formula, then it must be either true or false. But this is not the case in contemporary supervaluationism: we can define truth as super-truth, namely truth in all worlds with full valuations: all logical truths will then be super-true, including the law of excluded middle, even when both disjuncts are indeterminate (Varzi 2007, 647; Fine 1975). Frege did not distinguish derivation (the law of excluded middle: either A or not A) from semantics (what is now called bivalence: any proposition be either true or false). Thus, even if truth-functionality extended indeterminateness, logical laws could be saved.

c) The Indeterminate and higher-order vagueness

Frege's insistence that a concept should be either true or false at *all* objects can be objected to as damaging to science: there are scientific cases where it is reasonable to reserve judgement. Division by zero is undefined. Therefore, any mathematical sentence containing division by zero is to be undefined as well. This may lead to a three-valued logic in which the truth tables will have Indeterminate for any complex expression if any of component expressions had the Indeterminate. Remark that we cannot take the disjunction of Indeterminate with its negation as true. Also, where three-valuationists like Michael Tye say that the conjunction of False with Indeterminate results in the False, we would still have Indeterminate, in order to circumscribe the Fregean scientific project (Tye 1996, 282). But it is hard to see how Frege would accept such an object as the Indeterminate: it lacks the timelessness mark of Frege's Platonism expressed in the introduction of the True and the False. Moreover, Frege tests truth-values as references of propositions by their ability to be substituted[33] by one another, and, in plain speech 'A man of 170 cm is tall' does not seem replaceable with 'A man of 300 hairs is bald', in all contexts.

Yet, a three-valued logic or supervaluationism[34] may be acceptable formally to Frege, even if anachronistic. They would conserve all the truths of the begriffsschrift at classical truth-values, and simultaneously separate all propositions in either Fregean (or classical) or indeterminate.

It is an open issue whether any theory assigning a precise truth-value, or a precise lack of truth-value can do justice to indeterminateness. As Williamson puts it, "To fail to stipulate a value is not to stipulate that there be no value" (1994, 41). Indeterminateness consists in, maybe among others, being unable to say where the border lies. This is the phenomenon currently discussed as higher-order vagueness (Williamson 1994, 2): it being indeterminate whether a case is indeterminate or not. As we shall see, Frege's metaphor of figures on a plane hints at the problem. But Frege's work contains no discussion of higher-order vagueness, and he usually speaks as if sharp boundaries are right around the corner.

In "On Sinn and Bedeutung" Frege proposes securing a reference to the `*divergent infinite series'* which was mentioned above, by stipulating it as 0. This strategy, if applied ceteris paribus to ordinary-language predicates, would eliminate higher-order vagueness up until a new object appears, when the definition should be re-worked so that it covers it. Although Frege often states that functions should have one value at any possible object, the necessary redefinition may proceed, so to speak, in bulk, or by `fields`, as he writes: "Every widening of the field to which the objects indicated by *a* and

[33] For both see (Frege 1997i, 159).
[34] For a discussion of Frege's supervaluationist bend see (Weiner 2010, 48)

b belong obliges us to give a new definition of the plus sign" (1997c, 141)[35]. There is some tension between this proposal to extend a definition by cases and his criticism of piecemeal definition cited above that arguably does the same thing without so much rigor, which tensions reinforce my claim that Frege's criticism of piecemeal definitions is heuristic. Note that piecemeal definitions, while unprincipled and prone to error, need not go wrong in all cases. Great mathematicians gave piecemeal definitions, while avoiding contradiction (van Heijenoort 1986, 34). On the other hand, as results in mathematical logic after Frege's time showed, there is no guaranteed way to extend a theory while keeping all its properties, so Frege's case-by-case approach is heuristic as well. Take Presburger Arithmetic which is decidable and complete but does not contain multiplication. The addition of multiplication will get you Peano Arithmetic - undecidable and incomplete.

In conclusion, Frege's argument against indeterminate predicates both ignores higher-order vagueness and assumes that ordinary speakers should re-negotiate their usage – across the entire linguistic community – each time a new situation or class of situations appear, which seems unrealistic[36].

d) The region metaphor and incompleteness

Let us try to fix a precise sense for "sharp boundary of concept" mentioned above by Frege in his now-famous[37] metaphor of "a concept with respect to its extension as a region in a plane".

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[35] Contrast van Heijenoort (1986, 32), where he goes against Frege's explicit words.

[36] van Heijenoort writes "…not that such an enterprise cannot be carried out, but rather that neither mathematics nor ordinary language proceeds thus" (1986, 37). I don't think it can be carried out, at least for mathematics.

[37] Boundaries are of course an ubiquitous metaphor in the debate on vagueness (Keefe and Smith 1996b)

A first interpretation may go like this. Supposedly conceptual expressions are similar to differently colored areas, so that their color difference creates the boundary. That kind of boundary does not correspond to any real object. Suppose the points constituting the plane are the objects in the domain and those objects amenable to our concept-words cluster together. That is, if our concept-word is `*tall man*`, non-persons will be to the bottom, and persons will be the points of the plane from the top left to the top right, ordered by height in centimeters. And our concept-words are supposed to pick up, i.e., to color all points corresponding to tall people. Under this extensional view, if the color fades slowly into the background as Frege has it, we could say that vague concepts would lead to vague extensions. And it would be impossible to define a set without knowing whether any object is in it or not. Yet, there is an easy way out. The contemporary fuzzy theorist works with the concept of fuzzy set. She replaces extensions with a fuzzy set where a membership function associates a number on the real interval (from 0 to 1 - a degree) with each object (Machina 1996, 180). As with the three-valued approach discussed before, it is technically feasible[38], because all valuations involving classical truth-values stay the same, even though the objection related to higher-order vagueness will apply with the same strength[39]. Since ″exact laws concerning them″ would still be possible, contrary to Frege, this would be to simply understand Frege's stance as unimaginative of this further logical development and give up on it.

Another, more charitable explanation of the metaphor, rejects our ordering of persons according to their height, since nothing in fact requires that the area of the concept be compact. Frege does not indicate that

[38] I do not affirm that the whole begriffsschrift could be conservatively extended, I mean only first-order predicate logic, excluding the second-order quantification.

[39] A common philosophical objection to fuzzy logic is that it replaces vagueness with the maximum precision (Keefe and Smith 1996a, 46).

indeterminateness is in any way regular or gradual[40]. Then, we should simply color those points for which the conceptual expression has the True as value. When we cannot do it, the blurring into the background would simply expresses our indecision, not a gradual decrease in height that could be correlated with a decreasing fuzzy value or probabilistic verity (Edgington 1996, 302). Therefore, Frege's indeterminateness may not be gradual and, as we saw above, may not to be equivalent to introducing a third truth-value, at least for those predicates displaying higher-order vagueness.

Williamson writes that Frege briefly compares blurred borders with dashed (interrupted) borders, without making much of it (1994, 279). Indeed, two types of indeterminateness should be distinguished against Frege[41]. The first is vagueness proper, characterized by higher-order vagueness, a problem Frege's writings do not address. The other is incompleteness of definition. Namely, the definition of a conceptual expression is incomplete when we have some conceptual marks of the concept-words, but we have no ground to expect either that there are no other such marks or that there are. Metaphorically, the boundaries start strict, but we have no ground to foresee how or whether they continue. I think it is best to cite Fine's example, reminiscent, as he writes, of Carnap's meaning postulates:

(a) n is nice1 if n>15

(b) n is not nice1 if n <13 (Fine 1975, 266)

Frege gave himself one example of such incompleteness of definition: the Homeric μϖλυ ("mōly"), a magic plant characterized by

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[40] Kit Fine introduced such rules under the name of "penumbral connections" or "truths on a penumbra". Vague concepts are to be governed by some rules along the lines of "any blob redder than a pinkish blob is red if the latter is" (Fine 1975).

[41] "Frege's requirement of completude is intimately connected with that of sharpness. For him, in fact, the two requirements seem to fuse into one. Countless times in his writings, we find the words 'complete' and 'sharp' conjoined" (van Heijenoort 1986, 37).

Michael Beaney as ”having a black root and a milk-white flower“ (Frege 1997b, 178). Frege says of it that it is *bedeutungslos*”, although it is true that certain marks are supplied“ (idem). Let us remark that this seems to break the definition of conceptual marks given by Frege earlier, in that the proposition ”All μϖλυ s are black rooted plants“ cannot be true as long as μϖλυ is *bedeutungslos*. What we see is that Frege refuses to take the two conceptual marks he knows as the only ones. If he did so, `*μϖλυ*` would be a concept defined for all objects[42], and thus sharply defined. Still, `*μϖλυ*` does not illustrate the same higher-order indetermination that `*bald*` does, so it would not be vague in the same sense, it is just incomplete.

A third interpretation of the region metaphor starts from the fact discussed above that definitions and redefinitions of concept-words work in bulk, based on the properties of objects. Not each one is to be assigned individually. As Williamson puts it: for Frege ”to grasp as sense is to know where its boundary runs“ (1996, 276) . But the fragment above allows that ”it may not always be possible, for us humans, with our deficient knowledge, to decide the question.“ So the sharp boundary can be identified with the conceptual distinctions known, most probably by the scientific community as a whole or ideally, possibly on an extended timescale. Frege says in his “Foundations of Arithmetic” that if the manner of determining the pieces covered by a name changes, the objectivity of the determination will not. His example is that the objectivity of the North Sea will ”not [be] affected by the fact that it is a matter of our arbitrary choice which part of all the water on the earth’s surface we mark off and elect to call the `North Sea`” (Frege 1960, 34). This means again that as long as it is unambiguous, any definition will do, even if it is not presently used by the linguistic community. Then, what the metaphor shows is that the scientific community should know now or should be able to know in the future exactly what

[42] At least assuming the concept-words expressing the two conceptual marks correspond themselves to real concepts.

separates one concept from the rest, that potential criterion being serving itself as the sharp border of the metaphor, which is now blurred.

This third interpretation turns as well against Frege. He may not believe contemporary demographics a science, but there are contemporary surveys that count the numbers of Christians per country. Since Frege is committed to science ideally, there seems to be no a priori reason why such terms as `*Christian*` cannot at one time receive a definition to serve as a sharp border, be it by self-report. Such definitions, even for our current use of `Christian` may already be available, although not yet *discovered*. As stated, to bring rigor, Frege treated heuristic issues as constitutive. So, there is no wonder that the idea of *discovering* adequate definitions of natural language predicates did not arrive at him. This can be called Frege's third heuristic: that criteria should be given, not waited for to be discovered.

e) Do indeterminate expressions have a sense?

Frege asks the final question of the completeness fragment as if it may be possible for the predicate "Christian" to have sharp boundaries. Were those missing, the proposition "Are we still Christians" would have no graspable sense. We can even strengthen Frege's stance with some examples inspired from the contemporary debate on vagueness. Think of the persons of which most cannot say, and even most of them themselves cannot say whether they are Christian; they may identify themselves as "cultural Christians", since they admire churches on the outside but never go in. Or, even though Frege himself wrote in his "Foundations of Arithmetic" that "The number belonging to the concept `inhabitant of Germany at New Year 1883, Berlin time` is the same for all eternity" (1960, 60), we may suppose some German inhabitants in train of becoming foreign residents at about 12am on the 1st of Jan 1883, German inhabitants having their last seconds of life around that time or stateless persons lost on the mountainous Swiss border[43].

Let us now ask how would the presumed existence of some objects which are neither Christian nor not-Christian affect our ability to grasp the sense of the question ”Are we still Christians“? That sense is to be a Fregean thought, and Frege tells us that even fictional thoughts can be grasped. He then may say that, since \`Christian\` is not well-defined, it does not have a sense, so the whole thought does not have a sense.

But suppose two propositions:

EP: We are Elvish

CP: We are Christian

\`*Elvish*\` is fictional, thus no object falls under the concept. Suppose \`*Christian*\` is identical with \`*Elvish*\`, except that for ⊙ and some other planets we fail to stipulate whether they fall or not under the concept. According to Frege *EP* has a sense (while being *bedeutungslos*) and so should *CP*, since the difference does not affect the thought. By this I mean that there is no connection between whether some objects are stipulated to fall under a concept or not and my grasping[44] of the words of *CP* as a mode of determination (i.e., towards whether *CP* is true or false). In his “Introduction to Logic”, Frege states that if we had a fictional thought about a mythical person whose real existence we later come to accept, ”the thoughts would strictly remain the same” (1997d, 293), which means the sense is not, at least here, a question of existing in the world. And Frege absolutely does not require the whole domain (of quintillions of objects) to be grasped as a psychological act. Therefore, contrary to the allusion of the completeness

[43] Such examples originate with Esenin-Volpin (Wright 1996, 155).
[44] “My grasping” could be read as “my linguistic community\`s grasping” and the argument would stay the same.

fragment, senses of whole propositions should be graspable, even if they contain indeterminate concept-words.

Vague concept-words having a sense is more controversial. On one hand, Frege seems to say in "Concept and Object", regarding the example of `$\mu\varpi\lambda\upsilon$` discussed above, that it has a sense just as the fictional name "Nausicäa" has one. Against that, Williamson argues that Frege's theoretical apparatus cannot accommodate concept-words with sense but without reference. In the true proposition "There is a heap of sand on most building sites", if concept-words contributed to the overall thought, the truth-value of the whole would be determined by finding the referent of those concept-words, among other conditions (Williamson 1994, 45). The way out is to observe that for Frege, the entire proposition cannot be true, since it contains concept-words without a referring concept. Thus, vague predicates could have senses that contribute to the *thoughts* of their containing propositions, as long as we're prepared to read them similarly to fictional concepts. Unfortunately, there is no apparatus in Frege's work to distinguish further the fictional from the indeterminate.

5. Conclusion

At the beginning of "Begriffsschrift", Frege believed that natural language hides contents linked logically by pure thought but also that there exist "illusions that through the use of language often almost unavoidably arise concerning the relations of concepts' (1997a, 51). While he uncovered many of those illusions by revealing a hidden logical structure beneath, Frege did not find a place for vague predicates, up to the point of taking the entire ordinary language as inconsistent on their account.

While some of Frege's arguments seem stronger than others, some objectionable consequences include the following. First, Frege's theory commits us to enigmatic objects such as the referent of "the concept `man`" or value-ranges. Secondly, he does not distinguish incompleteness of definition from vagueness as illustrated by the common contemporary issue of higher-order vagueness. Thirdly, we are seemingly forced to treat ordinary language, full of vague expressions as it is, as akin to fiction.

Finally, the researcher can invoke one of the competing contemporary theories of vagueness to paste vagueness onto Frege's ontology, and I have illustrated briefly how epistemicism, fuzzy logic, trivaluationism, and supervaluationism can do that. The upshot is that vagueness does not raise, by itself, any important objection to Frege's project, even without going non-classical. Had he not treated heuristic issues as constitutive, he could accept that there are precise yet unwieldy definitions of `*Christian*` to be discovered, that `*Heap*` associate any individual group of sand to the True and the False - even though perhaps not by a simple gradual rule (Călborean 2020, 96) - and thus, that there is no parallel between fictional and vague predications. Capturing natural language in Frege's ontology is not impossible, it is only a little difficult.

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