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First-Order Dialogical Games and Tableaux

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Abstract We present a new proof of soundness/completeness of tableaux with respect to dialogical games in Classical First-Order Logic. As far as we know it is the first thorough result for dialogical games where finiteness of plays is guaranteed by means of what we call repetition ranks.

Keywords Dialogical semantics · Tableaux · Finite dialogues · Classical logic

1 Introduction

In dialogical semantics, the meaning of expressions is given by the way they are used in an argumentative debate. The debate is designed as a game between two players: the Proponent (**P**) and the Opponent (**O**). The main aim of this paper is to present a new proof of soundness and completeness of the tableau method with respect to dialogical semantics for Classical First-Order Logic. Section 2 presents the dialogical games we are interested in. Section 3 tackles the level of strategies and presents some useful properties. The proof itself is given in Section 4.

The connection between dialogues and tableaux has been discussed in various places, and we start with some remarks on the differences between our work and these.[1] First of all we study the relation between tableaux and dialogical

[1] Felscher [1], Krabbe [5] or Rahman/Keiff [11].

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games by considering the latter from the point of view of their extensive form. In Krabbe [5] and Rahman/Keiff [11], strategies of the Proponent are conceived as trees where material implication is treated as in tableaux. In particular, material implications played by the Opponent trigger ramifications in **P**-strategies. See [5, p. 308] and [11, p. 374]. But according to the local rule for material implication (see Section 2.1) the challenge by the Proponent and the defence by the Opponent are played within the same play (sequence of moves). Hence in the extensive form of the game they belong to the same branch. This difference was pointed out in Rahman et al. [10, p. 314–315], and the way to overcome it is given in our proof.[2]

More importantly, we use *repetition ranks* in the rules of dialogical games. Repetition ranks are positive integers bounding the number of challenges and defences which the players can perform in a play. The main motivation for using these is that it is one of the philosophical tenets of the dialogical theory of meaning that plays are finite: see Lorenzen/Lorenz [9] and Lorenz [8, p. 258]. It is of course possible to question this tenet and to accept infinite dialogues, for example for the sake of generality. Consider for example Felscher [1, 2], where the Proponent can trigger infinite dialogues because the number of challenges he can perform is not limited.[3] By contrast the number of authorized **O**-challenges is always bounded.[4] But this asymmetry in the rules—which by the way triggers an asymmetry in the winning conditions—is motivated by considerations pertaining to the level of strategies.[5] The point is that it does not change anything about the existence of winning strategies for **P**, and as long as one is only interested in this aspect the considerations show that it is possible to work with this restriction on authorized **O**-moves. Thus, in our view, there are two drawbacks in Felscher's formulation: finiteness of plays is thrown away, and the level of rules—which is about what the players can do—is not clearly separated from the level of strategies—which is about how the players should or do play. With repetition ranks we guarantee that any play is finite and we put the same constraints on the players. And we do so regardless of strategic considerations: the rule says nothing about which rank the players should choose.

Putting the focus on winning strategies aside, Felscher's rules lead to curious games where the debate about a contingent formula can be infinite, for example simply by challenging again and again a conjunction. And the situation is arguably even worse in Felscher's dialogues for Classical Logic where the number of defences **P** can perform is not limited either so that he can for example defend infinitely many times a contradiction such as $\exists x(Px \wedge \neg Px)$ (by changing or repeating infinitely many times the individual constant he uses to answer the challenge). Even if doing so cannot lead **P** to victory, it does not seem very convincing to have

[2] Actually Felscher [1, Section 4] noticed the difference and proposed a "deformation operation" to overcome it. But this operation is defined for dialogues where the E rule constraints the order of the Opponent's moves. By contrast, there is no such constraint in the dialogical games as we define them.

[3] It should be emphasized that this is of no harm on the Equivalence result proven in Felscher [1] because it includes finiteness of the play in the conditions for **P**'s victory.

[4] By rule D13 or its liberalized version D13$_k$ [1, p. 227].

[5] See Section 3.6 in Felscher [2] and in particular p. 359.

games of argumentation where one can play indefinitely after uttering a contradiction. An other advantage of repetition ranks is to get rid of such undesirable phenomena.

The device of repetition ranks comes from Lorenz [7] and is mentioned in Krabbe [5]. In Lorenz's setting each player is submitted to two, possibly different, repetition ranks *n* and *m*: one for challenges and the other for defences.[6] More importantly repetition ranks are, in Lorenz's original setting,[7] set at the level of structural rules. Thus, each combination of values for *n* and *m* defines a set of dialogical rules and one needs either to consider all these sets of rules or to use an auxiliary result[8] when proving soundness and completeness.

We propose a simpler account of repetition ranks. We do not use different ranks for challenges and defences,[9] and repetition ranks are not to be set at the level of the rules: their values are chosen by the players. A first consequence is that we don't need to add priority criteria over the rules: in Lorenz's formulation such criteria are needed, in particular for intuitionistic dialogues where the *Last Duty First* rule interferes with the repetition rule. This also means that we start with a fixed set of rules so that we can focus only on the level of strategies to prove most results.

2 Dialogical Games

First-Order dialogical games are defined by *local* and *structural* rules. We come to *strategies* in the next Section.

2.1 Basic Definitions and Local Rules

We let $\mathcal{L}$ denote a first-order language (without equality) where every term is either a variable or an individual constant. A *move* is of the form $\mathbf{X}\, e$, where $\mathbf{X} \in \{\mathbf{O}, \mathbf{P}\}$ and e is of one of the following form:

- Assertion: $!\varphi$, where φ is a sentence of $\mathcal{L}$.
- Request: $?[A_0, \ldots, A_n]$, where each A_i may be an assertion or a request.
- $(\mathtt{n} := \mathtt{r}_i)$; $(\mathtt{m} := \mathtt{r}_j)$, with $\mathtt{r}_i, \mathtt{r}_j \in \mathbb{N}^*$.

The illocutionary forces of assertion and request are denoted by the symbols '!' and '?'. These are the basic kinds of speech acts at stake in dialogues. The third possible

[6] Lorenz used this distinction for his claim that Intuitionistic Logic is characterized by the smallest repetition rank (for defences) because the so-called *Last Duty First* rule of intuitionistic dialogues bounds the possible number of defences to one.

[7] It seems that in more recent works Lorenz's account of repetition ranks came closer to the one we introduce. See Lorenz [8, p. 260].

[8] Such as the one mentioned in Krabbe [5, p. 324] or Lorenz [7, pp. 86–87].

[9] Because it is not needed in the cases we consider, not even to formulate the distinction between Classical and Intuitionistic Logic. See Section 2.2.

form for e represents choices of repetition ranks among the positive integers (see the structural rules).

Local (or *particle*) rules are triples of moves which provide the *local* semantics. They show how assertions can be challenged and defended. The description is abstract in the sense that it is independent from particular states of the game and also from the identity of the players. This is why they are formulated with the variables **X** and **Y** standing for **O** and **P**, with $\mathbf{X} \neq \mathbf{Y}$. The local rules are given in Fig. 1, where $\varphi(x/a_i)$ denotes the result of replacing every occurrence of x in φ by the individual constant a_i.

The notation in the rule for material implication is a bit uncommon, but the rule is actually the same as usual. We have added the request '$?[!\psi]$' between parentheses in the challenge in order to make explicit the fact that **X** is expected to assert the consequent when defending. If we don't keep this expectation in mind, it is arguably not clear why asserting the consequent should count as a defence at all.

Here are some additional definitions. A *play* is a sequence of moves which complies with the game rules. The *dialogical game* for a sentence φ is the set $\mathcal{D}(\varphi)$ of all plays with φ as the thesis (see SR0 below).

For every move M in a given sequence Σ of moves, $\mathrm{p}_\Sigma(M)$ denotes the position of M in Σ. Positions are counted starting with 0. We also use below a function F, where the intended interpretation of $\mathrm{F}_\Sigma(M) = \left[m', Z\right]$ is that in the sequence Σ, the move M is a challenge (if $Z = C$) or a defence (if $Z = D$) against the move of previous position m'.[10]

2.2 Structural Rules

Structural rules provide the global level of a dialogical semantics. They give the conditions for a sequence of moves to be a play in a given dialogical game. The rules define the way a play starts (SR0), how the players can play (SR1 and SR2) and how the winner of a play is decided (SR3).

SR0 - Starting rule Let φ be a complex sentence of $\mathcal{L}$.[11] For any play $\Delta \in \mathcal{D}(\varphi)$ we have:

(i) $\mathrm{p}_\Delta(\mathbf{P}\,!\varphi) = 0$,
(ii) $\mathrm{p}_\Delta(\mathbf{O}\,\mathtt{n} := \mathtt{r}_1) = 1$ and $\mathrm{p}_\Delta(\mathbf{P}\,\mathtt{m} := \mathtt{r}_2) = 2$.

Any play in $\mathcal{D}(\varphi)$ starts with **P** asserting φ (the *thesis*). Then the two players choose their repetition ranks among the positive integers.

SR1 - Classical Development rule

– For any move M in Δ such that $\mathrm{p}_\Delta(M) > 2$ we have $\mathrm{F}_\Delta(M) = \left[m', Z\right]$ where $Z \in \{C, D\}$ and $m' < \mathrm{p}_\Delta(M)$.

[10] This is inspired by and adapted from Felscher [1].
[11] The reason why atomic sentences are not included is related to SR2.

Assertion	$\mathbf{X}\,!\varphi \vee \psi$	$\mathbf{X}\,!\varphi \wedge \psi$	$\mathbf{X}\,!\varphi \rightarrow \psi$	$\mathbf{X}\,!\neg\varphi$
Challenge	$\mathbf{Y}\,?[!\varphi, !\psi]$	$\mathbf{Y}\,?[!\varphi]$ or $\mathbf{Y}\,?[!\psi]$	$\mathbf{Y}\,!\varphi\,(?[!\psi])$	$\mathbf{Y}\,!\varphi$
Defence	$\mathbf{X}\,!\varphi$ or $\mathbf{X}\,!\psi$	$\mathbf{X}\,!\varphi$ resp. $\mathbf{X}\,!\psi$	$\mathbf{X}\,!\psi$	$--$

Assertion	$\mathbf{X}\,!\forall x\varphi$	$\mathbf{X}\,!\exists x\varphi$
Challenge	$\mathbf{Y}\,?[!\varphi(x/a_i)]$	$\mathbf{Y}\,?[!\varphi(x/a_1), \ldots, !\varphi(x/a_n), \ldots]$
Defence	$\mathbf{X}\,!\varphi(x/a_i)$	$\mathbf{X}\,!\varphi(x/a_i)$

Fig. 1 Local rules

– Let $\mathtt{r}$ be the repetition rank of Player **X** and $\Delta \in \mathcal{D}(\varphi)$ such that:
  – the last member of Δ is a **Y**-move,
  – $M_0 \in \Delta$ is a **Y**-move of position m_0,
  – there are n moves $M_1, \ldots, M_n$ of player **X** in Δ such that $\mathrm{F}_\Delta(M_1) = \mathrm{F}_\Delta(M_2) = \ldots = \mathrm{F}_\Delta(M_n) = \left[m_0, Z\right]$ with $Z \in \{C, D\}$.

  Let N be an **X**-move such that $\mathrm{F}_{\Delta^\frown N}(N) = \left[m_0, Z\right]$. We have $\Delta^\frown N \in \mathcal{D}(\varphi)$ if and only if $n < \mathtt{r}$.

After the repetition ranks have been chosen, every move is a challenge or a defence of a previous move; players move alternately, and the number of challenges and defences they can perform in reaction to a same move is bounded by their repetition ranks.[12]

SR2 - Formal rule The sequence Δ is a play only if the following condition is fulfilled: if $N = \mathbf{P}\,!\psi$ is a member of Δ, for any atomic sentence ψ, then there is a $M = \mathbf{O}\,!\psi$ in Δ such that $\mathrm{p}_\Delta(M) < \mathrm{p}_\Delta(N)$.

In other words: **P** can assert an atomic sentence ψ only if **O** asserted it beforehand. For our last structural rule we need the following definition:

Definition 1 Let Δ be a play in $\mathcal{D}(\varphi)$ the last member of which is an **X**-move. If there is no **Y**-move N such that $\Delta^\frown N \in \mathcal{D}(\varphi)$ then Δ is said to be **X**-terminal.

SR3- Winning rule for plays Player **X** *wins* a play $\Delta \in \mathcal{D}(\varphi)$ if and only if Δ is **X**-terminal.

By Rule SR1, repetition ranks ensure that any play is of finite length.[13] However there are infinitely many possible plays in a given dialogical game. This is because players have infinitely many possible choices for repetition ranks and also for individual constants when applying the rules for quantifiers. This feature is very important to study the connection between first-order dialogical games and first-order tableaux.

[12]Intuitionistic dialogical games are defined simply by modifying SR1 so that the repetition ranks bounds only the number of challenges, and players can defend only once against the last non-answered challenge. In this way there is no interference with other rules and thus no need for priority criteria over structural rules.

[13]By bounding the total number of challenges and defences, and not only the identical ones.

Definition 2 Let φ be a complex sentence of $\mathcal{L}$. The *extensive form* $\mathfrak{E}(\varphi)$ of $\mathcal{D}(\varphi)$ is simply the tree representation of the game. Nodes are labelled with moves so that the root is labelled with the thesis, paths in $\mathfrak{E}(\varphi)$ are linear representations of plays and maximal paths represent terminal plays in $\mathcal{D}(\varphi)$.

The extensive form of a dialogical game is thus an infinitely generated tree where each branch is of finite length. It is then obvious that the connection with tableaux is not at the level of plays, since first-order tableaux are finitely generated with possibly infinite branches.

3 Strategies

It is at the level of strategies—and more particularly of winning strategies for the Proponent—that dialogical games meet tableaux.

Definition 3

(a) A *strategy* of player **X** in $\mathcal{D}(\varphi)$ is a function s_x which assigns a legal **X**-move to every non terminal play $\Delta \in \mathcal{D}(\varphi)$ the last member of which is a **Y**-move.
(b) An **X**-strategy is *winning* if it leads to **X**'s win no matter how **Y** plays.

A strategy for a player prescribes the way to play when it is this player's turn to move, provided he can play (he has at least one possible move).

Definition 4 Let s_x be a strategy of player **X** in $\mathcal{D}(\varphi)$ of extensive form $\mathfrak{E}(\varphi)$. The extensive form of s_x is the fragment $\mathfrak{S}_x$ of $\mathfrak{E}(\varphi)$ such that:

1. The root of $\mathfrak{E}(\varphi)$ is the root of $\mathfrak{S}_x$,
2. For any node t which is associated with an **X**-move in $\mathfrak{E}(\varphi)$, any immediate successor of t in $\mathfrak{E}(\varphi)$ is an immediate successor of t in $\mathfrak{S}_x$,
3. For any node t which is associated with a **Y**-move in $\mathfrak{E}(\varphi)$, if t has at least an immediate successor in $\mathfrak{E}(\varphi)$ then t has exactly one immediate successor in $\mathfrak{S}_x$ namely the one labelled with the **X**-move prescribed by s_x.

Ramifications representing different possible moves for **X** in $\mathfrak{E}(\varphi)$ are not preserved: for each such ramification, s_x selects exactly one move for **X** to play. On the contrary, **Y**-ramifications are preserved since by definition s_x prescribes a move for each of these **Y**-choices.

We now present various interesting properties which will be useful further on. We only give sketches of proofs for them because our main concern is the proof in Section 4. First, dialogical games are determined in the following sense:

Proposition 1 *Let φ be any sentence from $\mathcal{L}$. There is a winning **X**-strategy in $\mathcal{D}(\varphi)$ if and only if there is no winning **Y**-strategy in $\mathcal{D}(\varphi)$.*

Proof Dialogical games are *well-founded games*. There are infinitely many possible choices of repetition rank for the players but no unique upper bound for the length of all the plays in a game. Hence the Gale–Stewart Theorem applies to dialogical games.[14] The Gale–Stewart Theorem in turns implies Proposition 1. □

By $\mathcal{D}^1(\varphi)$ we mean the sub-game of $\mathcal{D}(\varphi)$ triggered by **O** choosing rank $\mathtt{1}$: the plays in $\mathcal{D}^1(\varphi)$ are the plays of $\mathcal{D}(\varphi)$ with initial segment $\big(\mathbf{P}\,!\varphi, \mathbf{O}\,\mathtt{n} := \mathtt{1}\big)$. Our next Proposition states that it is necessary and enough to consider the case where **O** chooses rank $\mathtt{1}$.

Proposition 2 *Let φ be any sentence from $\mathcal{L}$. There is a winning* **P***-strategy in $\mathcal{D}(\varphi)$ if and only if there is a winning* **P***-strategy in $\mathcal{D}^1(\varphi)$.*

Proof The "necessary" part is straightforward: if there is a way for the Proponent to win no matter how the Opponent plays, then there is one in particular when the Opponent chooses rank $\mathtt{1}$.

Because of Proposition 1, the "enough" part amounts to the idea that if **O** cannot win when she chooses rank $\mathtt{1}$ then she cannot win at all. The reason for this is that the Opponent chooses her repetition rank before the Proponent chooses his, and that the Opponent has no particular restriction (contrary to the Proponent who is subjected to the Formal Rule). This means that a repetition rank bigger than $\mathtt{1}$ does not give the Opponent possibilities she does not have with rank $\mathtt{1}$. □

Finally, the following property of extensive forms of winning **P**-strategy will be particularly useful in the next Section.

Proposition 3 *Let φ be a complex sentence of $\mathcal{L}$ such that there is a winning* **P***-strategy s_p in $\mathcal{D}(\varphi)$, and $\mathfrak{S}$ the extensive form of s_p.*

(*a*) *Every leaf of $\mathfrak{S}$ is labelled with a* **P***-assertion of an atomic sentence.*
(*b*) *Dialogical closure: for every branch $\mathcal{B}$ of $\mathfrak{S}$, there is an atomic sentence ψ such that $\mathcal{B}$ contains the moves* $\mathbf{O}\,!\psi$ *and* $\mathbf{P}\,!\psi$.

Proof Statement (b) follows from (a) by virtue of rule SR2. As for statement (a), we know by Definition 3 that every leaf of $\mathfrak{S}$ is labelled with a **P**-move M. Suppose M is not an assertion of an atomic sentence. Then it is either an assertion of a complex formula or a request. But in the first case **O** would be allowed by the rules to challenge the complex formula and in the second case **O** would be allowed to defend against the challenge, which would mean that the play is not terminal. Contradiction. □

[14] See Gale–Stewart [4].

4 Soundness and Completeness Result

In this Section we prove our main result:

Theorem 1 *Let φ be a complex sentence of $\mathcal{L}$. There is a winning* **P***-strategy in $\mathcal{D}(\varphi)$ if, and only if, there is a tableau proof of φ.*

The proof is given as an algorithm to go from a tableau proof to the extensive form of a winning **P**-strategy and back.

4.1 Forewords on Tableaux

The tableau generation rules are formulated with signed formulas as in Smullyan [12], except that we use **O** and **P** in place respectively of T and F.

- α formulas: $\frac{\alpha}{\alpha_1}$ $\quad\quad$ $\frac{\alpha}{\alpha_2}$
- β formulas: $\frac{\beta}{\beta_1 \mid \beta_2}$
- γ formulas: $\frac{\gamma}{\gamma(c)}$
  where c is any individual constant
- δ formulas: $\frac{\delta}{\delta(c)}$
  where c is a new constant

These are the standard generation rules for first-order logic with the flexible version for α formulas.[15] We use Smullyan's unifying notation. As usual, we say that a tableau T is atomically closed if every branch of T contains a pair of nodes labelled $\mathbf{P}\,\psi$, $\mathbf{O}\,\psi$, for some atomic ψ.

First-order tableaux can be infinite because of the rule for γ formulas, and a systematic procedure is required to guarantee that a tableau "supposed" to close actually closes if built according to this procedure. The procedure we use is a variant of the ones given in Smullyan [12] or Letz [6, pp. 150–151]: γ formulas are dealt with in the same way, after the other types of formulas have been used. But we give different priorities between the other types of formulas:

- β formulas get priority over δ formulas which have priority over α formulas, and γ formulas come last.[16]
- Stop only if (i) an atomically closed tableau is obtained and (ii) there is no **P**-complex formula to which a rule is not applied.

We borrow from Fitting the notion of *quantifier depth* of a tableau. Originally the quantifier depth is a positive integer Q bounding the number of possible applications

[15] Letz [6, Definition 74 and footnote 5].

[16] If there are several formulas of same type, priority is given according to complexity.

of the γ rule in a tableau construction. See Fitting [3, pp. 162–163]. From now on we use the term "quantifier depth" more particularly for the total number of applications of the rule for γ formulas in a closed tableau. In Section 4.2 we relate this notion to repetition ranks as part of our algorithm to go from tableau proofs to winning **P**-strategies.

We can now prove Theorem 1. The basic idea of the proof is to understand the players' identities as the tableau signatures, which is why we formulated tableaux with **P** and **O** as signatures. However, there are some complications triggered in particular by the fact that ramifications represent very different things in tableaux and in extensive forms.

4.2 Soundness

We show that if there is a tableau proof for φ then there is a winning **P**-strategy in $\mathcal{D}^1(\varphi)$, so that we can conclude by Proposition 2 that there is a winning **P**-strategy in $\mathcal{D}(\varphi)$.

The idea is to show how the extensive form of a winning **P**-strategy in $\mathcal{D}^1(\varphi)$ can be obtained from a systematic closed tableau for φ. After some transformations, we obtain a tree that represents a set of plays which can be seen as a fragment of an extensive form of a **P**-strategy. After some generalisations we get a full extensive form of a winning **P**-strategy.

4.2.1 Step A. Repetition Ranks and Requests

We first add nodes labelled with expressions which do not occur in tableaux, namely repetition ranks and requests.[17]

We add to $\mathcal{T}$ a node t labelled with "**O** $\mathtt{n} := \mathtt{1}$" as the immediate successor of the root. Indeed the tableau rules for **O**-cases can be read as observing the structural rule SR1 in the case where **O** chooses rank $\mathtt{1}$: during the generation of a tableau, no repetition can occur in applications of **O**-rules.[18]

As for the choice of **P**'s repetition rank, let $\mathtt{r}$ be a positive integer such that if $Q \leq 2$ then $\mathtt{r} = 2$ and if $Q > 2$ then $\mathtt{r} = Q$.[19] We add a node t' labelled with "**P** $\mathtt{m} := \mathtt{r}$" as the immediate successor of t. After that the resulting tree obeys the Starting Rule SR0.

Now dialogues feature requests whereas tableau rules never do. Indeed, one of the differences between tableaux and strategies in dialogical games is that there is no interaction in the former: simply replacing the signatures T and F by **O** and **P** is not enough for a tableau to become a strategy in a game. Our next task is thus to add nodes labelled with requests to $\mathcal{T}$ while observing the dialogical rules. This is

[17] We also add the symbol '!' for assertions.

[18] With the notable exception of **O**-implications which are dealt with in Step C.

[19] The quantifier depth of $\mathcal{T}$ is the total number of applications of the γ rule in $\mathcal{T}$. This is more than enough for **P** to play the needed number of challenges and defences against universally and existentially quantified sentences. But it is also obvious that **P**'s repetition rank should most of the time be at least $\mathtt{2}$ so that he can efficiently challenge conjunctions or defend disjunctions.

done in the following way: a request performed by **O** is inserted immediately after the challenged **P**-formula, and a request by **P** is performed immediately before the relevant **O**-defence.

4.2.2 Step B. Formal Rule

The systematic construction procedure for tableaux does not ensure that $\mathcal{T}$ observes the formal rule and our second task is to remedy this. We say that a node t in a formula tree is *weakly contentious* if it is labelled with an atomic **P**-formula χ and no predecessor of t is labelled with $\mathbf{O}\,!\chi$. We say that t is *strongly contentious* if in addition none of its successors is labelled with $\mathbf{O}\,!\chi$. Going top-down (and from the left to the right) in $\mathcal{T}$, we apply the following reordering operation for every weakly contentious node t. For every branch θ passing through t:

- if $\mathbf{O}\,!\chi$ is labelled to a successor t' of t in θ, then move t as the immediate successor of t'.
- if there is no successor t' of t in θ which is labelled with $\mathbf{O}\,!\chi$, then move t as the last member of θ.[20]

N.B.: if t has two immediate successors the ramification is simply moved to the immediate predecessor of t. Once there is no remaining weakly contentious node, we remove the strongly contentious nodes. We obtain a tree named D which by construction now also observes SR2. Moreover we have:

Proposition 4 *Every leaf of* D *is labelled with a* **P***-signed atomic formula.*

Proof Every leaf of the starting tableau $\mathcal{T}$ is labelled with a signed atomic formula producing atomic closure. If it is a **P**-signed formula, then it is left unmoved by the procedure we just gave. If it is an **O**-signed atomic formula, then the corresponding **P**-signed atomic formula is moved to become the leaf. □

4.2.3 Step C. Opponent's Implications

The tree D still does not observe the local rule for material implication in the **O**-case. The tableau rule for **O**-material implications is the following:

$$\frac{\mathbf{O}\,\varphi \rightarrow \psi}{\mathbf{P}\,\varphi \mid \mathbf{O}\,\psi}$$

In dialogical terms, the left-hand successor is the **P**-challenge against the material implication and the right-hand successor is the corresponding **O**-defence. But in the dialogical setting, the challenge and the defence are never made in different plays: they always belong to a same branch in extensive forms. This illustrates the fact that

[20] This does not make t strongly contentious: it simply means that the successor of t labelled with $\mathbf{O}\,!\chi$ is not in θ but in another branch passing through t.

ramifications are to be understood very differently in tableaux and in extensive forms: in the latter ramifications represent the different possibile moves for a player whereas in the former ramifications are used to consider only the cases which are relevant to prove (or refute) a formula.

We need to transform D so that it observes the rule for material implication. Take the (leftmost) maximal[21] pair of nodes t_1 and t_2 in D with $\ell(t_1) = \ell(t_2)$ such that t_1 is labelled with a **P**-formula and t_2 is labelled with an **O**-formula. We call $D(t_1)$ and $D(t_2)$ the sub-trees of D with respective roots t_1 and t_2.

For any node $t \neq t_2$ in $D(t_2)$, we say that t is *superfluous* if it is not labelled with an atomic **P**-formula and (1) it is the result of applying a sequence of rules starting from a predecessor s of t_2 in D and (2) there is a node t' in $D(t_1)$ labelled with the same expression which is also the result of applying this sequence of rules starting from s. Let $D'(t_2)$ be the tree obtained by removing from $D(t_2)$ each of its superfluous nodes. "Cut" $D(t_2)$ from D and "paste" $D'(t_2)$ as an extension of every branch of $D(t_1)$.[22]

If relevant, we apply this operation to all such pairs of the same level, starting with the leftmost one, etc. This concludes the first stage. Once this is done, we repeat the same procedure, going our way bottom-up, until there is no such nodes t_1 and t_2.

Since there are finitely many branches in D, this procedure terminates after a finite number of stages. We call the resulting tree $\mathcal{A}$. At each stage, the leafs of $D(t_2)$ are left unchanged. Therefore Proposition 4 holds for $\mathcal{A}$. We also have:

Proposition 5 *Every branch of $\mathcal{A}$ represents a* **P***-terminal play.*

Proof We must show that each branch of $\mathcal{A}$ actually represents a play and that each of these plays is **P**-terminal. By construction $\mathcal{A}$ observes the local rules as well as SR0 and SR2. Now since at each stage of Step C the node t_2 is moved to be a successor of t_1, we get that in every branch each node except the three first is labelled with a move which is a challenge or a defence of some previous move in the branch. Finally, the way we have settled the values of the players' repetition ranks ensure that the last part of SR1 is also observed in $\mathcal{A}$.

By Proposition 4 we know that every leaf is labelled with a **P**-move. But we have to show that there are no further possible moves for **O** to ensure that the branches represent terminal plays. Notice first that by the systematic construction procedure of $\mathcal{T}$ and Step A there is no unchallenged **P**-complex formula. Moreover, because of the way **P**-requests have been added in Step A and by the transformation we have just made on **O**-implications, there is no unanswered **P**-challenge. Since the Opponent's repetition rank is `1`, there is nothing more she can do. Hence, the play is **P**-terminal. □

[21] By this we mean "of maximal level", that is, as low as possible in the tree.

[22] Nodes labelled with atomic **P**-formulas may belong to $D(t_2)$ as the result of the previous Step of the procedure, and we cannot let them be considered as superfluous and removed.

4.2.4 Step D. Order of the Opponent's Moves

In general, a tree obtained after Steps A to C is not the extensive form of a **P**-strategy yet, because all the possible ways for the Opponent to play are not considered. Our next task is to take into account the different possible orders in which **O** can play her moves.

So far we have obtained a binary tree where ramifications represent when relevant the various ways the Opponent can challenge a given formula or defend against a given challenge. But the dialogical rules allow a player to postpone a move if he has other possible moves. This occurs as soon as a player can choose between challenging and defending. Now since her repetition rank is `1` in the trees we are interested in, there is only one situation where **O** can choose between challenging and defending: after **P** has challenged an **O**-implication. Indeed at this particular stage of a play the Opponent can either defend by asserting the tail of the implication or challenge the head. We call the node labelled with the **P**-challenge an *order-decision* node for **O**. Notice that each **P**-move after an order-decision node for **O** is also an order-decision node since the Opponent can then choose between the move she delayed so far and another. We must enrich our tree $\mathcal{A}$ so that such choices for the Opponent are represented.

The branches in $\mathcal{A}$ are sequences of moves. We consider the sequences which contain a **P**-challenge against an **O**-implication. For any such sequence σ, let t be its member labelled with the first such **P**-challenge in the sequence. Consider first the permutations σ' of σ such that:

– The initial proper segment of σ of which t is the last member is an initial proper segment of σ',
– Dialogical rules are observed: σ' represents a play.

If several permutations feature the same order for the Opponent's moves, we keep only one such permutation. The sequences we keep are called **O**-permutations. We then consider the set

$$\mathcal{A} \cup \{\sigma \mid \sigma \text{ is an } \mathbf{O}\text{-permutation of a branch of } \mathcal{A}\}.$$

The tree representation of this set is denoted as $\mathfrak{A}$. Notice that Proposition 5 holds for $\mathfrak{A}$ because of the requirement that **O**-permutations are plays: an **O**-permutation of a **P**-terminal play which leaf is not labelled with an atomic **P**-formula is not a play.

4.2.5 Step E. Full Liberalization for Constants

Finally, we have to add all the possible choices of individual constants by the Opponent.

For any branch $\mathcal{B}$ of $\mathfrak{A}$, we say that $\mathcal{B}'$ is an *alphabetic-variant* of $\mathcal{B}$ if $\mathcal{B}'$ is the result of uniformly substituting some individual constant chosen by **O** in $\mathcal{B}$ with a different individual constant. We let $\mathfrak{S}$ denote the tree obtained by enriching $\mathfrak{A}$ with all the branches which are alphabetic-variants of some branch of $\mathfrak{A}$. Since the language contains infinitely many individual constants, we obtain a tree with infinitely many branches. By construction, $\mathfrak{S}$ is the extensive form of a **P**-strategy and it is straightforward that Proposition 5 holds for $\mathfrak{S}$.[23] Hence we have obtained the extensive form of a winning **P**-strategy for φ from a tableau proof of φ, which concludes the proof of soundness.

4.3 Completeness

We now show how a tableau proof for φ can be extracted from the extensive form of a winning **P**-strategy in $\mathcal{D}(\varphi)$. Unsurprisingly, the procedure is the converse of the one we just gave. Using Proposition 2, we start with the extensive form $\mathfrak{S}$ of a winning **P**-strategy in $\mathcal{D}^1(\varphi)$.

Our first task is to get rid of the branches of $\mathfrak{S}$ which are not relevant for tableau closure. After that we deal with **O**-material implications so that the tableau rule for them is observed. We must ensure that dialogical closure is preserved throughout the process so that the tableau we obtain is atomically closed.

4.3.1 Step A. Infinite Ramifications

Whenever she is to challenge a **P**-signed universal formula or defend an **O**-signed existential formula, the Opponent can choose any individual constant. In the extensive form of a **P**-strategy, this is represented as an infinite ramification: a branch is open for each possible choice of individual constant by **O**. But in tableaux it is well known that it is enough, when we are interested in (atomic) closure, to require the individual constant to be new in the rules for δ formulas.

To overcome this difference, Felscher [1, 2] uses the notion of *eigenvariable* to define *skeletons* of strategies. We use a procedure which is related to Krabbe's Lemma 2 [5, p. 312] instead. The point is that if the Proponent can win no matter how the Opponent plays, then he can win in particular when **O** always chooses new individual constants. We thus simply have to retain such choices from $\mathfrak{S}$. We do so in the following way.

Call a node in $\mathfrak{S}$ *critical* if it has infinitely many immediate successors. For each critical node t in $\mathfrak{S}$, we partition the set $S(t)$ of its immediate successors in the following way:

1. For each predecessor t' of t labelled with a formula of the form **O**-$\exists x\psi$, we form the set $S_{t'}$ of all the immediate successors of t which result from applying the local rule for $\exists$ to t',
2. For each predecessor t'' of t labelled with a formula of the form **P**-$\forall x\psi$, we form the set $S_{t''}$ of all the immediate successors of t which result from applying the local rule for $\forall$ to t'',

[23] The possibility to add alphabetic-variants without changing anything in terms of existence of winning **P**-strategies is the reason why some works add as a condition that **O** always chooses *new* individual constants.

3. Finally, the set $S'(t)$ is the set of all the immediate successors of t which result from the application of any other local rule.

Because t has finitely many predecessors, there are finitely many sets of the first kind and finitely many sets of the second kind, although each of them is of infinite cardinality. Moreover, the set $S'(t)$ is finite. The idea is to select exactly one member—if any—of each set of the first and second kinds and to retain all the members of $S'(t)$—if any. All the other immediate successors are dropped and the branches they open are removed from $\mathfrak{S}$. By doing so, we obtain a fragment of $\mathfrak{S}$ where each node has finitely many immediate successors.

Moreover, we require that the retained members of sets of the first and second kind are nodes where the individual constant is new, so that the tableau rule for δ formulas is observed.

It is straightforward that the resulting tree—call it $\mathcal{S}$—is dialogically closed since we have removed most of the branches of $\mathfrak{S}$ but we have left the remaining branches unchanged: all the branches in $\mathcal{S}$ are branches in $\mathfrak{S}$.

4.3.2 Step B. Order of the Opponent's Moves

Many of the branches of $\mathcal{S}$ represent plays which are similar modulo the order in which **O** plays her moves. That is, the tree at hand represents the union of several strict partial orders on the same multiset of moves. As far as atomic closure of tableaux is concerned, only one of these orders is relevant. Thus, we extract from $\mathcal{S}$ one of its fragment S representing exactly one of these orders.

Again, no matter which fragment S we choose, each of its branches is also a branch of $\mathcal{S}$ and S is thus dialogically closed. We can freely select a fragment and choose one where, for each **P**-challenge against an **O**-implication, the Opponent starts by challenging the head and asserts the tail only when she has no other possible move.[24]

4.3.3 Step C. Opponent's Implications

We first delete from S every occurrence of the assertion sign, and all the nodes labelled with repetition ranks and requests, without changing the relative order of the remaining nodes.

Let us denote the resulting tree as D. It is obvious that D observes all the tableau generation rules of Section 3.1 except one. First, we have ensured that the tableau rules for δ formulas is observed in Step A. As for the other types of formulas, there is a one-to-one correspondence between the local rules and the tableau rules except, as we have already discussed, in the case of **O**-implications. Thus, it is necessary to device a procedure to transform D into a tableau by dealing with **O**-implications.

Take the leftmost branch $\mathcal{B}$ of the tree at hand containing a **P**-challenge against an **O**-implication. Take the first such challenge, call the node labelled with it t_1

[24] This makes it easier to deal with **O**-implications afterwards.

and its immediate predecessor t_0. The node labelled with the relevant **O**-defence is t_3 and its immediate predecessor is t_2. Let $\mathcal{B}_0$ denote the proper initial segment of $\mathcal{B}$ the last member of which is t_0. The branch $\mathcal{B}$ thus can be written as $\mathcal{B} = \langle \mathcal{B}_0, t_1, \sigma_1, t_2, t_3, \sigma_2, t_n \rangle$, where σ_1 and σ_2 stand respectively for the sequence of moves between t_1 and t_2 and the one between t_3 and the leaf t_n.

The operation then consists in replacing $\mathcal{B}$ in the tree at hand by two branches $B_1 = \langle \mathcal{B}_0, t_1, \sigma_1, t_2 \rangle$ and $B_2 = \langle \mathcal{B}_0, t_3, \sigma_2, t_n \rangle$. Graphically, this means that we replace $\mathcal{B}$ by the following branches:

$$
\begin{array}{cc}
\multicolumn{2}{c}{\mathcal{B}_0} \\
\diagup & \diagdown \\
t_1 & t_3 \\
\vdots & \vdots \\
\sigma_1 & \sigma_2 \\
\vdots & \vdots \\
t_2 & t_n
\end{array}
$$

Consider the sequence of formula trees the first member of which is D, such that each other member is obtained from the previous one by the operation above and that no further extension by this operation is possible. Since D has finitely many branches of finite length, the sequence is finite. We call T the last member of the sequence.

4.3.4 Atomic Closure

By construction, T is a tableau: the procedure in Step C creates ramifications for **O**-implications matching the corresponding tableau rule, but leaves everything else untouched. Notice that the Proponent may very well be able to win without using the two possible challenges against a conjunction or using the two possible defences for a disjunction. This is why we needed the flexible version of the rule for α formulas. It remains to check that T is atomically closed. We show that the creation of ramifications preserves dialogical closure, so that every member of the sequence of formula trees including T is atomically closed.

Proof Consider D and the branch $\mathcal{B}$. By construction of D, we know that t_n is labelled with an atomic **P**-formula χ and there is a predecessor of t_n which is labelled with **O**-χ. We also know that t_2 is labelled with an atomic **P**-formula ψ because it is the only case where the move associated with t_3 is the only possible one for the Opponent, as required in Step B. Hence, by the formal rule, there is a predecessor of t_2 labelled with **O**-ψ.

Atomic Closure of B_1: $B_1 = \langle \mathcal{B}_0, t_1, \sigma_1, t_2 \rangle$ therefore it is straightforward from the last remark that B_1 is atomically closed.

Atomic Closure of B_2: We show that there is a member of B_2 labelled with **O**-χ. Suppose it is not the case. Then it is the same for the sequence $\langle \mathcal{B}_0, t_1, t_3, \sigma_2, t_n \rangle$, since t_1 is labelled with a **P**-move. It follows that this sequence contravenes the formal rule and is not a play. Hence, the extension $\mathcal{B}' = \langle \mathcal{B}_0, t_1, t_3, \sigma_2, t_n, \sigma_1, t_2 \rangle$ of this sequence is not a play either. But $\mathcal{B}'$ is a branch in the original extensive form $\mathfrak{S}$.[25] Hence $\langle \mathcal{B}_0, t_1, t_3, \sigma_2, t_n \rangle$ must observe the formal rule. Contradiction. $\square$

5 Concluding Remarks

We have proven soundness and completeness of tableaux with respect to the existence of winning **P**-strategies in dialogical games, in the case of Classical First-Order Logic. To our knowledge, our proof is the first thorough result about dialogical games defined with repetition ranks and considered from the point of view of their extensive form. We have discussed the differences between our work and previous papers.

In Section 4.2 we have developed an algorithm to transform and expand a tableau proof into the extensive form of a winning **P**-strategy. This means that a tableau proof gives enough information to build a winning strategy of the Proponent for the formula at stake. In Section 4.3 we have given an algorithm for the converse result, showing how a tableau proof for a formula can be extracted from the extensive form of a winning **P**-strategy for this formula. We have seen that a winning **P**-strategy conveys more information than needed to prove a formula. This shows that the dialogical approach to meaning cannot be reduced to a proof-theoretical semantics. Indeed, the connection between dialogical games and proofs occurs at the level of strategies only. That is, tableaux ignore the level of plays.

Various refinements can probably be supplied to our proof. For example it may be worth the trouble to work with a more efficient systematic procedure to build tableaux than the one we use. Let us mention other directions in which the result of this article can be extended. We have considered the particular case of a first-order language without equality and without complex terms i.e., without function symbols. Furthermore, we have dealt with dialogical games and tableaux for sentences. Hence, future work shall consider a generalization of the result for arbitrary first-order languages and dialogical games and tableaux with free variables. This would be a chance to study the dialogical manifestation in strategical terms of the mechanism of unification[26] in tableaux.

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[25] $\mathcal{B}'$ simply represents a play in which **O** starts by defending and subsequently counter-attacks when she has no other possible move.

[26] See for example Fitting [3] and Letz [6].

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