Birkhoff Completeness in Institutions

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Abstract. We develop an abstract proof calculus for logics whose sentences are 'Horn sentences' of the form: $(\forall X)H \implies c$ and prove an institutional generalization of Birkhoff completeness theorem. This result is then applied to the particular cases of Horn clauses logic, the 'Horn fragment'[1] of preorder algebras, order-sorted algebras and partial algebras and their infinitary variants.

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1. Introduction

In 1935 Birkhoff [3] first prove a completeness theorem for equational logic, in the unsorted case. Goguen and Meseguer [14], giving a sound and complete system of proof rules for finitary many-sorted equational deduction, generalized the completeness theorem of Birkhoff to the completeness of finitary many-sorted equational logic and provided simultaneously a full algebraization of finitary many-sorted equational deduction. The unsorted rules can be unsound for many sorted algebras that may have empty carriers (as noticed in [14]), suggesting the idea that generalizations to other variants of equational logics may imply some difficulties.

The deduction rules introduced in [14] can be presented as in figure 1.

The concept of institution is a category-based model-oriented formalization of the concept of logical system, including syntax, semantics and satisfaction between them. Institution-independent model theory provides an abstract approach towards model theory, without a particular underlying logical system. This is important especially in the context of the recent proliferation of logics in computer science, mostly in the area of formal specification, where it is now a tradition to have an institution underlying each language or system. This perspective has the advantage of clarifying model theoretic phenomena and causality relationships between them, allowing thus new fundamental insights and results even in traditional areas of model theory.

$(Reflexivity)\ \vdash (\forall X)t = t$
$(Symmetry)\ (\forall X)t = t' \vdash (\forall X)t' = t$
$(Transitivity)\ \{(\forall X)t = t',\ (\forall X)t' = t''\} \vdash (\forall X)t = t''$
$(Congruence)\ \{(\forall X)t_i = t_i'\} \vdash (\forall X)\sigma(t_1, ..., t_n) = \sigma(t_1', ..., t_n')$
for any $\sigma \in F_{s_1 \dots s_n \to s}$
$(Substitutivity)\ \Gamma \cup \{(\forall X)\theta^{Sen}(c)\} \vdash (\forall X)\theta^{Sen}(t) = \theta^{Sen}(t')$
for each set of conditional equations Γ, for each equation $(\forall Y)c \implies t = t'$ in Γ and for each substitution $\theta : Y \to T_F(X)$

FIGURE 1. Complete rules for equational logic

We generalize the Birkhoff completeness result to arbitrary institutions obtaining in a unitary fashion sound and complete systems of proof rules for an entire class of logics used in computer science such as: Horn clause logic (**HCL**), the 'Horn fragment' of preorder algebras (**Horn**(**POA**)), order sorted algebra (**Horn**(**OSA**)) and partial algebras (**Horn**(**PA**)).

There are several aspects that motivate and justify our study. One of them is the importance for model theory. Our study isolates the particular aspects of the logics from general ones in order to obtain an abstract completeness which covers many examples such as the ones mentioned above and also their infinitary versions, $\mathbf{HCL}_\infty$, $\mathbf{Horn}(\mathbf{POA})_\infty$, $\mathbf{Horn}(\mathbf{OSA})_\infty$ and $\mathbf{Horn}(\mathbf{PA})_\infty$. This top-down approach is broader, in the sense that it can also be applied in the case of logics that are not necessarily Horn, such as "universal" institutions.

The present paper is also meaningful for computer science. Modern algebraic specification languages (such as CafeOBJ [10], CASL [2], Maude) are rigorously based on logic, in the sense that each feature and construct in a language can be expressed within a certain logic underlying it. Completeness results have great significance for an operational semantics of executable specification languages. In the context of proliferation of a multitude of specification languages, these abstract results provide complete systems of proof rules for the logical systems underlying them.

The paper is organized as follows: Section 2 presents the notions of institution-independent model theory used in the paper, illustrating each concept in the familiar first-order logic with equality (**FOL**). Section 3 introduces the abstract concepts of universal and Horn institution and a generic universal entailment system which is proved sound and complete, under conditions which are also investigated. Section 4 consists of the refinement of the universal entailment system to the particular case of Horn institutions, while Section 5 contains concrete instances of the Birkhoff completeness in several logics, like **HCL**, **PA**, **POA**, **OSA** and their infinitary versions. Section 6 concludes the paper and discusses the future work.

We assume that the reader is familiar with basic categorical notions like functor, natural transformation, colimit, comma category, etc. A standard textbook on the topic is [17]. We use both the terms "morphism" and "arrow" to refer morphisms of a category.

Composition of morphisms and functors is denoted using the symbol ";" and is considered in diagrammatic order.

2. Preliminaries

2.1. Institutions

Institutions were introduced in [13] with the original goal of providing an abstract framework for algebraic specifications of computer science systems. By isolating the essence of a logical system in the abstract *satisfaction relation*, which states that *truth is invariant to change of notation or enlargement of context*, and leaving open the details of signatures, sentences and models, these structure achieves an appropriate level of generality for the development of abstract model theory - i.e. independent of the specific nature of the underlying logic. Many logical notions and results can be developed in an institution-independent way, to mention just a few: ultraproducts [5], Craig interpolation [7], Robinson consistency [12], saturated models [11], Beth definability [20]. A textbook dedicated to this topic is under preparation [9].

Definition 1. *An institution [13] consists of:*

1. *a category* $\mathbb{S}ig$*, whose objects are called* signatures*.*
2. *a functor* $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$*, providing for each signature a set whose elements are called* (Σ-)sentences*.*
3. *a functor* $\mathbb{M}od : \mathbb{S}ig^{op} \longrightarrow Cat$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)morphisms*.*
4. *a relation* $\models_\Sigma \subseteq |\mathbb{M}od(\Sigma)| \times \mathbb{S}en(\Sigma)$ *for each* $\Sigma \in |\mathbb{S}ig|$*, called* ($\Sigma$-)satisfaction*, such that for each morphism* $\varphi : \Sigma \longrightarrow \Sigma'$ *in* $\mathbb{S}ig$*, the* satisfaction condition
$$M' \models_{\Sigma'} \mathbb{S}en(\varphi)(e) \text{ iff } \mathbb{M}od(\varphi)(M') \models_\Sigma e$$
holds for all $M' \in |\mathbb{M}od(\Sigma')|$ *and* $e \in \mathbb{S}en(\Sigma)$*.*

Following the usual notational conventions, we sometimes let $\_\upharpoonright_\phi$ denote the reduct functor $\mathbb{M}od(\varphi)$ and let φ denote the sentence translation $\mathbb{S}en(\varphi)$. When $M = M' \upharpoonright_\varphi$ we say that M' is a φ-*expansion* of M, and that M is the φ-*reduct* of M'; and similarly for model morphisms. When E and E' are sets of sentences of the same signature Σ, we let $E \models_\Sigma E'$ denote the fact that $M \models E$ implies $M \models E'$ for all Σ-models M. The relation $\models_\Sigma$ between sets of sentences is called (Σ-)*semantic consequence relation* (notice that it is written just like the satisfaction relation).

Example 1. *First order logic (***FOL***).* The signatures are triplets (S, F, P), where S is the set of sorts, $F = \{F_{w\to s}\}_{w\in S^*, s\in S}$ is the ($S^* \times S$ -indexed) set of operation symbols, and $P = \{P_w\}_{w\in S^*}$ is the (S^*-indexed) set of relation symbols. If $w = \lambda$, an element of $F_{w\to s}$ is called a *constant symbol*, or a *constant*. By a slight notational abuse, we let F and P also denote $\bigcup_{(w,s)\in S^*\times S} F_{w\to s}$ and $\bigcup_{w\in S^*} P_w$ respectively. A signature morphism between (S, F, P) and (S', F', P') is a triplet $\varphi = (\varphi^{sort}, \varphi^{op}, \varphi^{rel})$, where $\varphi^{sort} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rel} : P \to P'$ such that $\varphi^{op}(F_{w\to s}) \subseteq F'_{\varphi^{sort}(w)\to\varphi^{sort}(s)}$ and $\varphi^{rel}(P_w) \subseteq P'_{\varphi^{sort}(w)}$ for all $(w, s) \in$

$S^* \times S$. When there is no danger of confusion, we may let φ denote each of φ^{sort}, φ^{rel} and φ^{op}. Given a signature $\Sigma = (S,F,P)$, a Σ-model M is a triplet $M = (\{M_s\}_{s\in S}, \{M_\sigma^{w,s}\}_{(w,s)\in S^* \times S, \sigma\in F_{w,s}}, \{M_\pi^w\}_{w\in S^*, \pi\in P_w})$ interpreting each sort s as a set M_s, each operation symbol $\sigma \in F_{w\to s}$ as a function $M_\sigma^{w,s} : M_w \to M_s$ (where M_w stands for $M_{s_1} \times \ldots \times M_{s_n}$ if $w = s_1 \ldots s_n$), and each relation symbol $\pi \in P_w$ as a relation $M_\pi^w \subseteq M_w$. When there is no danger of confusion we may let M_σ and M_π denote $M_\sigma^{w,s}$ and M_π^w respectively. Morphisms between models are the usual Σ-homomorphisms, i.e., S-sorted functions that preserve the structure. The Σ-sentences are defined as the least set of sentences that contains all *atoms*, i.e., equality atoms $t_1 = t_2$, where $t_1, t_2 \in (T_F)_s$ [1] or relational atoms $\pi(t_1,\ldots,t_n)$, where $\pi \in P_{s_1\ldots s_n}$ and $t_i \in (T_F)_{s_i}$ for each $i \in \{1,\ldots,n\}$, and is closed under:

- negation, conjunction, disjunction;
- universal or existential quantification over finite sets of constants (variables).

Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms t as elements M_t in models M. The definitions of functors $\mathbb{S}en$ and $\mathbb{M}od$ on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $\mathbb{S}en(\varphi) : \mathbb{S}en(\Sigma) \to \mathbb{S}en(\Sigma')$ translates sentences symbol-wise, and $\mathbb{M}od(\varphi) : \mathbb{M}od(\Sigma') \to \mathbb{M}od(\Sigma)$ is the forgetful functor.

Universal sentences in **FOL** *(***UNIV***)* A *universal sentence* for a **FOL** signature (S,F,P) is a (S,F,P)-sentence of the form $(\forall\chi)\rho$ where ρ is a sentence formed without quantifiers.

Example 2. *Horn Clause logic (***HCL***). A universal Horn sentence* for a **FOL** signature (S,F,P) is a (universal) sentence of the form $(\forall X)H \Rightarrow C$, where H is a finite conjunction of (relational or equational) atoms, C is a (relational or equational) atom, and $H \Rightarrow C$ is the implication of C by H. In the tradition of logic programming universal Horn sentences are known as *Horn Clauses*. Thus **HCL** has the same signatures and models as **FOL** but only universal Horn sentences as sentences.

By considering the case of empty sets of relational symbols, we obtain the conditional equational logic, **CEQL**.

Example 3. *Preorder algebra (***POA***).* The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra homomorphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra homomorphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first order quantification.

Example 4. *Order sorted algebra (***OSA***)* An order sorted signature $(S,\leq,F)$ consists of an algebraic signature (S,F), with a partial ordering $(S,\leq)$ such that the following

[1] T_F is the ground term algebra over F.

monotonicity condition is satisfied $\sigma \in F_{w_1 \to s_1} \cap F_{w_2 \to s_2}$ and $w_1 \leq w_2$ imply $s_1 \leq s_2$. A morphism of **OSA** signatures $\varphi : (S, \leq, F) \to (S', \leq', F')$ is just a morphism of algebraic signatures $(S, F) \to (S', F')$ such that the ordering is preserved, i.e. $\varphi(s_1) \leq' \varphi(s_2)$ whenever $s_1 \leq s_2$.

An order sorted signature $(S, \leq, F)$ is *regular* if and only if for each $\sigma \in F_{w_1 \to s_1}$ and each $w_0 \leq w_1$ there is a unique least element in the set $\{(w, s) \mid \sigma \in F_{w \to s}$ and $w_0 \leq w\}$.

Fact 1. *For regular signatures $(S, \leq, F)$, any $(S, \leq, F)$ has a least sort and the initial $(S, \leq, F)$-algebra in $\mathbb{M}od(S, \leq, F)$ can be defined as a term algebra, cf. [16].*

A partial ordering $(S, \leq)$ is *filtered* if and only if for all $s_1, s_2 \in S$, there is some $s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$. A partial ordering is *locally filtered* if and only if every connected component of it is filtered. An order sorted signature $(S, \leq, F)$ is *locally filtered* if and only if $(S, \leq)$ is locally filtered, and it is *coherent signature* if and only if it is both locally filtered and regular. Hereafter we assume that all **OSA** signatures are coherent.

Given an order sorted signature $(S, \leq, F)$, an order sorted $(S, \leq, F)$-algebra is a (S, F)-algebra M such that

- $s_1 \leq s_2$ implies $M_{s_1} \subseteq M_{s_2}$, and
- $\sigma \in F_{w_1 \to s_1} \cup F_{w_2 \to s_2}$ and $w_1 \leq w_2$ imply $M_\sigma^{w_1, s_1} = M_\sigma^{w_2, s_2}$ on M_{w_1}.

Given order sorted $(S, \leq, F)$-algebras M and N, an order sorted $(S, \leq, F)$-homomorphism $h : M \to N$ is a (S, F)-homomorphism such that $s_1 \leq s_2$ implies $h_{s_1} = h_{s_2}$ on M_{s_1}.

Let $(S, \leq, F)$ be a an order sorted signature. We say that the sorts s_1 and s_2 are in the same *connected component* of S if and only if $s_1 \equiv s_2$, where $\equiv$ is the least equivalence on S that contains $\leq$. The atoms of the signature $(S, \leq, F)$ are equations of the form $t_1 = t_2$ such that the least sort of the terms t_1 and t_2 are in the same connected component. The sentences are formed from these equations by means of boolean connectives and quantification over (first order) variables. Order sorted algebras were extensively studied in [15] and [16].

Example 5. *Partial algebra (***PA***).* A partial algebraic signature is a tuple (S, TF, PF) such that $(S, TF \cup PF)$ is an algebraic signature. Then TF is the set of total operations and PF is the set of partial operations. A morphism of **PA** signatures $\varphi : (S, TF, PF) \to (S, TF', PF')$ is just a morphism of algebraic signatures $(S, TF \cup PF) \to (S, TF' \cup PF')$ such that $\varphi(TF) \subseteq TF'$ and $\varphi(PF) \subseteq PF'$.

A partial algebra M for a **PA** signature (S, TF, PF) is just like an ordinary algebra but interpreting the operations of PF as partial rather than total functions, which means that M_σ might be undefined for some arguments. A partial algebra homomorphism $h : M \to N$ is a family of (total) functions $\{h_s : M_s \to N_s\}_{s \in S}$ indexed by the set of sorts S of the signature such that $h_w(M_\sigma(a)) = N_\sigma(h_s(a))$ for each operation symbol $\sigma : w \to s$ and each string of arguments $a \in M_w$ for which $M_\sigma(a)$ is defined.

The sentences have three kinds of atoms: *definedness def*(_), *strong* equality $\stackrel{s}{=}$ and existence equality $\stackrel{e}{=}$. The *definedness* $def(t)$ of a term t holds in a partial algebra M when the interpretation M_t of t is defined. The strong equality $t_1 \stackrel{s}{=} t_2$ holds when both terms are undefined or both of them are defined and are equal. The existence equality

$t_1 \stackrel{e}{=} t_2$ holds when both terms are defined and are equal. The sentences are formed from these atoms by means of boolean connectives and quantification over total (first order) variables. Notice that each definedness atom $def(t)$ is semantically equivalent with $t \stackrel{e}{=} t$ and any strong equality $t_1 \stackrel{s}{=} t_2$ is semantically equivalent with $(def(t_1) \vee def(t_2)) \Rightarrow t_1 \stackrel{e}{=} t_2$. Partial algebras and their applications were extensively studied in [21] and [4].

Notation 1. *Notice that in most of the standard cases, a logic comes with a notion of atomic sentence, with the help of with the formulas of the logic are built. We can restrict the sentences of some logic I, formalized as an institution, to the atomic ones, and denote this sub-institution* $\mathbf{Atomic}(I)$. *We also denote by* $\mathbf{Horn}(I)$ *the sub-institution of I formed by restricting the sentences to universal Horn sentences built over the atomic formulae of I.*

Example 6. *Infinitary Horn clause logic (*$\mathbf{HCL}_\infty$*)* This is the infinitary extension of **HCL** obtained by allowing the hypothesis parts of a Horn clauses $(\forall X)H \Rightarrow C$ to consist of infinitary conjunctions of atoms.

Similarly we may define $\mathbf{Horn}(\mathbf{POA})_\infty$, $\mathbf{Horn}(\mathbf{OSA})_\infty$, $\mathbf{Horn}(\mathbf{PA})_\infty$.

Example 7. *Institution of presentations.* A *presentation* is a pair (Σ, E) consisting of a signature Σ and a set E of Σ-sentences. A *presentation morphism* $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ which maps the axioms of the source presentation to logical consequences of the target presentation: $E' \models \varphi(E)$. Presentation morphisms form a category, denoted $\mathbb{P}res^I$. The model functor $\mathbb{M}od$ of an institution can be extended from the category of its signatures $\mathbb{S}ig$ to a model functor from the category of its presentations $\mathbb{P}res$, by mapping a presentation (Σ, E) to the full subcategory $\mathbb{M}od^{pres}(\Sigma, E)$ of $\mathbb{M}od(\Sigma)$ consisting of all Σ-models s atisfying E. The correctness of the definition of $\mathbb{M}od^{pres}$ is guaranteed by the satisfaction condition of the base institution; this is easy to check. This leads to the *institution of presentations* $I^{pres} = (\mathbb{S}ig^{pres}, \mathbb{S}en^{pres}, \mathbb{M}od^{pres}, \models^{pres})$ over the base institution I where

- $\mathbb{S}ig^{pres}$ is the category $\mathbb{P}res^I$
- $\mathbb{S}en^{pres}(\Sigma, E) = \mathbb{S}en(\Sigma)$, and
- for each (Σ, E)-model M and any Σ-sentence e, $M \models^{pres} e$ iff $M \models e$.

Definition 2. *A set of sentences* $E \subseteq Sen(\Sigma)$ *is called* basic *if there exists a* Σ*-model* M_E *such that, for all* Σ*-models* M, $M \models E$ *if and only if there exists a homomorphism* $M_E \to M$.

A set of sentences E is epic basic *if it is basic and the morphism* $M_E \to M$ *is unique.*

Basic sentences were introduced in [24] under the name of "ground positive elementary sentences". We prefered to use the terminology from [5].

The concept of epic basic sentence constitute the best institution-independent approximation of the actual atoms of the logic. One of the important consequences is that, directly from the definition, we obtain that epic basic sets of sentences always admit initial models.

Lemma 1. *Any set of atomic sentences in* **FOL***,* **POA***,* **OSA** *and* **PA** *is epic basic.*

Proof. In any **FOL** the basic model M_E for a set E of atomic (S,F,P)-sentences is the initial model for E. This is constructed a follows: on the quotient $(T_F)_{=_E}$ of the term model T_F by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(M_E)_\pi = \{(t_1/_{=_E}, \ldots, t_n/_{=_E}) \mid \pi(t_1,\ldots,t_n) \in E\}$. A similar argument as the preceding holds for **POA** and **OSA**.

In **PA** for a set E of atomic (S,TF,PF)-sentences we define S_E to be the set if subterms appearing in E. We also define $T_{TF}(S_E)$ to be the partial algebra that is generated by the set of terms S_E. The basic model M_E will be the quotient of this algebra by the partial congruence induced by the equalities from E. □

Internal logic. The following institutional notions dealing with logical connectives and quantifiers were defined in [23].

Definition 3. *Let Σ be a signature in an institution,*

- a Σ-sentence $\neg e$ is a (semantic) negation of e when $M \models \neg e$ if and only if $M \not\models e$, for each Σ-model M, and

- a Σ-sentence $e_1 \wedge e_2$ is a (semantic) conjunction of e_1 and e_2 when $M \models e_1 \wedge e_2$ if and only if $M \models e_1$ and $M \models e_2$, for each Σ-model M.

- a Σ-sentence $(\forall\chi)e'$ is a (semantic) universal χ-quantification of e' over χ when $M \models (\forall\chi)e'$ if and only if $M' \models e'$ for all χ-expansion M' of M.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the finitary signature extensions with constants. For a class $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms, we say that the institution has universal $\mathcal{D}$-quantification when for each $\chi : \Sigma \to \Sigma'$ in $\mathcal{D}$, each Σ-sentence has a universal χ-quantification.

Representable signature morphisms. The institutional notion of *representable* signature morphisms is meant to capture the phenomena of quantification over (sets of) first-order variables. The notion starts from the fact that semantics of quantification in first-order-like logics can be given in terms of signature extensions: $M \models_{(S,F,P)} (\forall X)e$ ($M \models_{(S,F,P)} (\exists X)e$) iff $M' \models_{(S,F\cup X,P)} e$ for each (for some) $(S,F\cup X,P)$-expansion M' of M. Thus, in order to reach first-order quantification institutionally, one needs to define somehow what ”injective signature morphism that only adds constant symbols” (such as $\iota : (S,F,P) \hookrightarrow (S,F\cup X,P)$) means.

Definition 4. *([5]) A signature morphism $\chi : \Sigma \to \Sigma'$ is called:*

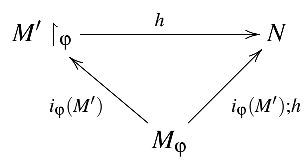
- representable*, if there exists a Σ-model M_χ (called the* representation *of χ) and an isomorphism of categories $i_\chi : \mathbb{M}od(\Sigma') \to M_\chi/\mathbb{M}od(\Sigma)$ such that $i_\chi;U = \mathbb{M}od(\chi)$, where $U : M_\chi/\mathbb{M}od(\Sigma) \to Mod(\Sigma)$ is the usual forgetful functor;*

- finitely representable*, if it is representable and M_χ is a finitely presented object*[2] *in $\mathbb{M}od(\Sigma)$;*

Since 1_{M_χ} is the initial object of the category $M_\chi/\mathbb{M}od(\Sigma)$, the Σ'-model $(i_\chi)^{-1}(1_{M_\chi})$ is the initial object of the category $\mathbb{M}od(\Sigma')$. We denote by $O_{\Sigma'}$ the Σ'-model $(i_\varphi)^{-1}(1_{M_\chi})$. The notion of representability is built on the intuition that, in **FOL**, an expansion of a $\Sigma = (S,F,P)$-model A over a signature inclusion $\iota : \Sigma \hookrightarrow \Sigma' = (S, F \cup X, P)$ that only adds constants can be viewed as a pair (M, v), where $v : X \to M$ is a function interpreting the new constants in X, and furthermore as a pair $(M, \overline{v})$, where $\overline{v} : T_\Sigma(X) \to M$ is a model-morphism. Hence ι is represented by $T_\Sigma(X)$. And ι is finitely representable, i.e., $T_\Sigma(X)$ is finitely presented in $\mathbb{M}od(\Sigma)$, if X is finite.

Remark 1. *If $\chi : \Sigma \to \Sigma'$ is a representable signature morphism then for each Σ'-model M', the canonical functor determined by the reduct functor $Mod(\chi)$ is an isomorphism of comma categories: $M'/\mathbb{M}od(\Sigma') \cong (M' \upharpoonright_\chi)/\mathbb{M}od(\Sigma)$. .*

This means that each Σ-homomorphism $M' \upharpoonright_\chi \xrightarrow{h} N$ admits an unique χ-expansion $M' \xrightarrow{h'} N' = i_\chi^{-1}(h)$ where $h : i_\varphi(M') \to (i_\varphi(M'); h)$ is regarded as a morphism in $M_\varphi/\mathbb{M}od(\Sigma)$.



Lemma 2. *Let $\Sigma \xrightarrow{\chi} \Sigma'$ be a representable signature morphism. Then the following presentation morphism $(\Sigma, E) \xrightarrow{\chi} (\Sigma', E')$ is also representable (as a signature morphism in I^{pres}) if the set of sentences E' is epic basic.*

Proof. Since E' is epic basic the Σ'-model $M_{E'}$ is the initial object in the category $\mathbb{M}od(\Sigma', E')$. Thus, $\mathbb{M}od(\Sigma', E') \cong M_{E'}/\mathbb{M}od(\Sigma')$ and by the above remark $M_{E'}/\mathbb{M}od(\Sigma') \cong M_{E'} \upharpoonright_\chi /\mathbb{M}od(\Sigma)$. We denote by $i_\chi^{E'}$ the composition of isomorphisms $\mathbb{M}od(\Sigma', E') \cong M_{E'}/\mathbb{M}od(\Sigma') \cong M_{E'} \upharpoonright_\chi /\mathbb{M}od(\Sigma)$ and we have $i_\chi^{E'}; U = \mathbb{M}od(\chi)$. □

The following Lemma is proven in [5].

Lemma 3. *The reduct functor corresponding to representable signature morphisms preserves directed co-limits of models.*

Substitutions. Given a **FOL** signature (S, F, P) and two sets of new constants, called *first order variables* X and Y, a *first order* (S, F, P)*-substitution* from X to Y consists of a mapping $\theta : X \to T_F(Y)$ of the variables X with F-terms over Y.

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[2] An object A in a category $\mathbb{C}$ is called *finitely presented* ([1]) if

- for each directed diagram $D : (J, \leq) \to \mathbb{C}$ with co-limit $\{Di \xrightarrow{\mu_i} B\}_{i \in J}$, and for each morphism $A \xrightarrow{g} B$, there exists $i \in J$ and $A \xrightarrow{g_i} Di$ such that $g_i; \mu_j = g$,

- for any two arrows g_i and g_j as above, there exists $i \leq k, j \leq k \in J$ such that $g_i; D(i \leq k) = g_j; D(j \leq k) = g$.

On the semantics side, each (S,F,P)-substitution $\theta : X \to T_F(Y)$ determines a functor $\mathbb{M}od(\theta) : \mathbb{M}od(S,F\cup Y,P) \to \mathbb{M}od(S,F\cup X,P)$ defined by

- $\mathbb{M}od(\theta)(M)_x = M_x$ for each sort $s \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$, and

- $\mathbb{M}od(\theta)(M)_x = M_{\theta(x)}$, i.e. the evaluation of the term $\theta(x)$ in M, for each $x \in X$.

On the syntax side, θ determines a sentence translation function $\mathbb{S}en(\theta) : \mathbb{S}en(S,F\cup X,P) \to \mathbb{S}en(S,F\cup Y,P)$ which in essence replaces all symbols from X with the corresponding $(F\cup Y)$-terms according to θ.

- $\mathbb{S}en(\theta)(t_1 = t_2)$ is defined as $\overline{\theta}(t) = \overline{\theta}(t')$ for each $(S,F\cup X,P)$-equation $t_1 = t_2$, where $\overline{\theta} : T_F(X) \to T_F(Y)$ is the unique extension of θ to an F-morphism ($\overline{\theta}$ is replacing variables $x \in X$ with $\theta(x)$ in each $F\cup X$-term t).

- $\mathbb{S}en(\theta)(\pi(t_1,\dots,t_n))$ is defined as $\pi(\overline{\theta}(t_1),\dots,\overline{\theta}(t_n))$ for each $(S,F\cup X,P)$-relational atom $\pi(t_1,\dots,t_n)$.

- $\mathbb{S}en(\theta)(\rho_1 \wedge \rho_2)$ is defined as $\mathbb{S}en(\theta)(\rho_1) \wedge \mathbb{S}en(\theta)(\rho_2)$ for each conjunction $\rho_1 \wedge \rho_2$ of $(S,F\cup X,P)$-sentences, and similarly for the case of any other logical connectives.

- $\mathbb{S}en(\theta)((\forall Z)\rho)$ is defined as $(\forall Z)\mathbb{S}en(\theta_Z)(\rho)$ for each $(S,F\cup X\cup Z,P)$-sentence ρ, where θ_Z is the trivial extension of θ to an $(S,F\cup Z,P)$-substitution.

Note that we have extended the notations used for the models functor $\mathbb{M}od$ and for the sentence functor $\mathbb{S}en$ from the signatures to the first order substitutions. This notational extension is justified by the following satisfaction condition given by Proposition 1

Proposition 1. *For each* **FOL**-*signature* (S,F,P) *and each* (S,F,P)-*substitution* $\theta : X \to T_F(Y)$,

$$Mod(\theta)(M) \models \rho \text{ if and only if } M \models \mathbb{S}en(\theta)(\rho)$$

for each $(S,F\cup Y,P)$-*model* M *and each* $(S,F\cup X,P)$-*sentence* ρ.

Proof. By noticing that $\mathbb{M}od(\theta)(M)_t = M_{\overline{\theta}(t)}$ for each $(F\cup X)$-term t, and by straightforward induction on the structure of the sentences. □

The satisfaction condition property expressed above permits the definition of a general concept of substitution by abstracting

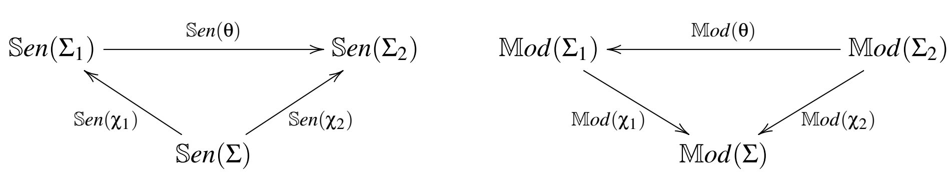
- **FOL** signatures (S,F,P) to signatures Σ in arbitrary institutions, and
- sets of first order variables X for (S,F,P) to signature morphisms $\Sigma \to \Sigma_1$.

For any signature Σ of an institution, and any signature morphisms $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$, a Σ-*substitution* [6] $\theta : \chi_1 \to \chi_2$ consists of a pair $(\mathbb{S}en(\theta), \mathbb{M}od(\theta))$, where

- $\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma_1) \to \mathbb{S}en(\Sigma_2)$ is a function and

- $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma_2) \to \mathbb{M}od(\Sigma_1)$ is a functor.

such that both of them preserving Σ, i.e. the following diagrams commute:



and such that the following Satisfaction Condition holds:

$$\mathbb{M}od(\theta)(M_2) \models \rho_1 \text{ if and only if } M_2 \models \mathbb{S}en(\rho_1)$$

for each Σ_2-model M_2 and each Σ_1-sentence ρ_1.

We sometimes let $\_ \upharpoonright_\theta$ denote the functor $\mathbb{M}od(\theta)$ and let θ denote the sentence translation $\mathbb{S}en(\theta)$.

Looking at the example of substitution in **FOL** we observe that a homomorphism between the term models $T_F(X)$ and $T_F(Y)$ determines a first order substitution. Abstractly we will want first order substitutions to be defined by morphisms between representants M_{χ_1} and M_{χ_2} of two representable signature morphisms $\chi_1 : \Sigma \to \Sigma_2$ and $\chi_2 : \Sigma \to \Sigma_2$. One should easily notice that given homomorphism $h : M_{\chi_1} \to M_{\chi_2}$ we can define the semantic part of a substitution θ_h by $\mathbb{M}od(\theta_h) = i_{\chi_2}; (h/\mathbb{M}od(\Sigma)); i_\chi^{-1}$. However, the syntactic part of the substitution must be defined as a characteristic of the institution.

Definition 5. *For any class of morphisms $\mathcal{D}$ in an institution, a D-substitution is just a substitution between signature morphisms in $\mathcal{D}$.*

An institution has representable $\mathcal{D}$-substitutions, for a class $\mathcal{D}$ of representable signature morphisms, if for every $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$ from $\mathcal{D}$ and every homomorphism $h : M_{\chi_1} \to M_{\chi_2}$ there exists a substitution $\theta_h : \chi_1 \to \chi_2$, that has the semantic part defined as $\mathbb{M}od(\theta_h) = i_{\chi_2}; (h/\mathbb{M}od(\Sigma)); i_\chi^{-1}$.

For sake of simplicity, we may refer to this notion just by saying that an institution has $\mathcal{D}$-substitution.

2.2. System of rules and entailment systems

Abstract systems of proof rules have been introduced in [8] which also developed the free proof system defined in [18]. Entailment systems are just proof systems such that the category of proofs for a given signature is a preorder. The results in this subsection can be found in [8].

Definition 6. *A* system of (proof) rules $(\mathbb{S}ig, \mathbb{S}en, Rl)$ *consists of*

- a category of "signatures" $\mathbb{S}ig$,

- a "sentence functor" $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$

- a family of relations $Rl = (Rl_\Sigma)_{\Sigma \in |\mathbb{S}ig|}$ between sets of sentences $Rl_\Sigma \subseteq \mathcal{P}(\mathbb{S}en(\Sigma)) \times \mathcal{P}(\mathbb{S}en(\Sigma))$ for all signatures $\Sigma \in |\mathbb{S}ig|$. such that the following property holds :

An entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ is just a systems of rules such that the entailment relation $\vdash_\Sigma$ of each signature

anti-monotonicity: $E_1 \vdash_\Sigma E_2$ if $E_2 \subseteq E_1$,

transitivity: $E_1 \vdash_\Sigma E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_2 \vdash_\Sigma E_3$,

unions: $E_1 \vdash_\Sigma E_2 \cup E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_1 \vdash_\Sigma E_3$, and

translation: $\varphi(E) \vdash_{\Sigma'} \varphi(E')$ if $E \vdash_\Sigma E'$, for all $\varphi : \Sigma \longrightarrow \Sigma'$

When we allow infinite unions, i.e. $E \vdash \bigcup_{i \in J} E_i$ if $E \vdash E_i$ for all $i \in J$, we call the entailment system infinitary.

In any institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, the semantic consequence relation $\models$ between sets of sentences gives an example of an infinitary entailment system $(\mathbb{S}ig, \mathbb{S}en, \models)$, which is called the *semantic entailment system* of the institution. Infinitary entailment systems are used for infinitary logics like $\mathbf{HCL}_\infty$, $\mathbf{Horn}(\mathbf{POA})_\infty$, $\mathbf{Horn}(\mathbf{PA})_\infty$ and $\mathbf{Horn}(\mathbf{OSA})_\infty$.

When is no danger of confusion we may omit the subscript Σ from $\vdash_\Sigma$ and for every signature morphism $\varphi \in \mathbb{S}ig$, we sometimes let φ denote the sentence translation $\mathbb{S}en(\varphi)$.

Fact 2. *Note that every systems of rules $(\mathbb{S}ig, \mathbb{S}en, Rl)$ generates freely an entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$, where $\vdash$ is the least entailment relation which contains Rl and it is closed under anti-monotonicity, transitivity, unions and translation property.*

The free infinitary entailment system is obtain by replacing unions with infinite unions in the above definition.

A proof rule $\langle h, c\rangle$ is *finitary* when both sets h and c are finite. We say that an entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ is *compact* whenever $\Gamma \vdash E_f$ for a finite set of sentences $E_f \subseteq \mathbb{S}en(\Sigma)$, there exists $\Gamma_f \subset \Gamma$ finite such that $\Gamma_f \vdash E_f$. For each entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ one can easily construct the *compact entailment sub-system* $(\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ by defining the entailment relation $\vdash^c$ as follows:

$\Gamma \vdash^c E$ if and only if for each $E_f \subseteq E$ finite there exists $\Gamma_f \subseteq \Gamma$ finite such that $\Gamma_f \vdash E_f$

Proposition 2. $(\mathbb{S}ig, \mathbb{S}en, \vdash^c)$ *is an entailment system.*

The result bellow is a corollary of Proposition 2.

Corollary 1. *The entailment system freely generated by a system of finitary proof rules is compact.*

An entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ has universal $\mathcal{D}$-quantification, for a sub-category $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms if the entailment system satisfies the following property (also called the meta-rule of 'Generalization').

$$\Gamma \vdash_\Sigma (\forall\chi)e' \Leftrightarrow \chi(\Gamma) \vdash_{\Sigma'} e'$$

for each set of sentences $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and any sentence $(\forall\chi)e' \in \mathbb{S}en(\Sigma)$, where $\chi : \Sigma \to \Sigma' \in \mathcal{D}$.

The entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ of an institution $(\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is *sound* when $E_1 \vdash E_2$ implies $E_1 \models E_2$. Likewise for the entailment system of an institution, the system of rules $(\mathbb{S}ig, \mathbb{S}en, Rl)$ of an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is sound when for each rule $\langle h, c\rangle \in Rl_\Sigma$, we have that $h \models c$. Notice that all the definitions regarding soundness may be extended to the infinitary case and actually all the results bellow hold for both finitary and infinitary case.

The following lemma shows that the free construction of entailment systems from systems of rules preserve the soundness property and explains the actual practice of establishing soundness of the entailment systems which consists only of checking the soundness of the rules.

Lemma 4. *The (infinitary) entailment system of an institution is sound whenever it is freely generated by a sound system of rules.*

Note that the semantic entailment system of an institution which admits universal quantifications over a sub-category $\mathcal{D}$ of signature morphisms satisfies the meta-rule of Generalization. The result below shows that by adding the meta-rule of Generalization to a system of rules the soundness property is preserved.

Lemma 5. *The (infinitary) entailment system with $\mathcal{D}$-universal quantifications of an institution is sound whenever it is freely generated by a sound system of rules.*

3. Universal institutions

Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and

- let $\mathbb{S}en_1$ be a sub-functor of $\mathbb{S}en$ (i.e. $\mathbb{S}en_1 : \mathbb{S}ig \to \mathbb{S}et$ such that $\mathbb{S}en_1(\Sigma) \subseteq \mathbb{S}en(\Sigma)$ and $\varphi(\mathbb{S}en_1(\Sigma)) \subseteq \mathbb{S}en_1(\Sigma')$, for each signature morphism $\varphi : \Sigma \to \Sigma'$), and
- let $\mathcal{D} \subseteq \mathbb{S}ig$ be a sub-category of signature morphisms.

We say that I is a $\mathcal{D}$-universal institution over I_1, where $I_1 = (\mathbb{S}ig, \mathbb{S}en_1, \mathbb{M}od, \models)$ when

- I admits all sentences of form $(\forall\chi)\rho$, where $\chi : \Sigma \to \Sigma'$ is any signature morphism in $\mathcal{D}$ and ρ is any $\mathbb{S}en_1(\Sigma')$ sentence, and
- any sentence of I is semantically equivalent to a sentence of form $(\forall\chi)\rho$ as in the item above.

For example, **UNIV** is a $\mathcal{D}$-universal institution over the restriction of **FOL** to the quantifier-free sentences (i.e. sentences without quantifiers), where $\mathcal{D}$ is the class of all signature extensions with a finite number of constants. Another example is **HCL** which is a $\mathcal{D}$-universal institution over the restriction of **FOL** to sentences of the form $H \Rightarrow C$ where H is a finite conjunction of atoms and C is an atom. Similarly, infinitary versions $\mathbf{UNIV}_\infty$ and $\mathbf{HCL}_\infty$ are also examples of $\mathcal{D}$-universal institutions, but in this case $\mathcal{D}$ is the class of all signature extensions with constants (i.e. $\mathcal{D}$ might contain infinitary extensions).

Horn institutions. An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is a (finitary) $\mathcal{D}$-Horn institution over $I_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$ when I is a $\mathcal{D}$-universal i nstitution over $I_1 = (\mathbb{S}ig, \mathbb{S}en_1, \mathbb{M}od, \models)$ and $\mathbb{S}en_0$ is a sub-functor of $\mathbb{S}en_1$ such that

- for each signature Σ, the institution I_1 admits all sentences of the form $H \Rightarrow C$ where H is any (finite) set of $\mathbb{S}en_0(\Sigma)$ sentences and C is any $Sen_0(\Sigma)$ sentence, and
- any sentence of I_1 is semantically equivalent to a sentence of the form $H \Rightarrow C$ as in the item above.

For example, **HCL** is a finitary $\mathcal{D}$-Horn institution over **Atomic**(**FOL**), where $\mathcal{D}$ is the class of all signature extensions with a finite number of constants. Similarly, $\mathbf{HCL}_\infty$ is an infinitary $\mathcal{D}$-Horn institution over **Atomic**(**FOL**), but in this case $\mathcal{D}$ is the class of all signature extensions with constants (i.e. $\mathcal{D}$ might contain infinitary extensions).

The generic entailment system for Horn institutions developed in this section consists of three layers:

1. The 'atomic' layer is that of the entailment system of I_0, which in the abstract setting is assumed but which is to be developed in the concrete examples.

2. The layer of the entailment system for I_1 which is obtained by adding the so-called 'Modus-Ponens' meta-rule in a restricted form involving the sentences of I_0.
3. The upmost layer is that of the entailment system for I, which is obtained by adding the so-called 'Substitutivity' rule and 'Generalization' meta-rule to the entailment system of I_1.

The soundness and the completeness at each layer is obtained relatively to the soundness and completeness of the layer immediately below.

Such layered decomposition of the entailment system of I leads also to sound and complete entailment systems for universal institutions which are not necessarily Horn institutions. For this it is enough to start with a sound and complete entailment system for I_1. For example, a sound and complete entailment system for quantifier-free sentences in **FOL** determines automatically a sound and complete entailment system for **UNIV**.

3.1. The generic universal entailment system

Let us assume a $\mathcal{D}$-universal institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ over I_1 with $\mathbb{S}en_1$ the sub-functor of $\mathbb{S}en$. We also assume a sub-functor Sen_0 of Sen_1, and denote the corresponding institution by I_0, such that

- for each (finite) set of sentences $B \subseteq \mathbb{S}en_0(\Sigma)$ and any sentence $e \in \mathbb{S}en_1(\Sigma)$ there exists a sentence in $Sen_1(\Sigma)$ which is semantically equivalent to $B \Rightarrow e$.

Remark 1. *This condition is significantly more general than if we assumed that I is a Horn institution over I_0. Indeed, if I were a Horn institution over I_0 for any I_1-sentence $e = (H \Rightarrow C)$ and any (finite) set of I_0-sentences B we have that $B \Rightarrow e$ is semantically equivalent to the I_1-sentence $(H \cup B) \Rightarrow C$. However, the above condition holds also for non-Horn settings such as when $I =$* **UNIV***, the institution of the* **FOL** *universal sentences, $\mathbb{S}en_1$ is a (sub-)functor of the quantifier-free sentences, and $\mathbb{S}en_0$ is the (sub-)functor of the atomic sentences.*

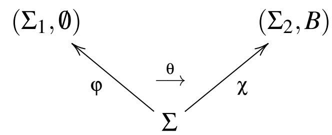
Note also that the above condition comes in two variants: a finitary and an infinitary one. The infinitary variant is applicable only to the infinitary variants of institutions, such as $\mathbf{HCL}_\infty$ or $\mathbf{UNIV}_\infty$.

We also assume another rather mild technical condition, namely that :

- For each $\mathcal{D}$-substitution $\theta : (\varphi : \Sigma \to (\Sigma_1, E_1)) \to (\chi : \Sigma \to (\Sigma_2, E_2))$ in I_0^{pres}, the institution of I_0-presentations, there exists a $\mathcal{D}$-substitution $\varphi \to \chi$ in I_1^{pres} which 'extends'θ. Since there is no danger of confusion we denote this latter substitution by θ too. This means that $\mathbb{S}en_0(\theta)$ extends to a function $\mathbb{S}en_1(\theta) : \mathbb{S}en_1(\Sigma_1) \to \mathbb{S}en_1(\Sigma_2)$ such that the pair $(\mathbb{S}en_1(\theta), \mathbb{M}od^{pres}(\theta))$ constitute a substitution in I_1^{pres}.

In all the examples mentioned above, this condition is fulfilled rather easily since the I_1-sentences are Boolean expressions of I_0-sentences. For example, if I is a $\mathcal{D}$-Horn institution over I_0, then $\mathbb{S}en_0(\theta)$ extends canonically to $\mathbb{S}en_1(\theta)$ by defining $Sen_1(\theta)(H \Rightarrow C)$ by $\mathbb{S}en_0(H) \Rightarrow \mathbb{S}en_0(C)$.

The rule of substitutivity. For all $\mathcal{D}$-universal sentences $(\forall\varphi)\rho$ and all $\mathcal{D}$-substitutions $\theta : (\Sigma \xrightarrow{\varphi} (\Sigma_1, \emptyset)) \to (\Sigma \xrightarrow{\chi} (\Sigma_2, B))$ in I_0^{pres}



we consider the following rules of $\mathcal{D}$-substitutivity

$$(\forall\varphi)\rho \vdash (\forall\chi)(B \Rightarrow \theta(\rho))$$

where $\theta(\rho)$ denotes $\mathbb{S}en_1(\theta)(\rho)$.

Note that the rule of Substitutivity may also be considered in either a finitary and in an infinitary variant. The above formulation corresponds to the infinitary variant since B may be infinite. If in addition we only consider those I_0-presentations (Σ_2, B) for which B is finite, we get the rule of *finitary $\mathcal{D}$-substitutivity.*

Below we will see that under some technical conditions, which are often met in the applications, the rule of Substitutivity may be rephrased to the familiar form

$$(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$$

for $\mathcal{D}$-substitutions θ in I_0 rather than I_0^{pres}. However, our formulation is more general and can be applied in more actual situations.

Proposition 3. *The rule of $\mathcal{D}$-substitutivity is sound.*

Proof. Let M be a Σ-model such that $M \models (\forall\varphi)\rho$ and let M_2 be any χ-expansion of M such that $M_2 \models B$. Because $\mathbb{M}od(\theta)(M_2)$ is a φ-expansion of M (since $\mathbb{M}od(\theta)(M_2)\upharpoonright_{\varphi} = M_2 \upharpoonright_{\chi}$) which by hypothesis satisfies $(\forall\varphi)\rho$, we have that $\mathbb{M}od(\theta)(M_2) \models \rho$. By the satisfaction condition for substitutions, we obtain that $M_2 \models \theta(\rho)$. Since M_2 was an arbitrary expansion of M, we have thus proved that $M \models (\forall\chi)(B \Rightarrow \theta(\rho))$. □

Universal entailment systems. Given a compact entailment system $(\mathbb{S}ig, \mathbb{S}en_1, \vdash^1)$ for I_1, the *$\mathcal{D}$-universal entailment system* of I consists of the free entailment system

- with universal $\mathcal{D}$-quantification (i.e. the meta-rule of 'Generalization')
- over $(\mathbb{S}ig, \mathbb{S}en_1, \vdash^1)$ plus the rules of finitary $\mathcal{D}$-Substitutivity.

This is the finitary version of the universal entailment system. Its *infinitary* variant is obtained by considering the rules of (infinitary) $\mathcal{D}$-Substitutivity, by dropping off the compactness condition, and by considering the infinitary entailment system for I.

The soundness of $\mathcal{D}$-substitutivity given by Proposition 3 has the following consequence.

Corollary 2. *If the entailment system of I_1 is sound then the corresponding universal entailment system for I is sound too.*

Proof. By Lemma 4 we lift the soundness from the entailment system of I_1 to the entailment system which adds the rules of $\mathcal{D}$-substitutivity. Then by Lemma 5 we lift soundness further to the free entailment system with universal $\mathcal{D}$-quantification. □

3.2. Universal completeness

Completeness of the universal entailment systems is significantly more difficult than the soundness property and therefore requires more conceptual infrastructure. The universal completeness result below comes both in a finite and an infinite variant, the finite one being obtained by assuming the finitary version for the entailment system of I_1 and by adding (to the hypotheses of the infinite one) all the finiteness hypotheses marked in brackets.

Theorem 1. *(Universal completeness) The (finitary) universal entailment system for I determined by the entailment system of I_1 as defined above is complete if*

1. *the entailment system of I_1 is complete,*
2. *every signature morphism in $\mathcal{D}$ is (finitely) representable,*
3. *every set of sentences in I_0 is epic basic,*
4. *I_0^{pres} has representable $\mathcal{D}$-substitutions and*
5. *for each set of sentences $E \subseteq \mathbb{S}en_1(\Sigma)$ and each sentence $e \in \mathbb{S}en_1(\Sigma)$, we have that*

$$E \models e \; if \; and \; only \; if \; M_B \models (\wedge E \Rightarrow e) \; for \; each \; set \; of \; sentences \; B \subseteq \mathbb{S}en_0(\Sigma)$$

(where M_B are the models defining B as basic sets of sentences).

Proof. Assume that $\Gamma \models (\forall\chi)e'$ for a set $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and $e' \in \mathbb{S}en_1(\Sigma')$, where $(\chi : \Sigma \to \Sigma') \in \mathcal{D}$. We want to show that $\Gamma \vdash (\forall\chi)e'$. Suppose towards a contradiction that $\Gamma \not\vdash (\forall\chi)e'$.

We define the set of Σ'-sentences

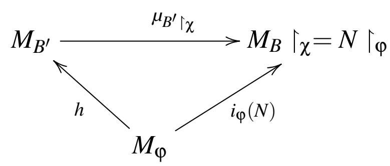
$$\Gamma_1^\chi = \{\rho' \in \mathbb{S}en_1(\Sigma') | \Gamma \vdash (\forall\chi)\rho'\}$$

Suppose $\Gamma_1^\chi \vdash e'$. For the infinitary case we take $\Gamma' = \Gamma_1^\chi$. For the finitary case, since the entailment system of I_1 is compact, there exists a finite $\Gamma' \subseteq \Gamma_1^\chi$ such that $\Gamma' \vdash e'$. Because the universal entailment system of I has $\mathcal{D}$-universal quantification, we have that $\chi(\Gamma) \vdash \rho'$, for all $\rho' \in \Gamma_1^\chi$, thus $\chi(\Gamma) \vdash \Gamma'$. Therefore $\chi(\Gamma) \vdash e'$ and again by the universal $\mathcal{D}$-quantification property for the entailment system of I we obtain $\Gamma \vdash (\forall\chi)e'$, which contradicts our assumption. Thus $\Gamma_1^\chi \not\vdash e'$.

By the completeness of I_1, $\Gamma_1^\chi \not\vdash e'$ implies $\Gamma_1^\chi \not\models e'$. By the hypothesis there exists an epic basic set of sentences $B \subseteq \mathbb{S}en_0(\Sigma')$ such that $M_B \models \Gamma_1^\chi$ but $M_B \not\models e'$. This implies $M_B \upharpoonright_\chi \not\models (\forall\chi)e'$. If we proved that $M_B \upharpoonright_\chi \models \Gamma$ we reached a contradiction with $\Gamma \models (\forall\chi)e'$. We will therefore focus on proving that $M_B \upharpoonright_\chi \models \Gamma$.

Let $(\forall\varphi)e_1 \in \Gamma$, where $(\varphi : \Sigma \to \Sigma_1) \in \mathcal{D}$ and let N be any φ-expansion of $M_B \upharpoonright_\chi$. We have to show that $N \models e_1$. For this we use the following Lemma (which we prove later):

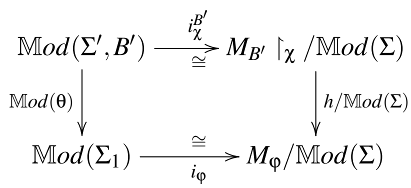
Lemma 6. *There exists a (finite) subset of sentences $B' \subseteq B$ and a homomorphism $h : M_\varphi \to M_{B'} \upharpoonright_\chi$ such that the diagram below commutes:*



where $\mu_{B'}$ is the unique homomorphism $M_{B'} \to M_B$ (because B' is epic basic).

Because $\chi : \Sigma \to \Sigma'$ is representable and B' is epic basic by Lemma 2 $\chi : \Sigma \to (\Sigma', B')$ is representable as presentations morphism.

Because I_0^{pres} has representable $\mathcal{D}$-substitutions, the homomorphism $h : M_\varphi \to M_{B'} \upharpoonright_\chi$ given by the Lemma 6 determines a substitution $\theta : (\varphi : \Sigma \to \Sigma_1) \to (\chi : \Sigma \to (\Sigma', B'))$ such that the following diagram commutes:



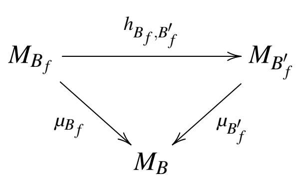
We have that

$$\mathbb{M}od(\theta)(M_B) = i_\varphi^{-1}(h; i_\chi^{B'}(M_B)) = i_\varphi^{-1}(h; \mu_{B'} \upharpoonright_\chi) = i_\varphi^{-1}(i_\varphi(N)) = N$$

By (finitary) $\mathcal{D}$-substitutivity we obtain $\Gamma \vdash (\forall\chi)(B' \Rightarrow \theta(e_1))$. This implies $B' \Rightarrow \theta(e_1) \in \Gamma_1^\chi$. Since $M_B \models \Gamma_1^\chi$, we obtain $M_B \models B' \Rightarrow \theta(e_1)$. Because $M_B \models B'$ we get $M_B \models \theta(e_1)$. By the satisfaction condition for substitutions we obtain that $N \models e_1$.

Proof of Lemma 6.

The infinitary case is rather simple: we take $B' = B$ and consequently $h = i_\varphi(N)$. For the finitary case, first note by using the fact that each subset of B is epic basic we have that $(\mu_{B_f})_{B_f \subseteq B, B_f \text{ finite}}$ is the directed colimit of $(h_{B_f, B'_f})_{B_f \subseteq B'_f \subseteq B \text{ finite}}$



where μ_{B_f} and h_{B_f, B'_f} are the unique model homomorphisms given by the fact that each subset of B is epic basic. Because the reduct functors corresponding to representable signature morphisms preserve directed co-limits(cf. Lemma 3), we have that $(\mu_{B_f} \upharpoonright_\chi)_{B_f \subseteq B, \text{ finite}}$ is also a directed co-limit. Because φ is finitary representable, M_φ is finitely presented. Hence there exists a finite set of sentences $B' \subseteq B$ and a model homomorphism $h : M_\varphi \to M_{B'} \upharpoonright_\chi$ such that $h; \mu_{B'} \upharpoonright_\chi = i_\varphi(N)$. □

Representable substitutions for presentations. The only condition of the completeness theorem which has a rather technical nature is the existence of representable substitutions for presentations. However, in many situations this can be reduced to a simpler form.

Lemma 7. *Let I_0 be an institution with a sub-category $\mathcal{D}$ of representable signature morphisms such that every set of sentences is epic basic. Then the institution of presentations I_0^{pres} has $\mathcal{D}$-substitutions whenever for each signature morphism $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$ in $\mathcal{D}$ and any set E of Σ_2-sentences every homomorphism $h : M_{\chi_1} \to M_E \upharpoonright_{\chi_2}$ determines a I_0^{pres}-substitution $\theta_h : (\chi_1 : \Sigma \to (\Sigma_1, \emptyset)) \to (\chi_2 : \Sigma \to (\Sigma_2, E))$ such that $\mathbb{M}od(\theta_h) = i^E_{\chi_2}; h/\mathbb{M}od(\Sigma); i^{-1}_{\chi}$.*

Proof. Note that because the institution has only epic basic sets of sentences, each presentation (Σ, E) has an initial model $0_{(\Sigma,E)}$ which is precisely M_E, the model defining E as a basic set of sentences.

Let $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$ and let $h : M_{E_1} \upharpoonright_{\chi_1} \to M_{E_2} \upharpoonright_{\chi_2}$ be a Σ-model homomorphism where E_i are sets of Σ_i sentences. We have to show that h determines a I_0^{pres}-substitution $\theta : (\chi_1 : \Sigma \to (\Sigma_1, E_1)) \to (\chi_2 : \Sigma \to (\Sigma_2, E_2))$ such that the diagram below commutes

Let $g = i_{\chi_1}(M_{E_1}); h$. By hypothesis the Σ-homomorphism g generates a substitution $\theta_g : (\Sigma \to (\Sigma_1, \emptyset)) \to (\Sigma \to (\Sigma_2, E_2))$ such that $\mathbb{M}od(\theta_g) = i^{E_2}_{\chi_2}; g/\mathbb{M}od(\Sigma); i^{-1}_{\chi_1}$. We define the substitution $\theta_h : (\Sigma \to (\Sigma_1, E_1)) \to (\Sigma \to (\Sigma_2, E_2))$ as follows

- $\mathbb{S}en(\theta_h) = \mathbb{S}en(\theta)$
- $Mod(\theta_h) = i^{E_2}_{\chi_2}; h/\mathbb{M}od(\Sigma); (i^{-1}_{\chi_1})$.

In order to show that θ_h is a substitution, i.e. the satisfaction condition holds, it suffices to prove that for each (Σ_2, E_2)-model M_2, $\mathbb{M}od(\theta_h)(M_2) = \mathbb{M}od(\theta_g)(M_2)$. We have that $\mathbb{M}od(\theta_h)(M_2) = (i^{E_1}_{\chi_1})^{-1}(h; (M_{E_2} \to M_2)) = i^{-1}_{\chi_1}(i_{\chi_1}(M_{E_1}); h; (M_{E_2} \to M_2)) = i^{-1}_{\chi_1}(g; (M_{E_2} \to M_2)) = \mathbb{M}od(\theta_g)(M_2)$. □

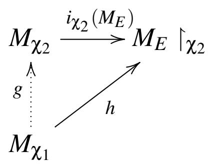
Proposition 4. *In any institution I_0 with a sub-category $\mathcal{D}$ of representable signature morphisms such that*

1. *every set of sentences is epic basic and,*
2. *the representation M_φ of any signature morphism $\varphi \in \mathcal{D}$ is projective with respect to $\mathcal{D}$-reducts of model homomorphisms of the form $0_\Sigma \to M_E$ for all sets E of sentences,*

then the institution of presentations I_0^{pres} has representable $\mathcal{D}$-substitutions whenever I_0 has representable $\mathcal{D}$-substitutions.

Proof. By Lemma 7 it suffices to prove that for every signature morphisms $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$, each Σ-homomorphism $h : M_{\chi_1} \to M_E \upharpoonright_{\chi_2}$, where E is any set of Σ_2-sentences, determines a substitution $\theta_h : (\Sigma \to (\Sigma_1, \emptyset)) \to (\Sigma \to (\Sigma_2, E))$ such that $\mathbb{M}od(\theta_h) = i^E_{\chi_2}; h/\mathbb{M}od(\Sigma); i^{-1}_{\chi_1}$.

Because M_{χ_1} is projective with respect to $M_{\chi_2} = 0_{\Sigma_2} \upharpoonright_{\chi_2} \to M_E \upharpoonright_{\chi_2}$, there exists a homomorphism g such that the diagram below commutes



Because I_0 has representable $\mathcal{D}$-substitutions there exists a $\mathcal{D}$-substitution $\theta_g : \chi_1 \to \chi_2$ in I_0 such that $\mathbb{M}od(\theta_g) = i_{\chi_2}; g/\mathbb{M}od(\Sigma); i_{\chi_1}^{-1}$. We define the substitution $\theta_h : (\Sigma \to (\Sigma_1, \emptyset)) \to (\Sigma \to (\Sigma_2, E))$

- $\mathbb{S}en(\theta_h) = \mathbb{S}en(\theta_g)$.
- $\mathbb{M}od(\theta_h) = i^E_{\chi_2}; h/\mathbb{M}od(\Sigma); i_{\chi_1}^{-1}$.

In order to show that θ_h is a I_0^{pres}-substitution, i.e. the satisfaction condition holds it suffices to prove that $\mathbb{M}od(\theta_h)(M_2) = \mathbb{M}od(\theta_g)(M_2)$ for each (Σ_2, E)-model M_2. We have that $\mathbb{M}od(\theta_h)(M_2) = i_{\chi_1}^{-1}(h; (M_E \to M_2) \upharpoonright_{\chi_2}) = i_{\chi_1}^{-1}(g; i_{\chi_2}(M_E); (M_E \to M_2) \upharpoonright_{\chi_2}) = i_{\chi_1}^{-1}(g; (O_{\Sigma_2} \to M_E) \upharpoonright_{\chi_2}; (M_E \to M_2) \upharpoonright_{\chi_2}) = i_{\chi_1}^{-1}(g; (O_{\Sigma_2} \to M_2) \upharpoonright_{\chi_2}) = i_{\chi_1}^{-1}(g; i_{\chi_2}(M_2)) = \mathbb{M}od(\theta_g)(M_2)$. □

The first condition of the proposition is one of the conditions on I_0 from Birkhoff completeness Theorem 2. The second condition of Proposition 4 (closely related to the concept of 'projectively representable') is very easy to establish in institutions where the model homomorphisms $0_\Sigma \to M_E$ are surjective. One rather typical example is **Atomic**(**FOL**) with $\mathcal{D}$ being the class of all signature extensions with constants.

Corollary 3. **Atomic**(**FOL**)^{pres} *has representable $\mathcal{D}$-substitutions.*

Proof. Each set E of atoms is epic basic and the model homomorphism $0_\Sigma \to M_E$ is surjective. The reducts of surjective model homomorphisms are surjective too. For each signature extension with constants χ the model M_χ (which represents χ) is a free model (i.e. term model) hence it is projective with respect to any surjective homomorphism. **Atomic**(**FOL**) has representable substitutions because each model homomorphism between free models $h : T_\Sigma(X) \to T_\Sigma(Y)$ determines the Σ-substitution θ defined by $\theta(x) = h(x)$ for each $x \in X$. Thus all conditions of the Proposition 4 are fulfilled, hence **Atomic**(**FOL**)^{pres} has representable $\mathcal{D}$-substitutions. □

This type of argument can be replicated in many institutions such as the atomic sub-institutions of **POA** and **OSA**, one notable exception being the institution of existence equations in partial algebra (**PA**). In this example, the model homomorphisms $0_\Sigma \to M_E$ are not necessarily surjective.

The substitutivity rule revisited. The conditions underlying Proposition 4 have also another important consequence: they permit a significantly simpler formulation of the Substitutivity rule which uses substitutions in the base institution rather than in the institution of the presentations. As usually, the finitary variant of the result below requires the conditions in the brackets.

Proposition 5. *Under the conditions of Proposition* 4 *and if the entailment system corresponding to the proof system of* I_1 *has (finitary) Modus Ponens for* $\mathbb{S}en_0$*, meaning that*

$$\Gamma \cup B \vdash_\Sigma e \text{ if and only if } \Gamma \vdash_\Sigma B \Rightarrow e$$

for any sets of sentences $\Gamma \subseteq \mathbb{S}en_1(\Sigma)$ *and (finite)* $B \subseteq \mathbb{S}en_0(\Sigma)$ *and each sentence* $e \in \mathbb{S}en_1(\Sigma)$*, then we may use only Substitutivity rules of the form*

$$(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$$

where θ *is any* $\mathcal{D}$*-substitution in* I_0.

Proof. Let us note that the Substitutivity rules of the form

$$(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$$

for θ any $\mathcal{D}$-substitution in I_0 are just special cases of the full Substitutivity rules by considering $B = \emptyset$. Therefore we have only to show that for any $\mathcal{D}$-substitution $\theta : (\varphi : \Sigma \to (\Sigma_1, \emptyset)) \to (\chi : \Sigma \to (\Sigma_2, B))$ in I_0^{pres} we can have a proof

$$(\forall\varphi)\rho \vdash (\forall\chi)(B \Rightarrow \theta(\rho))$$

by using the Substitutivity rule in the simpler form proposed above.

The key to obtaining such proof lies in proof of Proposition 4 which shows that each I_0^{pres}-substitution $\theta : (\Sigma \xrightarrow{\varphi} (\Sigma_1, \emptyset)) \to (\Sigma \xrightarrow{\chi} (\Sigma_2, B))$ determines a I_0-substitution $\theta' : (\Sigma \xrightarrow{\varphi} \Sigma_1) \to (\Sigma \xrightarrow{\chi} \Sigma_2)$ such that $Sen_0(\theta) = Sen_0(\theta')$. By hypothesis we have that $(\forall\varphi)\rho \vdash (\forall\chi)\theta(\rho)$. Because $\theta(\rho) \bigcup B \vdash \theta(\rho)$ and because I_1 has *Modus Ponens for* $\mathbb{S}en_0$, we have that $\theta(\rho) \vdash B \Rightarrow \theta(\rho)$. Because $(\forall\chi)\theta(\rho) \vdash (\forall\chi)\theta(\rho)$ and because the entailment system of I has universal $\mathcal{D}$-quantification we have that $\chi((\forall\chi)\theta(\rho)) \vdash \theta(\rho)$. This implies $\chi((\forall\chi)\theta(\rho)) \vdash B \Rightarrow \theta(\rho)$ and again by the universal $\mathcal{D}$-quantification property we obtain $(\forall\chi)\theta(\rho) \vdash (\forall\chi)B \Rightarrow \theta(\rho)$ which leads to $(\forall\varphi)\rho \vdash (\forall\chi)B \Rightarrow \theta(\rho)$. □

4. General Birkhoff entailment systems

Birkhoff entailment systems for Horn institutions refine the universal entailment systems defined above by assuming a entailment system for I_0 and defining a entailment system for I_1 rather than assuming an entailment system for I_1. Thus

- we assume an entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash^0)$ for I_0 and
- for I_1 we consider the free entailment system $(\mathbb{S}ig, \mathbb{S}en_1, \vdash)$ over $(\mathbb{S}ig, \mathbb{S}en_0, \vdash^0)$ with (finitary) Modus Ponens for $\mathbb{S}en_0$, i.e.

$$\Gamma \cup B \vdash e \text{ if and only if } \Gamma \vdash B \Rightarrow e$$

  for any $\Gamma \subseteq \mathbb{S}en_1(\Sigma)$, any (finite) $B \subseteq \mathbb{S}en_0(\Sigma)$ and each $e \in \mathbb{S}en_1(\Sigma)$.

The Birkhoff entailment system is finitary if and only if it is generated by using the finitary version of Modus Ponens for $\mathbb{S}en_0$, and $(\mathbb{S}ig, \mathbb{S}en, \vdash^0)$ is generated by finitary rules, otherwise it is infinitary.

Fact 3. *The entailment system of* I_1 *is sound if the entailment system of* I_0 *is sound.*

In order to instantiate the general universal completeness Theorem 1 to the Birkhoff entailment system we need to address the first and the last conditions of the theorem.

The following result addresses the first condition plus the compactness condition from the definition of the finitary $\mathcal{D}$-universal entailment systems. As usual, the result comes in a finitary version (with the information contained within brackets included) and in an infinitary version.

Proposition 6. *Let us assume that*

1. *each set of I_0-sentences is basic, and*
2. *the entailment system of I_0 is complete (and compact).*

Then the entailment system of I_1 is complete (and compact).

Proof. Because the entailment system of I_1 has (finitary) *Modus Ponens for* $\mathbb{S}en_0$ it is enough to prove that

$$\Gamma \models \rho \text{ implies } \Gamma \vdash \rho$$

for each $\Gamma \subseteq \mathbb{S}en_1(\Sigma)$ and each $\rho \in \mathbb{S}en_0(\Sigma)$. Let M_{Γ_0} be the model defining the set of sentences $\Gamma_0 = \{e \in \mathbb{S}en_0(\Gamma) | \Gamma \vdash e\}$ as basic. We use the following couple of lemmas.

Lemma 8. *$M_{\Gamma_0} \models e$ if and only if $\Gamma \vdash e$ for each sentence $e \in \mathbb{S}en_0(\Sigma)$.*

Lemma 9. *$M_{\Gamma_0} \models \Gamma$.*

If $\Gamma \models \rho$ then by Lemma 9 we have that $M_{\Gamma_0} \models \rho$. Now by Lemma 8 we obtain $\Gamma \vdash \rho$.

For the compactness of the entailment system of I_1 first let us recall that the sentences of I_1 are of the form $H \Rightarrow C$ where C is an I_0 sentence and H is a finite set of I_0 sentences. We consider the sub-system of the compact entailments of I_1; this is an entailment (sub-)system by Proposition 2. It contains the entailment system of I_0 since the latter is compact by the hypotheses. It also has the finitary *Modus Ponens for* $\mathbb{S}en_0$ because for any finite $B \subseteq \mathbb{S}en_0(\Sigma)$ the entailment $\Gamma \cup B \vdash e$ is compact if and only if the entailment $\Gamma \vdash B \Rightarrow e$ is compact. Indeed, $\Gamma \cup B \vdash e$ compact means that there exists finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \cup B \vdash e$ which by the *Modus Ponens for* $\mathbb{S}en_0$ property is equivalent to $\Gamma_0 \vdash B \Rightarrow e$ which means $\Gamma \vdash B \Rightarrow e$ is compact. Now because the entailment system of I_1 is the least one containing the entailment system of I_0 and satisfying the finitary *Modus Ponens for* $\mathbb{S}en_0$ property we may conclude that this is exactly the compact sub-system of the entailment system of I_1, which just means that the entailment system of I_1 is compact.

Proof of Lemma 8. The implication from right to left holds by the definition of Γ_0. For the other implication let us consider a sentence e such that $M_{\Gamma_0} \models e$. For any model M such that $M \models \Gamma_0$, because Γ_0 is basic there exists a model homomorphism $M_{\Gamma_0} \to M$. Since $M_{\Gamma_0} \models e$ and e is basic, there exists another model homomorphism $M_e \to M_{\Gamma_0}$. These give a model homomorphism $M_e \to M$ which means $M \models e$. We have thus shown that $\Gamma_0 \models e$.

By the completeness of I_0 we obtain that $\Gamma_0 \vdash e$. For the infinitary case let us take $\Gamma'_0 = \Gamma_0$. For the finitary case, since the entailment system of I_0 is compact, there exists $\Gamma'_0 \subseteq \Gamma_0$ finite such that $\Gamma'_0 \vdash e$. By the definition of Γ_0 we obtain that $\Gamma \vdash \Gamma'_0$ hence $\Gamma \vdash e$.

Proof of Lemma 9. Let us consider that we have an I_1-sentence $H \Rightarrow C \in \Gamma$ and let us assume that $M_{\Gamma_0} \models H$. By Lemma 8 we have that $\Gamma \models H$ and because $H \Rightarrow C \in \Gamma$ and the entailment system for I_1 has (finitary) *Modus Ponens for* $\mathbb{S}en_0$ we obtain that $\Gamma \vdash C$. By Lemma 8 again we deduce $M_{\Gamma_0} \models C$. □

The following shows that the last condition of Theorem 1 is fulfilled.

Proposition 7. *Under the conditions of the Proposition* 6*, for each set of sentences* $E \subseteq \mathbb{S}en_1(\Sigma)$ *and each sentence* $e \in \mathbb{S}en_1(\Sigma)$ *we have that*

$$E \models e \text{ if and only if } M_B \models (E \Rightarrow e) \text{ for each set of sentences } B \subseteq \mathbb{S}en_0(\Sigma)$$

(where M_B *are the models defining* B *as basic sets of sentences).*

Proof. Let $e = H \Rightarrow C$ with $H \subseteq \mathbb{S}en_0(\Sigma)$ and $C \in \mathbb{S}en_0(\Sigma)$. Consider the model $M_{(E \cup H)_0}$ defining $(E \cup H)_0 = \{\rho \in \mathbb{S}en_0(\Sigma) | E \cup H \models \rho\}$. By Lemma 9 we have that $M_{(E\cup H)_0} \models E \cup H$. By the hypothesis this implies $M_{(E\cup H)_0} \models H \Rightarrow C$. Because $M_{(E\cup H)_0} \models H$ too, it follows that $M_{(E\cup H)_0} \models C$. Since C is basic there exists a homomorphism $M_C \to M_{(E\cup H)_0}$.

Now let M be any model such that $M \models E \cup H$. By Lemma 8 we obtain that $M \models (E \cup H)_0$. Because $(E \cup H)_0$ is basic, there exists a homomorphism $M_{(E\cup H)_0} \to M$. We obtain thus a homomorphism $M_C \to M$, which means $M \models C$. □

The Propositions 6 and 7 lead to the following completeness result for Horn institutions obtained as an instance of the general universal completeness theorem.

Theorem 2. *The (finitary) Birkhoff entailment system for a (finitary) $\mathcal{D}$-Horn institution is complete if*

1. *the entailment system of* I_0 *is complete (and compact),*
2. *every signature morphism in* $\mathcal{D}$ *is (finitely) representable,*
3. *every set of sentences in* I_0 *is epic basic and*
4. I_0^{pres} *has representable* $\mathcal{D}$*-substitutions.*

5. Instances of Birkhoff completeness

In order to develop concrete sound and complete Birkhoff entailment systems we need to set the parameters of the completeness theorem for each example.

5.1. The Birkhoff entailment system of HCL

We set the parameters of the completeness theorem for **HCL** as follows:

- the institution I is **HCL**
- the institution I_0 is **Atomic**(**FOL**)
- $\mathcal{D}$ is the class of all signature extensions with a finite number of constants
- the system of proof rules for **Atomic**(**FOL**) is given by the following set of rules for any **FOL** signature (S, F, P):
  - $(R)\emptyset \vdash t = t$ for each term t
  - $(S)t = t' \vdash t' = t$ for any terms t, t'
  - $(T)\{t = t', t' = t''\} \vdash t = t''$ for any terms t, t', t''

- $(F)\{t_i = t'_i | 1 \le i \le n\} \vdash \sigma(t_1,...,t_n) = \sigma(t'_1,...,t'_n)$ for any $\sigma \in F$
- $(P)\{t_i = t'_i | 1 \le i \le n\} \cup \{\pi(t_1,...,t_n)\} \vdash \pi(t'_1,...,t'_n)$ for any $\pi \in P$

Proposition 8. Atomic(FOL) *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of atoms for a signature (S,F,P) we define

$$\equiv_E = \{(t,t') | E \vdash t = t'\}$$

By (R), (S), (T) and (F) this is a congruence on T_F. Then we define a model M_E as follows:

- the (S,F)-algebra part of M_E is defined as the quotient of the initial algebra (term algebra) T_F by $\equiv_E$, and
- for each relation symbol $\pi \in P$, we define $(M_E)_\pi = \{x/_{\equiv_E} | E \vdash \pi(x)\}$

The definition of $(M_E)_\pi$ is correct because of the rule (P). Now we note that for each (S,F,P)-atom ρ we have $E \vdash \rho$ if and only if $M_E \models \rho$. Now if $E \models \rho$ then $M_E \models \rho$ which means $E \vdash \rho$.

□

We are now able to formulate the following corollary of the general Birkhoff completeness theorem.

Corollary 4. *The finitary entailment system for* **HCL** *is sound and complete. Moreover, this entailment system is obtained as the free entailment system*

- *with universal quantification and*
- *with implication at the quantifier-free level i.e. for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences*

$$\Gamma \cup H \vdash C \text{ if and only if } \Gamma \vdash H \Rightarrow C$$

generated by the following system of finitary rules for a signature (S,F,P)

- $(R)\emptyset \vdash t = t$ *for each term* t
- $(S)t = t' \vdash t' = t$ *for any terms* t,t'
- $(T)\{t = t', t' = t''\} \vdash t = t''$ *for any terms* t,t',t''
- $(F)\{t_i = t'_i | 1 \le i \le n\} \vdash \sigma(t_1,...,t_n) = \sigma(t'_1,...,t'_n)$ *for any* $\sigma \in F$
- $(P)\{t_i = t'_i | 1 \le i \le n\} \cup \{\pi(t_1,...,t_n)\} \vdash \pi(t'_1,...,t'_n)$ *for any* $\pi \in P$
- $(Subst)(\forall Y)\rho \vdash (\forall X)\theta(\rho)$ *for any (S,F,P)-sentence $(\forall Y)\rho$ and for any substitution* $\theta : Y \to T_F(X)$.

The following is the infinitary version of the above corollary.

Corollary 5. *The infinitary Birkhoff entailment system* $\mathbf{HCL}_\infty$ *is sound and complete for the same rules as in the corollary above.*

Remark 2. *We can also consider the case when the set of relation is empty, obtaining thus a completeness result for equational logic,* **CEQL**.

5.2. The Birkhoff entailment system of Horn(POA)

We set the parameters of the completeness theorem for **Horn**(**POA**).

- the institution I is **Horn**(**POA**)
- the institution I_0 is **Atomic**(**POA**)
- $\mathcal{D}$ is the class of all signature extensions with a finite number of constants
- the system of proof rules for **Atomic**(**POA**) is given by the following set of rules for any **POA** signature (S,F):
  - $(R)\emptyset \vdash t = t$ for each term t
  - $(S) t = t' \vdash t' = t$ for any terms t, t'
  - $(T)\{t = t', t' = t''\} \vdash t = t''$ for any terms t, t', t''
  - $(F)\{t_i = t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1,...,t_n) = \sigma(t'_1,...,t'_n)$ for any $\sigma \in F$
  - $(R')\emptyset \vdash t \leq t$ for each term t
  - $(T')\{t \leq t', t' \leq t''\} \vdash t \leq t''$ for any terms t, t', t''
  - $(F')\{t_i \leq t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1,...,t_n) \leq \sigma(t'_1,...,t'_n)$ for any $\sigma \in F$
  - $(ET)\{t_1 = t_2,\ t_2 \leq t_3,\ t_3 = t_4\} \vdash t_1 \leq t_4$ for any terms t_1, t_2, t_3, t_4

For the readers not familiar with preorder algebras we give the following definition:

Definition 7. *A (preorder) congruence relation on a (S,F)-preorder algebra M is a pair $(\equiv, \sqsubseteq)$ where $\equiv$ is a (S,F)-congruence relation and $\sqsubseteq$ is a preorder on M which*

- preserve the preorder structure of M, i.e. $m \leq m'$ implies $m \sqsubseteq m'$ for all elements $m, m' \in M$,

- is compatible with operations in F, i.e. $m \leq m'$ implies $M_\sigma(m) \leq M_\sigma(m')$ for all operations $\sigma \in F_{w,s}$ and all elements $m, m' \in M_w$, and

- is compatible with the congruence $\equiv$, i.e. $m_1 \equiv m_2$, $m_2 \sqsubseteq m_3$ and $m_3 \equiv m_4$ implies $m_1 \sqsubseteq m_4$ for all elements $m_1, m_2, m_3, m_4 \in M$.

Proposition 9. **Atomic**(**POA**) *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of atoms for a signature (S,P) we define $(\equiv_E, \sqsubseteq_E)$

- $\equiv_E = \{(t,t') \mid E \vdash t = t'\}$
- $\sqsubseteq_E = \{(t,t') \mid E \vdash t \leq t'\}$

By the above rules $(\equiv_E, \sqsubseteq_E)$ is a preorder congruence on the term algebra T_F. Then we define the preorder algebra M_E as the quotient of the term algebra by $(\equiv_E, \sqsubseteq_E)$. We note that for each equational or transitional (S,F)-atom ρ

$$E \vdash \rho \text{ if and only if } M_E \models \rho$$

Now if $E \models \rho$ then $M_E \models \rho$ which means $E \vdash \rho$. □

Corollary 6. *The finitary entailment system for* **Horn**(**POA**) *is sound and complete. Moreover, this entailment system is obtained as the free entailment system*

- *with universal quantification and*
- *with implication at the quantifier-free level i.e. for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences $\Gamma \cup H \vdash C$ if and only if $\Gamma \vdash H \Rightarrow C$.*

generated by the following system of finitary rules for a signature (S,F)

- $(R)\emptyset \vdash t = t$ *for each term* t
- $(S)t = t' \vdash t' = t$ *for any terms* t,t'
- $(T)\{t = t', t' = t''\} \vdash t = t''$ *for any terms* t,t',t''
- $(F)\{t_i = t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1,...,t_n) = \sigma(t'_1,...,t'_n)$ *for any* $\sigma \in F$
- $(R')\emptyset \vdash t \leq t$ *for each term* t
- $(T')\{t \leq t', t' \leq t''\} \vdash t \leq t''$ *for any terms* t,t',t''
- $(F')\{t_i \leq t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1,...,t_n) \leq \sigma(t'_1,...,t'_n)$ *for any* $\sigma \in F$
- $(ET)\{t_1 = t_2,\ t_2 \leq t_3,\ t_3 = t_4\} \vdash t_1 \leq t_4$ *for any terms* t_1,t_2,t_3,t_4
- $(Subst)(\forall Y)\rho \vdash (\forall X)\theta(\rho)$ *for any* (S,F,P)*-sentence* $(\forall Y)\rho$ *and for any substitution* $\theta : Y \to T_F(X)$.

Corollary 7. *The infinitary Birkhoff entailment system* $\mathbf{Horn}(\mathbf{POA})_\infty$ *is sound and complete for the same rules as in the corollary above.*

5.3. The Birkhoff entailment system of Horn(OSA)

We set the parameters of the completeness theorem for $\mathbf{Horn}(\mathbf{OSA})$ as follows:

- the institution I is $\mathbf{Horn}(\mathbf{OSA})$,
- the institution I_0 is $\mathbf{Atomic}(\mathbf{OSA})$,
- $\mathcal{D}$ is the class of all signature extensions with a finite number of constants, and
- the system of proof rules for $\mathbf{Atomic}(\mathbf{OSA})$ is given by the following set of rules for any **OSA** signature $(S,\leq,F)$:
  - $(R)\emptyset \vdash t = t$ for each term t
  - $(S)t = t' \vdash t' = t$ for any terms t,t'
  - $(T)\{t = t', t' = t''\} \vdash t = t''$ for any terms t,t',t''
  - $(F)\{t_i = t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1,...,t_n) = \sigma(t'_1,...,t'_n)$ for any $\sigma \in F$

We give the definition of congruence relation on an order sorted model.

Definition 8. *A congruence relation* $\equiv$ *on a* $(S,\leq,F)$*-model* M *is a* (S,F)*-congruence relation* $\equiv = (\equiv_s)_{s\in S}$ *such that if* $s \leq s'$ *in* $(S,\leq)$ *and* $a,a' \in M_s$ *then* $a \equiv_s a'$ *if and only if* $a \equiv_{s'} a'$.

Proposition 10. $\mathbf{Atomic}(\mathbf{OSA})$ *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of equations for a signature $(S,\leq,F)$ we define

$$\equiv_E = \{(t,t') | E \vdash t = t'\}$$

Since the signature $(S,\leq,F)$ is regular the term algebra T_F is the initial $(S,\leq,F)$-algebra in $\mathbb{M}od(S,\leq,F)$. By (R), (S), (T) and (F) this is an F-congruence on T_F. $\equiv_E$ is also an order sorted congruence on T_F, because the definition of $\equiv_E$ does not depend upon a sort. Since the signature $(S,\leq,F)$ is locally filtered we may define a model M_E as the quotient of the initial algebra (term algebra) T_F by order sorted congruence $\equiv_E$.

Notice that for each $(S,\leq,F)$-equation $t = t'$, $E \vdash t = t'$ if and only if $M_E \models t = t'$. Now if $E \models t = t'$ then $M_E \models t = t'$ which means $E \vdash t = t'$. □

We are now able to formulate the following corollary of the general Birkhoff completeness theorem.

Corollary 8. *The finitary entailment system for* **Horn**(OSA) *is sound and complete. Moreover, this entailment system is obtained as the free entailment system*

- *with universal quantification and*
- *with implication at the quantifier-free level i.e. for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences $\Gamma \cup H \vdash C$ if and only if $\Gamma \vdash H \Rightarrow C$.*

generated by the following system of finitary rules for a signature $(S, \leq, F)$

- $(R)\emptyset \vdash t = t$ *for each term* t
- $(S)t = t' \vdash t' = t$ *for any terms* t, t'
- $(T)\{t = t', t' = t''\} \vdash t = t''$ *for any terms* t, t', t''
- $(F)\{t_i = t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1, ..., t_n) = \sigma(t'_1, ..., t'_n)$ *for any* $\sigma \in F$
- $(Subst)(\forall Y)\rho \vdash (\forall X)\theta(\rho)$ *for any (S, F, P)-sentence $(\forall Y)\rho$ and for any substitution* $\theta : Y \to T_F(X)$.

Corollary 9. *The infinitary Birkhoff entailment system* **Horn(OSA)**$_\infty$ *is sound and complete for the same rules as in the corollary above.*

5.4. The Birkhoff entailment system of Horn(PA)

We set the parameters of the completeness theorem for **Horn**(**PA**) as follows:

- the institution I is **Horn**(**PA**),
- the institution I_0 is **Atomic**(PA) (the restriction of **PA** to the existence equations),
- $\mathcal{D}$ is the class of all signature extensions with a finite number of total constants, and
- the system of proof rules for **Atomic**(PA) is given by the following set of rules for any **PA** signature (S, TF, PF):
  - (S) $t \stackrel{e}{=} t' \vdash t' \stackrel{e}{=} t$ for any terms t, t'
  - (T) $\{t \stackrel{e}{=} t',\ t' \stackrel{e}{=} t''\} \vdash t \stackrel{e}{=} t''$ for any terms t, t', t''
  - (C) $\{t_i \stackrel{e}{=} t'_i,\ def(\sigma(t_1, \ldots, t_n)),\ def(\sigma(t'_1, \ldots, t'_n))\} \vdash \sigma(t_1, \ldots, t_n) \stackrel{e}{=} \sigma(t'_1, \ldots, t'_n)$ for any $\sigma \in TF \cup PF$
  - $(Totality)$ $\{def(t_i) \mid i = \overline{1,n}\} \vdash def(\sigma_t(t_1, \ldots, t_n))$ for any $\sigma_t \in TF$
  - $(Subterm)$ $def(\sigma(t_1, \ldots, t_n)) \vdash \{def(t_i) \mid i \in \overline{1,n}\}$ for any $\sigma \in TF \cup PF$

We give the definition of partial congruence relation.

Definition 9. *A congruence relation $\equiv$ on a (S, TF, PF)-model M is a S-sorted equivalence relation $\equiv = (\equiv_s)_{s \in S}$ such that for every operation symbol $\sigma \in TF \cup PF$ and elements m, $m' \in M$ with $m \equiv m'$ if both $M_\sigma(m)$ and $M_\sigma(m')$ are defined then $M_\sigma(m) \equiv M_\sigma(m')$.*

For each set E of existence (S, TF, PF)-equations we denote by S_E the set of all subterms of the terms which occur in E and by $def(E)$ the set of sentences $\{def(t) \mid t \in S_E\}$. We define a model $M_{def(E)}$ as follows

- $(M_{def(E)})_s = (T_{TF}(S_E))_s$ for each sort $s \in S$, where $T_{TF}(S_E)$ is the free (S, TF)-algebra of terms with elements from S_E,

- $(M_{def(E)})_{\sigma_t}(t_1,\ldots,t_n) = \sigma_t(t_1,\ldots,t_n)$ for each total operation symbol $\sigma_t \in TF$ and any terms $t_1,\ldots,t_n \in T_{TF}(S_E)$, and

- $(M_{def(E)})_{\sigma}(t_1,\ldots,t_n) = \sigma(t_1,\ldots,t_n)$ if $\sigma(t_1,\ldots,t_n) \in S_E$ for each partial operation symbol $\sigma \in PF$.

Proposition 11. **Atomic**(**PA**) *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of atoms for a signature (S,TF,PF) we define

$$\equiv_E = \{(t,t')|E \vdash t \stackrel{e}{=} t'\}$$

We use the following Lemma (which we prove later).

Lemma 10. *For every set of existence equations $E \subseteq \mathbb{S}en(S,TF,PF)$ we have that $E \vdash def(t)$ if and only if $t \in M_{def(E)}$.*

Firstly we prove that $\equiv_E$ is a congruence relation on $M_{def(E)}$. The reflexivity of $\equiv_E$ is given by the above Lemma. The first two rules ensure the symmetry and the transitivity of $\equiv_E$. By the rule (C) we have that $\equiv_E$ is a congruence relation on $M_{def(E)}$.

For each existence equation $t \stackrel{e}{=} t'$ we have $E \vdash t \stackrel{e}{=} t' \Leftrightarrow t \equiv_E t' \Leftrightarrow M_{def(E)}/_{\equiv_E} \models t \stackrel{e}{=} t'$. If $E \models t \stackrel{e}{=} t'$ then $M_{def(E)}/_{\equiv_E} \models t \stackrel{e}{=} t'$ which implies $E \vdash t \stackrel{e}{=} t'$.

Proof of Lemma 10.

"the only if part" one can easily prove by induction in the definition of $\vdash$ that $E \vdash t \stackrel{e}{=} t'$ implies $t,t' \in M_{def(E)}$.

"the if part" We prove this by induction on the structure of the term t. Let $\sigma(t_1,\ldots,t_n)$ be a term such that $t_i \in M_{def(E)}$, for all $i \in \{1,\ldots,n\}$. By the hypothesis induction we have $E \vdash def(t_i)$, for all $i \in \{1,\ldots,n\}$.

- if $\sigma \in TF$ then by *Totality* rule we obtain $E \vdash def(\sigma(t_1,\ldots,t_n))$

- if $\sigma \in PF$ then by the definition of $M_{def(E)}$ we have $\sigma(t_1,\ldots,t_n) \in S_E$. By the definition of S_E there exists an existence equation $t_1 \stackrel{e}{=} t_2 \in E$ such that $t \in S_{t_1 \stackrel{e}{=} t_2}$. Without loss of generality we assume that $t \in S_{def(t_1)}$. We have $E \vdash t_1 \stackrel{e}{=} t_2$ and $E \vdash t_2 \stackrel{e}{=} t_1$ which implies $E \vdash def(t_1)$. By *Subterm* rule, $def(t_1) \vdash def(t)$. So $E \vdash def(t)$. □

The case of **Horn**(**PA**) requires more technical constructions. We address the conditions of Theorem 2.

First order substitutions in PA. Given a **PA** signature (S,TF,PF) and two sets of new total constants X and Y, a *first order* (S,TF,PF)*-substitution* from X to Y consists of a mapping $\theta : X \to T_{TF\cup PF}(Y)$ of the variables X with $(TF \cup PF)$-terms over Y. Let $def(\theta)$ to denote the set $\{def(\theta(x)) \mid x \in X\}$ of $(S,TF \cup Y,PF)$-sentences.

On the semantics side, each (S,TF,PF)-substitution $\theta : X \to T_{TF\cup PF}(Y)$ determines a functor $\mathbb{M}od(\theta) : \mathbb{M}od((S,TF \cup Y,PF), def(\theta)) \to \mathbb{M}od(S,F \cup X,P)$ defined by

- $\mathbb{M}od(\theta)(M)_x = M_x$ for each sort $x \in S$, or operation symbol $x \in TF \cup PF$, and

- $\mathbb{M}od(\theta)(M)_x = M_{\theta(x)}$, i.e. the evaluation of the term $\theta(x)$ in M, for each $x \in X$. Notice that since $M \models def(\theta)$ the term $\theta(x)$ which may contain partial operation symbols is evaluated in the model M.

On the syntax side, θ determines a sentence translation function $\mathbb{S}en(\theta) : \mathbb{S}en(S, TF \cup X, PF) \to \mathbb{S}en(S, TF \cup Y, PF)$ which in essence replaces all symbols from X with the corresponding $(TF \cup Y \cup PF)$-terms according to θ.

- $\mathbb{S}en(\theta)(t_1 \stackrel{e}{=} t_2)$ is defined as $\overline{\theta}(t) \stackrel{e}{=} \overline{\theta}(t')$ for each $(S, TF \cup X, PF)$-existence equation $t_1 = t_2$, where $\overline{\theta} : T_{TF \cup PF}(X) \to T_{TF \cup PF}(Y)$ is the unique extension of θ to an $(TF \cup PF)$-homomorphism ($\overline{\theta}$ is replacing variables $x \in X$ with $\theta(x)$ in each $(TF \cup X \cup PF)$-term t).

- $\mathbb{S}en(\theta)(\rho_1 \wedge \rho_2)$ is defined as $\mathbb{S}en(\theta)(\rho_1) \wedge \mathbb{S}en(\theta)(\rho_2)$ for each conjunction $\rho_1 \wedge \rho_2$ of $(S, TF \cup X, PF)$-sentences, and similarly for the case of any other logical connectives.

- $\mathbb{S}en(\theta)((\forall Z)\rho)$ is defined as $(\forall Z)\mathbb{S}en(\theta_Z)(\rho)$ for each $(S, TF \cup X \cup Z, PF)$-sentence ρ, where θ_Z is the trivial extension of θ to an $(S, TF \cup Z, PF)$-substitution.

The satisfaction condition is given by the proposition bellow.

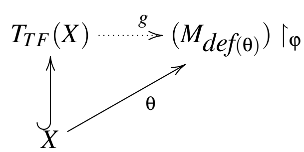
Proposition 12. *For each* **PA***-signature and each (S, TF, PF)-substitution* $\theta : X \to T_{TF \cup PF}(Y)$

$$\mathbb{M}od(\theta)(M) \models t_1 \stackrel{e}{=} t_2 \text{ if and only if } M \models \mathbb{S}en(\theta)(t_1 \stackrel{e}{=} t_2)$$

for each $(S, TF \cup Y, PF)$-model M which satisfies $def(\theta)$ and each existence $(S, TF \cup X, PF)$-equation $t_1 \stackrel{e}{=} t_2$.

Proof. By noticing that $\mathbb{M}od(\theta)(M)_t = M_{\overline{\theta}(t)}$ for each $(TF \cup X \cup PF)$-term t. □

We denote by χ and φ the inclusion signature morphisms $(S, TF, PF) \hookrightarrow (S, TF \cup X, PF)$ and $(S, TF, PF) \hookrightarrow (S, TF \cup Y, PF)$. Notice that there exists an unique homomorphism $g : T_{TF}(X) \to (M_{def(\theta)}) \restriction_{\varphi}$ such that $g(x) = \theta(x)$ for all $x \in X$.



Proposition 13. $\mathbb{M}od(\theta) = i_{\varphi}^{def(\theta)}; g/\mathbb{M}od(S, TF, PF); i_{\chi}^{-1}$.

Proof. For any $(S, TF \cup Y, PF)$-model M which satisfies $def(\theta)$ we have

- $(i_{\varphi}^{def(\theta)}; g/\mathbb{M}od(S, TF, PF); i_{\chi}^{-1})(M)_x = M_x = \mathbb{M}od(\theta)(M)_x$ for each sort $x \in S$, or operation symbol $x \in TF \cup PF$

- $(i_{\varphi}^{def(\theta)}; g/\mathbb{M}od(S, TF, PF); i_{\chi}^{-1})(M)_x = i_{\chi}^{-1}(g; (M_{def(\theta)} \to M) \restriction_{\varphi})_x$
$= (g; (M_{def(\theta)} \to M) \restriction_{\varphi})(x) = (M_{def(\theta)} \to M) \restriction_{\varphi} (g(x)) = (M_{def(\theta)} \to M)(g(x)) =$
$(M_{def(\theta)} \to M)(\theta(x)) = M_{\theta(x)} = \mathbb{M}od(\theta)(M)_x$ for each $x \in X$. □

Proposition 14. **Atomic**$(\mathbf{PA})^{pres}$ *has representable $\mathcal{D}$-substitutions.*

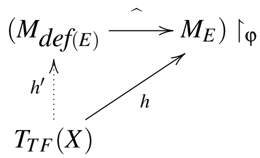
Proof. Let $\chi : (S,TF,PF) \hookrightarrow (S,TF\cup X,PF)$ and $\varphi : (S,TF,PF) \hookrightarrow (S,TF\cup Y,PF)$ be two extensions with total constants of the signature (S,TF,PF) and let $h : T_{TF}(X) \to M_E \upharpoonright_\varphi$ be a (S,TF,PF)-homomorphism where E is a set of existence $(S,TF\cup Y,PF)$-equations. By Lemma 7 it suffices to show that there exists a substitution
$\theta : X \to T_{TF\cup PF}(Y)$ such that

$$(i^E_\varphi; h/\mathbb{M}od(S,TF,PF); i^{-1}_\chi) : \mathbb{M}od((S,TF\cup Y,PF),E) \to \mathbb{M}od((S,TF\cup X,PF),\emptyset)$$

is a sub-functor of

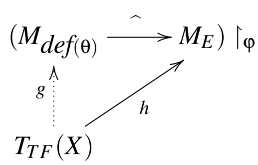
$$\mathbb{M}od(\theta) : \mathbb{M}od((S,TF\cup Y,PF),def(\theta)) \to \mathbb{M}od((S,TF\cup X,PF),\emptyset)$$

The reducts of surjective model homomorphisms are surjective too and since the algebra of terms $T_{TF}(X)$ is projective with respect to all surjections there exists a model homomorphism $h' : T_{TF}(X) \to M_{def(E)} \upharpoonright_\varphi$ such the following diagram commutes.



We define the substitution $\theta : X \to T_{TF\cup PF}(Y)$ as the restriction of h' to the set of variables X, i.e. $\theta(x) = h'(x)$, for each $x \in X$. There exists an unique (S,TF,PF)-homomorphism $g : T_{TF}(X) \to (M_{def(\theta)}) \upharpoonright_\varphi$ such that $g(x) = \theta(x)$, for each $x \in X$. Because $g(x) = \theta(x) = h'(x)$, for each $x \in X$, we have that

- $\theta(X) \subseteq S_E$ which implies $M_{def(\theta)} \subseteq M_{def(E)}$, and
- the following diagram commutes



For each $(S,TF\cup Y,PF)$-model $(i^E_\varphi; h/\mathbb{M}od(S,TF,PF); i^{-1}_\chi)(M) = i^{-1}_\chi(h; (M_E \to M) \upharpoonright_\varphi) = i^{-1}_\chi(g; (M_{def(\theta)} \to M) \upharpoonright_\varphi) = (i^{def(\theta)}_\varphi; g/\mathbb{M}od(S,TF,PF); i^{-1}_\chi)(M) = \mathbb{M}od(\theta)(M)$. □

The Substitutivity rules for Horn(PA).

Proposition 15. *In* **Horn(PA)** *we may use substitutivity rules of the form*

$$(\forall Y)\rho \vdash (\forall X)def(\theta) \Rightarrow \theta(\rho)$$

Proof. Notice that the Substitutivity rules of the form

$$(\forall Y)\rho \vdash (\forall X)def(\theta) \Rightarrow \theta(\rho)$$

where $\theta : Y \to T_{TF\cup PF}(X)$ is a any (S,TF,PF)-substitution are just special cases of Substitutivity rules by considering $B = def(\theta)$. Therefore we have only to show that for any substitution $\theta : Y \to T_{TF\cup PF}(X)$ we can have a proof

$$(\forall Y)\rho \vdash (\forall X)B \Rightarrow \theta(\rho)$$

by using the Substitutivity rule in the simpler form proposed above. Since $\mathbb{M}od(\theta) : \mathbb{M}od((S,TF\cup X,PF),def(\theta)) \to \mathbb{M}od(S,TF\cup Y,PF)$, $B \models def(\theta)$ which implies (by Proposition 11) $B \vdash def(\theta)$. Because the institution I_1 has implications we have $def(\theta) \cup (def(\theta) \Rightarrow \theta(\rho)) \vdash \theta(\rho)$. We obtain $B \cup (def(\theta) \Rightarrow \theta(\rho)) \vdash \theta(\rho)$ and using again the fact that I_1 has implications $def(\theta) \Rightarrow \theta(\rho) \vdash B \Rightarrow \theta(\rho)$.

Because $(\forall\chi)def(\theta) \Rightarrow \theta(\rho) \vdash (\forall\chi)def(\theta) \Rightarrow \theta(\rho)$ and because the entailment system of I has universal $\mathcal{D}$-quantification we have that $\chi((\forall\chi)def(\theta) \Rightarrow \theta(\rho)) \vdash def(\theta) \Rightarrow \theta(\rho)$. This implies $\chi((\forall\chi)def(\theta) \Rightarrow \theta(\rho)) \vdash B \Rightarrow \theta(\rho)$ and again by the universal $\mathcal{D}$-quantification property we obtain $(\forall\chi)def(\theta) \Rightarrow \theta(\rho) \vdash (\forall\chi)B \Rightarrow \theta(\rho)$ which leads to $(\forall\varphi)\rho \vdash (\forall\chi)B \Rightarrow \theta(\rho)$. □

We are now able to formulate the following corollary of the general Birkhoff completeness theorem.

Corollary 10. *The finitary entailment system for* **Horn**(**PA**) *is sound and complete. Moreover, this entailment system is obtained as the free entailment system*

- *with universal quantification and*
- *with implication at the quantifier-free level i.e. for each quantifier-free Horn sentence $H \Rightarrow C$ and all sets Γ of quantifier-free Horn sentences $\Gamma \cup H \vdash C$ if and only if $\Gamma \vdash H \Rightarrow C$.*

generated by the following system of finitary rules for a signature (S,TF,PF)

- (S) $t \overset{e}{=} t' \vdash t' \overset{e}{=} t$ *for any terms* t,t'
- (T) $\{t \overset{e}{=} t',\ t' \overset{e}{=} t''\} \vdash t \overset{e}{=} t''$ *for any terms* t,t',t''
- (C) $\{t_i \overset{e}{=} t'_i,\ def(\sigma(t_1,\dots,t_n)),\ def(\sigma(t'_1,\dots,t'_n))\} \vdash \sigma(t_1,\dots,t_n) \overset{e}{=} \sigma(t'_1,\dots,t'_n)$ *for any* $\sigma \in TF \cup PF$
- $(Totality)$ $\{def(t_i) \mid i = \overline{1,n}\} \vdash def(\sigma_t(t_1,\dots,t_n))$ *for any* $\sigma_t \in TF$
- $(Subterm)$ $def(\sigma(t_1,\dots,t_n)) \vdash \{def(t_i) \mid i \in \overline{1,n}\}$ *for any* $\sigma \in TF \cup PF$
- $(Subst)$ $(\forall Y)\rho \vdash (\forall X)def(\theta) \Rightarrow \theta(\rho)$ *for any* (S,TF,PF)*-sentence* $(\forall Y)\rho$ *and for any substitution* $\theta : Y \to T_{TF\cup PF}(X)$.

The following is the infinitary version of the above corollary.

Corollary 11. *The infinitary Birkhoff entailment system* $\mathbf{Horn}(\mathbf{PA})_\infty$ *is sound and complete for the same rules as in the corollary above.*

6. Conclusions and future work

The present result is the first generalization of Birkhoff Completeness to the institutional case. We developed a common framework that allows one to formulate and prove complete deduction for variants and generalizations of equational logic. The paper captures uniformly both finitary and infinitary case and it shows the connection between the structure of the sentences and the proof of completeness. The approach distinguishes clearly the specific aspects of an institution from the general ones yielding to the decomposition of the completeness on three layers. Such an approach leads to unexpected results.

- Theorem 1 is applicable also to universal institutions which are not necessarily Horn institutions, such as **UNIV**.
- the conditions of Proposition 6 do not require that the sentences of I_0 to be epic basic. In this case we may start with an institution I_0 which has sentences of the form $(\exists X)e$ where e is a an atom or a conjunction of atoms.

The practical aspect of the present paper is that it provides complete calculus for the logics underlying the modern algebraic specification languages.

An interesting direction of future work is to extend the result to modal logics. Another direction of further research is to provide sound and complete systems of proof rules for first order institutions like **FOL**, **POA** and **PA**. An institutional version of Godel's completeness theorem by redoing Henkin's proof may be found in [19] but this is applicable only to the finitary case. We suggest a different approach by forcing techniques which covers also the infinitary case in classical first order logic.

Concerning related work, another abstract calculus for equational logics is developed in [22], in a categorial framework, based on satisfaction as injectivity.

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