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Cross-Term Conservation Relationships for Electromagnetic Energy, Linear Momentum, and Angular Momentum

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Cross-term conservation relationships for electromagnetic energy, linear momentum, and angular momentum are derived and discussed here. When two or more sources of electromagnetic fields are present, these relationships connect the cross terms that appear in the traditional expressions for the electromagnetic (1) energy, (2) linear momentum, and (3) angular momentum, over to, respectively, (1) the sum of the rates of work, (2) the sum of the forces, and (3) the sum of the torques, that are due to the fields of each charge or current source acting upon the other charge and current sources. These relationships, although not new, appear to be rarely recognized and used in the physics literature. As shown here, they can be extremely helpful for solving and gaining a deeper physical understanding into a rather diverse range of interesting problems in electrodynamics, including (1) aspects of Poynting's theorem when applied to charged point particles, (2) the detailed physical basis of electrostatic analysis, (3) understanding the connection between different techniques used in the past for solving Casimir force problems, and (4) reconciling the invalidity of Newton's third law in electrodynamics.

1. INTRODUCTION

A set of conservation relationships are derived and discussed here for the "cross-terms" that appear in the conventional expressions for electromagnetic energy, momentum, and angular momentum. These "cross-term," as will be referred to here, entail terms such as $(\mathbf{E}_1 \cdot \mathbf{E}_2 + \mathbf{B}_1 \cdot \mathbf{B}_2)$, that arises when expanding the expression

$$(\mathbf{E}_1 + \mathbf{E}_2)^2 + (\mathbf{B}_1 + \mathbf{B}_2)^2$$

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or, for example, $(\mathbf{E}_1 \times \mathbf{B}_2) + (\mathbf{E}_2 \times \mathbf{B}_1)$, that arises when expanding

$$(\mathbf{E}_1 + \mathbf{E}_2) \times (\mathbf{B}_1 + \mathbf{B}_2)$$

where $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{B}_1$, and $\mathbf{B}_2$, are electric and magnetic fields associated with two different charge and/or current sources. The three relationships for the cross-terms of electromagnetic energy, momentum, and angular momentum that will be discussed here relate these cross-term expressions to, respectively, the sum of the rates of work, the sum of the forces, and the sum of the torques, due to each charge and current distribution acting on the other one.

These cross-tenn relationships, although certainly known, appear to be rarely discussed and used in the physics literature, but, as discussed here, are often extremely helpful when solving various electrodynamic problems that involve more than one electromagnetic source. Indeed, a number of classical problems are treated here by making direct use of these cross-term relationships, often resulting in exact solutions and usually enabling a deeper insight and easier means for obtaining these solutions. One key advantage to these relationships is that they can be used to solve certain classes of problems while avoiding the singularities encountered when dealing with classical charged point particles and point multipole charge distributions. Thus these relationships enable one to glean aspects of information, such as the sum of the rates of work of two electromagnetic sources acting on each other, without needing to encounter complicated singularity and renormalization issues. Moreover, these relationships also certainly apply to nonsingular, continuous charge and current sources, and are helpful here as well for easily extracting the same type of information.

The outline of this article is the following. In Sec. 2, these cross-term relationships are motivated and discussed further by noting how they can help to supplement the normal discussion on Poynting's theorem[1–4] when considering charged point particles.[2] Here, the singularity issues that can be avoided with the cross-term relationships are explicitly discussed. In Sec. 3 of this article, these cross-term relationships are derived and briefly discussed. As shown here, the derivations are close to, but not the same as, those used in standard textbooks[1–4] for obtaining relationships for the time-rate of change of the full electromagnetic quantities; they differ in the use of cross-term quantities. Section 4 turns to a number of examples of applications of the cross-term relationships. Most of the problems discussed here have been analyzed and published in the physics literature

[2] At the beginning of Ref. 3 (see p. 2), Stratton states that only continuous charge distributions will be considered. He discusses only the macroscopic Maxwell equations, while the present article deals with the microscopic Maxwell equations. Reference 4 deals primarily with the macroscopic Maxwell equations, at least until Chap. 19.

before but have not made use of the cross-term relationships. As will be seen, the cross-term relationships enable the desired results to be obtained by easier means, and often enable exact solutions to be obtained. The problems discussed here were chosen to attempt to provide an interesting range of electrodynamic examples, from such fundamental topics as (1) the invalidity (except in the electrostatic case) of Newton's third law for classical electromagnetic forces[(5–7)] and (2) the connection between work done by electrostatic forces and changes in electrostatic energy due to small displacements of charged bodies,[3] to more advanced and specialized topics such as (3) the calculation of van der Waals forces in quantum electrodynamics, as well as (4) the calculation of the classical lag effect, which is closely tied to the Aharonov–Bohm effect in quantum mechanics.[(8, 9)]

2. POYNTING'S THEOREM AND THE CROSS-TERM RELATIONSHIPS

Poynting's theorem, as derived in standard textbooks on electromagnetism,[(1–4), 2] states that the rate of work done by electromagnetic fields upon a charge distribution in a volume V is related to the rate of change of electromagnetic energy in V plus the rate of electromagnetic energy radiated out of V. Similar relationships hold for electromagnetic linear momentum and angular momentum, as well as more complicated quantities.[(10, 11), 4] However, these conservation laws, as usually derived, hold only for continuous charge distributions. In the case of point charges, singularities arise that prevent the finite evaluation of the quantities appearing in these relationships.[(12), 5]

Most of the standard textbooks on electromagnetism do not explicitly state that their steps in deriving Poynting's theorem hold only for continuous charge and current distributions. Indeed, upon reading some of the derivations, one cannot help but feel that the author is implying that the derivation is intended to include the case of point charges.[6] Undoubtedly

[3] See, for example, Ref. 1, Sec. 1.11.

[4] See, in particular, Ref. 10. Also see Secs. 3.17 and 4.9–4.12 in Ref. 11.

[5] For an interesting related discussion, see Ref. 12.

[6] For an example where Poynting's theorem is inappropriately implied to hold for point charges, see p. 76 of Ref. 2. The assumption of $e\mathbf{v}\cdot\mathbf{E}=(d/dt)\,E_{\text{kin}}$, just after Eq. (31.3), implies that point charges are being considered. A similar problem occurs in Ref. 1 with the consideration of Eq. (6.113), $\mathbf{F}=q(\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}$. Also, at several other points in the discussion of Sec. 6.7 in Ref. 1, "particles" are referred to rather than continuous charge and current sources. As long as the particles are extended charged particles, there exists no problem with the derivation; however, this restriction is not explicitly made, except for one brief mention of continuous charge and current sources at the beginning of Sec. 6.7. Consequently, many students will undoubtedly think in terms of point particles when reading the derivation.

this implication is just an oversight and is unintentional, or possibly it is done for pedagogical reasons to avoid raising the confusing difficulties associated with point charges too early in the text. Alternatively, the authors may want the students studying the text to intuitively feel comfortable with the central quantity

$$\int \mathbf{J} \cdot \mathbf{E}\, d^3x$$

that appears in Poynting's theorem, or the quantity

$$\int \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) d^3x$$

that appears in the related theorem on conservation of momentum. By thinking in terms of charged point particles, then the student will identify the above two quantities as being the sum of the rates of work

$$\sum_{i\,(\text{sum over particles})} q_i \mathbf{E}[\mathbf{z}_i(t), t] \cdot \frac{d\mathbf{z}_i(t)}{dt}$$

and the sum of the forces

$$\sum_{i\,(\text{sum over particles})} q_i \left\{ \mathbf{E}[\mathbf{z}_i(t), t] + \frac{1}{c} \frac{d\mathbf{z}_i(t)}{dt} \times \mathbf{B}[\mathbf{z}_i(t), t] \right\}$$

on a system of point charges. Unfortunately, the danger exists here that if the student thinks more deeply about the above quantities and realizes that the fields **E** and **B** in Poynting's theorem are the *total* electromagnetic fields, including the fields at the particle itself, then more confusion may arise since the above two quantities do not in general have finite values for point charges.

Of course, the authors of these textbooks are well aware of the difficulties involved with dealing with point charges; indeed, later in their texts they usually devote a large portion of material specifically to the appropriate equation of motion governing the charged point particle. The reasoning involved with deducing this equation of motion is directly tied to ideas about conservation of energy and momentum, as in Poynting's theorem and the related electromagnetic theorem on conservation of momentum. However, in the case of point charges, any discussion on conservation of energy and momentum must confront the singularities arising from the self-fields of the particles. The usual derivations of Poynting's theorem in standard textbooks make no mention of this issue; resolving this issue can be done, but only with additional physical arguments, such as are found in the renormalization steps discussed in Refs. 13 and 14.

Consider, for example, the usual expression for the electromagnetic energy in a volume V, due to two point charges:

$$U_{\text{E\&M}} = \frac{1}{8\pi} \int_V d^3x [(\mathbf{E}_\text{A} + \mathbf{E}_\text{B})^2 + (\mathbf{B}_\text{A} + \mathbf{B}_\text{B})^2] \tag{1}$$

where the fields labelled by A and B will be taken here to be the full retarded fields of the two particles. From Eq. (1), $U_{\text{E\&M}}$ can be written as the sum of three terms: namely, two single-particle electromagnetic energy terms,

$$U_{\text{A(B)}} = \frac{1}{8\pi} \int_V d^3x [(\mathbf{E}_{\text{A(B)}})^2 + (\mathbf{B}_{\text{A(B)}})^2] \tag{2}$$

where either the label A or B would be used in Eq. (1), plus the electromagnetic energy due to the cross-term

$$U_{\text{A-B}} = \int_V d^3x \, u_{\text{A-B}} \tag{3}$$

where

$$u_{\text{A-B}}(\mathbf{x}, t) = \frac{1}{8\pi} [(\mathbf{E}_\text{A} \cdot \mathbf{E}_\text{B} + \mathbf{E}_\text{B} \cdot \mathbf{E}_\text{A}) + (\mathbf{B}_\text{A} \cdot \mathbf{B}_\text{B} + \mathbf{B}_\text{B} \cdot \mathbf{B}_\text{A})] \tag{4}$$

(Writing $u_{\text{A-B}}$ in the particular way given above emphasizes its correspondence with the structure of other cross-terms encountered later.)

For points in space at a small distance R from a particle, the integrand in Eq. (2) varies as $1/R^4$. Since the volume of a spherical shell equals $4\pi R^2 \, dR$, then the single particle energy terms in Eq. (2) are divergent. Consequently, the usual formulation of Poynting's theorem cannot be directly applied to point charges.[7] Moreover, analogous singularities occur

[7] One might wonder why a simple use of the Dirac delta formalism doesn't help to get around these problems, by assigning $\rho(\mathbf{x}) = q\delta^3(\mathbf{x} - \mathbf{z}(t))$ and $\mathbf{J}(\mathbf{x}) = q\dot{\mathbf{z}}(t)\,\delta^3(\mathbf{x} - \mathbf{z}(t))$ to the charge and current densities, respectively, of a point charged particle at position $\mathbf{z}(t)$. A moment's reflection, though, reveals the difficulty. Upon following the usual derivations for energy or momentum conservation, one encounters terms like $\int d^3x \, \mathbf{J} \cdot \mathbf{E}$ and $\int d^3x (\rho \mathbf{E} + (1/c)\, \mathbf{J} \times \mathbf{B})$. Upon inserting $\rho(\mathbf{x}) = q\delta^3(\mathbf{x} - \mathbf{z}(t))$ and $\mathbf{J}(\mathbf{x}) = q\dot{\mathbf{z}}(t)\,\delta^3(\mathbf{x} - \mathbf{z}(t))$ into these expressions, one sees the very singular nature of the resulting integrals, due to the fields $\mathbf{E}$ and $\mathbf{B}$ being singular at $\mathbf{x} = \mathbf{z}(t)$.

in the electromagnetic linear momentum and angular momentum of point charges, resulting in the inapplicability of the standard conservation relationships associated with these quantities.

When considering energy and linear momentum, a renormalization procedure for removing singularities can be carried out by appropriately combining mechanical kinetic energy and mechanical linear momentum with the singular terms that arise in the electromagnetic energy and momentum expressions. Dirac first formulated this procedure, which resulted in a renormalized mass for a classical charged point particle and an equation of motion that is now usually referred to as the Lorentz–Dirac equation:[11, 13, 14]

$$m\frac{d^2z^\mu}{d\tau^2}=\frac{2}{3}\frac{q^2}{c^3}\left[\frac{d^3z^\mu}{d\tau^3}-\frac{1}{c^2}\left(\frac{d^2z^\lambda}{d\tau^2}\frac{d^2z_\lambda}{d\tau^2}\right)\frac{dz^\mu}{d\tau}\right]+\frac{q}{c}F^{\mu\nu}\frac{dz_\nu}{d\tau}+F^\mu_{\text{ext}} \tag{5}$$

where $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$ represents the usual electromagnetic tensor due to incident radiation plus the retarded fields of other electromagnetic sources, and F^μ_{ext} represents other external four-forces. A similar, but more complicated renormalization procedure also exists for removing singularities in the angular momentum case.[14]

As will be shown here, relationships for point charges, as well as for continuous charge distributions, can be derived without encountering singularities for the time-rate of change of the cross-term quantities associated with electromagnetic energy, linear momentum, and angular momentum. For example, note that unlike Eq. (2), $U_{\text{A-B}}$ in Eq. (3) is nonsingular since the integrand varies as $1/R^2$ near each particle. Moreover, the cross-term relationships may be applied to charge distributions of even a more singular nature than point charges, namely, point multipole charge distributions. This point is explained at the end of Sec. 3, as its validity is certainly not apparent from the present discussion.

This derivation of these cross-term relationships does not require the use of a renormalization procedure. In general, however, the physical information embodied by these cross-term relationships cannot serve as a substitute for the energy–momentum relationship of Eq. (5) or for angular momentum relationships obtained via renormalization procedures.[14] The latter two relationships, obtained via renormalization, impose constraints that restrict particle motion to specific trajectories. As described in the following section, the cross-term relationships do not contain these constraints, but rather they apply to whatever trajectory is prescribed for the particles. Hence, the cross-term relationships certainly do not include all the information that is embodied, for example, within the Lorentz–Dirac equation, such as being able to predict the trajectory of a particle.

Nevertheless, as will be shown, the cross-term relationships enable one to obtain a variety of other useful information about the electromagnetic interactions between charge and current distributions, while avoiding singularity problems. In particular, these relationships directly provide information on the sum of the rates of work, the sum of the forces, and the sum of the torques, for charge and current sources acting on each other.

3. DERIVATION AND DISCUSSION OF ELECTROMAGNETIC CROSS-TERM RELATIONSHIPS

Let $\mathbf{E}_A$ and $\mathbf{B}_A$ be any electric and magnetic fields that are solutions of Maxwell's equations, where the electromagnetic sources are given by a charge density $\rho_A(\mathbf{x}, t)$ and a current density $\mathbf{J}_A(\mathbf{x}, t)$. Usually $\mathbf{E}_A$ and $\mathbf{B}_A$ are chosen as the retarded fields associated with the sources $\rho_A(\mathbf{x}, t)$ and $\mathbf{J}_A(\mathbf{x}, t)$. In the derivations to follow, however, one could just as well choose the electromagnetic fields to be, for example, advanced fields instead of retarded ones. If ρ_A and $\mathbf{J}_A$ both equal zero, then $\mathbf{E}_A$ and $\mathbf{B}_A$ are free fields; whether they are, for example, incident free fields or advanced free fields, also does not matter in the derivations that follow.

Now consider any second set of fields $\mathbf{E}_B$ and $\mathbf{B}_B$ that are solutions of Maxwell's equations associated with the electromagnetic sources $\rho_A(\mathbf{x}, t)$ and $\mathbf{J}_A(\mathbf{x}, t)$. The following steps then immediately apply:

$$\mathbf{J}_A(\mathbf{x}, t) \cdot \mathbf{E}_B(\mathbf{x}, t) = \frac{c}{4\pi}\left(\nabla \times \mathbf{B}_A - \frac{1}{c}\frac{\partial \mathbf{E}_A}{\partial t}\right) \cdot \mathbf{E}_B$$

$$= -\frac{c}{4\pi}\nabla \cdot (\mathbf{E}_B \times \mathbf{B}_A) - \frac{1}{4\pi}\left(\mathbf{B}_A \cdot \frac{\partial \mathbf{B}_B}{\partial t} + \frac{\partial \mathbf{E}_A}{\partial t} \cdot \mathbf{E}_B\right)$$

By reversing A and B in Eq. (6) and adding the two equations together, one obtains

$$\mathbf{J}_A(\mathbf{x}, t) \cdot \mathbf{E}_B(\mathbf{x}, t) + \mathbf{J}_B(\mathbf{x}, t) \cdot \mathbf{E}_A(\mathbf{x}, t) = -\frac{\partial}{\partial t} u_{A\text{-}B}(\mathbf{x}, t) - \nabla \cdot \mathbf{S}_{A\text{-}B}(\mathbf{x}, t) \tag{7}$$

where

$$\mathbf{S}_{A\text{-}B}(\mathbf{x}, t) = \frac{c}{4\pi}(\mathbf{E}_A \times \mathbf{B}_B + \mathbf{E}_B \times \mathbf{B}_A) \tag{8}$$

and $u_{A\text{-}B}$ is given in Eq. (4).

Integrating over a volume V that contains the charge distributions immediately yields the result that the work per time

$$\int_V d^3x\, \mathbf{J}_A(\mathbf{x}, t) \cdot \mathbf{E}_B(\mathbf{x}, t)$$

by the fields of source B upon the charge distribution A, plus the work per time

$$\int_V d^3x\, \mathbf{J}_B(\mathbf{x}, t) \cdot \mathbf{E}_A(\mathbf{x}, t)$$

of the analogous quantity, equals the negative of the sum of (1)

$$\frac{d}{dt}\int_V d^3x\, u_{A\text{-}B}(\mathbf{x}, t)$$

the time-rate of change of electromagnetic energy within V due to the cross terms of the total electromagnetic energy density, plus (2)

$$\oint_S d^2x\, \hat{\mathbf{n}} \cdot \mathbf{S}_{A\text{-}B}(\mathbf{x}, t)$$

the electromagnetic energy per unit time leaving the volume V through the surface S due to the cross terms of the total Poynting vector. If one set of fields, for example, $\mathbf{E}_B$ and $\mathbf{B}_B$, is free incident fields, then $\mathbf{J}_B \cdot \mathbf{E}_A = 0$; hence, the only work done would be by these free fields acting on the A-distribution. If more than two sets of fields and sources are present, i.e., $\mathbf{E}_{total} = \mathbf{E}_A + \mathbf{E}_B + \mathbf{E}_C + \cdots$, then every pair of fields and sources will satisfy the above energy-rate cross-term relationship.

Thus, the above relationship governs the sum of the rates of work that are done by the electromagnetic fields of each charge distribution acting on the other charge distribution and expresses them in terms of the cross-term parts of the remaining quantities in Poynting's theorem. This result is typical of the three cross-term relationships that will be discussed in this article. These relationships are obtained by breaking the total field into parts, $\mathbf{E}_{total} = \mathbf{E}_A + \mathbf{E}_B + \mathbf{E}_C + \cdots$, and physically attributing the field parts $\mathbf{E}_A$, $\mathbf{E}_B$, etc., to different charge and current sources. This procedure follows the usual practice of identifying fields with sources.

Since Eq. (7) does not consider the rate of work done by the fields of a charge distribution upon itself, then the singularities encountered for point charges in Poynting's theorem are not found here. To apply Eq. (7) to point charges, let

$$\rho_{A(B)}(\mathbf{x}, t) = q_{A(B)}\delta^3[\mathbf{x} - \mathbf{z}_{A(B)}(t)] \tag{9}$$

$$\mathbf{J}_{A(B)}(\mathbf{x}, t) = q_{A(B)}\dot{\mathbf{z}}_{A(B)}(t)\,\delta^3[\mathbf{x} - \mathbf{z}_{A(B)}(t)] \tag{10}$$

where $\mathbf{z}_{A(B)}(t)$ equals the position of the A(B) particle at time t, and $\dot{\mathbf{z}}_{A(B)} = (d/dt)\,\mathbf{z}_{A(B)}$. Integrating over a volume V that contains the particles then yields

$$q_A\dot{\mathbf{z}}_A(t)\cdot\mathbf{E}_B[\mathbf{z}_A(t), t] + q_B\dot{\mathbf{z}}_B(t)\cdot\mathbf{E}_A[\mathbf{z}_B(t), t]$$

$$= -\frac{d}{dt}\int_V d^3x\, u_{A\text{-}B} - \oint_S d^2x\,\hat{\mathbf{n}}\cdot\mathbf{S}_{A\text{-}B} \tag{11}$$

In a sense, the derivation of the energy-rate cross-term relationship of Eq. (7) is a generalization of the usual derivation of Poynting's theorem as applied to the microscopic Maxwell equations. Thus, the usual theorem follows immediately from Eq. (7) by (1) considering only continuous charge distributions, (2) by letting A equal B in Eq. (7), and (3) by treating $\mathbf{E}_A$, $\mathbf{B}_A$, ρ_A, and $\mathbf{J}_A$ as the total fields and charge sources that are present. The equation that results was first derived in 1884 by Poynting[15] for the case of the macroscopic Maxwell equations.

Often in standard textbooks, one proceeds from Poynting's theorem to a conservation law for total energy. (See, e.g., Refs. 1 and 2.) If all forces are electromagnetic, then one can write for continuous charge distributions

$$\int_V d^3x\,\mathbf{J}_{\text{total}}\cdot\mathbf{E}_{\text{total}} = \frac{d}{dt}\,U_{\text{mech}} \tag{12}$$

where U_{mech} represents the mechanical kinetic energy associated with the charge distribution. In this way, a full conservation relationship is obtained for the sum of mechanical kinetic energy plus electromagnetic energy:

$$\frac{d}{dt}(U_{\text{mech}} + U_{\text{E\&M}}) = -\oint_S d^2x\,\hat{\mathbf{n}}\cdot\mathbf{S}_{\text{total}} \tag{13}$$

The step made in Eq. (12) involves a somewhat subtle point. Here, an assumption was made concerning the rate of change of mechanical kinetic energy; this assumption cannot be deduced from Maxwell's equations. A similar statement applies to the derivations of the analogous electrodynamic conservation theorems of linear and angular momentum, where an assumption must be introduced concerning the time-rate of change of mechanical linear momentum and mechanical angular momentum, respectively. All the other steps in the derivations of these three conservation relationships follow entirely from Maxwell's equations, as they involve only electromagnetic quantities. In particular, the following quantities appear: (1) the rate of work, (2) the force, and (3) the torque on a charge distribution due to the Lorentz force per unit volume

$$\rho(\mathbf{x}, t)\,\mathbf{E}(\mathbf{x}, t) + \frac{1}{c}\,\mathbf{J}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t)$$

By making the additional assumptions that equate these three quantities to, respectively, the time-rate of change of (1) mechanical kinetic energy [as in Eq. (12)], (2) mechanical linear momentum, and (3) mechanical angular momentum, then the three conservation relationships follow. (The implicit assumption is made in these derivations that only electromagnetic forces are present.)

The importance of noting the above point lies in understanding a characteristic of the cross-term relationships for point charges that may at first seem peculiar. Equation (11) illustrates this feature: no matter what trajectories the two point charges may follow, Eq. (11) is perfectly valid. Indeed, Eq. (11) places no restrictions on the motions of the two charges. This result arises because Maxwell's equations are perfectly valid for arbitrary trajectories of charges, and only Maxwell's equations were used in obtaining Eq. (11). However, if other constraints are imposed, as in Eq. (12), then the allowed motions become restricted. In particular, when demanding that the total electromagnetic and mechanical energy and linear momentum be conserved for a single point charge in an incident electromagnetic field, and after carrying out a particular renormalization procedure, then a specific equation of motion can be deduced: namely, the Lorentz–Dirac equation corresponding to this situation.[8]

Finally, the same technique used to obtain the electromagnetic cross-term energy-rate relationship of Eq. (7) can be used to obtain other electromagnetic cross-term relationships. Below, the electromagnetic linear momentum- and angular momentum-rate cross-term relationships are given, where all quantities are evaluated at position $\mathbf{x}$ and time t:

[8] For further explanation on the indicated renormalization procedure, see Ref. 14, Sec. 8.1, as well as the comment at the end of Sec. 7.2. This renormalization method removes the dependence of the sum of the material four-momentum plus the bound electromagnetic four-momentum upon the orientation of the surface $\sigma(\tau)$ described in Ref. 14.

$$\left[\rho_A \mathbf{E}_B + \frac{1}{c}(\mathbf{J}_A \times \mathbf{B}_B)\right] + \left[\rho_B \mathbf{E}_A + \frac{1}{c}(\mathbf{J}_B \times \mathbf{B}_A)\right]$$
$$= -\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{S}_{A\text{-}B} + \nabla \cdot \overleftrightarrow{\mathbf{T}}_{A\text{-}B} \quad (14)$$

$$\mathbf{x} \times \left(\rho_A \mathbf{E}_B + \frac{1}{c}\mathbf{J}_A \times \mathbf{B}_B\right) + \mathbf{x} \times \left(\rho_B \mathbf{E}_A + \frac{1}{c}\mathbf{J}_B \times \mathbf{B}_A\right)$$
$$= -\frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{x} \times \mathbf{S}_{A\text{-}B}) - \nabla \cdot (\overleftrightarrow{\mathbf{T}}_{A\text{-}B} \times \mathbf{x}) \quad (15)$$

where

$$T_{A\text{-}B,ij} = \frac{1}{4\pi}\left[(E_{Ai}E_{Bj} + E_{Bi}E_{Aj}) + (B_{Ai}B_{Bj} + B_{Bi}B_{Aj})\right.$$
$$\left. - \delta_{ij}(\mathbf{E}_A \cdot \mathbf{E}_B + \mathbf{B}_A \cdot \mathbf{B}_B)\right] \quad (16)$$

It should be noted that the results of this section can be written in a covariant form. For example, just as the usual electromagnetic energy-rate and momentum-rate relationships can be written in the covariant form of[9]

$$\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c}F^{\beta\gamma}J_\gamma \quad (17)$$

so also can the corresponding cross-term relationships of Eqs. (7) and (14) be written as[(16), 10]

$$\partial_\alpha \Theta^{\alpha\beta}_{A\text{-}B} = -\frac{1}{c}F^{\beta\gamma}_{A}J_{B\gamma} - \frac{1}{c}F^{\beta\gamma}_{B}J_{A\gamma} \quad (18)$$

where $\Theta^{\alpha\beta}_{A\text{-}B}$ represents the cross terms of the electromagnetic symmetric energy–momentum tensor.

Finally, before leaving this section, a clarification needs to be made on the applicability of the electromagnetic cross-term relationships and the usual full-field relationships, when singular charge distributions are present. Poynting's theorem holds for continuous charge distributions and can be applied when moving surface charges are present. However, Poynting's theorem does not hold in general for regions of space that include higher-order singular charge distributions, due to, for example, moving (1) line charges, (2) point charges, and (3) point multipoles.[11]

[9] See, for example, p. 611 in Ref. 1.

[10] Rowe presented this cross-term relationship in Ref. 16, following Eq. (64) on p. 3653.

[11] See, for example, Secs. 1.6–1.9 in Ref. 4 for a discussion on singular charge distributions.

In contrast to the above restriction, the cross-term relationships are generally valid, provided that two charge distributions are considered that do not have singularities at the same point in space. At first glance, the volume integral of the right-hand side of either Eq. (7), Eq. (14), and Eq. (15) might appear to yield a singular result when a region of space is considered that contains, for example, an electric dipole moving at a constant velocity in the presence of incident fields. In particular, the integrals $\int d^3x\, \dot{u}_{\text{A-B}}$, $\int d^3x\, \dot{\mathbf{S}}_{\text{A-B}}$, and $\int d^3x(\mathbf{x}\times\dot{\mathbf{S}}_{\text{A-B}})$, might cause some concern. Nevertheless, although some care must be taken in carrying out the volume integral on the right-hand sides of Eqs. (7), (14), and (15), the integrals do exist, and are finite. This result is seen most easily by considering the left-hand sides. For a stationary electric dipole in the presence of incident fields, the volume integral on the left-hand side of Eq. (14) yields

$$\begin{aligned}\int d^3x\, \rho_{\mathrm{A}}(\mathbf{x}, t)\, \mathbf{E}_{\mathrm{B}}(\mathbf{x}, t) &= \int d^3x[-\mathbf{p}_{\mathrm{A}}\cdot\nabla\delta^3(\mathbf{x}-\mathbf{R}_{\mathrm{A}})]\, \mathbf{E}_{\mathrm{in}}(\mathbf{x}, t)\\ &= +(\mathbf{p}_{\mathrm{A}}\cdot\nabla_{\mathbf{R}_{\mathrm{A}}})\, \mathbf{E}_{\mathrm{in}}(\mathbf{R}_{\mathrm{A}}, t) \qquad (19)\end{aligned}$$

which illustrates the operation that must be carried out in general when dealing with such singular charge distributions. Thus, the cross-term relationships hold in general provided that the singularities of each of the charge distributions lie at points where the fields of the other charge distribution are continuous, and continuous in all spatial derivatives up to a sufficiently high order of derivative that the volume integrals, as in Eq. (19), can be evaluated by repeated operations of integration by parts.

4. APPLICATIONS

Several examples are given here to illustrate how the electromagnetic cross-term relationships can provide additional physical insight into electromagnetic interactions between charge distributions, as well as aid in carrying out calculations for various problems. In several of the examples discussed here, although not all of them, references and comparisons are made to problems tackled by other researchers. A reasonable understanding of these problems should be obtainable from the material described here without excessive referencing to these other articles and texts; however, for all the details, the reader will need to consult these other references. This situation seems unavoidable in order to show how the cross-term relationships can simplify, and often generalize, the previous approaches to these problems, without lengthening this article excessively by repeating all the details of the problems.

As a starting point, the cross-term relationships may be applied to several examples of interacting point charges discussed by Page and Adams in Refs. 5 and 6. Using an expansion of the fields in powers of $(1/c)$, they calculated the sum of the Lorentz forces due to each particle acting on the other. The point was made that the sum does not equal zero, thereby violating the Newtonian idea of action and reaction being equal in magnitude and opposite in direction. Upon calculating the rate of change of momentum associated with the electromagnetic cross terms, they found that the sum of this quantity plus the Lorentz force terms, equaled zero. Page and Adams evaluated the quantities explicitly, but limited their analysis to order $1/c^2$. They gave a similar demonstration for the sum of torques due to Lorentz forces plus the rate of change of angular momentum associated with the electromagnetic cross terms.

The cross-term relationships derived in Sec. 3 involve just the ideas illustrated by Page and Adams but show that the ideas hold exactly, not just to terms of order $1/c^2$. For example, by substituting Eqs. (9) and (10) in Eq. (14) and then integrating over all space, the left-hand side becomes equal to what Page and Adams investigated to order $1/c^2$. The surface term at infinity that arises from the right-hand side of Eq. (14) represents radiated linear momentum. This term becomes evident only at order $1/c^4$ and higher, since the fields at infinity that contribute to this surface term are the acceleration fields, where each contain a factor of $1/c^2$. Thus, the contribution of this radiated linear momentum term is beyond the range of the $1/c^2$ approximation in Refs. 5 and 6. In the analysis of the sum of the torques, the radiation of angular momentum is similarly beyond the range of their expansion.

Thus the analysis of Page and Adams is indeed correct, but their results could have been carried out exactly and by a means that is considerably less involved than their analysis to order $1/c^2$. This $1/c^2$ analysis was done by evaluating all terms explicitly, which is advantageous if the explicit forms of the terms are desired, but not required if the only point is to verify the general ideas represented by the cross-term relationships. Moreover, these general physical ideas are not fully reflected in the relationships verified by Page and Adams, since (1) the cross-term relationships hold for an arbitrary volume V of space, and not just for the case of infinite space examined in Refs. 5 and 6; (2) the cross-term relationships immediately apply for continuous charge and current distributions as well as for point charge; and (3) the physical contribution of the radiated energy, linear momentum, and angular momentum does not show up in the $1/c^2$ analysis. To illustrate more explicitly, upon integrating Eq. (14) over any volume V that contains two charge distributions, one obtains the result that the sum of the Lorentz forces of each charge distribution acting on the other one is equal to the negative of the cross-term contributions due to (1) the time-rate of change of the electromagnetic linear momentum in V, plus (2) the electromagnetic momentum flow out of V. Similar statements hold for the sum of the torques, as well as for the case of the sum of the rates of work.

A similar analysis to that of Page and Adams was carried out by Boyer in Ref. 7. Here, the interactions of particles traveling at constant velocities were investigated exactly (i.e., not just to order $1/c^2$). Boyer's main point was the discussion and explicit verification of these three relationships, namely, Eqs. (46)–(48) in Ref. 7. These three equations are just the cross-term relationships derived here for electromagnetic linear momentum, energy, and angular momentum; however, Boyer deduced them by quite different reasoning[12] and verified them for particles moving only at constant velocity and not for arbitrary trajectories. Moreover, the need for the elimination of singular terms, as described in Ref. 7, is completely removed here by dealing only with the cross-term components.

As another application, the electromagnetic energy-rate cross-term relationship of Eq. (7) can be shown to provide a different viewpoint and basis, from conventional discussions, on the connection between electrostatic forces and changes in electrostatic energy. The following example illustrates this point.

Consider a charge distribution ρ_A that is finite in extent and that is stationary in some inertial frame. The calculations that follow are carried out in this frame. Consequently, $\mathbf{J}_A = 0$ and $\mathbf{B}_A = 0$; also, $\mathbf{E}_A$ must fall off as fast, or faster, at large distances than $1/R^n$, for some n where $n \geqslant 2$. Let a point charge q_B be displaced from $\mathbf{z}_B(t)$ to $\mathbf{z}_B(t+\delta t)$ during the time interval $t \to t+\delta t$. From Eq. (7),

$$q_B \dot{\mathbf{z}}_B(t) \cdot \mathbf{F}_{\text{Lor, A} \to \text{B}}(t) = -\frac{d}{dt}\left(\int_V d^3x \, \frac{2}{8\pi} \mathbf{E}_A \cdot \mathbf{E}_B\right) - \oint_S d^2x \, \hat{\mathbf{n}} \cdot \frac{c}{4\pi} (\mathbf{E}_A \times \mathbf{B}_B) \tag{20}$$

where $\mathbf{F}_{\text{Lor, A} \to \text{B}}$ is the Lorentz force due to ρ_A acting on q_B, and q_B is contained in the volume V. If V is taken to be all of space, then the surface term in Eq. (20) goes to zero, since $\mathbf{E}_A \times \mathbf{B}_B$ can vary at most like $1/R^3$ on S. Consider Eq. (20) after being multiplied by δt. Let $\Delta \mathbf{z}_B = \dot{\mathbf{z}}_B(t)\,\delta t$. Since

[12] Boyer dealt with the total electromagnetic quantities of energy and momentum. Cutoff procedures were then used, for particles moving at constant velocity, to extract the change in relevant quantities: namely, the cross terms of the present article. It should be noted that the surface terms are missing in Eqs. (46)–(48) of Ref. 7, as they were in Refs. 5 and 6. For particles moving at constant velocity, this omission is correct, since the surface term at infinity equals zero.

Eq. (20) holds for arbitrary trajectories of q_B, then Δz_{Bi} can be independently varied, yet still the equation must hold. Hence,

$$F_{\text{Lor, A}\to\text{B},\,i}(t) = -\frac{\Delta}{\Delta z_{\text{Bi}}} \int d^3x \frac{2}{8\pi} [\mathbf{E}_A(\mathbf{x}, t) \cdot \mathbf{E}_B(\mathbf{x}, t)] \tag{21}$$

Of course, the force of the stationary charge distribution ρ_A on q_B is given by

$$\mathbf{F}_{\text{Lor, A}\to\text{B}}(t) = q_B \mathbf{E}_A[\mathbf{z}_B(t), t] = -\nabla_{\mathbf{z}_B}\{q_B \phi_A[\mathbf{z}_B(t)]\} \tag{22}$$

where

$$\phi_A(\mathbf{x}) = \int d^3x' \frac{\rho_A(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{23}$$

thereby connecting the change in the potential energy $q_B \phi_A[\mathbf{z}_B(t)]$ to the change in the cross-term energy in Eq. (21). This connection can be explicitly shown by substituting $\mathbf{E}_A(\mathbf{x}, t) = -\nabla\phi_A(\mathbf{x})$ in Eq. (21), followed by

$$\begin{aligned} F_{\text{Lor, A}\to\text{B},\,i}(t) &= -\frac{2}{8\pi}\frac{\Delta}{\Delta z_{Bi}} \int d^3x [-\nabla\phi_A(\mathbf{x}) \cdot \mathbf{E}_B(\mathbf{x}, t)] \\ &= \frac{1}{4\pi}\frac{\Delta}{\Delta z_{Bi}} \int d^3x\, \nabla \cdot [\phi_A(\mathbf{x})\, \mathbf{E}_B(\mathbf{x}, t)] \\ &\quad -\frac{1}{4\pi}\frac{\Delta}{\Delta z_{Bi}} \int d^3x\, \phi_A(\mathbf{x}, t)\, \nabla \cdot \mathbf{E}_B(\mathbf{x}, t) \\ &= -\frac{1}{4\pi}\frac{\Delta}{\Delta z_{Bi}} \int d^3x\, \phi_A(\mathbf{x})\, 4\pi\rho_B(\mathbf{x}, t) = -\frac{\Delta}{\Delta z_{Bi}} q_B \phi_A(\mathbf{z}_B(t)] \end{aligned} \tag{24}$$

The first term in the second line can be shown to vanish.[13] In the last line, Eq. (9) was used.

[13] The following rough outline shows why the first term in the second line vanishes. First, taking into account possible "acceleration fields," $\mathbf{E}_B(\mathbf{x}, t)$ must vary roughly as $1/|\mathbf{x} - \mathbf{z}_B|^{n_B}$, for $n_B \geqslant 1$, at large distances from $\mathbf{z}_B$. Second, the electrostatic potential $\phi_A(\mathbf{x})$ must vary as $1/|\mathbf{x}|^{n_A}$, for $n_A \geqslant 1$, for large $|\mathbf{x}|$. Treating Δz_{Bi} as a small change, bringing $\Delta/\Delta z_{Bi}$ inside the integral sign, noting that it will operate only on $\mathbf{E}_B(\mathbf{x}, t)$, and noting $(\Delta/\Delta z_{Bi})\, \mathbf{E}_B(\mathbf{x}, t)$ will vary as $1/|\mathbf{x} - \mathbf{z}_B|^{(1+n_B)}$ for large $|\mathbf{x}|$, enables one to deduce that the surface term below will vanish for S at infinity:

$$\begin{aligned} \frac{\Delta}{\Delta z_{Bi}} \int_V d^3x\, \nabla \cdot [\phi_A(\mathbf{x})\, \mathbf{E}_B(\mathbf{x}, t)] &= \frac{\Delta}{\Delta z_{Bi}} \int_S d^2x\, \hat{\mathbf{n}} \cdot [\phi_A(\mathbf{x})\, \mathbf{E}_B(\mathbf{x}, t)] \\ &= \int_S d^2x\, \hat{\mathbf{n}} \cdot \left[\phi_A(\mathbf{x}) \frac{\Delta}{\Delta z_{Bi}} \mathbf{E}_B(\mathbf{x}, t) \right] \end{aligned}$$

In the above analysis, the assumption was not required that q_B move quasistatically. Thus, $\mathbf{E}_B$ in the integrand on the right-hand side of Eq. (21) need not be the electric field of a stationary charge, even though, mathematically, the correct result is indeed obtained if the fields of q_B are treated as being purely Coulombic at times t and $t+\delta t$.

Here it seems interesting to note that in order to physically realize the situation usually considered where the electromagnetic fields before and after moving q_B are indeed electrostatic ones, then the process of moving q_B must satisfy a number of additional conditions not yet specified and not needed in the previous analysis. If the charge q_B is brought to rest after being displaced a short distance and Eq. (20) is integrated in time to a point approximately R/c beyond the time where q_B is brought to rest, where R is the approximate diameter of the volume V, then the fields of q_B will again be Coulombic within V. The change in electromagnetic energy within V then explicitly becomes a change in the cross terms of electrostatic energy. As before, if V is sufficiently large enough, then the cross-term energy flow out of V may be ignored in the time integral of Eq. (20). Equation (21) then explicitly represents the physical situation usually discussed involving a change in electrostatic energies. Thus, the present analysis provides deeper insight into the treatment of elementary electrostatics where self-energies are simply dropped.[14]

Another application of the cross-term relationships can be found by using Eq. (14) to establish direct connections, under various conditions, between (1) the surface integral of the cross terms of Maxwell's stress energy tensor and (2) the Lorentz force acting on a charge distribution due to external sources. Such a result does not appear in the standard textbooks on electrodynamics, nor apparently in the physics literature; however, it falls out easily from the cross-term relationship of Eq. (14). As will be seen, one interesting result of this analysis will be a deeper insight

[14] To see why the above analysis complements the usual treatment on electrostatics in standard textbooks, see the discussion in Ref. 1, Chap. 1 (e.g., pp. 42–43), and the discussion in Ref. 2 involving Eqs. (37.4)–(37.8). An improved treatment from these two references, involving cutoffs that are eventually removed around each charge, is given by Boyer in Sec. III A of Ref. 7, as well as in Sec. II C of Ref. 17. Rowe, in Ref. 16, Sec. VII, discusses electrostatics from another point of view, by using distribution theory to deal with the singular self-energies. Nevertheless, unlike the discussion in the present article, which is based on the cross-term relationship of Eq. (7), the discussions in these references on changes in electrostatic energies do not analyze the physical operation of actually displacing a charged particle and, consequently, do not explicitly account for electromagnetic energy flow, as in the surface term in Eq. (20). Moreover, note that Eq. (20) is valid for any volume V and surface S enclosing the two charge sources.

into certain calculations on van der Waals forces that have been carried out in the past by some researchers.

Consider two spatially separated charge or current distributions. Upon integrating Eq. (14) over a volume V containing just one of the distributions, then one of the Lorentz force terms drops out. For several classes of problems that are of general interest, the term

$$\dot{\mathbf{P}}_{\text{A-B}}(t) \equiv \frac{1}{c^2}\frac{d}{dt}\int_V d^3x\, \mathbf{S}_{\text{A-B}}(\mathbf{x}, t) \tag{25}$$

can be ignored. Three such cases are discussed shortly.[15] For these problems,

$$\int_V d^3x \left(\rho_{\text{A}}\mathbf{E}_{\text{B}} + \frac{1}{c}\mathbf{J}_{\text{A}} \times \mathbf{B}_{\text{B}}\right)_i = \oint_S d^2x \sum_{j=1}^{3} n_j T_{\text{A-B},\, ij} \tag{26}$$

Thus, here the Lorentz force on the A charge (or current) distribution due to the B one becomes directly related to the above surface integral of the cross terms of the Maxwell stress–energy tensor. [Equation (26) also holds if $\mathbf{E}_{\text{B}}$ and $\mathbf{B}_{\text{B}}$ represent free fields acting on the A distribution, as often treated in solving Casimir force related problems. Consequently, Eq. (26) is an important one, since I am unaware of this direct relationship being published in the Casimir literature.]

The first case, and the most obvious one, where $\dot{\mathbf{P}}_{\text{A-B}}$ can be dropped is when the fields are time independent, as in electrostatic or magnetostatic problems. Then, $(\partial/\partial t)\, \mathbf{S}_{\text{A-B}} = 0$, so that Eq. (26) becomes an exact relationship for any surface S enclosing the A, but not the B source distribution. Also, if the fields are slowly varying in time, then $\dot{\mathbf{P}}_{\text{A-B}}$ can generally be ignored, particularly so because of the $1/c^2$ factor that it contains.

A second class of problems where the $\dot{\mathbf{P}}_{\text{A-B}}$ term can be dropped is when the volume V is specifically chosen in such a way as to make $\dot{\mathbf{P}}_{\text{A-B}}$ negligible. This case occurs, for example, when considering a thin Gaussian pillbox, with area δA and thickness d, that encloses charge $\sigma\, \delta A$ and current element $\mathbf{K}\, \delta A$ due to a surface charge σ and surface current density $\mathbf{K}$, respectively, on the surface of a perfectly conducting material. If $\mathbf{E}_{\text{B}}$ and $\mathbf{B}_{\text{B}}$ are fields from another source, then for $d \to 0$, $\dot{\mathbf{P}}_{\text{A-B}}$ indeed becomes negligible. The right-hand side of Eq. (26) then allows another means of calculating the Lorentz force per unit area due to, for example, an electromagnetic plane wave incident upon the plane surface of a perfectly conducting material.

[15] See the related interesting discussion by Lorentz in Ref. 18, Secs. 19–22.

The third class of problems mentioned here involves charge and current distributions that fluctuate with time, but in such a way that they obey a stationary stochastic process in time. Then, $\langle \dot{\mathbf{P}}_{\text{A-B}}(t) \rangle = (d/dt) \langle \mathbf{P}_{\text{A-B}}(t) \rangle = 0$, where $\langle \mathbf{P}_{\text{A-B}}(t) \rangle$ represents the expectation value of $\mathbf{P}_{\text{A-B}}(t)$. Equation (26) then holds, provided the expectation value brackets are included on each term.

The right-hand side of Eq. (26), with the brackets included, has been used in a number of articles[(19–27), 16] for calculating van der Waals forces between parallel plates made of various materials, as well as between polarizable particles. The articles listed in Refs. 19–27, however, used the full Maxwell stress–energy tensor, rather than just the cross-terms, thereby masking the immediate connection to the Lorentz force given by the expectation value of Eq. (26). The additional terms that arise in using the full expression are related to the Lorentz force of the charge distribution acting upon itself. For the problems considered in these articles, one would expect the average of these forces to be zero, since the charge distributions in question are not undergoing a net acceleration (for example, the self-force of a charged particle is zero unless the particle accelerates).

The connection can be made very clearly and easily between the expectation value of Eq. (26) and the van der Waals forces calculated in Refs. 25, 28, and 29. These calculations involve only classical physics, so that Maxwell's equations immediately apply. Here, nonrelativistic electric dipole harmonic oscillators were considered that were situated in classical electromagnetic zero-point radiation. These calculations were first carried out for all distances between two oscillators[(28)] and later extended to nonzero temperatures [(25)] and to N oscillators.[(29)] The total Lorentz force on an oscillator was calculated by summing the Lorentz forces due to (1) the incident radiation fields, plus (2) the electromagnetic dipole fields due to the other oscillators.[17] Consequently, Eq. (26) can be immediately applied, by taking any surface S that encloses only one of the oscillators. The cross terms must be summed that arise between (1) the oscillator's electromagnetic dipole fields and the incident fields, and (2) the oscillator's dipole

[16] Calculations of van der Waals forces using the Maxwell stress–energy tensor have been carried out via semiclassical means for (1) dielectric plates in Refs. 19 and 20; (2) conducting plates in Ref. 21; and (3) polarizable particles at large distances in Ref. 22. The method of calculating the Casimir force between conducting plates via the Maxwell stress–energy tensor is discussed from a more traditional quantum electrodynamical viewpoint in Ref. 23. Also, calculations using the Maxwell stress–energy tensor have been carried out from an entirely classical point of view by assuming that classical electromagnetic zero-point radiation must be present. These classical calculations have been worked out for (1) conducting parallel plates in Ref. 24 and 25, (2) dielectric parallel plates in Ref. 26, and (3) dielectric and permeable parallel plates in Ref. 27.

[17] See for example, Eq. (41) in Ref. 29.

fields and the dipole fields of the other oscillators. A precise connection is then obtained[18] between the expectation value of the terms corresponding to the right-hand side of Eq. (26) and the van der Waals force calculated in Refs. 25, 28, and 29.

This work was used in Refs. 30 and 31 by combining the energy cross-term relationship relationships with single-particle energy relationships, to analyze the case of no heat flowing during a reversible displacement operation of harmonic dipole oscillators at absolute zero temperature. This analysis enabled the derivation of the spectral form of classical electromagnetic zero-point radiation, from a purely classical electrodynamic analysis.

Finally, the last application of the cross-term relationships discussed here has interesting ties with the Aharonov–Bohm effect in quantum mechanics. Reasoning similar to that in Ref. 7 was again used by Boyer in Ref. 8 to analyze the conservation of energy, linear momentum, and angular momentum for a charged point particle passing a solenoid. As was true in Ref. 7, the cross-term relationships of the present article are indeed relevant for the analysis[19] in Ref. 8. An interesting application of these results is the observation raised by Boyer[(9)] that just as a classical lag effect will produce interference fringe shifts similar to those arising from the electrostatic Aharonov–Bohm effect, so also may this arise in the solenoid Aharonov–Bohm case. Experimental confirmation of the former case exists,[(32)] but experimental investigation of the latter has not yet been carried out.

Thus, a classical explanation exists that appears to provide a shift in the fringe patterns observed in the electrostatic and solenoid Aharonov–Bohm experiments. As Boyer has emphasized, to ferret out whether the fringe pattern shift observed in these experiments is due to this classical lag effect, or to the quantum mechanical explanation by Aharanov and Bohm, requires a very carefully controlled experimental setup. The cross-term relationships can be used in the analysis of this classical lag effect.[19]

[18] More specifically, let $\rho_A(\mathbf{x}, t) = -\mathbf{p}_A(t) \cdot \nabla\delta(\mathbf{x} - \mathbf{R}_A)$, and $\mathbf{J}_A(\mathbf{x}, t) = \dot{\mathbf{p}}_A(t)\, \delta^3(\mathbf{x} - \mathbf{R}_A)$, be the charge and current density, respectively, of the "A" labeled oscillator, which is treated in the electric dipole limit and is located at position $\mathbf{R}_A$. Here, the electric dipole of the A oscillator is given by $\mathbf{p}_A(t) = e\mathbf{z}_A(t)$, where $\mathbf{z}_A(t)$ is given by Eq. (40) in Ref. 29. For $|\mathbf{x} - \mathbf{R}_A| > 0$, the retarded electromagnetic fields due to ρ_A and $\mathbf{J}_A$ are given by Eqs. (17) and (23) in Ref. 29, which were the expressions used for the fields of the electric dipole oscillators when calculating the van der Waals force in Eq. (41) in Ref. 29.

[19] See, for example, Eqs. (14), (15), (22), (38), (41), (49), and (56) in Ref. 8. These results may be viewed as special cases of the cross-term relationships derived here.

5. CONCLUDING REMARKS

Relationships were derived and discussed that involve the time-rate of change of the cross terms that appear in the traditional expressions for the electromagnetic energy, linear momentum, and angular momentum, when two or more sources of electromagnetic fields are present. These relationships are valid for point charges, as well as for continuous charge and current distributions, unlike the analogous theorems in electromagnetic theory that involve the total fields. In a sense, the latter theorems (e.g., Poynting's theorem) are special cases of the cross-term relationships described here. [The comments after Eq. (11) specify in what sense this special case can be viewed.]

Now these cross-term relationships are certainly not *new*; undoubtedly many researchers have, at least privately, noted the validity of, for example, Eq. (18). (Indeed, see Rowe in Ref. 16.) However, precisely how these relationships can be applied does appear to have been largely missed in the literature. As a strong example, the momentum cross-term relationship shows that Newton's third law does not hold in general between two charged particles; instead, the sum of the forces between the two particles exactly equals the negative of the rate of change of the cross-term part of the electromagnetic momentum carried by the fields of the particles. This result has been noted by Page and Adams[5, 6] to hold for terms up to order $1/c^2$ and by Boyer[7] to hold exactly in the case where particles move at constant velocity. However, from the cross-term relationship, this result falls out exactly for the general case of arbitrary motion of the particles.

Many other problems in electromagnetism involving work, forces, and torques can be carried out by using these relationships. In particular, when systems of charged particles are involved, then the cross-term relationships are useful as they provide the general rules that govern the sum of the rates of work, the sum of the forces, and the sum of the torques, by the fields of each particle acting upon the other particles.

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