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The Race Model Inequality: Interpreting a Geometric Measure of the Amount of Violation

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An inequality by J. O. Miller (1982) has become the standard tool to test the race model for redundant signals reaction times (RTs), as an alternative to a neural summation mechanism. It stipulates that the RT distribution function to redundant stimuli is never larger than the sum of the distribution functions for 2 single stimuli. When many different experimental conditions are to be compared, a numerical index of violation is very desirable. Widespread practice is to take a certain area with contours defined by the distribution functions for single and redundant stimuli. Here this area is shown to equal the difference between 2 mean RT values. This result provides an intuitive interpretation of the index and makes it amenable to simple statistical testing. An extension of this approach to 3 redundant signals is presented.

Keywords: redundant signals, race model inequality, negative dependence

In the redundant signals paradigm for simple reaction time (RT), the observer must initiate a response as quickly as possible following the detection of any stimulus onset. A typical finding is that of redundancy gain: Responses are faster, on average, when two or more signals are presented simultaneously than when a single signal appears. Since the pioneering study by Todd (1912), this *redundant signals effect* (RSE) has been replicated many times for both manual and saccadic RTs, and under different experimental settings, for example, comparing uni- versus multimodal stimulation (Amlôt, Walker, Driver, & Spence, 2003; Diederich, 1995; Diederich & Colonius, 1987; Diederich, Colonius, Bockhorst, & Tabeling, 2003; Gielen, Schmidt, & Van den Heuvel, 1983; Hughes, Nelson, & Aronchick, 1998; Miller, 1982, 1986; Molholm, Ritter, Javitt, & Foxe, 2004), single versus multiple stimuli within the same modality (e.g., Schwarz & Ischebeck, 1994), or monocular versus binocular stimulation (Blake, Martens, & DiGianfillipo, 1980; Westendorf & Blake, 1988) and also for specific populations (e.g., Corballis, 1998; Marzi et al., 1996, for hemianopics; Miller, 2004, for individuals who have undergone split-brain surgery; Reuter-Lorenz, Nozawa, Gazzaniga, & Hughes, 1995; Savazzi & Marzi, 2004).

Raab (1962) was the first to propose a *race model* for simple RT such that (a) each individual stimulus elicits a detection process performed in parallel to the others and (b) the winner's time determines the observable RT. This model suggests that RSE is generated by statistical facilitation: If detection latencies are interpreted as (nonnegative) random variables, the time to detect the first of several redundant signals is faster, on average, than the detection time for any single signal. A generalization of Raab's model was recently developed in Miller and Ulrich (2003).

Testing the race model amounts to testing whether an observed RT speed-up is too large to be attributed to statistical facilitation (viz., probability summation). The race model inequality (RMI) proposed in Miller (1982) has become the standard testing tool in many RT studies.[1] It stipulates that the RT distribution function for redundant stimuli is never larger than the sum of the RT distributions for the single stimuli. A violation of this inequality is interpreted as an indicator of an underlying neural summation (or coactivation) mechanism. When many different experimental conditions are to be compared, a numerical index of the amount of violation is very desirable. A widespread practice is to take a certain area with contours defined by the distribution functions for single and redundant stimuli. Here we show, for the first time, that this area can be interpreted in terms of mean RT differences, thus providing both a simple intuitive interpretation of the area and a means for statistical testing.[2] We also present a partial extension of this result to the trimodal stimulation condition.

We need the following notation. Let RT_X and RT_Y denote the processing time for the detection of signal s_X, and, respectively, s_Y, when presented alone, and let RT_{XY} denote the processing time when both signals s_X and s_Y are present. For simplicity, it is assumed here that detection latencies are identical to the observable RTs. Note that, because RT_X, RT_Y, and RT_{XY} are measured under different experimental conditions, there is no natural probability space to define their joint distribution. However, the race model assumptions can be stated explicitly using the "equal-in-distribution" notion: Two random variables U and V are equal in distribution ($U =_{st} V$) when they have distribution functions of identical form. The race model assumes that (a) there is a nonnegative random vector (X, Y) (defined by a distribution with respect to some probability space) such that $RT_{XY} =_{st} \min(X, Y)$, and (b) $X =_{st} RT_X$, $Y =_{st} RT_Y$. The latter assumption is often referred to as *context invariance,* stipulating that the signal detec-

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[1] According to the Web of Science (May, 2005), Miller's (1982) article has 198 citations, with 60 of them over the last 3 years.

[2] Miller (1986) introduced the use of this geometric measure to assess the degree of violation of the inequality and also pioneered a bootstrapping test. For reasons unbeknown to us, the latter seems to have been ignored completely in the subsequent literature.

tion latency distributions for s_X and s_Y are identical in single and redundant signal trials (Ashby & Townsend, 1986; Luce, 1986).

With E standing for the expected value of random variables, it follows, as a special case of Jensen's inequality (e.g., Billingsley, 1979), that

$$E[\min(X,Y)] \leq \min[E(X),E(Y)] \quad (1)$$

for any distribution of (X, Y). Random variables X and Y are not observable (only their minimum is, in the redundant signal condition), but from the equal-in-distribution assumptions a testable analogue of Inequality 1 follows:

$$E(RT_{XY}) \leq \min[E(RT_X),E(RT_Y)] \quad (2)$$

Testing the Race Model

The latter inequality has been used to test the race model on the level of average RTs. For example, in an RT stimulation, Gielen et al. (1983) obtained mean RTs for bimodal (visual–auditory and visual–kinesthetic) and unimodal stimuli. In order to derive the race model's prediction of mean bimodal RT, they assumed stochastic independence between the two detection times in the bimodal condition and found average bimodal RTs to be smaller than predicted by the model, leading them to a rejection of the race model.

Note, however, that the validity of the inequality in Equation 2 is not restricted to the case of stochastic independence. This is important because dependent processing does affect the predictions of the race model. Indeed, assuming negative dependence—that is, relatively fast detection latencies for signal s_X co-occur with relatively slow detection latencies for signal s_Y and vice versa—it is obvious that the smaller of the two random latencies RT_X and RT_Y tends to be small as compared with the smaller of two independent latencies, as long as the individual latencies' means do not vary.[3] The difficulty Gielen et al. (1983) faced was how to derive predictions of a dependent race model without restricting the model by specific distributional assumptions.

A more general test of the race model was developed by Miller (1978, 1982) in showing that

$$P(RT_{XY} \leq t) \leq P(RT_X \leq t) + P(RT_Y \leq t) \quad (3)$$

must hold for all $t \geq 0$. This RMI follows from

$$P[\min(X,Y) \leq t] \leq P(X \leq t) + P(Y \leq t) \quad (4)$$

a special case of Boole's inequality (Billingsley, 1979). RMI and some of its variations and generalizations have been the subject of numerous theoretical and methodological studies (Ashby & Townsend, 1986; Colonius, 1986, 1990, 1999; Colonius & Ellermeier, 1997; Colonius & Townsend, 1997; Colonius & Vorberg, 1994; Diederich, 1992; Miller, 1986, 1991, 2004; Miller & Ulrich, 2003; Mordkoff & Yantis, 1991; Townsend & Nozawa, 1995, 1997; Townsend & Wenger, 2004; Ulrich & Giray, 1986; Ulrich & Miller, 1997).

Violations of RMI have been observed in many multimodal stimulation experiments but also under unimodal stimulation (e.g., Turatto, Mazza, Savazzi, & Marzi, 2004). A common way to depict the amount of RMI violation is to subtract the single signal distributions from the redundant signals distribution

$$P(RT_{XY} \leq t) - P(RT_X \leq t) - P(RT_Y \leq t) \quad (5)$$

and to plot this as a function R^*_{XY}, say, of t (Miller, 1986):

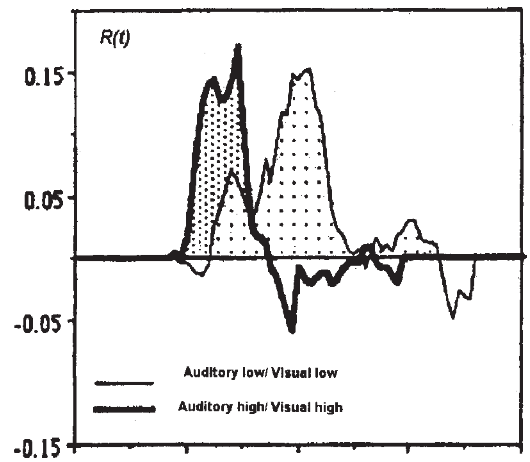

Figure 1. Areas above horizontal line represent the amount of violation of race model inequality for saccadic reaction times in two different stimulus conditions (after Nozawa et al., 1994). From "Parallel and serial processes in the human oculomotor system: Bimodal integration and express saccades," by G. Nozawa, P. A., Reuter-Lorenz, and H. C. Hughes, 1994, *Biological Cybernetics, 72,* 19–34. Copyright 1994 by Springer-Verlag. Adapted with permission.

$$R^*_{XY}(t) = P(RT_{XY} \leq t) - P(RT_X \leq t) - P(RT_Y \leq t). \quad (6)$$

By the inequality shown in Equation 3, positive values of R^*_{XY} (t) indicate violations of RMI. For example, Figure 1 presents functions R^*_{XY} (t) from two different stimulus conditions in a visual–auditory saccadic RT study by Nozawa, Reuter-Lorenz, and Hughes (1994).

Given that the left-hand side of the inequality shown in Equation 3 is always bounded by 1, the inequality can be rewritten as

$$P(RT_{XY} \leq t) \leq \min[P(RT_X \leq t) + P(RT_Y \leq t),1], \quad (7)$$

resulting in a slightly modified function

$$R_{XY}(t) = P(RT_{XY} \leq t) - \min[P(RT_X \leq t) + P(RT_Y \leq t),1]. \quad (8)$$

Violations of RMI will again result in positive values of $RT_{XY}(t)$, whereas negative or zero values of $RT_{XY}(t)$ are compatible with the race model.[4]

Assessing the Amount of RMI Violation

The amount of violation is typically interpreted as the strength of neural summation or coactivation, that is, the amount of response facilitation that is not reducible to probability summation (viz., statistical facilitation). If many different experimental conditions are to be compared with respect to their degree of RMI violation, reducing the information contained in $RT_{XY}(t)$ or $RT^*_{XY}(t)$ to a single numerical index of neural summation is very desirable. It has become common practice to interpret the area

[3] Positive dependence has the opposite effect: In the extreme case of perfect positive dependence, the smaller of the two random latencies will have the same mean as the one with the smaller mean.

[4] Because there is some arbitrariness, slightly different definitions of $R_{XY}(t)$ occur in the literature. The version chosen here is best suited for our purposes.

under positive $RT_{XY}(t)$ or $RT^*_{XY}(t)$ values V^+, say, as a quantitative measure of the amount of violation of RMI and, thereby, of neural summation. For example, in a study on summation, Hughes et al. (1998) plotted V^+ values as a function of four different spatial positions of the auditory and three different positions of the visual stimulus (see Figure 2).

This geometric measure of RMI violation is simple and attractive, but it may seem a bit arbitrary. The following proposition, on the other hand, shows that a slight modification of this geometric measure has a deeper interpretation relating it directly to the aforementioned race model test on the level of average RTs.

Proposition 1: Let $E^{(-)}[\min(RT_X, RT_Y)]$ be the mean RT predicted by a race model with maximal negative dependence between the detection latencies RT_X and RT_Y. Then the integral over function $RT_{XY}(t)$

$$V_{XY} = \int_0^\infty R_{XY}(t)dt = E^{(-)}[\min(RT_X, RT_Y)] - E(RT_{XY}) \quad (9)$$

where $E(RT_{XY})$ is the (observed) mean RT in the redundant signals condition.

This proposition is based on the fact that the right-hand side of the RMI in Equation 7,

$$\min[P(RT_X \leq t) + P(RT_Y \leq t), 1], \quad (10)$$

is a distribution function, namely, the distribution function of $\min(RT_X, RT_Y)$ with maximal negatively dependent RT_X, RT_Y (for a proof, see Appendix A). A negative or zero value of V_{XY} indicates that the amount of observed RT facilitation is completely attainable by a race model, possibly with negative dependence between RT_X and RT_Y.[5] The larger the positive values of V_{XY}, the larger the amount of facilitation not explainable by the race model even if extreme negative dependence between the detection processing times is assumed.

Moreover, assuming that observable RTs include a variable base time, it can be shown that this implies a moderating effect on negative dependence leading to a possible underestimation of V_{XY} (see Colonius, 1990, Proposition 5.1) .[6] In order to gauge the size of this underestimation, however, one would need to have an estimate of the base time variability, which may be difficult to obtain in practice.

Given that function $R_{XY}(t)$ may be positive or negative depending on the value of t, index V_{XY} is equal to the area under $R_{XY}(t)$ above the abscissa minus the area below the abscissa but above $RT_{XY}(t)$. Numerical estimates of these areas can, in principle, be obtained through numerical integration. An attractive alternative, not requiring any area estimations, is using the method of *antithetic variates* (e.g., Thompson, 2000) to generate a pair of maximally negative dependent random variables from the two single signal distributions and to compute the mean of their minima (Colonius, 1990; Miller, 1986). We illustrate the aforementioned proposition and the numerical estimation of $E^{(-)}[\min(RT_X, RT_Y)]$ by a hypothetical visual–auditory interaction experiment.

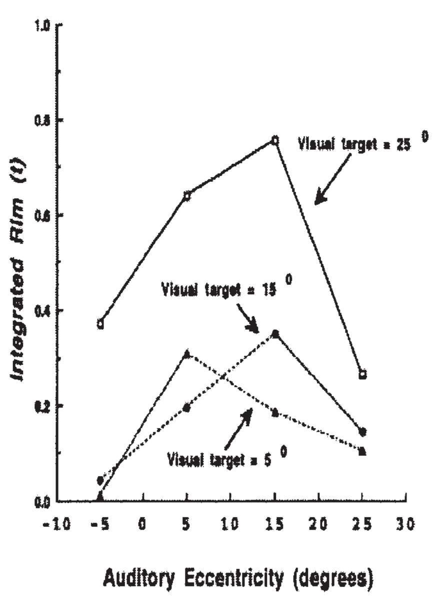

Figure 2. Positive areas under $R^*(t)$ as a function of visual and auditory stimulus position (after Hughes et al., 1998). From "Spatial characteristic of visual–auditory summation in human saccades," by H. C. Hughes, M. D. Nelson, and D. M. Aronchick, 1998, *Vision Research, 38,* 3955–3963. Copyright 1998 by Elsevier. Adapted with permission.

Example: Visual–Auditory Interaction in RT

Using an artificial data set with known underlying RT distributions allows us to study the effect of different degrees of neural summation on the geometric index V_{VA}. For computational simplicity, we assume exponentially distributed visual and auditory processing times with intensity parameters λ_V and λ_A, respectively, for the unimodal stimulus conditions. Bimodal processing time is also exponentially distributed,[7] with parameter λ_{VA}. Obviously, for $\lambda_{VA} = \lambda_V + \lambda_A$, we have an independent race model, but for $\lambda_{VA} > \lambda_V + \lambda_A$, violations of RMI occur, as illustrated by function $R_{VA}(t)$'s being positive for a large range of t values (see Figure 3).

Computation of area value V_{VA} requires determination of the winner's mean in a race model with maximally negative dependence, $E^{(-)}[\min(RT_V, RT_A)]$, which—at the population level—is done by simple integration (cf. Appendix A). Each curve in Figure 3 corresponds to a different value of λ_{VA}, and area V_{VA} increases monotonically with λ_{VA}, indicating an increasing amount of neural coactivation.

Numerical estimates of $E^{(-)}[\min(RT_V, RT_A)]$ from sample data are computed by the method of antithetic variates. Basically, the procedure is to take pairs of RT values from the single signal distributions as follows: Take the fastest RT from the RT_V sample and the slowest RT from the RT_A sample as the first pair, the next-to-the-fastest from the RT_V sample and the next-to-the-slowest from the RT_A sample, and so on. For each pair, determine

[5] It should be noted, however, that nonviolation of RMI does not automatically validate a race model explanation. In fact, Ulrich and Miller (1997) developed a test that, in principle, may rule out race models even when RMI is not violated.

[6] We are grateful to two of the reviewers, Jim Townsend and Christopher Honey, for pointing this out.

[7] This model derives from the Marshall-Olkin bivariate exponential distribution, an important model in reliability theory (cf. Galambos & Kotz, 1978).

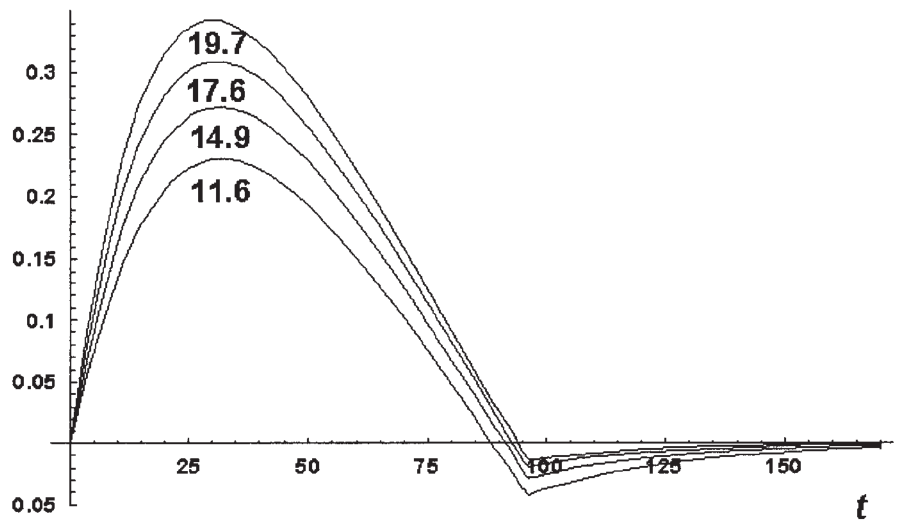

Figure 3. $R_{VA}(t)$ functions of visual–auditory interaction example with $\lambda_V = .005$, $\lambda_A = .01$, and $\lambda_{VA} = .018$, .022, .026, and .03, respectively. The corresponding V_{VA} areas are indicated by the inscribed numbers.

the minimum and take the mean over these minima. Table 1 presents ordered samples of size $n = 10$ from the RT_V distribution ($\lambda_V = .005$) and from the RT_A distribution ($\lambda_A = .01$).

The mean of the minima equals 59 as an estimate of $E^{(-)}[\min(RT_V, RT_A)] = 42$. This estimate can, of course, be improved by taking larger, more realistic sample sizes.

Statistical Testing of the Amount of RMI Violation

In order to go beyond a descriptive measure of RMI violation, a statistical test for evaluating the null hypothesis of the inequality shown in Equation 3's being true is desirable. The index V_{XY}, as a simple difference of independent means, is amenable to statistical testing of the null hypothesis of $V_{XY} \leq 0$ by a conventional *t* test (with nonhomogeneous variances) or a nonparametric (Mann–Whitney) *U* test.[8] Another possible approach, already taken in Miller (1986), is to compute confidence intervals for the mean RT predicted by the (maximal negatively dependent) race model via bootstrapping from the observed single signal RT distributions (for details, see Miller, 1986).

Table 1
Ordered Samples of Size 10 From RT_v and RT_A Distributions (Columns 1 and 2) and Their Minima (Column 3)

Ordered RT_V	Reverse-ordered RT_A	Minima
68	349	68
83	251	83
86	156	86
141	106	106
147	90	90
153	44	44
154	42	42
209	41	41
380	22	22
678	9	9

Note. Average of the minima (third column) is 59.

Note that local violations of RMI may occur, although the corresponding V_{XY} value, as a global measure, may not show a significant violation. Thus, nonparametric tests at the level of the distribution functions will, in general, be more sensitive to violations of the race model. In this vein, Maris and Maris (2003) developed an interesting Kolmogorov–Smirnov-type test, but it is restricted to experimental paradigms where the single signal responses are drawn from a mixture distribution of the single signal distributions. No general solution in this direction is available yet.

Extension to Three Redundant Signals

The notion of a race easily extends to the case of more than two processes unfolding in time. A prominent example is the paradigm of multimodal stimulation with stimuli from the visual, auditory, and somatosensory modality (as early as Todd, 1912). Assuming (a) $RT_{XYZ} =_{st} \min(X, Y, Z)$; (b) $RT_{XY} =_{st} \min(X, Y)$, $RT_{YZ} =_{st} \min(Y, Z)$, $RT_{XZ} =_{st} \min(X, Z)$; and (c) $X =_{st} RT_X$, $Y =_{st} RT_Y$, and $Z =_{st} RT_Z$, an extension of the inequality in Equation 2,

$$E(RT_{XYZ}) \leq \min[E(RT_X), E(RT_Y), E(RT_Z)], \qquad (11)$$

predicts statistical facilitation again. There is a dramatic difference, however, between the bivariate and the multivariate (greater than 2) situation as far as the role of statistical dependence is concerned. By an elementary observation, three random variables cannot be pairwise negatively dependent to an arbitrarily high degree. Thus, although the direct extension of RMI,

$$P(RT_{XYZ} \leq t) \leq \min[P(RT_X \leq t) + P(RT_Y \leq t) + P(RT_Z \leq t), 1], \qquad (12)$$

[8] Independence can be assumed by constructing a joint probability space from the three separate experimental conditions (single and redundant stimuli). The sample estimate for $E^{(-)}[\min(RT_X, RT_Y)]$ is a function of the order statistics of the two single stimulus conditions and is thus independent of the estimate for $E(RT_{XY})$ from the redundant stimuli condition.

obviously holds, its right-hand side does not, in general, constitute a distribution function for $\min(RT_X, RT_Y, RT_Z)$ (cf. Joe, 1997), thus preventing a direct generalization of Proposition 1.

Nevertheless, alternative distribution inequalities exist that lend themselves to geometric interpretation. One example is (Diederich, 1992)[9]

$$P(RT_{XYZ} \leq t) \leq P(RT_{XY} \leq t) + P(RT_{YZ} \leq t) - P(RT_Y \leq t), \quad (13)$$

which follows from

$$P[\min(X,Y,Z) \leq t] \leq P[\min(X,Y) \leq t] + P[\min(Y,Z) \leq t] - P(Y \leq t). \quad (14)$$

This inequality has recently been tested in a multimodal stimulation experiment in Diederich and Colonius (2004). Figure 4 presents an example from a trimodal condition where the area between the upper curve (observed trimodal RT) and the middle one (right-hand side of the inequality in Equation 13) suggests a violation of the inequality.

An analogue to Proposition 1 gives the following interpretation of this area.

Proposition 2 (for proof see Appendix B): The integral over function

$$R_{XYZ}(t) = P(RT_{XYZ} \leq t) - [P(RT_{XY} \leq t) + P(RT_{YZ} \leq t) - P(RT_Y \leq t)] \quad (15)$$

equals

$$V_{XYZ} = \int_0^\infty R_{XYZ}(t)dt = [E(RT_{XY}) + E(RT_{YZ}) - E(RT_Y)] - E(RT_{XYZ}). \quad (16)$$

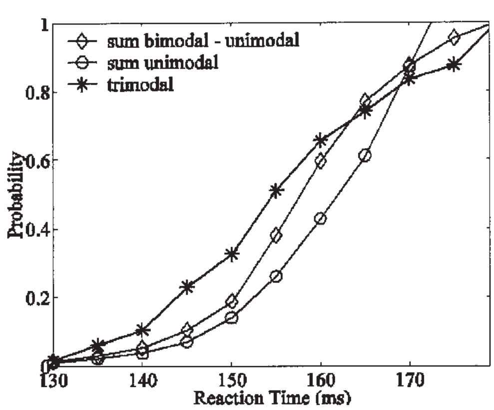

Figure 4. Trimodal stimulation result: Area between the upper and middle curves represents the value of V_{XYZ} from Proposition 2. From "Bimodal and trimodal multisensory enhancement of reaction time: Effects of stimulus onset and intensity," by A. Diederich and H. Colonius, 2004, *Perception & Psychophysics, 66,* 1388–1404. Copyright 2004 by the Psychonomic Society. Adapted with permission.

Thus, a positive value of V_{XYZ} indicates that mean RT with three signals is faster than predicted from the race model. It is interesting to note that Equation16 measures violation over and above that possibly caused with two signals. Indeed, let us assume that there are $V_{XY} > 0$ and $V_{YZ} > 0$ such that

$$E(RT_{XY}) = E^{(-)}[\min(RT_X,RT_Y)] - V_{XY} \quad (17)$$

and

$$E(RT_{YZ}) = E^{(-)}[\min(RT_Y, RT_Z)] - V_{YZ}. \quad (18)$$

Inserting these expression into Equation 16 suggests that the RMI violations with two signals have already been discounted in the computation of V_{XYZ}:

$$V_{XYZ} = \{E^{(-)}[\min(RT_X,RT_Y)] - V_{XY}\} + \{E^{(-)}[\min(RT_Y,RT_Z)] - V_{YZ}\} - E(RT_Y) - E(RT_{XYZ}). \quad (19)$$

Conclusion

We have shown that a commonly used geometric measure of the amount of violation of the race model relates performance to the case of maximal negative dependence between the two processing times and, specifically, that it equals a simple difference of mean RTs amenable to statistical testing. A direct generalization of this result to the processing of three or more signals was shown to be impossible in principle, but alternative geometric measures assessing race model violations, again expressible as mean RT differences, can be developed as demonstrated here for the trivariate situation.

[9] Two more inequalities of the same type follow from symmetry, with *X*, or *Z*, taking over the role of *Y*. Replacing the right-hand side of the inequality in Equation 13 by the minimum over all three possible upper bounds leads to a possibly sharper inequality generalizing the subsequent development. However, for ease of exposition, we abstain from presenting the more general case.

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(*Appendix follows*)

Appendix A

Proof of Proposition 1

Writing $F_{XY}(t) = P(RT_{XY} \leq t)$, $F_X(t) = P(RT_X \leq t)$, and $F_Y(t) = P(RT_Y \leq t)$,

$$\begin{aligned} R_{XY}(t) &= F_{XY}(t) - \min[F_X(t) + F_Y(t),1] \\ &= 1 - \min[F_X(t) + F_Y(t),1] - [1 - F_{XY}(t)] \\ &= \max[1 - F_X(t) - F_Y(t),0] - [1 - F_{XY}(t)]. \end{aligned} \quad \text{(A1)}$$

Integrating yields

$$\int_0^\infty R_{XY}(t)dt = \int_0^\infty \max[1 - F_X(t) - F_Y(t),0]dt - \int_0^\infty [1 - F_{XY}(t)]dt = E^{(-)}[\min(RT_X,RT_Y)] - E(RT_{XY}), \quad \text{(A2)}$$

where $E^{(-)}$refers to the mean RT under maximally negative dependence between RT_X and RT_Y (cf. Colonius, 1990) and $E(RT_{XY})$ is the observed redundant signals mean RT.

The last step follows from the equality

$$\int_0^\infty [1 - F_X(t)]dt = E(X) \quad \text{(A3)}$$

holding for any positive (continuous) random variable X with distribution function F_X.

Appendix B

Proof of Proposition 2

Writing $F_{XYZ}(t) = P(RT_{XYZ} \leq t)$, and so on,

$$\begin{aligned} R_{XYZ}(t) &= P(RT_{XYZ} \leq t) - [P(RT_{XY} \leq t) + P(RT_{YZ} \leq t) - P(RT_Y \leq t)] \\ &= [1 - F_{XY}(t)] + [1 - F_{YZ}(\text{t})] - [1 - F_Y(t)] - [1 - F_{XYZ}(t)]. \end{aligned} \quad \text{(B1)}$$

Integrating yields

$$\int_0^\infty R_{XYZ}(t)dt = [E(RT_{XY}) + E(RT_{YZ}) - E(RT_Y)] - E(RT_{XYZ}). \quad \text{(B2)}$$

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