A Proof that Quantum Interference Arises in a Clifford Algebraic Formulation of Quantum Mechanics.

Elio Conte

Department of Pharmacology and Human Physiology – TIRES – Center for Innovative Technologies for Signal Detection and Processing, University of Bari- Italy;
School of Advanced International Studies for Applied Theoretical and Non Linear Methodologies of Physics, Bari, Italy.

Abstract: We review a rough scheme of quantum mechanics using the Clifford algebra. Following the steps previously published in a paper by another author [17], we demonstrate that quantum interference arises in a Clifford algebraic formulation of quantum mechanics.

1. Introduction

It is well known that in 1936 Birkhoff and von Neumann extended conventional quantum mechanics by using quaternions in place of complex numbers in order to represent the wave-functions and probability amplitudes in such a theory [1]. Starting with 1972 [4], we began to apply Clifford algebra to a quantum mechanical framework, and our effort culminated in 2000 [3], when we were able to reformulate the whole standard framework of quantum mechanics by a rough algebraic scheme, showing in detail that, by this rough formulation, we may re-obtain all the standard quantum theory, and in particular we may give proof of the existing quantization of systems, of basic features of harmonic oscillator, of orbital angular momentum, of hydrogen atom, as well as of time evolution of quantum systems, arriving to consider also the well known EPR problem of quantum mechanics [4], the non locality, the Kochen and Specker theorem [5] as well as the more recent Bell approach [6]. We covered all the standard features of quantum mechanics. Clifford algebra gives an unifying framework of physical knowledge here including quantum mechanics, relativity, electromagnetism and other physical matter. When we introduce a Clifford rough scheme of quantum mechanics, we cannot ignore the emerging salient feature of this formulation. It is that in this case we obtain a quantum mechanical theoretical framework invoking only an algebraic structure that does not contain any further specific requirement. This is a very important and salient feature of this algebraic structure.

Under a restricted and more methodological profile, it must be outlined with clearness that such our approach to quantum mechanics does not give a formulation with alternative and entirely new ideas in quantum mechanics. In fact, recalling as example the contributions given in [1], we outline that our elaboration results at least consistent with old ideas that were known from the earliest days of quantum theory, although it contains some, few, but very interesting new features that possibly deserve careful consideration. In our effort there is, first of all, a research finality that is evident. There is also a didactic finality. Starting with 2003 [7], we have performed a number of experiments showing that the well known quantum interference effect may be observed in perceptive-cognitive processes of human subjects, and, in particular, during their perception and cognition of ambiguous figures. We will not enter here in the theoretical and experimental details of such new obtained results [7], but their importance under the perspective to establish if or not the psychological functions of human subjects involve also the domain of quantum mechanics, results rather evident. In a recent contribution we tested the possible Bell quantum violation in mental states, and we introduced for the first time Clifford algebraic elements as basic quantum mental observables [8]. Such new results in psychological and neurophysiological studies open interesting perspectives, and lead as consequence that a whole set of researchers as psychologists and neurologists could be interested to approach the whole scheme of quantum mechanics in order to fully understand the possible potentialities of such theory when we attempt to explain by it some basic features of our mind and of our human thinking. In brief, a rough approach to quantum mechanics is required since it is well known that such researchers in some cases may have not the necessary mathematical competence to approach the sophisticated and standard formalism of the traditional quantum mechanics. In these cases, our simplified, rough , Clifford scheme of quantum mechanics, without violating the requirements of the scientific rigour that is required, but allowing at the same time some derogation under the strict formal profile, enables as counterpart to acquire the basic conceptual foundations of a theory, and thus to apply such new knowledge in the proper but distant sphere of competence. This is the basic objective that the effort, given in [3], may substantially reach at the previously mentioned didactic profile. Clifford algebra uses few basic rules and algebraic elements. Therefore, the advantage to use such so restricted algebraic framework in a didactic perspective to acquire knowledge of the basic foundations of quantum mechanics, may be of importance for the researchers who unfortunately may have not direct competence in the standard language of quantum mechanics that, as it is well known, holds about the abstract field of Hilbert space and of acting linear operators. Of course, the finality to introduce such alternative didactic patterns is not new here. We remember, as example, the excellent book of T.F. Jordan [9], that reaches the same objective using the simple matrix form, and that inspired so much our elaboration. In any manner, our principal objective is to evidence the profound existing link between quantum mechanics and Clifford algebra since this specific link opens some basic questions that are of relevant and basic interest for the same widening of the basic foundations, nature and meaning of quantum mechanics in the whole complex of unifying physical knowledge. It offers also didactic opportunities. Therefore, our selected objective remains confined to the analysis and examination of such existing link. To this purpose, it may be of interest the result that we obtain in the present paper. We proof that quantum interference leads necessarily to a Clifford algebraic formulation of quantum mechanics. Consequently, it adds still rigour to the Clifford algebraic formulation of quantum mechanics .

2. A Clifford algebraic rough scheme of quantum mechanics

Let us explain briefly the basic framework of our approach [3].

Let us give a proper definition of Clifford algebra. By using the Clifford algebra in our rough quantum mechanical scheme it is intended that, specifically, a Clifford algebra is a unital associative algebra which contains and is generated by a vector space V equipped with a quadratic form Q. The Clifford algebra $C\ell(V,Q)$ is the algebra generated by V subjected to the condition

$$v^2 = Q(v) \quad \text{for all} \quad v \in V$$

where

$$uv + vu = 1/2(Q(u,v) - Q(u) - Q(v))$$

is the symmetric bilinear form associated to Q.

We will utilize and follow the work that, starting with 1981, was developed by Y. Ilamed and N. Salingaros [10], using sometimes the same technique that these authors introduced in their work.

Let us anticipate that only two basic assumptions, quoted as (a) and (b), are required in order to formulate such rough scheme of quantum mechanics.

Let us consider three abstract basic elements, e_i, with $i = 1,2,3$, and let us admit the following two assumptions :

a) it exists the scalar square for each basic element:

$e_1e_1 = k_1$, $e_2e_2 = k_2$, $e_3e_3 = k_3$ with $k_i \in \Re$. (2.1)

In particular we have also that

$e_0e_0 = 1$.

b) The basic elements e_i are anticommuting elements, that is to say:

$e_1e_2 = -e_2e_1$, $e_2e_3 = -e_3e_2$, $e_3e_1 = -e_1e_3$. (2.2)

In particular it is

$e_ie_0 = e_0e_i = e_i$.

Note that, owing to the axioms (a) and (b), we consider the given basic elements $e_i\,(i = 1,2,3)$ as abstract entities that we call potentialities, given in a numerical field since do not exist actual numerical entities satisfying both the (1) and the (2) simultaneously. In detail, by the (2.1), the e_i have the potentiality to simultaneously assume the numerical values $\pm k_i^{1/2}$. According to [10], let us introduce the necessary and the sufficient conditions to derive all the basic features of the algebra that we have just introduced. To give proof, let us consider the general multiplication of the three basic elements e_1, e_2, e_3, using scalar coefficients $\omega_k, \lambda_k, \gamma_k$ pertaining to some field:

$e_1e_2 = \omega_1e_1 + \omega_2e_2 + \omega_3e_3$; $e_2e_3 = \lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3$; $e_3e_1 = \gamma_1e_1 + \gamma_2e_2 + \gamma_3e_3$. (2.3)

Let us introduce left and right alternation:

$e_1e_1e_2 = (e_1e_1)e_2$; $e_1e_2e_2 = e_1(e_2e_2)$; $e_2e_2e_3 = (e_2e_2)e_3$; $e_2e_3e_3 = e_2(e_3e_3)$; $e_3e_3e_1 = (e_3e_3)e_1$;

$e_3e_1e_1 = e_3(e_1e_1)$. (2.4)

Using the (2.4) in the (2.3) it is obtained that

$$k_1e_2 = \omega_1k_1 + \omega_2e_1e_2 + \omega_3e_1e_3; \qquad k_2e_1 = \omega_1e_1e_2 + \omega_2k_2 + \omega_3e_3e_2;$$
$$k_2e_3 = \lambda_1e_2e_1 + \lambda_2k_2 + \lambda_3e_2e_3; \qquad k_3e_2 = \lambda_1e_1e_3 + \lambda_2e_2e_3 + \lambda_3k_3;$$
$$k_3e_1 = \gamma_1e_3e_1 + \gamma_2e_3e_2 + \gamma_3k_3; \qquad k_1e_3 = \gamma_1k_1 + \gamma_2e_2e_1 + \gamma_3e_3e_1 . \tag{2.5}$$

From the (2.5), using the assumption (b), we obtain that

$$\frac{\omega_1}{k_2}e_1e_2 + \omega_2 - \frac{\omega_3}{k_2}e_2e_3 = \frac{\gamma_1}{k_3}e_3e_1 - \frac{\gamma_2}{k_3}e_2e_3 + \gamma_3;$$
$$\omega_1 + \frac{\omega_2}{k_1}e_1e_2 - \frac{\omega_3}{k_1}e_3e_1 = -\frac{\lambda_1}{k_3}e_3e_1 + \frac{\lambda_2}{k_3}e_2e_3 + \lambda_3;$$
$$\gamma_1 - \frac{\gamma_2}{k_1}e_1e_2 + \frac{\gamma_3}{k_1}e_3e_1 = -\frac{\lambda_1}{k_2}e_1e_2 + \lambda_2 + \frac{\lambda_3}{k_2}e_2e_3 \tag{2.6}$$

For the principle of identity , we have that it must be

$$\omega_1 = \omega_2 = \lambda_2 = \lambda_3 = \gamma_1 = \gamma_3 = 0 \tag{2.7}$$

and

$$-\lambda_1k_1 + \gamma_2k_2 = 0 \qquad \gamma_2k_2 - \omega_3k_3 = 0 \qquad \lambda_1k_1 - \omega_3k_3 = 0 \tag{2.8}$$

The (2.8) is an homogeneous system admitting non trivial solutions since its determinant $\Lambda = 0$, and the following set of solutions is given:

$k_1 = -\gamma_2\omega_3$, $k_2 = -\lambda_1\omega_3$, $k_3 = -\lambda_1\gamma_2$ (2.9).

Admitting $k_1 = k_2 = k_3 = +1$, it is obtained that

$$\omega_3 = \lambda_1 = \gamma_2 = i \tag{2.10}$$

In this manner, using the (2.3), a theorem, showing the existence of such algebra, is proven. The basic features of this algebra are given in the following manner

$e_1e_2 = -e_2e_1 = ie_3$; $e_2e_3 = -e_3e_2 = ie_1$; $e_3e_1 = -e_1e_3 = ie_2$; $i = e_1e_2e_3$ (2.11).

The content of this theorem is thus established: given three abstract basic elements as defined in (a) and (b), an algebraic structure is established with four generators (e_0, e_1, e_2, e_3).

Of course, as counterpart, the (2.11) are well known also in quantum mechanics and the isomorphism with Pauli's matrices at various orders is well known and discussed in detail in [10]. Here, they have been derived only on the basis of two algebraic assumptions, given respectively in (a) and (b).

We may now add some comments to the previous formulation.

Let us attempt to identify the phenomenological counterpart of the algebraic structure given in (2.1), (2.2), and (2.11) with

$$e_1^2 = 1 \quad , \quad e_2^2 = 1 \, , \; e_3^2 = 1 \tag{2.12}$$

A generic member of our algebra is given by

$$x = \sum_{i=0}^{3} x_i e_i \tag{2.13}$$

with x_i pertaining to some field $\Re$ or C. The (2.12) evidences that the e_i are abstract potential entities, having the potentiality that we may attribute them the numerical values, or ± 1. Admitting to be $p_1(+1)$ the probability to attribute the value $(+1)$ to e_1 and $p_1(-1)$ the probability to attribute (-1), considering the same corresponding notation for the two remaining basic elements, we may introduce the following mean values:

$$< e_1 >= (+1)p_1(+1) + (-1)p_1(-1) \qquad , \qquad < e_2 >= (+1)p_2(+1) + (-1)p_2(-1) \, ,$$

$$< e_3 >= (+1)p_3(+1) + (-1)p_3(-1). \tag{2.14}$$

Selected the generic element of the algebra, given in (2.13), its mean value results

$$< x >= x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \tag{2.15}$$

Let us call

$$a = x_1^2 + x_2^2 + x_3^2 \tag{2.16}$$

so that

$$-a \le x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \le a \tag{2.17}$$

and

$$-1 \le < e_i > \le +1 \quad i = (1,2,3) \tag{2.18}$$

The (17) must hold for any real number x_i, and, in particular, for

$$x_i = < e_i >$$

so that we have the fundamental relation

$$< e_1 >^2 + < e_2 >^2 + < e_3 >^2 \le 1 \tag{2.19}$$

See details of this proof in ref. [9] for quantum mechanics in simple matrix form and for its extension in Clifford algebra in ref. [3]

Let us observe some important things:

1) The (2.19), owing to the (2.14), says that probabilities for basic elements e_i are not independent and this is of basic importance to acknowledge the essential features of a rough quantum mechanical scheme.
2) The (2.19) still says that also mean values of e_i are not independent. In detail, the (2.19) may be considered to represent a general principle of ontic potentialities. We have here a formulation of a basic, irreducible, ontic randomness. In particular, it affirms that we never can attribute simultaneously, definite numerical values to two basic elements e_i. Let us consider, as example, $< e_3 >= +1$, that is to say that $e_3 \to +1$, we have consequently that $< e_1 >=< e_2 >= 0$, that is to say that e_1 and e_2 are both in a complete condition of randomness. The values are equally probable, there is full indetermination. We have a condition of ontic potentiality.

In conclusion, by using only the axioms (a) and (b), by the (2.11), the (2.14) and the (2.19), we have delineated a rough scheme of quantum theory using only an algebraic structure. Let us observe that the elective role in our formulation is performed in particular from the axiom (b) that relates non commutativity of the basic elements. In this algebraic scheme some principles of the basic quantum theoretical framework result to be represented. In particular, this algebraic structure reflects the intrinsic indetermination and the ontic potentiality that are basic components of quantum mechanics. This means that, in absence of a direct numerical attribution, such basic elements are abstract entities that act having an intrinsic, irreducible, indetermination, an ontic randomness, an ontic potentiality. Therefore, by using such rough quantum mechanical scheme, we may explore what is the actual role of potentiality in nature, what is its manner to combine with actual elements of our reality and what is the manner in which potentiality may contribute to the general dynamics of systems in Nature.

Let us add still some other feature of the scheme that we have in consideration. Let us consider two generic elements of our algebra, given as in the (2.13), and let us indicate them by x and y. Owing to the (2.11), they will result in general not commutative, that is to say

$$xy \neq yx \tag{2.20}$$

However, under suitable conditions, non-commutativity may fail and such abstract entities return to have the actual and traditional numerical role in some selected field. In this condition we have that $xy = yx$

Starting with 1974, [2] we introduced a theorem showing that necessary and sufficient condition for two given algebraic elements, x and y, to be commutative is that

$$xy = yx \leftrightarrow x_j = \lambda\, y_j \ , \forall \lambda \,(j = 1,2,3) \tag{2.21}$$

This theorem regulates the passage from potentiality of abstract elements in this algebra to actual numerical values relating instead any numerical field of our direct experience. In quantum mechanics this passage from potentiality to actualisation is called the collapse of wave function. An important feature of the theorem given in (2.21) is that the algebraic structure given in (2.1), (2.2), (2.11), and (19) admits idempotents. Let us consider two of such idempotents:

$$\psi_1 = \frac{1+e_3}{2} \quad \text{and} \quad \psi_2 = \frac{1-e_3}{2} \tag{2.22}$$

It is easy to verify that $\psi_1^2 = \psi_1$ and $\psi_2^2 = \psi_2$. Let us examine now the following algebraic relations:

$$e_3\psi_1 = \psi_1 e_3 = \psi_1 \tag{2.23}$$

$$e_3\psi_2 = \psi_2 e_3 = -\psi_2 \tag{2.24}$$

Similar relations hold in the case of e_1 or e_2. The relevant result is that the (2.23) establishes that the given algebraic structure, with reference to the idempotent ψ_1, attributes to e_3 the numerical value of $+1$ while the (2.24) establishes that, with reference to ψ_2, the numerical value of -1 is attributed to e_3.

The conclusion is very important: the conceptual counter part of the (2.23) and (2.24) is that we are in presence of a self-referential process. On the basis of such self-referential process, as given in (2.23) and in (2.24), this algebraic structure is able to attribute a precise numerical value to its basic elements. Each of the three basic elements may “ transitate” from the condition of pure potentiality to a condition of actualization, that is to say, in mathematical terms, from the pure, symbolic representation of their being abstract elements to that one of a real number. Let us remember that, on the basis of the (2.19), this self-referential process may regard each time one and only one of the three basic elements. It is well known that self-referential processes relate the basic phenomenology of our mind and consciousness.

In conclusion, for the first time we have an algebraic structure that represents a rough quantum mechanical scheme and that, at the same time, evidences, on the basis of a self-referential process, that it is possible a transition from potentiality to actualization. Other features of our formulation are given in [2,3,11]. It remains to evidence that a profound link exists between the idempotents prospected as example in the (2.22) and the traditional wave function that is introduced in standard quantum mechanics.

Let us consider the mean values of (2.22). We have that

$$2 < \psi_1 >= 1 + < e_3 > \quad \text{and} \quad 2 < \psi_2 >= 1 - < e_3 > \tag{2.25}$$

Using the last equation in (2.14) we obtain that

$$p_3(+1) = \frac{1 + < e_3 >}{2} \quad \text{and} \quad p_3(-1) = \frac{1 - < e_3 >}{2} \tag{2.26}$$

Therefore, considering the (2.22), we have that

$$p_3(+1) =< \psi_1 > \quad \text{and} \quad p_3(-1) =< \psi_2 > \tag{2.27}$$

The same result holds obviously when considering the basic elements e_1 or e_2. Considering that in quantum mechanics (Born probability rule), given the wave function $\varphi_{+,-}$, we have

$$\left|\varphi_{+,-}\right|^2 = p_{+,-} \tag{2.28}$$

we conclude that

$$\varphi_3(+) = \sqrt{< \psi_1 >} e^{i\vartheta_1} \quad \text{and} \quad \varphi_3(-) = \sqrt{< \psi_2 >} e^{i\vartheta_2} \tag{2.29}$$

and we have given proof that our rough scheme of quantum mechanics foresees the existence of wave functions as exactly traditional quantum mechanics makes.

We need here to make an important digression. Quantum mechanics runs usually about some fixed axioms. States of physical systems are represented by vectors in Hilbert spaces : historically, theoretical physicists as Planck, Bohr, Heisenberg, Pauli, Born, Dirac, established the rather general and consistent quantum mechanics in the form that is presently known to day. The question on the manner in which systems behave sometimes like particles and sometimes like waves as well as the question about the exact meaning of the complex wave functions are usually retained to represent examples of open question in the theory. In our opinion there is often no matter for such questions , and this is evidenced in our formulation about the rough quantum mechanical scheme by Clifford algebra. We consider the quantum wave function as the first evidence of the strong link existing between cognitive performance and linked physical description at some stages of our reality. Of course, we retain that superposition and interference effects by wave functions play a key role. We support that wave intensities and probability densities are not a matter of simple interpretation, that is added to quantum mechanics as it may be established evaluating that the Born probability rule was in fact introduced and thus added to quantum mechanics for purposes of probabilistic interpretation of quantum theory. It is no matter of a so simple Born interpretation. There is instead a precise theorem, proved and published well before quantum mechanics, that shows the fundamental role of the superposition principle and the profound link existing between quantum wave functions and probability densities. The theorem was published in 1915 by Fejer and by Riesz [12]. There is an excellent paper by F.H. Frohner that, time ago, properly evidenced the profound existing link between probability theory and quantum mechanics [13]. For any purpose, we retain of importance to report here this theorem that states

$$0 \leq \rho(x) \equiv \sum_{l=-n}^{n} c_l e^{ilx} \equiv \left|\sum_{k=0}^{n} a_k e^{ikx}\right|^2 \equiv |\psi(x)|^2$$

where the complex Fourier polynomial $\psi(x)$ has not restrictions, where instead to the Fourier polynomial $\rho(x)$ is imposed the requirement of its reality and non-negativity.

So, in conclusion, such required link exists and it is mathematically established. This is the matter in spite of the continuous claims that in quantum mechanics such link holds only on the basis of a given Born's interpretation.

Let us look now to another link existing between standard quantum mechanics and our rough quantum mechanical scheme. It is well known the central role that is developed in traditional quantum mechanics from density matrix operator . In our scheme of quantum mechanics, we have the corresponding algebraic member that is given in the following manner

$$\rho = a + be_1 + ce_2 + de_3 \tag{2.30}$$

with

$$a = \frac{|c_1|^2}{2} + \frac{|c_2|^2}{2}, \; b = \frac{c_1^* c_2 + c_1 c_2^*}{2}, \; c = \frac{i(c_1 c_2^* - c_1^* c_2)}{2}, \quad d = \frac{|c_1|^2 - |c_2|^2}{2} \tag{2.31}$$

where in matrix notation, e_1, e_2, and e_3 are the well known Pauli matrices. The complex coefficients c_i $(i = 1,2)$ are the well known probability amplitudes for the considered quantum state

$$\psi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{and} \quad |c_1|^2 + |c_2|^2 = 1 \tag{2.32}$$

For a pure state in quantum mechanics it is $\rho^2 = \rho$. In our scheme a theorem may be demonstrated that

$$\rho^2 = \rho \leftrightarrow \; a = \frac{1}{2} \; \text{ and } \; a^2 = b^2 + c^2 + d^2 \tag{2.33}$$

The details of this our theorem are given in [14]. Written in matrix form we have also $Tr(\rho) = 2a = 1$. In this manner we have the necessary and sufficient conditions for ρ to represent a potential state or, in traditional quantum mechanics, to have a superposition of states.

We have to examine now quantum time evolution.

It is clear that the quantum like scheme we are discussing is based on the two dimensional abelian subalgebra of the four dimensional Clifford algebra. Of course, generally speaking, we are considering our quantum rough scheme using quantum like operators acting on vectors of a given Hilbert space.

For time evolution, we consider Heisenberg description. Given the operator α connected to some observable A, the mean value at a given time t will be given as

$$< \alpha_t >= (\psi_0, U^{-1} \alpha U \psi_0)$$

with U time evolution operator.

It is well known that we have

$$ih \frac{d < \alpha >}{dt} = ih < \frac{\partial \alpha}{\partial t} + \frac{1}{ih}[\alpha, H]> \tag{2.34}$$

and

$$\frac{d\alpha}{dt} = \frac{\partial \alpha}{\partial t} + \frac{1}{ih}[\alpha, H] \tag{2.35}$$

where H is the Hamiltonian of the system. The manner in which such Hamiltonian may be constructed for psychological states in the Clifford algebra framework is given in [15].

It is well known that members of Clifford algebra transform according to

$$e_i^{'} = U^+ e_i U \quad , \quad U^+ U = 1 \tag{2.36}$$

In [3] we give a rigorous proof of the (2.34) and the (2.35) using the Clifford algebra.

Still we have to remember here that in the past there were attempts to go beyond the linear Schrodinger equation [16], but, as well as we know, nobody tried to do the same thing in the Heisenberg's picture. It is very important to outline here that in the non linear case, such two, Heisenberg and Schrodinger, representations, no more result to be equivalent.

We have in fact that

$$U = \exp(-ihHt) = 1 - \frac{i}{h}Ht + (\frac{iH}{h})^2 \frac{t^2}{2!} - (\frac{iH}{h})^3 \frac{t^3}{3!} + ... \quad (2.36)$$

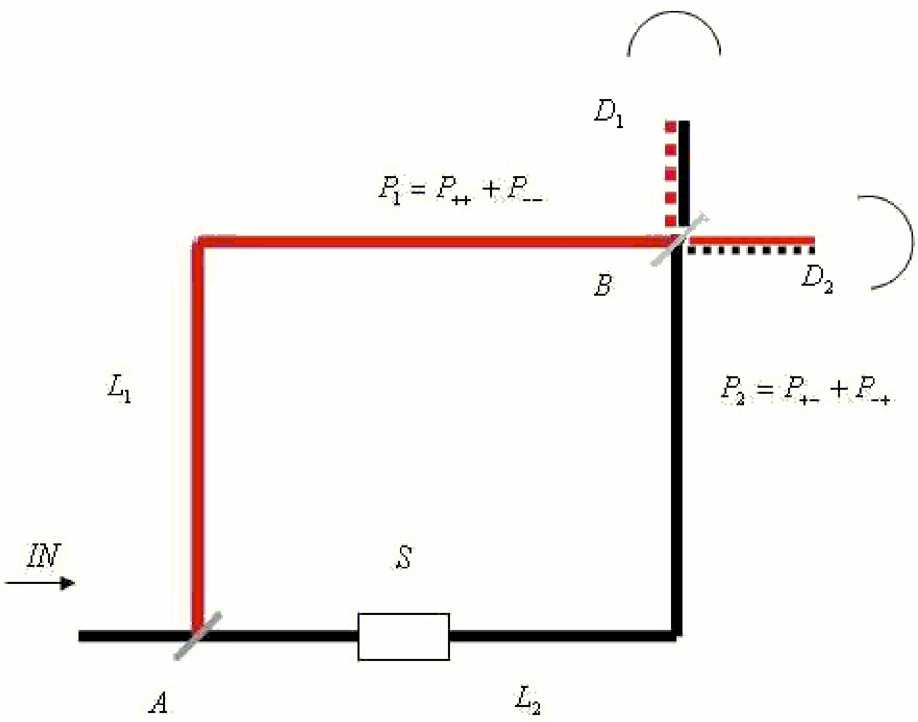
and

$$\frac{dU}{dt} = -\frac{iH}{h}\left[1 - \frac{i}{h}Ht + (\frac{iH}{h})^2 \frac{t^2}{2!} + ...\right] \quad (2.37)$$

By using the Clifford rough scheme of quantum mechanics we are in the condition to take account also for such possible non linear processes in Heisenberg like quantum representation.

3. Proof that Quantum Interference Arises in a Clifford Algebraic Formulation of Quantum Mechanics and the irreducible, ontic randomness of basic Clifford algebraic elements.

Consider a beam of particles impinging on a beam splitter A so that randomly may be either reflected to proceed a path L_1 or transmitted to proceed along the path L_2 (Fig.1). At the end of L_1, the particles impinge on the upper side of a second beam splitter , B, and it may be either reflected and detected by the detector D_1 or transmitted and detected by the detector D_2.The particles arriving from path L_2, impinge on the opposite side of to be either transmitted reaching the detector D_1 or reflected to reach the counter D_2. As it is well known we are considering here the interference pattern of a beam of particles passing through a Mach Zender interferometer.



The considered random variable A assumes the value $a = +1$ in the case of reflection and the value $a = -1$ in the case of transmission. The random variable B assumes the value $b = +1$ in the case of reflection and the value $b = -1$ in the case of transmission. We have a third variable $C = AB$ that is determined by the product of the values of A and B.

In analogy with the rough quantum scheme previously developed we call still write the mean value of A by $<A>$ and

$$<A>=(a=+1)p_{ab}+(a=-1)p_{ab} \qquad (3.1)$$

the mean value of B by $<B>$ and

$$<B>=(b=+1)p_{ab}+(b=-1)p_{ab} \qquad (3.2)$$

and the mean value of C by $<C>$ and

$$<C>=(ab;a=+1,b=+1)p_{ab}+(ab;a=+1,b=-1)p_{ab}+(ab;a=-1,b=+1)p_{ab}+(ab;a=-1,b=-1)p_{ab}$$
(3.3)

Let us follow directly the argument as it was recently developed in [17]. According to this interesting paper , we may write easily the expression of the probability for the corresponding four alternatives ($a=\pm1, b=\pm1$) in the following manner

$$p_{ab}=\frac{1}{4}(1+ax+by+abz) \qquad (3.4)$$

where

$$x\equiv<A>,\; y\equiv<B>, z\equiv<C>. \qquad (3.5)$$

Still according to ref. [17] let us calculate the probability for counting the detector D_1. We have that

$$p_{++}=\frac{1}{4}(1+x+y+z) \quad \text{and} \quad p_{--}=\frac{1}{4}(1-x-y+z) \qquad (3.6)$$

so that in the detector D_1 we have

$$p_{D_1}=p_{++}+p_{--}=\frac{1}{2}(1+z)=\frac{1}{2}(1+<C>) \qquad (3.7)$$

In the case of the detector D_2, we have

$$p_{+-}=\frac{1}{4}(1+x-y-z) \quad \text{and} \quad p_{-+}=\frac{1}{4}(1-x+y-z) \qquad (3.8)$$

and

$$p_{D_2}=p_{+-}+p_{-+}=\frac{1}{2}(1-z)=\frac{1}{2}(1-<C>) \qquad (3.9)$$

This is of course the classical statistical argument holding on an epistemic interpretation of randomness . In order to introduce the quantum like elaboration the author in ref.[17] correctly introduced three new variables:

$$U=\alpha A+\beta B+\gamma C \quad \text{with} \quad \alpha^2+\beta^2+\gamma^2=1; \qquad (3.10)$$

$$V=\lambda A+\mu B+\nu C\;, \quad \text{with } \lambda^2+\mu^2+\nu^2=1\;, \quad \alpha\lambda+\beta\mu+\gamma\nu=0 \qquad (3.11)$$

and

$$W=\delta A+\omega B+\vartheta C \quad \text{with} \quad \delta=\beta\nu-\gamma\mu\;;\; \omega=\gamma\lambda-\alpha\nu\;, \vartheta=\alpha\mu-\beta\lambda\;, \qquad (3.12)$$

and considered

$$<U>=u \qquad (3.13)$$

plus

$$<V>=<W>=0 \qquad (3.14)$$

in order to take into account a complete indetermination in the case of variables V and W.
Following this argument one obtains

$$\alpha<A>+\beta<B>+\gamma<C>=u\;; \qquad (3.15)$$
$$\lambda<A>+\mu<B>+\nu<C>=0;$$
$$\delta<A>+\omega<B>+\vartheta<C>=0$$

that admits solutions

$$<A>=\alpha u\;,<B>=\beta u\;,<C>=\gamma u\;. \qquad (3.16)$$

Inserting the (3.16) in the (3.4), one obtains

$$p_{ab}=\frac{1}{4}\left[1+(\alpha\, a+\beta\, b+\gamma\, ab)u\right] \qquad (3.17)$$

and

$$p_{D_1}=p_{++}+p_{--}=\frac{1}{2}(1+\gamma\, u) \qquad (3.18)$$

and

$$p_{D_2}=p_{+-}+p_{-+}=\frac{1}{2}(1-\gamma\, u) \qquad (3.19)$$

Let us comment the obtained results.

First consider the classical case.

Probability given in (3.4) must be between the well known limits

$$0\le p_{ab}\le 1 \qquad (3.20)$$

Consequently, still according to the findings in ref.[17], $<A>,<B>,<C>$, are the coordinates of a point inside the equilateral octahedron having the vertices

$<A>=\pm 1$, $<B>=0$, $<C>=0$; $<B>=\pm 1$, $<A>=0$, $<C>=0$; $<C>=\pm 1$, $<A>=0$, $<C>=0$

The author in ref. [17] correctly argues that the first limiting values correspond to the case of pure reflection (transmission) by A and equally probable reflection and transmission by B and zero correlation. The second limiting values correspond to equally probable reflection and transmission by A followed by pure reflection by B and zero correlation, and the third limiting values correspond to the case of complete correlation between the two splitters with equally probable transmission and reflection by A and B. These are the limiting cases while for the other possible conditions we have average values of the considered random variables having values less than one. This means that always particles with both the values (± 1) are present. We have that

$$-1\le <A>+<B>+<C>\le +1 \qquad (3.22)$$

According to the (3.16) in the case of the (3.10)-(3.12) and (3.13)-(3.14) , we have that

$$-1\le(\alpha+\beta+\gamma)u\le +1 \qquad (3.23)$$

which implies that the absolute value of u is always smaller than one. Particles with both values (± 1) of A,B,C are always present , [17].

We may now explore the quantum case. Instead of the (3.7) and the (3.9) , of the (3.18) and the (3.19) , the correct probabilities in quantum theory result to be

$$p_{D_1}=p_{++}+p_{--}=\frac{1}{2}(1+\gamma) \qquad (3.24)$$

and

$$p_{D_2}=p_{+-}+p_{-+}=\frac{1}{2}(1-\gamma) \qquad (3.25)$$

that result to be

$$p_{D_1}=p_{++}+p_{--}=\frac{1}{2}(1+\cos\phi) \qquad (3.26)$$

and

$$p_{D_2}=p_{+-}+p_{-+}=\frac{1}{2}(1-\cos\phi) \qquad (3.27)$$

This is to say that we must have $u=1(u=-1)$, and

$$<A>=\alpha\,,\ <B>=\beta\,,\ <C>=\gamma \qquad (3.28)$$

with

$$<A>^2+<B>^2+<C>^2=1 \qquad (3.29)$$

and

$\gamma=\cos\phi$

as polar angle of the unit vector on the sphere given in (3.29). This is to say that it must be

$< U^2 >= 1$ (3.30)

and

$< V >=< W >= 0$ (3.31)

to assure complete indetermination.

Let us consider again the variable U as given in the (3.10). It results that

$$U^2 = (\alpha A + \beta B + \gamma C)(\alpha A + \beta B + \gamma C) = \alpha^2 + \beta^2 + \gamma^2 + \alpha\beta(AB + BA) + \alpha\gamma(AC + CA) + \beta\gamma(BC + CB) = 1 + \alpha\beta(AB + BA) + \alpha\gamma(AC + CA) + \beta\gamma(BC + CB). \quad (3.32)$$

It is

$$< U^2 >= 1 + < \alpha\beta(AB + BA) + \alpha\gamma(AC + CA) + \beta\gamma(BC + CB) >. \quad (3.33)$$

The only way to obtain the (3.30) is that

$$AB = -BA\ ,\ AC = -CA,\ BC = -CB \quad (3.34)$$

and this is to say that the variables A, B, C must be the basic elements of the Clifford algebra, the e_i ($i = 1,2,3$) basic elements that we introduced in the previous section in the (2.1),(2.2),(2.11).

It is

$A \equiv e_1$, $B \equiv e_2$, $AB \equiv e_1 e_2 = ie_3$

Therefore, A, B, C, as given in the (3.10),(3.11), and the (3.12) are members of the Clifford algebra. So we reach the following conclusion.

Quantum mechanics holds about the basic phenomenon of quantum interference. We may realize it using the basic elements, and the structure of the Clifford algebra. The author in [17] concluded that typical objects of the required kind are Hermitean matrices with eigenvalues (± 1).

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