A Reformulation of von Neumann's postulates on quantum measurement by using two theorems in Clifford algebra

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Abstract: According to a procedure previously introduced from Y. Ilamed and N. Salingaros, we start giving proof of two existing Clifford algebras, the Si that has isomorphism with that one of Pauli matrices and the $N_{i,\pm 1}$ where N_i stands for the dihedral Clifford algebra. The salient feature is that we show that the $N_{i,\pm 1}$ may be obtained from the Si algebra when we attribute a numerical value (+1 or –1) to one of the basic elements (e_1, e_2, e_3) of the Si. We utilize such result to advance a criterium under which the Si algebra has as counterpart the description of quantum systems that in standard quantum mechanics are considered in absence of observation and quantum measurement while the $N_{i,\pm 1}$ attend when a quantum measurement is performed on such system with advent of wave function collapse. The physical content of the criterium is that the quantum measurement with wave function collapse induces the passage in the considered quantum system from the Si to $N_{i,+1}$ or to the $N_{i,-1}$ algebras, where each algebra has of course its proper rules of commutation. After a proper discussion on the difference between decoherence and wave function collapse, we re-examine the von Neumann postulate on quantum measurement, and we give a proper justification of such postulate by using the Si algebra. Soon after we study some applications of the above mentioned criterium to some cases of interest in standard quantum mechanics, analyzing in particular a two state quantum system , the case of time dependent interaction of such system with a measuring apparatus and finally the case of a quantum system plus measuring apparatus developed at the order n=4 of the considered Clifford algebras and of the corresponding density matrix in standard quantum mechanics. In each of such cases examined , we find that the passage from the algebra Si to $N_{i,\pm 1}$, considered during the quantum measurement of the system, actually describes the collapse of the wave function. Therefore we conclude that the actual quantum measurement has as counterpart in the Clifford algebraic description, the passage from the Si to the $N_{i,\pm 1}$ Clifford algebras, reaching in this manner the objective to reformulate von Neumann postulate on quantum measurement and proposing a self-consistent formulation of quantum theory.

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INTRODUCTION

Quantum mechanics has had a so great success to leave very little reasons to doubt its intrinsic validity. It has never been found in disagreement with experimental data, and in explaining a very large variety of physical processes and in predicting basic results also in other fields. Nevertheless, we cannot ignore that some questions concerning fundamental features of this theory remained unsolved, and some historic debates among scientists deeply influenced the early development of the theory. These basic issues were and often continue to be prevalently discussed mainly in philosophical contexts[1]. They will not receive here our direct consideration. In our opinion the object of direct investigation is to understand where the foundations of the theory lie, and why so many deep questions are still unanswered.
The first important question concerns the problem of the wave-function collapse by measurement. Its solution would be of relevant significance because it would provide us with a *self-consistent* formulation of the quantum-mechanical formalism. This result might be of importance also to foresee the way to be followed in order to understand and to explain also the other basic issues that remained often understandable in the story of this theory. As we know , they gave origin to a profound debate. On this basis the completeness of quantum mechanics as a physical theory was discussed, and the very validity of quantum mechanics was often questioned.
The aim of the present paper is to reformulate the basic von Neumann's postulate on quantum measurement on the basis of two theorems that we proof in the framework of the Clifford algebra. The results that we obtain seem to be of some relevance for the problem of wave function collapse since, based on two algebraic theorems, we are supported from the asepsis language of an algebraic framework and thus without resource to philosophical or to epistemological indications.

SOME FEATURES ABOUT THE COLLAPSE OF THE WAVE FUNCTION

In quantum mechanics we have the well known phenomenon of quantum interference.
We consider a quantum-mechanical particle to be a "physical entity" represented by a quantum wave function $\psi(x,t)$, which depends on the space coordinate x and the time variable *t* of this particle. Consider the well known interference experiment of the Young type, in which a beam of particles hits a target with two open slits . Also the theoretical description of this experiment is well known. It holds about two basic postulates:
(a) The total outgoing wave function $\psi(x,t)$ behind the slits is written as

$$\psi = \psi_1 + \psi_2 \qquad (1)$$

where ψ_1 and ψ_2 are the two waves originating from slits 1 and 2, respectively. This is the so-called superposition principle.

(b) The intensity of the wave function ψ is proportional to $|\psi|^2$. We understand the above experimental facts by means of a purely probabilistic interpretation of the wave function. It is assumed that

$$P = |\psi(x,t)|^2 \qquad (2)$$

is proportional to the probability of finding a particle at a space-point x at time *t,* when it is in a state represented by ψ .

The intensity observed at the screen is proportional to

$$P = |\psi|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\,\mathrm{Re}(\psi_1^* \psi_2) = P_1 + P_2 + 2\,\mathrm{Re}(\psi_1^* \psi_2) \qquad (3)$$

where there is the presence of the characteristic interference term $2\,\mathrm{Re}(\psi_1^* \psi_2)$ which is responsible for the observed interference pattern.

Suppose we find a particle at point X on the screen .On the basis of the probabilistic interpretation, we can state that the particle state immediately after the observation must be represented by a wave function $\psi_X(t)$, distributed only around X so that we conclude that the measurement has caused the change

$$\psi \to \psi_X \qquad (4)$$

of the wave function.

We call this change the wave-function collapse by measurement. The wave-function collapse is not a causal wave motion, continuously shrinking from ψ to ψ_X or to $\psi_{X'}$, but it is an acausal and purely probabilistic event. Quantum mechanics only gives the probabilistic prediction that the probability of finding each event is proportional to $|\psi_X|^2$ or to $|\psi_{X'}|^2$. Of course, the wave-function collapse cannot be described by the Schrödinger equation which gives only deterministic changes. Consequently, quantum mechanics becomes a non self-contained theory since the measuring process cannot be described by quantum mechanics itself.

These are only some preliminary features regarding the more articulated problem of measurement in quantum mechanics. For a complete examination of the actual problems that are involved, we refer to the several reviews that may be found in literature[2]. In particular, we intend to hint here only at some recent developments as the theory of quantum decoherence, a term that was used for the first time by Bohr, while an articulated elaboration was introduced more recently by Zurek[3]. It considers the mechanism by which quantum systems interact with their environments giving the appearance of wave function collapse. Still we mention here the theory of Ghirardi, Rimini and Webber (GRW)[4] who claim that particles undergo spontaneous wave-function collapses. The leading idea of the theory is to eradicate observers from the picture and view state reduction as a process that occurs as a consequence of the basic laws of nature. The theory achieves this by adding to the fundamental equation of quantum mechanics, the Schrödinger equation, a stochastic term which describes the state reduction occurring in the system.

After such preliminary remarks, we may now set the basis for our discussion. As previously stated, we consider the measurement of a given observable F on a quantum-mechanical system S in a normalized superposed state

$$\psi = \sum_i c_i \varphi_i \quad ; \quad c_i = (\varphi_i, \psi) \quad ; \quad \sum_i |c_i|^2 = 1 \qquad (5)$$

where φ_i is a normalized eigenstate of F , relative to an eigenvalue λ_i ,so that $F\varphi_i = \lambda_i \varphi_i$ and $(\varphi_i, \varphi_j) = \delta_{ij}$.

The probabilistic interpretation means that the probability of finding the eigenvalue λ_i (i.e. the corresponding eigenstate φ_i*)* in the measurement of *F* on a state ψ is equal to $|c_i|^2$.

The wave-function collapse is expressed in this case as

$$\psi \to \varphi_i \qquad (6)$$

The previous equation still does not describe the wave-function collapse by measurement, intended as an acausal and purely probabilistic event. A complete expression for the wave-function collapse must be formulated in terms of density matrix as it was initiated by von Neumann [5]

$$\rho_S = |\psi \rangle\langle \psi| = \sum_i \sum_j c_i c_j^* |\varphi_i \rangle\langle \varphi_j| \to \rho_{S,F} = \sum_k |c_k|^2 |\varphi_k \rangle\langle \varphi_k| \qquad (7)$$

The above expression describes rather well a process in which all the phase correlations among different eigenstates are erased. We obtain a sum of exclusive probabilities of finding each eigenstate.

However, also such formulation may still give origin to contradictions. In order to avoid such possible difficulties, we have to modify the previous expression for the wavefunction collapse, by introducing the states of a given measurement apparatus system A obtaining in this case

$$\rho = \rho_S \otimes \rho_A = \sum_i \sum_j c_i c_j^* \left|\varphi_i >< \varphi_j\right| \otimes \rho_A \to \rho_{S,A,t} = \sum_k \left|c_k\right|^2 \left|\varphi_k >< \varphi_k\right|_t \otimes \rho_{A(k),t} \tag{8}$$

Let us see in more detail the von Neumann's postulate about quantum measurement.
If a quantum system is in an eigenstate of the operator corresponding to the observable being measured, the outcome of the measurement will be the eigenvalue associated with that eigenstate. However, if the system is in a superposition of such eigenstates, the outcome will be unpredictable, and all that quantum theory can give, are the probabilities for the different outcomes. If the system is not destroyed by the measurement, and if the interaction fits into the so called 'measurement of the first kind', then the quantum state after the measurement will be the eigenstate associated to the measurement outcome, or more generally (to include degenerancies), the normalized projection of the original state onto the eigensubspace associated with the outcome. This rule is known as the *projection postulate*. It originated with Dirac and von Neumann, and was later formalized in degenerate cases by Luders and Ludwig[6].
According to such *projection postulate* the complete phase-damping way for a two state system may be written

$$D(\rho) = \left|0 >< 0\right|\rho\left|0 >< 0\right| \quad + \quad \left|1 >< 1\right|\rho\left|1 >< 1\right| \tag{9}$$

where the effect of this mapping is to zero-out the off-diagonal entries of a density matrix:

$$D\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \tag{10}$$

If we have a set of mutually orthogonal projection operators $(P_1, P_2, \ldots, P_m)$ which complete to identity, i.e., $P_i P_j = \delta_{ij} P_j$ and $\sum_i P_i = 1$ when a measurement is carried out on a system with state $|\psi>$ then

(1) The result *i* is obtained with probability $p_i = <\psi|P_i|\psi>$

(2) The state collapses to

$$\frac{1}{\sqrt{p_i}} P_i |\psi>$$

In detail, note that von Neumann's projection postulate only relates the vanishing of interference terms or decoherence. It does not explain the collapse of a pure state to another pure state associated to individual object systems. The formal distinction between decoherence and collapse is substantial as authors as Sussmann[7] in 1957 and Bell[8] more recently outlined. They distinguished clearly between what we call 'division' (decoherence) and 'reading' (the collapse which follows decoherence). *Decoherence* is a statistical concept, involving the transition from a pure state to a 'mixture', and the disappearance of interference terms. *Collapse* refers to an individual system, and it describes a transition from a pure state to another pure state.
For a single quantum object, we may therefore write:

$$\left|\sum_i a_i \varphi_i >\to \varphi_k\right. \tag{11}$$

with probability

$$|a_k|^2$$

For an ensemble of measurements of the same observable performed on the same initial pure state (that is, each measurement being performed on a different single object, all prepared in the same pure state), one may represent the statistical transition described by the projection postulate as

$$P\sum_i a_i \varphi_i \to \sum_k |a_k|^2 P(\varphi_k) \tag{12}$$

In brief, *an explanation for collapse implies an explanation for decoherence, but an explanation for decoherence doesn't imply an explanation for collapse*[9].

TWO THEOREMS IN CLIFFORD ALGEBRA

Let us start with a proper definition of the 3-D space Clifford (geometric) algebra Cl_3.
It is an associative algebra generated by three vectors e_1, e_2, and e_3 that satisfy the orthonormality relation

$$e_j e_k + e_k e_j = 2\delta_{jk} \tag{13}$$

for $j, k, \lambda \in [1,2,3]$.
That is,

$e_\lambda^2 = 1$ and $e_j e_k = -e_k e_j$ for $j \neq k$

Let **a** and **b** be two vectors spanned by the three unit spatial vectors in $Cl_{3,0}$. By the orthonormality relation the product of these two vectors is given by the well known identity: $ab = a \cdot b + i(a \times b)$ where $i = e_1 e_2 e_3$ is a Clifford algebraic representation of the imaginary unity that commutes with vectors.
To give proofs, let us follow the approach that, starting with 1981, was developed by Y. Ilamed and N. Salingaros[10].
Let us consider the three abstract basic elements, e_i, with $i = 1,2,3$, and let us admit the following two postulates:

a) it exists the scalar square for each basic element:

$$e_1 e_1 = k_1 \ , \ e_2 e_2 = k_2 \, , \ e_3 e_3 = k_3 \quad \text{with} \quad k_i \in \Re \ . \tag{14}$$

In particular we have also the unit element, e_0, such that that

$$e_0 e_0 = 1.$$

b) The basic elements e_i are anticommuting elements, that is to say:

$$e_1 e_2 = -e_2 e_1 \ , \ e_2 e_3 = -e_3 e_2 \, , \ e_3 e_1 = -e_1 e_3 . \tag{15}$$

It is

$$e_i e_0 = e_0 e_i = e_i \, .$$

Theorem n.1.

Assuming the two postulates given in (a) and (b) with $k_i = 1$, the following commutation relations hold for such algebra :

$$e_1 e_2 = -e_2 e_1 = i e_3 \ ; \ e_2 e_3 = -e_3 e_2 = i e_1 ; \ e_3 e_1 = -e_1 e_3 = i e_2 \ ; i = e_1 e_2 e_3 \, , (e_1^2 = e_2^2 = e_3^2 = 1) \tag{16}$$

They characterize the Clifford Si algebra . We will call it the algebra A(Si).
Proof.

Consider the general multiplication of the three basic elements e_1, e_2, e_3, using scalar coefficients $\omega_k, \lambda_k, \gamma_k$ pertaining to some field:

$$e_1 e_2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \ ; \ e_2 e_3 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \ ; \ e_3 e_1 = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 \, . \tag{17}$$

Let us introduce left and right alternation: for any (i, j), associativity exists $e_i e_i e_j = (e_i e_i) e_j$ and $e_i e_j e_j = e_i (e_j e_j)$ that is to say

$$e_1 e_1 e_2 = (e_1 e_1) e_2 \, ; \ e_1 e_2 e_2 = e_1 (e_2 e_2) \, ; \ e_2 e_2 e_3 = (e_2 e_2) e_3 \, ; \ e_2 e_3 e_3 = e_2 (e_3 e_3) \, ; e_3 e_3 e_1 = (e_3 e_3) e_1 \, ;$$

$$e_3 e_1 e_1 = e_3 (e_1 e_1) \, . \tag{18}$$

Using the (15) in the (18) it is obtained that

$$k_1 e_2 = \omega_1 k_1 + \omega_2 e_1 e_2 + \omega_3 e_1 e_3 \, ; \qquad k_2 e_1 = \omega_1 e_1 e_2 + \omega_2 k_2 + \omega_3 e_3 e_2 \, ;$$

$$k_2 e_3 = \lambda_1 e_2 e_1 + \lambda_2 k_2 + \lambda_3 e_2 e_3 \, ; \qquad k_3 e_2 = \lambda_1 e_1 e_3 + \lambda_2 e_2 e_3 + \lambda_3 k_3 \, ;$$

$$k_3 e_1 = \gamma_1 e_3 e_1 + \gamma_2 e_3 e_2 + \gamma_3 k_3 \, ; \qquad k_1 e_3 = \gamma_1 k_1 + \gamma_2 e_2 e_1 + \gamma_3 e_3 e_1 \ . \tag{19}$$

From the (19), using the assumption (b), we obtain that

$$\frac{\omega_1}{k_2} e_1 e_2 + \omega_2 - \frac{\omega_3}{k_2} e_2 e_3 = \frac{\gamma_1}{k_3} e_3 e_1 - \frac{\gamma_2}{k_3} e_2 e_3 + \gamma_3 \, ;$$

$$\omega_1 + \frac{\omega_2}{k_1} e_1 e_2 - \frac{\omega_3}{k_1} e_3 e_1 = -\frac{\lambda_1}{k_3} e_3 e_1 + \frac{\lambda_2}{k_3} e_2 e_3 + \lambda_3 \, ;$$

$$\gamma_1 - \frac{\gamma_2}{k_1} e_1 e_2 + \frac{\gamma_3}{k_1} e_3 e_1 = -\frac{\lambda_1}{k_2} e_1 e_2 + \lambda_2 + \frac{\lambda_3}{k_2} e_2 e_3 \tag{20}$$

By the principle of identity , we have that it must be

$$\omega_1 = \omega_2 = \lambda_2 = \lambda_3 = \gamma_1 = \gamma_3 = 0 \tag{21}$$

and

$$-\lambda_1 k_1 + \gamma_2 k_2 = 0 \qquad \gamma_2 k_2 - \omega_3 k_3 = 0 \qquad \lambda_1 k_1 - \omega_3 k_3 = 0 \tag{22}$$

The (22) is an homogeneous algebraic system admitting non trivial solutions since its determinant $\Lambda = 0$, and the following set of solutions is given:

$$k_1 = -\gamma_2\omega_3,\ k_2 = -\lambda_1\omega_3\ , k_3 = -\lambda_1\gamma_2 \tag{23}$$

Admitting $k_1 = k_2 = k_3 = +1$, it is obtained that

$$\omega_3 = \lambda_1 = \gamma_2 = i \tag{24}$$

In this manner, using the (14) and the (15), as a theorem, the existence of such algebra is proven. The basic features of this algebra are given in the following manner

$$e_1^2 = e_2^2 = e_3^2 = 1 \quad ; \quad e_1e_2 = -e_2e_1 = ie_3 \quad ; \quad e_2e_3 = -e_3e_2 = ie_1; \quad e_3e_1 = -e_1e_3 = ie_2 \quad ; \quad i = e_1e_2e_3 \tag{25}$$

The content of the theorem n.1 is thus established: given three abstract basic elements as defined in (a) and (b) ($k_i = 1$), an algebraic structure is established with four generators (e_0, e_1, e_2, e_3).

Let us go on now to give proof of theorem n.2.

Before let us note that the algebra A(Si), now given, admits idempotents.

Let us consider two of such idempotents:

$$\psi_1 = \frac{1+e_3}{2} \quad \text{and} \quad \psi_2 = \frac{1-e_3}{2} \tag{26}$$

It is easy to verify that $\psi_1^2 = \psi_1$ and $\psi_2^2 = \psi_2$.

Let us examine now the following algebraic relations:

$$e_3\psi_1 = \psi_1e_3 = \psi_1 \tag{27}$$

$$e_3\psi_2 = \psi_2e_3 = -\psi_2 \tag{28}$$

Similar relations hold in the case of e_1 or e_2. From a conceptual point of view , looking at the (27) and (28) we reach only a conclusion. With reference to the idempotent ψ_1, the algebra A(Si) (see the (27)), attributes to e_3 the numerical value of $+1$ while, with reference to the idempotent ψ_2, the algebra A(Si) attributes to e_3 (see the (28)), the numerical value of -1 .

However, assuming the attribution $e_3 \to +1$, from the (25) we have that new commutation relations should hold in a new Clifford algebra given in the following manner :

$$e_1^2 = e_2^2 = 1\, , i^2 = -1\, ; e_1e_2 = i\, , e_2e_1 = -i\, , e_2i = -e_1\, , ie_2 = e_1\, , e_1i = e_2\, , ie_1 = -e_2 \tag{29}$$

with three new basic elements (e_1, e_2, i) instead of (e_1, e_2, e_3) .

In other terms, in the case in which we attribute to e_3 the numerical value +1, a new algebraic structure should arise with new generators whose rules should be given in (29) instead of in (25). Therefore, the arising central problem is that we should be able to proof the real existence of such new algebraic structure with rules given in the (29). We repeat: in the case of the starting algebraic structure, the algebra A(Si), we showed by theorem n.1 that it exists in the following manner

$$e_1^2 = e_2^2 = e_3^2 = 1\, ;$$

$$e_1e_2 = -e_2e_1 = ie_3\ ;\ e_2e_3 = -e_3e_2 = ie_1;\ e_3e_1 = -e_1e_3 = ie_2\ ; i = e_1e_2e_3 \tag{30}$$

In the present case in which we attribute to e_3 the numerical value +1 , we should show that it exists a new algebra given in the following manner

$$e_1^2 = e_2^2 = 1\, ;\ i^2 = -1\, ;$$

$$e_1e_2 = i\ ,\ e_2e_1 = -i\, ,\ e_2i = -e_1\, , ie_2 = e_1\, , e_1i = e_2\, ,\ ie_1 = -e_2 \tag{31}$$

So we arrive to give proof of the theorem n.2.

Theorem n.2 .

Assuming the postulates given in (a) and (b) with $k_1 = 1,\ k_2 = 1, k_3 = -1$, the following commutation rules hold for such new algebra:

$$e_1^2 = e_2^2 = 1\, ;\ i^2 = -1\, ;$$

$$e_1e_2 = i\ ,\ e_2e_1 = -i\, ,\ e_2i = -e_1\, , ie_2 = e_1\, , e_1i = e_2\, ,\ ie_1 = -e_2 \tag{32}$$

They characterize the Clifford Ni algebra. We will call it the algebra $N_{i,+1}$

Proof

To give proof, rewrite the (17) in our case, and performing step by step the same calculations of the previous proof, we arrive to the solutions of the corresponding homogeneous algebraic system that in this new case are given in the following manner:

$$k_1 = -\gamma_2\omega_3\,;\; k_2 = -\lambda_1\omega_3\,; k_3 = -\lambda_1\gamma_2 \tag{33}$$

where this time it must be $k_1 = k_2 = +1$ and $k_3 = -1$. It results

$$\lambda_1 = -1\,;\gamma_2 = -1\,;\; \omega_3 = +1 \tag{34}$$

and the proof is given.

The content of the theorem n.2 is thus established. **When we attribute to e_3 the numerical value +1 we pass from the Clifford algebra Si (algebra A) to a new Clifford algebra $N_{i,+1}$ whose algebraic structure is no more given from the (30) of the algebra A but from the following new basic rules:**

$$e_1^2 = e_2^2 = 1\,;\; i^2 = -1\,;$$

$$e_1e_2 = i\ ,\ e_2e_1 = -i\,,\ e_2i = -e_1, ie_2 = e_1, e_1i = e_2\,,\ ie_1 = -e_2 \tag{35}$$

that are totally different from the basic commutation rules that we have in the case of the algebra A(Si)**.**

The theorem n.2 also holds in the case in which we attribute to e_3 the numerical value of -1 .

Assuming the postulates given in (a) and (b) with $k_1 = 1,\ k_2 = 1, k_3 = -1$, the following commutation rules hold for such new algebra

$$e_1^2 = e_2^2 = 1\,;\; i^2 = -1\,;$$

$$e_1e_2 = -i\ ,\ e_2e_1 = i\,,\ e_2i = e_1\,, ie_2 = -e_1\,, e_1i = -e_2\,,\ ie_1 = e_2 \tag{36}$$

They characterize the Clifford Ni algebra. We will call it the algebra $N_{i,-1}$

To give proof , consider the solutions of the (33) that are given in this new case by

$$\lambda_1 = +1\,;\gamma_2 = +1;\; \omega_3 = -1 \tag{37}$$

and the proof is given.

The content of the theorem n.2 is thus established. When we attribute to e_3 the numerical value –1, we pass from the Clifford algebra Si (algebra A) to a new Clifford algebra $N_{i,-1}$ whose algebraic structure is not given from the (30) of the algebra A and not even from the (35) but from the following new basic rules:

$$e_1^2 = e_2^2 = 1\,;\; i^2 = -1\,;$$

$$e_1e_2 = -i\ ,\ e_2e_1 = i\,,\ e_2i = e_1\,, ie_2 = -e_1\,, e_1i = -e_2\,,\ ie_1 = e_2 \tag{38}$$

In a similar way, proofs may be obtained when we consider the cases attributing numerical values (± 1) to e_1 or to e_2 .

Of course, the Clifford algebra, $N_{1,\pm 1}$, given in the (35) and in the (36) are well known . They are the dihedral Clifford algebra N_i (for details, see ref.10 page 2093 Table II).

In conclusion, in this section, using a Clifford algebraic framework, we have shown two basic theorems, the theorem n.1 and the theorem n.2. As any mathematical theorem they have maximum rigour, and an aseptic mathematical content that cannot be questioned. The basic statement that we reach by the proof of such two theorems is that in Clifford algebraic framework, we have the Clifford algebra A(Si) and inter-related Clifford algebras $N_{i,\pm 1}$. When we consider (e_1, e_2, e_3) as the three abstract elements with rules given in (30) , we are in the Clifford algebra A(Si) .When we attribute to e_3 the numerical value +1, we pass from the algebra A (the Clifford algebra Si, with basic features given in (30)), to the algebra B, the Clifford $N_{i,+1}$, with basic algebraic rules given in the (31). Instead, when we pass from the Clifford algebra A, (the Clifford algebra Si) to the Clifford algebra $N_{i,-1}$, the basic features are given in the (38) and we attribute to e_3 the numerical value –1 .

The same conceptual facts hold when we reason for Clifford basic elements e_1 or to e_2 , attributing in this case a possible numerical value (± 1) or to e_1 or to e_2 , respectively.

A POSSIBLE IMPLICATION FOR QUANTUM MECHANICS

If one looks at the algebraic rules and commutation relations given in the (30), the algebra A(Si) shown by theorem n.1, immediately acknowledges that they are universally valid in quantum mechanics. We called the algebra A as the Si

Clifford algebra because it links the Pauli matrices that are sovereign in quantum mechanics. Still the isomorphism between Pauli matrices and Clifford algebras is well established at any order.

Passing from the algebra A(Si) to $N_{i,\pm 1}$ it happens an interesting feature. Consider the case , as example, of e_3. While in A(Si) e_3 is an abstract algebraic element that has the potentiality to assume or the value +1 or the value –1(in correspondence, in quantum mechanics it is an operator with possible eigenvalues ± 1) , when we pass in the algebra $N_{i,\pm 1}$, e_3 is no more an abstract element in this algebra , it becomes a parameter to which we may attribute the numerical value +1 , and we have $N_{i,+1}$ whose three abstract element now are (e_1, e_2, i) with commutation rules given in the (35) . If we attribute to e_3 the numerical value –1 , we are in $N_{i,-1}$ whose three abstract elements are still (e_1, e_2, i) , and the commutation rules are given in (38). Reading this statement in the language and in the logic of quantum mechanical measurement , it means that if we are measuring the given quantum system S with a measuring apparatus and , as result of the actualized and performed measurement, we read the result +1 , we are in the corresponding algebraic case , in the algebra $N_{i,+1}$. If instead , performing the measurement , we read the result –1, in this case we are in the algebra $N_{i,-1}$. In each of the two cases this means that a collapse of the wave function has happened.

During a process of quantum measurement , speaking in terms of Clifford algebraic framework, we could have the passage from the Clifford algebra A,(Si), having such fundamental basic commutation rules :

$$e_1e_2 = -e_2e_1 = ie_3 \;;\; e_2e_3 = -e_3e_2 = ie_1;\; e_3e_1 = -e_1e_3 = ie_2 \;; i = e_1e_2e_3$$
$$e_1^2 = e_2^2 = e_3^2 = 1; \tag{39}$$

to the new $N_{i,+1}$ Clifford algebra having the following and totally new commutation rules:

$$e_1e_2 = i \;,\; e_2e_1 = -i,\; e_2i = -e_1, ie_2 = e_1, e_1i = e_2,\; ie_1 = -e_2$$
$$e_1^2 = e_2^2 = 1;\; i^2 = -1; \tag{40}$$

in the case in which the result of the measurement of e_3 is +1 (read on the instrument), and instead we could have the passage to the new $N_{i,-1}$ Clifford algebra , having the following and totally new commutation rules:

$$e_1e_2 = -i \;,\; e_2e_1 = i,\; e_2i = e_1, ie_2 = -e_1, e_1i = -e_2,\; ie_1 = e_2$$
$$e_1^2 = e_2^2 = 1;\; i^2 = -1 \tag{41}$$

in the case in which the result of the quantum measurement of e_3 gives value –1 (read on the instrument).

In such way it seems that a reformulation of von Neumann's projection postulate may be suggested. The reformulation is that, during a quantum measurement (wave-function collapse), we have the passage from the Clifford algebra A(Si), to the new Clifford algebra $N_{i,\pm 1}$.

In other terms

Quantum Measurement (wave-function collapse) = passage from algebra A (Si) to B ($N_{i,\pm 1}$).

In conclusion we think that the two previously shown theorems in Clifford algebraic framework give justification of the von Neumann's projection postulate and they seem to suggest , in addition , that we may use the passage from the algebra A(Si) to $N_{i,\pm 1}$ to describe actually performed quantum measurements.

APPLICATIONS OF THE PREVIOUS CRITERIUM TO SOME CASES OF QUANTUM MECHANICAL INTEREST

Let us start discussing a trivial application. It is important only to illustrate better the sense in which we must intend the present formulation.

Assume a two –level microscopic quantum system S with two states u_+, u_- corresponding to energy eigenvalues ε_+, ε_-. The Hamiltonian operator H_S can be written

$$H_S = \frac{1}{2}\varepsilon_+(1+e_3) + \frac{1}{2}\varepsilon_-(1-e_3) = \frac{1}{2}(\varepsilon_+ + \varepsilon_-) + \frac{1}{2}(\varepsilon_+ - \varepsilon_-)e_3 \tag{42}$$

The standard quantum methodological approach is also well known . We have that

$$u_+ = \begin{pmatrix}1\\0\end{pmatrix} \;,\; u_- = \begin{pmatrix}0\\1\end{pmatrix} \;,\; \text{and} \quad H_S u_i = \varepsilon_i u_i . \tag{43}$$

We may also choose $\varepsilon_+ = \varepsilon$ and $\varepsilon_- = 0$ simplifying the (42) to

$$H_S = \frac{1}{2}(1+e_3)\varepsilon \quad (44)$$

Indicate an arbitrary state of such quantum microsystem as

$$\psi_S = c_+ u_+ + c\_u_- \quad (45)$$

where, according to Born's rule, we have

$$c_+ = \sqrt{p_+}e^{i\delta_1} \quad , \quad c_- = \sqrt{p_-}e^{i\delta_2} \quad (46)$$

with

$$p_j \ \ (j = +,-) \quad (47)$$

corresponding probabilities with $p_+ + p_- = 1$.

This is the standard quantum mechanical formulation of the system.

Let us admit now that we want to measure the energy of S using a proper apparatus . The rules of quantum mechanics tell us that we will obtain the value ε with probability p_+, and the value zero with probability p_-. After the measurement the state of S will be either u_+ or u_- according to the measured value of the energy. The experiment will enable us also to estimate p_+ as well as p_-.

In such simple quantum mechanical example we have , as known, the (42), e_3, the (44) that are linear Hermitean operators with quantum states acting on the proper Hilbert space.

Let us see instead the question from a different point of view.

The e_3, and H_S given in the (42) or in the (44) are members of the Clifford algebra. They are Clifford algebraic members of what we have called the algebra A(Si), with basic rules given in the following manner:

$$e_1^2 = e_2^2 = e_3^2 = 1$$

$$e_1e_2 = -e_2e_1 = ie_3 ; \ e_2e_3 = -e_3e_2 = ie_1 \ ; \ \ e_2e_3 = -e_3e_2 = ie_1 ; \ i = e_1e_2e_3 \quad (48)$$

However, on the basis of theorems n.1 and n.2 shown in the previous sections, starting with the Clifford algebra A(Si), we must use the existing Clifford, dihedral algebra B, $N_{i,\pm1}$ when we arrive to attribute (by a measurement) as example to e_3 in one case the numerical value +1 and, in the other case, the numerical value –1.

In the first case we have a dihedral Clifford N_i algebra that is given in the following manner:

$$e_1^2 = e_2^2 = 1 \ i^2 = -1$$

$$e_1e_2 = i , \ e_2e_1 = -i , \ e_2i = -e_1 , \ e_2i = -e_1 , \ ie_2 = e_1 , \ e_1i = e_2 , \ ie_1 = -e_2 \quad (49)$$

that holds when we are attributing to e_3 the numerical value +1 (in analogy with quantum mechanics: the quantum measurement process has given as result +1). In the second case, we have instead that

$$e_1^2 = e_2^2 = 1 ; \ \ i^2 = -1 ;$$

$$e_1e_2 = -i \ , \ e_2e_1 = i , \ e_2i = e_1 , ie_2 = -e_1 , e_1i = -e_2 , \ ie_1 = e_2 \quad (50)$$

that holds when we have arrived to attribute to e_3 the numerical value –1 by a direct measurement

Reasoning in terms of a Clifford algebraic framework, we are authorized to apply the passage from algebra A(Si) to algebra B in the (42) . From it , we obtain:

$$H_{S(Clifford-element)} = \varepsilon_+ \quad (51)$$

if the instrument has given as result of the measurement , the value +1 to e_3 (Clifford algebraic parameter of dihedral $N_{i,+1}$ algebra), and

$$H_{S(Clifford-element)} = \varepsilon_- \quad (52)$$

if the instrument has given as result of the measurement , the value -1 to e_3. During the measurement we have had the passage from algebra A(Si) to the dihedral $N_{i,\pm1}$ algebra in which , with given probabilities, e_3 has assumed or the +1 or the value –1 , respectively.

In the first case, we have

$$H_{S(Clifford-element)} = \varepsilon$$

and in the second case, we have

$$H_{S(Clifford-element)} = 0$$

Consider now the second application .

Let us introduce a two state quantum system S with connected quantum observable $\sigma_3(e_3)$. We have

$$|\psi\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle \ , \ \varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{53}$$

and

$$|c_1|^2 + |c_2|^2 = 1$$

As we know, the density matrix of such system is easily written

$$\rho = a + be_1 + ce_2 + de_3 \tag{54}$$

with

$$a = \frac{|c_1|^2 + |c_2|^2}{2} \ , \ b = \frac{c_1^* c_2 + c_1 c_2^*}{2} \ , \ c = \frac{i(c_1 c_2^* - c_1^* c_2)}{2}, \ d = \frac{|c_1|^2 - |c_2|^2}{2} \tag{55}$$

where in matrix notation, e_1 , e_2, and e_3 are the well known Pauli matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{56}$$

Of course , the analogy still holds. The (54) is still an element of the Clifford algebra, and precisely of Clifford algebra A(Si). As Clifford algebraic member, the (54) satisfies the requirement to be $\rho^2 = \rho$ and Tr(ρ) $= 1$ under the conditions $a = 1/2$ and $a^2 - b^2 - c^2 - d^2 = 0$ as shown in detail elsewhere in ref11. In the algebraic framework previously outlined, let us admit that we attribute to e_3 the value +1 (that is to say … the quantum observable σ_3 assumes the value +1 during quantum measurement) or to e_3 the numerical value –1 (that is to say… the quantum observable σ_3 assumes the value –1 during the quantum measurement). As previously shown, in such two cases the algebra A, (Si) no more holds, and it will be replaced from the Clifford $N_{i,\pm 1}$. To examine the consequences , starting with the algebraic element (54), write it in the two equivalent algebraic forms that are obviously still in the algebra A(Si).

$$\rho = \frac{1}{2}(|c_1|^2 + |c_2|^2) + \frac{1}{2}(c_1 c_2^*)(e_1 + e_2 i) + \frac{1}{2}(c_1^* c_2)(e_1 - ie_2) + \frac{1}{2}(|c_1|^2 - |c_2|^2)e_3 \tag{57}$$

and

$$\rho = \frac{1}{2}(|c_1|^2 + |c_2|^2) + \frac{1}{2}(c_1 c_2^*)(e_1 + ie_2) + \frac{1}{2}(c_1^* c_2)(e_1 - e_2 i) + \frac{1}{2}(|c_1|^2 - |c_2|^2)e_3 \tag{58}$$

Both such expressions contain the following interference terms.

$$\frac{1}{2}(c_1 c_2^*)(e_1 + e_2 i) + \frac{1}{2}(c_1^* c_2)(e_1 - ie_2) \tag{59}$$

and

$$\frac{1}{2}(c_1 c_2^*)(e_1 + ie_2) + \frac{1}{2}(c_1^* c_2)(e_1 - e_2 i) \tag{60}$$

Let us consider now that the quantum measurement gives as result +1 for e_3. In this case there are the (57) and the (59) that we take in consideration. On the basis of our principle, we know that the previous Clifford algebra A(Si) no more holds, but instead it is valid the $N_{1,+1}$ that has the following new commutation rules:

$$e_1 e_2 = i \ , \ e_2 e_1 = -i \, , \ e_2 i = -e_1 \, , ie_2 = e_1 \, , e_1 i = e_2 \, , \ ie_1 = -e_2 \tag{61}$$

Inserting such new commutation rules in the (57) and the (59), remembering that here e_3 is now a parameter that has value +1, one sees that the interference terms are erased and the density matrix, given in the (57), now becomes

$$\rho \to \rho_M = |c_1|^2 \tag{62}$$

The collapse has happened.

In the same manner let us consider instead that the quantum measurement gives as result -1 for e_3. In this case there are the (58) and the (60) that we take in consideration On the basis of our principle, we know that the previous Clifford algebra A(Si) no more holds, but instead it is valid the $N_{1,-1}$ that has the following new commutation rules

$$e_1e_2 = -i \ , \ e_2e_1 = i \, , \ e_2i = e_1 \, , ie_2 = -e_1 \, , e_1i = -e_2 \, , \ ie_1 = e_2 \tag{63}$$

Inserting such new commutation rules in the (58) and (60), remembering that the parameter e_3 now assumes value –1, one sees that the interference terms are erased and the density matrix, given in the (54) or in the (58), now becomes

$$\rho \to \rho_M = |c_2|^2 \tag{64}$$

The collapse has happened.

Let us examine now von Neumann results.

In order to formulate in detail von Neumann's projection postulate , consider the spinor basis given in (53). Outer products give projection operators that are the idempotents in the A(si) Clifford algebra as explicitly given in (26). Consider again the (9).

Reasoning in terms of Clifford algebra

$$|0><0|$$

and

(65)

$$|1><1| \tag{66}$$

are respectively the idempotents

$$\frac{1+e_3}{2}$$

and (67)

$$\frac{1-e_3}{2} \tag{68}$$

Considering the first term on the right in the (9) one has that

$$(\frac{1+e_3}{2})\,\rho\,(\frac{1+e_3}{2}) \tag{69}$$

that , in terms of the matrix given in the (10) , gives

$$(\frac{1+e_3}{2})\,\rho\,(\frac{1+e_3}{2}) = \alpha \ (\frac{1+e_3}{2})$$

and explicitly (70)

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \tag{71}$$

Applying the same procedure in the case of

$$\frac{1-e_3}{2} \tag{72}$$

(the second term in the (9)) , one obtains as result

$$\beta(\ \frac{1-e_3}{2}) \tag{73}$$

and explicitly

$$\begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix} \tag{74}$$

The sum , as indicated in the (9), gives

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \tag{75}$$

In conclusion we have given full justification of von Neumann's projection postulate in Clifford A(Si) algebra. As expected, there is total equivalence between von Neumann postulate and corresponding A)Si) formulation . It is important to reaffirm here that it has been obtained using only the framework of A(Si) algebra. In accord with von Neumann we obtain

$$\alpha \ (\frac{1+e_3}{2}) \tag{76}$$

and

$$\beta(\frac{1-e_3}{2}) \tag{77}$$

Note now that , in application of our criterium, quantum measurement is obtained passing from algebra A(Si) to $N_{i,\pm 1}$. In this case we no more obtain the (76) and the (77) as it happens remaining in the framework of the A(Si) algebra, but we obtain respectively the (62) or the (64), that is to say,

$$\rho_M = |c_1|^2 \tag{78}$$

or

$$\rho_M = |c_2|^2 \tag{79}$$

as it must be when the collapse has happened.

The nature of such result obviously does not change if we explore a time dependent situation.

The reader is advised that we will use a lightly modified formalism that however does not alter the significance of our application.

Consider the quantum system S and indicate by ψ_0 the state at the initial time 0. The state at any time t will be given by

$$\psi(t) = U(t)\psi_0 \ \text{ and } \ \psi_0 = \psi(t=0) \tag{80}$$

An Hamiltonian H must be constructed such that the evolution operator U(t), that must be unitary, gives $U(t) = e^{-iHt}$.

It is well known that, given a finite N-level quantum system described by the state ψ , its evolution is regulated according to the time dependent Schrödinger equation

$$i\eta \frac{d\psi(t)}{dt} = H(t)\psi(t) \quad \text{with } \psi(0) = \psi_0 . \tag{81}$$

Let us introduce a model for the hamiltonian H(t). Details of this formalism may be found in reff.12 and 13. We express by H_0 the hamiltonian of the system S, and we add to H_0 an external time varying hamiltonian, $H_1(t)$, representing the perturbation to which the system S is subjected by action of the measuring apparatus. In conclusion we write the total hamiltonian as

$$H(t) = H_0 + H_1(t) \tag{82}$$

so that the time evolution will be given by the following Schrödinger equation

$$i\eta \frac{d\psi(t)}{dt} = [H_0 + H_1(t)]\psi(t) \tag{83}$$

and $\psi(0) = \psi_0$. We have that

$$\psi(t) = U(t)\psi_0 \tag{84}$$

where U(t) pertains to the special group SU(N). We will write that

$$i\eta \frac{dU(t)}{dt} = H(t)U(t) = [H_0 + H_1(t)]U(t) \qquad \text{and } U(0)=I \tag{85}$$

Let $A_1, A_2, \ldots, A_n$, ($n=N^2-1$), are skew-hermitean matrices forming a basis of Lie algebra SU(N). In this manner one arrives to write the explicit expression of the hamiltonian H(t). It is given in the following manner

$$-iH(t) = -i[H_0 + H_1(t)] = \sum_{j=1}^{n} a_j A_j + \sum_{j=1}^{n} b_j A_j \tag{86}$$

where a_j and $b_j = b_j(t)$ are respectively the constant components of the free hamiltonian and the time-varying control parameters characterizing the action of the measuring apparatus. If we introduce T, the time ordering parameter (for details see reff. 12 and 13), we arrive also to express U(t) that will be given in the following manner

$$U(t) = T\exp(-i\int_0^t H(\tau)d\tau) = T\exp(-i\int_0^t (a_j + b_j(\tau))A_j d\tau) \tag{87}$$

that is the well known Magnus expansion. Locally U(t) may be expressed by exponential terms as it follows

$U(t) = \exp(\gamma_1 A_1 + \gamma_2 A_2 + \dots\dots + \gamma_n A_n)$ (88)

on the basis of the Wein-Norman formula

$$\Xi(\gamma_1, \gamma_2, \dots\dots, \gamma_n)\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dots \\ \dot{\gamma}_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \dots \\ a_n + b_n \end{pmatrix} \qquad (89)$$

with Ξ n x n matrix, analytic in the variables γ_i. We have $\gamma_i(0) = 0$ and $\Xi(0) = I$, and thus it is invertible. We obtain

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dots \\ \dot{\gamma}_n \end{pmatrix} = \Xi^{-1}\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \dots \\ a_n + b_n \end{pmatrix} \qquad (90)$$

Consider a simple case based on the superposition of only two states. We have

$\psi = [y_1, y_2]^T$ and $|y_1|^2 + |y_2|^2 = 1$ (91)

As previously said, we have here an SU(2) unitary transformation, selecting the skew symmetric basis for SU(2). We will have that

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (92)$$

Now we consider the following matrices

$A_j = \frac{i}{2} e_j$, j = 1,2,3

The reader may now ascertain that the previously developed formalism is moving in direct correspondence with our Clifford algebra A(Si).

We are now in the condition to express H(t) and U(t) in our case of interest. The most simple situation we may examine is that one of fixed and constant control parameters b_j. The hamiltonian H will become fully linear time invariant and its exponential solution will take the following form

$$e^{t(\sum_{j=1}^{3}(a_j + b_j)A_j)} = \cos(\frac{k}{2}t)I + \frac{2}{k} sen(\frac{k}{2}t)\left(\sum_{j=1}^{3}(a_j + b_j)A_j\right) \qquad (93)$$

with $k = \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2}$. In matrix form it will result

$$U(t) = \begin{pmatrix} \cos\frac{k}{2}t + \frac{i}{k} sen\frac{k}{2}t(a_3 + b_3) & \frac{1}{k} sen\frac{k}{2}t[a_2 + b_2 + i(a_1 + b_1)] \\ \frac{1}{k} sen\frac{k}{2}t[-a_2 - b_2 + i(a_1 + b_1)] & \cos\frac{k}{2}t - \frac{i}{k} sen\frac{k}{2}t(a_3 + b_3) \end{pmatrix} \qquad (94)$$

and, obviously, it will result to be unimodular as required.

Starting with this matrix representation of time evolution operator U(t), we may deduce promptly the dynamic time evolution of quantum state at any time t writing

$\psi(t) = U(t)\psi_0$ (95)

assuming that we have for ψ_0 the following expression

$$\psi_0 = \begin{pmatrix} c_{true} \\ c_{false} \end{pmatrix} \qquad (96)$$

having adopted for the true and false states (or yes-not states, +1 and –1 corresponding eigenvalues of such states) the following matrix expressions

$\varphi_{true} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi_{false} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (97)

Finally, one obtains the expression of the state $\psi(t)$ at any time

$$\psi(t)=\left[c_{true}\left[\cos\frac{k}{2}t+\frac{i}{k}sen\frac{k}{2}t(a_3+b_3)\right]+c_{false}\left[\frac{1}{k}sen\frac{k}{2}t[(a_2+b_2)+i(a_1+b_1)]\right]\right]\varphi_{true}+$$
$$\left[c_{true}\left[\frac{1}{k}sen\frac{k}{2}t[i(a_1+b_1)-(a_2+b_2)]\right]+c_{false}\left[\cos\frac{k}{2}t-\frac{i}{k}sen\frac{k}{2}t(a_3+b_3)\right]\right]\varphi_{false} \quad (98)$$

As consequence, the two probabilities $P_{true}(t)$ and $P_{false}(t)$, will be given at any time t by the following expressions

$$P_{true}(t)=(A^2+B^2)\cos^2\frac{k}{2}t+\frac{1}{k^2}sen^2\frac{k}{2}t(P^2+Q^2)+\frac{senkt}{k}(AP+BQ)$$

and (99)

$$P_{false}(t)=(C^2+D^2)\cos^2\frac{k}{2}t+\frac{1}{k^2}sen^2\frac{k}{2}t(S^2+R^2)+\frac{senkt}{k}(RC+DS)$$

where

$A=\mathrm{Re}\, c_{true}$,$B=\mathrm{Im}\, c_{true}$, $C=\mathrm{Re}\, c_{false}$, $D=\mathrm{Im}\, c_{false}$,
$P=-D(a_1+b_1)+C(a_2+b_2)-B(a_3+b_3)$,
$Q=C(a_1+b_1)+D(a_2+b_2)+A(a_3+b_3)$, (100)
$R=-B(a_1+b_1)-A(a_2+b_2)+D(a_3+b_3)$,
$S=A(a_1+b_1)-B(a_2+b_2)-C(a_3+b_3)$

Until here we have developed only standard quantum mechanics. The reason to have developed here such formalism has been to evidence that at each step it has its corresponding counterpart in Clifford algebraic framework A(Si) , and thus we may apply to it the two theorems developed in the previous section and the previously introduced criterium, passing from the algebra Si to $N_{i,\pm 1}$. In fact , to this purpose , it is sufficient to multiply the (94) by the (96) to obtain the final forms of $c_{true}(t)$ and $c_{false}(t)$

In the final state we have that

$$\psi_t=\begin{pmatrix} c_{true}(t) \\ c_{false}(t) \end{pmatrix} \quad (101)$$

We may now write the density matrix that will result to have the same structure of the previously case given in the (54) but obviously with explicit evidence of time dependence. . In the Clifford algebraic framework it will pertain still to the Clifford algebra A(Si). In order to describe the wave-function collapse we have to repeat the same procedure that we developed previously from the (54) to the (64) , considering that , in accord to our criterium ,we have to pass from the algebra A(Si) to $N_{i,\pm 1}$, and obtaining

$$\rho\to\rho_M=\left|c_{true}(t)\right|^2 \quad (102)$$

in the case $N_{i,+1}$

and

$$\rho\to\rho_M=\left|c_{false}(t)\right|^2 \quad (103)$$

in the case $N_{i,-1}$, as required in the collapse.

Let us examine now the fourth application of our criterium.

Until here we considered only examples of two state quantum systems. Let us expand our formulation at any order n. First consider Clifford Si algebra at order n=4 (for details see ref.14) . One has

$$E_{0\,i}=I^1\otimes e_i; \qquad E_{i\,0}=e_i\otimes I^2 \quad (104)$$

The notation $\otimes$ denotes direct product of matrices, and I^i is the *i*th 2x2 unit matrix. Thus, in the case of $n=4$ we have two distinct sets of Clifford basic unities, $E_{0\,i}$ and $E_{i\,0}$, with

$$E_{0i}^2=1\ ; \qquad E_{i0}^2=1,\ i=1,2,3; \quad (105)$$
$$E_{0\,i}E_{0\,j}=i\,E_{0\,k}; \qquad E_{i\,0}E_{j\,0}=i\,E_{k\,0}\ ,\ j=1,2,3; \qquad i\neq j$$

and

$$E_{i0}\,E_{0\,j}=E_{0\,j}\,E_{i\,0} \quad (106)$$

with (i, j, k) cyclic permutation of $(1, 2, 3)$.

Let us examine now the following result

$$(I^1\otimes e_i)\,(e_j\otimes I^2)=E_{0\,i}\,E_{j\,0}=E_{j\,i} \quad (107)$$

It is obtained according to our basic rule on cyclic permutation required for Clifford basic unities. We have that $E_{0\,i}E_{j0}=E_{j\,i}$ with $i=1, 2, 3$ and $j=1, 2, 3$, with $E^2_{j\,i}=1$, $E_{i\,j}E_{k\,m}\neq E_{k\,m}E_{i\,j}$, and $E_{i\,j}E_{k\,m}=E_{p\,q}$ where p results from the cyclic permutation (i, k, p) of $(1, 2, 3)$ and q results from the cyclic permutation (j, m, q) of $(1, 2, 3)$.

In the case $n=4$ we have two distinct basic set of unities $E_{0\,i}$, $E_{i\,0}$ and, in addition, basic sets of unities $(E_{i\,j}, E_{i\,p}, E_{0\,m})$ with (j, p, m) basic permutation of $(1, 2, 3)$.

This is the Clifford algebra A at order n=4.
In the other more general cases we have $E_{0\,0\,i}$, $E_{0\,i\,0}$, and $E_{i\,0\,0}$, $i = 1, 2, 3$ and

$$E_{0\,0\,i} = I^1 \otimes I^1 \otimes e_i\,; \qquad E_{0\,i0} = I^2 \otimes e_i \otimes I^2\,; \qquad E_{i\,0\,0} = e_i \otimes I^3 \otimes I^3$$

and

$$(I^1 \otimes I^1 \otimes e_i)\cdot(I^2 \otimes e_i \otimes I^2)\cdot(e_i \otimes I^3 \otimes I^3) = e_i \otimes e_i \otimes e_i =$$

$$= E_{0\,0\,i}\, E_{0\,i\,0}\, E_{i\,0\,0} = E_{i\,i\,i} \tag{108}$$

Still we will have that

$$E_{0\,0\,i}\, E_{0\,i\,0} = E_{0\,i\,0}\, E_{i\,0\,0}\,; \quad E_{0\,0\,i} E_{i\,0\,0} = E_{i\,0\,0} E_{0\,0\,i}\,; \qquad E_{0\,i\,0} E_{i\,0\,0} = E_{i\,0\,0} E_{0\,i0} \tag{109}$$

Generally speaking, fixed the order n of the Si Clifford algebra in consideration , we will have that

$$\begin{aligned}
\Gamma_1 &= \Lambda_n \\
\Gamma_{2m} &= \Lambda_{n-m} \otimes e_2^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots\ldots\ldots \otimes I^n \\
\Gamma_{2m+1} &= \Lambda_{n-m} \otimes e_3^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots\ldots\ldots \otimes I^n \\
\Gamma_{2n} &= e_2 \otimes I^{(2)} \otimes \ldots\ldots\ldots \otimes I^n
\end{aligned} \tag{110}$$

with

$$\Lambda_n = e_1^{(1)} \otimes e_1^{(2)} \otimes \ldots\ldots \otimes e_1^{(n)} = (e_1 \otimes I^{(1)} \otimes \ldots\ldots \otimes I^n)\cdot(\ldots\ldots\ldots)\cdot(I^{(1)} \otimes I^{(2)} \ldots \otimes I^{(n)} \otimes e_1);$$

$m = 1, \ldots\ldots, n - 1$

according to the n-possible dispositions of e_1 and $I^1, I^2, \ldots, I^n$ in the distinct direct products.
We may now give the explicit expressions of $E_{0\,i}$, $E_{i\,0}$, and $E_{i\,j}$ at the order n=4.

$$E_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad E_{02} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad E_{03} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{111}$$

$$E_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; E_{20} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; E_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$E_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad E_{22} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; E_{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$E_{12} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; E_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; E_{21} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix};$$

$$E_{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; E_{23} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}; E_{32} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

Note the following basic feature: we have now some different sets of Clifford algebras A, (Si). In detail, we have the following sets of basic Si Clifford algebras:

(E_{01}, E_{12}, E_{13}), (E_{01}, E_{22}, E_{23}), (E_{01}, E_{32}, E_{33}), (E_{02}, E_{11}, E_{13}),
(E_{02}, E_{21}, E_{23}), (E_{02}, E_{31}, E_{33}), (E_{03}, E_{11}, E_{12}),
(E_{03}, E_{21}, E_{22}), (E_{03}, E_{31}, E_{32}), (E_{10}, E_{23}, E_{33}), (E_{10}, E_{22}, E_{32}),
(E_{10}, E_{21}, E_{31}), (E_{20}, E_{13}, E_{33}), (E_{20}, E_{12}, E_{32}), (E_{20}, E_{11}, E_{31}),
(E_{30}, E_{13}, E_{23}), (E_{30}, E_{12}, E_{22}), (E_{30}, E_{11}, E_{21}) (112)

All these are the sets of Clifford algebras A (Si) that we have at order n=4. To each of these sets we may apply the theorems n.1 and n.2 previously shown and we may apply the criterium of the passage from the Si to the $N_{1,\pm 1}$ that we have just used in the other previous cases of application.

Fixed such algebraic features, we may now consider the problem that we formulated in the introduction of the present paper. It is that, in order to avoid possible contradictions, we should still modify the previous expression for the wave-function collapse, by introducing the states of a given measurement apparatus system A obtaining in this case

$$\rho = \rho_S \otimes \rho_A = \sum_i \sum_j c_i c_j^* \left|\varphi_i\right>\left<\varphi_j\right| \otimes \rho_A \rightarrow \rho_{S,A,t} = \sum_k \left|c_k\right|^2 \left|\varphi_k\right>\left<\varphi_k\right|_t \otimes \rho_{A(k),t} \tag{113}$$

See the previous discussion that we introduced by the (8).

We may refer the algebraic sets E_{0i} to the quantum system S to be measured , and consider the algebraic sets E_{i0} to the measuring apparatus A. Still we have the basic algebraic set E_{ij} that relates the coupling of S with A. Let us write the density matrix ρ at such order n=4. To simplify, we may write it in the following general form

$$\rho = \begin{pmatrix} a & b_1 + ib_2 & c_1 + ic_2 & d_1 + id_2 \\ b_1 - ib_2 & e & f_1 + if_2 & q_1 + iq_2 \\ c_1 - ic_2 & f_1 - if_2 & h & t_1 + it_2 \\ d_1 - id_2 & q_1 - iq_2 & t_1 - it_2 & s \end{pmatrix} \tag{114}$$

Obviously, the correspondence between Clifford algebra and quantum mechanics still holds also at the present order. The ρ of the (114) is still a member of the Clifford algebra A(Si) that in fact, on the basis of the (111) may be written in the following manner

$$\rho = a\left(\frac{E_{00} + E_{03} + E_{30} + E_{33}}{4}\right) + e\left(\frac{E_{00} + E_{30} - E_{03} - E_{33}}{4}\right) + h\left(\frac{E_{00} + E_{03} - E_{30} - E_{33}}{4}\right) +$$
$$s\left(\frac{E_{00} - E_{03} - E_{30} + E_{33}}{4}\right) +$$
$$\left[b_1\left(\frac{E_{01} + E_{31}}{2}\right) - b_2\left(\frac{E_{02} + E_{32}}{2}\right)\right] + \left[c_1\left(\frac{E_{10} + E_{13}}{2}\right) - c_2\left(\frac{E_{23} + E_{20}}{2}\right)\right] + \left[d_1\left(\frac{E_{11} - E_{22}}{2}\right) - d_2\left(\frac{E_{12} + E_{21}}{2}\right)\right] +$$
$$\left[f_1\left(\frac{E_{11} + E_{22}}{2}\right) + f_2\left(\frac{E_{12} - E_{21}}{2}\right)\right] + \left[q_1\left(\frac{E_{10} - E_{13}}{2}\right) + q_2\left(\frac{E_{23} - E_{20}}{2}\right)\right] + \left[t_1\left(\frac{E_{01} - E_{31}}{2}\right) + t_2\left(\frac{E_{32} - E_{02}}{2}\right)\right] \tag{115}$$

It is in A(Si). Applying the previous criterium we must now pass from A(Si) to $N_{i,\pm 1}$. Let us start considering for E_{33} the numerical value +1 and this is to say that or $E_{03} = E_{30} = +1$ or $E_{03} = E_{30} = -1$.

On the basis of such condition of the measuring instrument, by inspection of the (115) it is seen that the terms by e and h go to zero. It remains the term by a for $E_{03} = E_{30} = +1$ and the term in s for $E_{03} = E_{30} = -1$. All the other terms containing $b_i, c_i, d_i, f_i, q_i, t_i$ ($i = 1,2$) go to zero and the wave function collapse has happened.

Let us explain as example as the term

$$\frac{E_{02} + E_{32}}{2} \tag{116}$$

pertaining to b_2, goes to zero.

Remember that we have attributed to E_{33} the value +1. By inspection of the (112) ,one sees that the basic algebraic A (Si) set in which E_{33} enters is (E_{01}, E_{32}, E_{33}). Passing from the algebra A to the algebra $N_{i,+1}$ (in fact we have attributed to E_{33} the numerical value +1) we obtain the new commutation rule that

$$E_{01}E_{32} = i \,. \tag{117}$$

On the other hand, considering the basic algebraic A(Si) set (E_{01}, E_{02}, E_{03}) of the (112) with attribution to E_{03} the numerical value -1, we have the new commutation rule that

$$E_{01}E_{02} = -i \tag{118}$$

In conclusion we have that

$$E_{32} = E_{01}i \tag{119}$$

and

$$\frac{E_{02} + E_{32}}{2} = \frac{E_{02} + E_{01}i}{2} = \frac{-E_{01}i + E_{01}i}{2} = 0 \tag{120}$$

Following the same procedure , one obtains that also the other interference terms are erased and in conclusion, passing from the algebra A (Si) to the $N_{i,\pm1}$, one obtains that the wave-function collapse has happened.

If $E_{03} = E_{30} = +1$ $(E_{33} = +1)$ from the (115) we obtain

$$\rho \to \rho_M = a \tag{121}$$

If $E_{03} = E_{30} = -1$ $(E_{33} = +1)$ from the (115) we obtain

$\rho \to \rho_M = s$ (122)

and the collapse has happened.

CONCLUSION

In section three , following Y. Ilamed and N. Salingaros [10], we have given proof of two theorems (n.1 and n.2) on two existing Clifford algebras , the Si and the N_i. Such two algebras are of course well known in Clifford algebraic framework[10], the first holding with isomorphism with Pauli matrices , the second representing the well known dihedral Clifford algebra N_i. We also gave previously a very preliminary proof of such theorems by exposition at the conference on Reconsideration of Quantum mechanics Foundations in Vaxjio – Sweeden [15]. The substance of the results that we obtain in the present paper is that we may pass from the algebra Si to $N_{i,\pm1}$ attributing to one of the abstract elements (e_1, e_2, e_3) a direct numerical value (as example , consider e_3 attributing to it the value +1 and thus passing from Si to $N_{i,+1}$ or attributing to e_3 the value –1 and thus passing from the algebra Si to $N_{i,-1}$. The algebra Si has its commuation rules based on the abstract elements (e_1, e_2, e_3), the algebra $N_{i,+1}$ has its three abstract elements $(e_1, e_2; i)$ and its basic commutation rules while the algebra $N_{i,-1}$ has its three abstract elements $(e_1, e_2; i)$ and its basic commutation rules . We foresee the possibility of a profound implication for the quantum measurement problem based on existence of such two Clifford algebras Si and $N_{i,\pm1}$, and , in particular, on the basic feature that has been shown in section three, that $N_{i,\pm1}$, may be obtained from Si by direct attribution , as example to e_3, of a direct numerical value (+1 or –1) . The reason is that when , given a quantum system S, we arrive to attribute to S a definite numerical value for some selected quantum observable , say e_3, actually this happens because we measure S with a proper measuring apparatus “reading” the numerical value +1 or –1 , respectively. This reason has motivated us to introduce a criterium. A quantum system without direct observation and actualization , induced from a proper measuring apparatus, has its Clifford algebraic counterpart in the Clifford algebraic structure Si while the collapse , happening on the considered system during the proper actualization by an instrument apparatus, may be described passing from the algebra Si to $N_{i,\pm1}$.

We have given three cases of application of such criterium showing in detail that it holds . On the other hand, there are still other basic considerations that in some manner legitimate the choice of such criterium. In section three, in the (65)-(77) we have re-obtained , as expected, the results of von Neumann projection postulate in quantum measurement. It is important to observe that we have re-obtained von Neumann projection postulate using only the Si algebra. Therefore, we have given a justification of von Neumann projection postulate showing that it is articulated only in the Si Clifford algebra. On the other hand , by using only the Si Clifford algebra one shows (see Appendix A) that one may obtain a rough scheme of quantum mechanics as shown in detail elsewhere[16]. Finally, in order to confirm still that , passing from Si to $N_{i,\pm 1}$, we have a description of quantum wave function collapse, we may also add two final considerations. The first is that remaining in a geometric interpretation of Clifford algebra one has that

1s 1v D = 1

1s 2v 1b D = 2

1s 3v 3b 1t D = 3

where s means scalar, v means vector, b means bivector, t means trivector. To describe standard Si one needs D=3 that is 1 scalar, 3 vectors , 3 bivectors and 1 trivectors that is the imaginary unity i of complex numbers . This is a classical two state quantum system with quantum dimension d=2, Hilbert space.
When we pas to $N_{i,\pm 1}$, we have

1s 2v 1b D = 2
one needs 1 scalar, 2 vectors , 1 bivector . The dimension has been decreased to D= 2 . In correspondence the dimension d of the quantum system has become d=1, Hilbert space. The system has collapsed.
The second consideration is based on the following reasoning.
In the Si , $Cl(3)$ Clifford algebra , we have two elements

$$\varepsilon_{\pm} = \frac{1}{2}(1 \pm e_3) \tag{123}$$

that are idempotent, better they are primitive idempotents , as we outlined in the (26). The sets $Cl(3)\ \varepsilon_{\pm}$ and $\varepsilon_{\pm}Cl(3)$ are left and right ideals in $Cl(3)$ in Si. They are vector spaces of complex dimension 2 and the identification of i with complex imaginary unity makes each of them identical to C^2. A spinor is precisely an element of a two dimensional representation space for the group SL(2;C), which is C^2.
Let us first consider $Cl(3)\varepsilon_{+}$. If we chose an arbitrary frame :

$$\begin{pmatrix}1\\0\end{pmatrix} = \varepsilon_{+} \quad \text{and} \quad e_1\varepsilon_{+} = \begin{pmatrix}0\\1\end{pmatrix} \tag{124}$$

we may decompose any arbitrary element, that is to say

$$\forall \varphi \in Cl(3)\varepsilon_{+},\ \varphi = \begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix} \in C^2 \tag{125}$$

A similar procedure applies to $\varepsilon_{+}Cl(3)$, choosing the basis (1 0) = ε_{+} and (0 1) = $\varepsilon_{+}e_1$.
We may also look at ref. 17 for further details.
This is in Si. Now , if we calculate

$$e_1\varepsilon_{+} = e_1(\frac{1+e_3}{2}) = \frac{e_1 - ie_2}{2} \tag{126}$$

that holds in Si, it gives

$$e_1\varepsilon_{+} = \begin{pmatrix}0\\1\end{pmatrix} \tag{127}$$

When instead we pass to $N_{1,+1}$, since we have in $N_{1,+1}$ that $i = e_1 e_2$, we obtain that

$$e_1\varepsilon_{+} = e_1(\frac{1+e_3}{2}) = \frac{e_1 - ie_2}{2} = 0 \tag{128}$$

that is to say

$$e_1\varepsilon_{+} = \begin{pmatrix}0\\1\end{pmatrix} \equiv 0 \tag{129}$$

The collapse has happened.

APPENDIX A

We may now derive a rough scheme of quantum mechanics using the Si Clifford algebraic framework[16,18].
Consider in Si the three abstract basic elements, e_i, with $i = 1,2,3$ that , as we know, are submitted to the following basic postulate:

$$e_1^{\;2} = 1 \;,\; e_2^{\;2} = 1 \,,\; e_3^{\;2} = 1 \qquad \text{(A.1)}$$

If we consider the $e_i\,(i = 1,2,3)$ as abstract quantum entities, we may conclude that they have an intrinsic randomness that is their essential irreducible nature. This of course happens also for quantum events. In the algebra A (Si) the $e_i\,(i = 1,2,3)$ have the intrinsic potentiality that we may attribute them or the numerical value +1 or the numerical value –1 .
A generic member of our algebra A(Si) is given by

$$x = \sum_{i=0}^{4} x_i e_i \qquad \text{(A.2)}$$

with x_i pertaining to some field $\Re$ or C. Since the e_i are abstract quantum entities, having the potentiality that we may attribute them the numerical values, or ± 1, and they have an intrinsic and irreducible randomness , we may admit to be $p_1(+1)$ the probability that e_1 assumes the value $(+1)$ and $p_1(-1)$ the probability that it assumes the value -1, so that we have its mean value that is given by

$$< e_1 >= (+1)p_1(+1) + (-1)p_1(-1) \qquad \text{(A.3)}$$

Considering the same corresponding notation for the two remaining basic elements, we may introduce the following mean values:

$$< e_2 >= (+1)p_2(+1) + (-1)p_2(-1)\,, \qquad \text{(A.4)}$$
$$< e_3 >= (+1)p_3(+1) + (-1)p_3(-1).$$

We have

$$-1 \le < e_i > \le +1 \quad i = (1,2,3) \qquad \text{(A.5)}$$

Selected the following generic element of the algebra A(Si):

$$x = \sum_{i=1}^{3} x_i e_i \qquad x_i \in \Re \qquad \text{(A.6)}$$

Note that

$$x^2 = x_1^2 + x_2^2 + x_3^2 \qquad \text{(A.7)}$$

Its mean value results to be

$$< x >= x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \qquad \text{(A.8)}$$

Let us call

$$a = (x_1^2 + x_2^2 + x_3^2)^{1/2} \qquad \text{(A.9)}$$

so that we may attribute to x the value $+a$ or $-a$
We have that

$$-a \le x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \le a \qquad \text{(A.10)}$$

The (A.8) must hold for any real number x_i, and, in particular, for

$$x_i =< e_i >$$

so that we have that

$$x_1^2 + x_2^2 + x_3^2 \le a$$

that is to say

$$a^2 \le a \quad \to a \le 1$$

so that we have the fundamental relation

$$< e_1 >^2 + < e_2 >^2 + < e_3 >^2 \le 1 \qquad \text{(A.11)}$$

This is the basic relation we are writing in our Clifford algebraic quantum like scheme of quantum theory.
Let us observe some important things:

(a) First of all it links the Clifford algebra A(Si) with the $N_{1,\pm 1}$. In absence of measurement, that is to say in absence of direct observation of one quantum entity e_i the (A.11) holds.

(b) If we attribute instead a definite numerical value to one of the three quantum entities, as example we attribute to e_3 the numerical value +1 , we have $< e_3 >= 1$, the (A.11) operates now in the $N_{i,+1}$ algebra , reduced to

$$< e_1 >^2 + < e_2 >^2 = 0 \ , \ < e_1 >=< e_2 >= 0, \tag{A.12}$$

and we have complete, irreducible, indetermination for e_1 and for e_2 . This is an excellent example of the profound link existing between quantum phenomenology with and without direct observation expressed in a pure algebraic framework.

(c) Finally, the (A.11) affirms that we never can attribute simultaneously definite numerical values to two basic non commutative elements e_i

Still let us examine another important consequence of our rough quantum mechanical scheme. As previously evidenced, in Clifford algebra A we have idempotents. Let us consider again two of such idempotents:

$$\psi_1 = \frac{1+e_3}{2} \quad \text{and} \quad \psi_2 = \frac{1-e_3}{2} \tag{A.13}$$

Let us consider the mean values of (A.13). We have that

$$2 < \psi_1 >= 1 + < e_3 > \quad \text{and} \quad 2 < \psi_2 >= 1 - < e_3 > \tag{A.14}$$

Using the last equation in (A.4) we obtain that

$$p_3(+1) = \frac{1 + < e_3 >}{2} \quad \text{and} \quad p_3(-1) = \frac{1 - < e_3 >}{2} \tag{A.15}$$

Therefore, we have that

$$p_3(+1) =< \psi_1 > \quad \text{and} \quad p_3(-1) =< \psi_2 > \tag{A.16}$$

This is to say that probabilities $p_{3,+1,-1}$ are the mean values of the idempotents. The same result holds obviously when considering the basic elements e_1 or e_2 . Considering that in quantum mechanics (Born probability rule), given wave functions $\varphi_{+,-}$, we have

$$\left|\varphi_{+,-}\right|^2 = p_{+,-} \tag{A.17}$$

we conclude that

$$\varphi_3(+) = \sqrt{< \psi_1 >} e^{i\vartheta_1} \quad \text{and} \quad \varphi_3(-) = \sqrt{< \psi_2 >} e^{i\vartheta_2} \tag{A.18}$$

and we have given proof that our rough scheme of quantum mechanics foresees the existence of wave functions as exactly traditional quantum mechanics makes.

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