Exact Elliptic Compactons in Generalized Korteweg-DeVries Equations

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We derive a general theorem relating the energy, momentum and velocity of any solitary wave solution of the generalized KdV equation which enables us to relate the amplitude, width, and momentum to the velocity of these solutions. We obtain the general condition for linear and Lyapunov stability. We then obtain a two parameter family of exact solutions to these equations which include elliptic and hyper-elliptic compacton solutions. For this general family we explicitly verify both the theorem and the stability criteria.

In a variety of physical contexts one finds nonlinear field equations for which a wide class of solitary wave solutions can exist. However, in many cases one is not able to obtain the solution in a closed form and thus it is not very easy to study the stability of such solutions. In this Letter we derive a general theorem relating the energy, momentum and velocity of any solitary wave of a generalized Korteweg-De Vries (KdV) equation of Cooper, Shephard and Sodano (CSS)[1]. The important point is that this particular generalization of the KdV equation is derivable from an Action Principle. Using the theorem, we are able to relate the amplitude, width and velocity of any of the solitary wave solution even if such a solution is not known in a closed form and also study its stability. Secondly, we obtain a two-parameter family of solutions to these equations which include elliptic and hyper-elliptic compacton solutions. For this general family we explicitly verify the theorem as well as the stability criteria.

Compactons were discovered originally in an extension of the KdV equation by Rosenau and Hyman in Ref. 2. Compactons are fundamental excitations (i.e., solitary waves with a compact support) of KdV-like equations that possess a nonlinear dispersion and collide quasi-elastically. They play an important role in pattern formation and emergence of nonlinear structures in physical systems. Other physical contexts in which compactons are relevant include fluid dynamics, optical waveguides and the field of intrinsic localized modes[3]. Breather compactons[4,5] and compactons in other nonlinear dispersive equations[6] are also known. In addition, KdV equations with higher power of nonlinearity and dispersion may lead to the phenomema of blow-up and collapse. Therefore, the study of compactons and their stability in this class of equations is important in its own right.

Rosenau and Hyman showed that in a particular generalization of the KdV equation, defined by parameters (m, n) (with m, n integers), namely

$$K(m,n) : u_t + (u^m)_x + (u^n)_{xxx} = 0\,, \tag{1}$$

a new form of solitary wave with compact support is a solution of this equation. For the case $m = n$ (m integer) these compactons had the property that the width was independent of the amplitude. In Ref. 2 it was stated that $K(3, 2)$ had an elliptic function solution. In a later work, Rosenau[7] obtained elliptic function compactons for the case of $K(4, 2)$ and $K(5, 3)$. Phase compactons have also been investigated[8]. Because the equations of Rosenau and Hyman were not equivalent to a Hamiltonian dynamical system, CSS[1] considered instead a related generalization of the KdV equation

$$K^*(l,p) : u_t + u_x u^{l-2} + \alpha[2u_{xxx}u^p + 4pu^{p-1}u_x u_{xx} \\ +p(p-1)u^{p-2}(u_x)^3] = 0\,. \tag{2}$$

Equation (2) has the property that it is derivable from a Lagrangian. CSS showed that the two equations have the same *class* of solitary wave solutions when $l = m+1$ and $p = n-1$. Because of this connection, the two parameter family of new solutions that we will find here will also be solutions of the $K(m, n)$ equation with slightly different coefficients. Cooper and Khare[9] later showed that these equations with $l = p + 2$, i.e. $m = n \le 3$ and m continuous, had compacton solutions of the form

$$u(x,t) = A[\cos(\beta y)]^{2/p}\,, \quad y = x - ct\,, \tag{3}$$

and for all these compactons also the width was independent of the amplitude.

The CSS equation has three conserved quantities, i.e. the Hamiltonian H, momentum P and mass M given by

$$H = \int \left[\alpha u^p (u_x)^2 - \frac{u^l}{l(l-1)}\right] dx\,, \tag{4}$$

$$P = \frac{1}{2}\int u^2(x,t)\,dx; \quad M = \int u(x,t)\,dx\,. \tag{5}$$

On the other hand, the Rosenau-Hyman equation has in general two conserved quantities (except when $m = n$ in which case there are four conservation laws[2,10]) given by

$$M = \int u(x,t)\,dx; \quad Q = \int u^{n+1}(x,t)\,dx\,. \tag{6}$$

The fact that the CSS equations were derivable from an Action Principle allowed CSS to consider time-dependent variational approximations based on simple time-dependent trial functions $u^v(x,t)$. By using post-gaussian trial wave functions $u^v(x,t) = A(t)\exp\left[-\beta(t)|x+q(t)|^\gamma\right]$ it was shown that these trial wave functions satisfy the relationship

$$\dot{q} = r\frac{H}{P}\,, \tag{7}$$

with

$$r = -\frac{p+l+2}{p+6-l}\,. \tag{8}$$

Here $\dot{q}$ is the velocity of the compacton. However, it was not known at the time if the result was true in general. In this Letter we will show that this relationship is entirely general and does not depend on any trial wave function approximation. We will show explicity that this relationship can be derived for all solitary wave solutions of the CSS equations of the form $u(x,t) = AZ[\beta(x+q(t))]$. This relationship will also allow us to relate the amplitude, width and velocity of the solitary wave. Having a Hamiltonian formulation also simplifies the discussion of stability, and using general arguments we shall prove for the CSS equations that the compacton solutions are stable provided $2 < l < p+6$.

Energy-momentum relation theorem: To derive the relationship between the conserved energy and the conserved momentum, the starting point is the action

$$\Gamma = \int L dt\,, \tag{9}$$

where L is given by (note $\phi_x = u$)

$$L(l,p) = \int \left(\frac{1}{2}\varphi_x \varphi_t + \frac{(\varphi_x)^l}{l(l-1)} - \alpha(\varphi_x)^p (\varphi_{xx})^2 \right) dx\,. \tag{10}$$

If we assume the exact solitary wave solution is of the generic form

$$\phi_x = u = AZ(\beta(x+q(t)))\,, \tag{11}$$

then the value of the Hamiltonian for the solitary wave solution can be shown to be

$$H = -C_1(l)\frac{A^l}{\beta l(l-1)} + \alpha\beta A^{p+2} C_2(p), \tag{12}$$

where

$$C_1(l) = \int Z^l(z)dz; \quad C_2(p) = \int [Z'(z)]^2 Z^p(z) dz\,. \tag{13}$$

Since H and P are conserved, therefore we can rewrite the parameter A in terms of the conserved momentum P and obtain

$$P = \frac{1}{2}\int dx u^2 = \frac{A^2}{2\beta} C_5\,, \quad C_5 = \int dz Z^2(z)\,. \tag{14}$$

Replacing A by P, we now have

$$H = -C_3(l) P^{l/2} \beta^{(l-2)/2} + C_4(p) P^{(p+2)/2} \beta^{(p+4)/2}, \tag{15}$$

where

$$C_3(l) = \frac{C_1(l)}{l(l-1)}\left[\frac{2}{C_5}\right]^{l/2}; \quad C_4 = \alpha C_2(p)\left[\frac{2}{C_5}\right]^{(p+2)/2}. \tag{16}$$

The exact solutions have the property that they are the functions of β that minimize the Hamiltonian with respect to β. Explicit examples showing this for both the CSS equation as well as a quintic generalization of this equation are found in the appendix of Ref. 11. On using $\partial H/\partial\beta = 0$, we obtain

$$\beta = P^{\frac{p-l+2}{l-p-6}} \left[\frac{C_4}{C_3}\frac{p+4}{l-2}\right]^{2/(l-p-6)}. \tag{17}$$

This leads to

$$H = f(l,p) P^{-r}\,, \tag{18}$$

where r is given by Eq. (7), and

$$f(l,p) = -\left(\frac{p-l+2}{p+4}\right) C_3(l) \left[\frac{C_4(p)}{C_3(l)}\frac{p+4}{l-2}\right]^{(l-2)/(l-p-6)}. \tag{19}$$

Hamilton's equation, $\dot{q} = \partial H/\partial P$ now yields the relationship as given by Eqs. (7) and (8).

Using Eqs. (7), (14) and (17) to (19) it is easy to show that the momentum P, amplitude A and the width parameter β functionally depend on the velocity c (note $c = -\dot{q}$) by

$$P \propto c^{\frac{p+6-l}{2(l-2)}}\,, \;\; A \propto c^{\frac{1}{l-2}}\,, \;\; \beta \propto c^{\frac{l-p-2}{2(l-2)}}\,. \tag{20}$$

Several comments are in order.

1. Notice that when $l = p+2$ then β is independent of the velocity c and momentum P and hence the amplitude A of the solitary wave.

2. Note that the c dependence of the amplitude A solely depends on the parameter l and is independent of the parameter p.

3. In the special case when $p = 0$ and $l = d+2$, the CSS equation reduces to d-th order KdV equation. In particular, in that case $d = 1$ corresponds to the KdV equation while $d = 2$ corresponds to the modified Korteweg-De Vries (mKdV) equation. For this case, a well known exact solution is $f(y) = A\mathrm{sech}^{2/d}(\beta y)$, where $\beta = (d/2)\sqrt{v}$ and $2A^d = (d+1)(d+2)v$ which indeed is consistent with relation (20). For that case we notice from above that for any d, the width parameter β varies as $c^{1/2}$ while the amplitude A varies as $c^{1/d}$.

Stability of Solutions. The stability problem at $l = p+2$ was studied in Ref. 10, using the results of Karpman[12,13]. Their analysis is in fact also valid for arbitrary l, p. The result of detailed analysis is that the criterion for linear stability is equivalent to the condition,

$$\frac{\partial P}{\partial c} > 0. \tag{21}$$

Since for all of our solutions $P \propto c^{(p+6-l)/2(l-2)}$, it immediately follows that the solutions are stable provided $2 < l < p+6$. Analysis of Lyapunov stability following Refs. 10,12,13 also leads to the same restriction on p.

Exact solitary wave and compacton solutions: If we assume a solution to (2) in the form of a travelling wave:

$$u(x,t) = f(y) = f(x-ct)\,, \tag{22}$$

we then obtain

$$cf' = f'f^{l-2} + \alpha\left(2f'''f^p + 4pf^{p-1}f'f'' + p(p-1)f^{p-2}f'^3\right). \tag{23}$$

Integrating twice we obtain:

$$\frac{c}{2}f^2 - \frac{f^l}{l(l-1)} - \alpha f'^2 f^p = K_1 f + K_2. \tag{24}$$

For compactons, the integration constants, K_1 and K_2 are zero. The general theorem derived above is valid in the case that the integration constants K_1, K_2 are zero. Unless stated otherwise, throughout this paper we shall consider the case when K_1, K_2 are both zero. On demanding that $f''f^p \to 0, f'^2 f^{p-1} \to 0$ at edges where $f \to 0$, while f' is finite at edges gives us the following bounds on l and $p : l > 1, 0 < p \leq 2, p \leq l$.

It is worth noting that Eq. (24) is very similar to the equation obtained for the Rosenau-Hyman case:

$$(f')^2 = \frac{2v}{n(n+1)}f^{3-n} - \frac{2}{n(n+m)}f^{m-n+2}\,. \tag{25}$$

Thus, we see that with $l = m+1$ and $p = n-1$ the equations for finding solutions are identical in form, with only differing coefficients. Therefore we expect to find similar solutions to the two sets of equations.

Let us now look at the various different compacton solutions to Eq. (24). For the particular case $l = p+2$ ($m = n$), Cooper and Khare[9] were able to show that the CSS equation has solutions of the form (3) and that for all these compacton solutions the width is independent of the amplitude. We now show that if instead $l = 2p+2$ then one gets a one parameter family of elliptic compacton solutions. In particular, it is easily shown that

$$f = A\,\mathrm{cn}^\gamma(\beta y, k^2 = 1/2)\,, \tag{26}$$

for

$$-K(k^2 = 1/2) \leq \beta y \leq K(k^2 = 1/2)\,, \tag{27}$$

and zero elsewhere is an exact elliptic compacton solutions to the field Eq. (23) provided

$$\gamma = 2/p\,,\ \ l = 2p+2\,,\ \ A^{2p} = c(p+1)(2p+1)\,,$$
$$\beta^4 = \frac{cp^4}{16\alpha^2(p+1)(2p+1)}\,. \tag{28}$$

The $(m,n) = (3,2)$ and $(m,n) = (5,3)$ are two special cases of solution (26) with $\gamma = 1, 2$. Here $cn(y,k)$ is a Jacobi elliptic function and $K(k)$ denotes the complete elliptic integral of the first kind with modulus k.

If instead $l = 3p+2$, then we obtain a one parameter family of elliptic solutions of the form

$$f = A\left[\frac{1-\mathrm{cn}(2(3)^{1/4}\beta y, k^2)}{(1+\sqrt{3}) + (\sqrt{3}-1)\mathrm{cn}(2(3)^{1/4}\beta y, k^2)}\right]^\gamma\,, \tag{29}$$

with the modulus

$$k^2 = \frac{1}{2} - \frac{\sqrt{3}}{4}\,. \tag{30}$$

Here

$$\gamma = 1/p\,,\ \ l = 3p+2\,,\ \ 2A^{3p} = c(3p+1)(3p+2)\,,$$
$$\beta^6 = \frac{c^2p^6}{256\alpha^3(3p+1)(3p+2)}\,. \tag{31}$$

This is a compacton solution in the range

$$0 \leq w = 2(3)^{1/4}\xi \leq 4K\left(k^2 = \frac{1}{2} - \frac{\sqrt{3}}{4}\right). \tag{32}$$

and zero elsewhere, where $\xi = \beta y$. The way this solution is obtained is by starting with the ansatz $f = AZ^a(\xi)$ and demanding that Z satisfies the differential equation $(dZ/d\xi)^2 = 1 - Z^6$. The above integral can be evaluated by converting the differential equation into the standard differential equation for the Weierstrass elliptic function[14] $\mathcal{P}(y,k)$. On simplifying, we obtain the explicit solution as given by Eq. (28). The $(m,n) = (4,2)$ solution of Ref. 7 is a special case of our general solution (29) with $p = 1$.

We now show that all the above solutions are in fact special cases of the two parameter (a,t) family of solutions obtained by assuming

$$f = AZ^a(\xi = \beta y)\,, \tag{33}$$

and demanding that

$$(Z')^2 = 1 - Z^{2t}\,. \tag{34}$$

We find the conditions:

$$l = pt+2;\ \ a = 2/p\,,\ \ 2A^{pt} = (pt+2)(pt+1)c\,,$$
$$\beta^{2t} = \frac{c^{t-1}p^t}{\alpha^t 2^{3t-1}(pt+1)(pt+2)}\,. \tag{35}$$

It may be noted that here l, p, t are all continuous (i.e. real) parameters. The various solutions discussed above correspond to $t = 1, 2, 3$. In fact an explicit compacton solution can also be found in the case $t = 3/2$. The solution for $t = 3/2$ is essentially the same as that for $t = 3$, Eq. (29), but with Eq. (30) replaced by the complementary modulus $k^2 = \frac{1}{2} + \frac{\sqrt{3}}{4}$. For $t > 3$, the compacton solutions are related to hyperelliptic functions[14].

It is interesting to note that even though none of the hyperelliptic solutions can be obtained in an analytic form ($t > 3$), still their momentum and energy can be obtained analytically. In particular, on using Eq. (34) in the expressions for H, P as given by Eqs. (4) and (5) it is easily shown that

$$H = -\frac{A^2c}{2\beta t}\frac{(6+p-l)}{(l+p+2)}\mathrm{B}\left(\frac{p+4}{2pt}, \frac{1}{2}\right), \tag{36}$$

$$P = \frac{A^2}{2\beta t}\mathrm{B}\left(\frac{p+4}{2pt}, \frac{1}{2}\right), \tag{37}$$

where B denotes the Beta function[14]. For all these solutions with p, l continuous (i.e. real) variables, the relationship $H/P = c/r$ is always satisfied.

For the hyper-elliptic compacton solutions we have that l, p and t are related by $l = pt + 2$ so that the general stability criterion can be rewritten as $0 < pt < p + 4$, or for $t > 1$

$$0 < p(t-1) < 4. \tag{38}$$

The requirement for non-singular solutions is that $0 < p \leq 2$. This means for $t \leq 3$, compactons with arbitrary p in the allowed range are linearly stable while when $t > 3$, the compactons are stable only for $0 < p < 4/(t-1)$. Analysis of Lyapunov stability following Refs. 10,12,13 also leads to the same restrictions on p.

From the stability analysis we also recover the well known result that the higher order KdV equations, characterized by $p = 0, l = d+2$, have stable soliton solutions provided $d < 4$ while one has unstable soliton solutions in the case[15,16,17] $d > 4$.

General remarks. In conclusion, we have obtained explicit exact compacton solutions in terms of Jacobi elliptic functions which exist at particular values of the elliptic modulus k. Note that at $k = 0$ these solutions do not exist. In addition, we derived a quite general theorem that relates the energy momentum and velocity of solitary wave solutions without knowing the explicit form of the solution. We note that the above analysis should also hold for the generalized quintic KdV equation. We also notice that the radial part of the generalized nonlinear Schrödinger equation with nonlinear power κ

$$i\frac{\partial\psi}{\partial t} + \nabla^2\psi + g|\psi * \psi|^\kappa = 0 \tag{39}$$

obeys the same equation as (24) with $p = 0$ and $l = 2\kappa + 2$, so that the stability analysis of these two problems are related. We believe that it should be possible to derive a similar theorem in other nonlinear systems.

Finally, as a byproduct of our results, we are able to obtain analytic expressions for $K(k)$ and the complete elliptic integral of the second kind, $E(k)$, at $k^2 = 1/2 - \sqrt{3}/4$ (see Eq. 30) or at $k = \sin(\pi/12)$. Specifically,

$$K\left(k = \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{3^{1/4}\sqrt{\pi}\Gamma(1/6)}{6\Gamma(2/3)}\,, \tag{40}$$

where $\Gamma(a)$ denotes the Gamma function[14]. In addition, at $k = \sin(\pi/12)$ using the relations $K' = \sqrt{3}K$ and

$$E = \frac{\pi\sqrt{3}}{12K} + \sqrt{\frac{2}{3}}\,k'K, \quad E' = \frac{\pi\sqrt{3}}{4K'} + \sqrt{\frac{2}{3}}\,k'K', \tag{41}$$

we also have the explicit analytic expressions for E and the two complete elliptic integrals with complementary modulus $k' = \sqrt{1-k^2} = \cos(\pi/12)$, namely K' and E'.

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