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To Continue with Continuity

ABSTRACT

The metaphysical concept of continuity is important, not least because physical continua are not known to be impossible. While it is standard to model them with a mathematical continuum based upon set-theoretical intuitions, this essay considers, as a contribution to the debate about the adequacy of those intuitions, the neglected intuition that dividing the length of a line by the length of an individual point should yield the line's cardinality. The algebraic properties of that cardinal number are derived pre-theoretically from the obvious properties of a line of points, whence it becomes clear that such a number would cohere surprisingly well with our elementary number systems.

1. Introduction.

Were there physical continua, e.g. space-time, there would be an objective fact of the matter about the truth of our hypotheses about continuity. One hypothesis that has shaped modern mathematics (and thence logic and metaphysics) to a very great extent is that the geometrical line and the real number line are isomorphic, which I will call C-D, as it is due to Cantor and Dedekind. That hypothesis is assumed by almost all scientists nowadays, but nonetheless its philosophical analysis might one day benefit from our having available the widest possible range of alternative hypotheses (e.g. see Ehrlich 1994, not to mention such category-theoretic possibilities as synthetic differential geometry). In this essay I take an informal (pre-theoretic) look at one neglected hypothesis. I cannot consider any of its philosophical ramifications in any depth, not as well as defining it (in §2) and exhibiting its structural coherence (in succeeding sections), but I will at least be enabling that to be done.

My hypothesis may be introduced as an extrapolation from the familiarly finite. If we consider sand grains to be cubic millimetres of silicate, to keep things simple, then a sandstone mountain, say *M*, composed entirely of such grains and occupying a cubic kilometre, would contain 10^9 m^3 ÷ 10^{-9} m^3 = 10^{18} grains. That may be expressed, using Kessler's (1980, 69) empiricistic notation, as 10^{18}(*M*, being-a-grain). By extrapolating, it is not hard to imagine that, if lines were composed of points, each of length 0, then a line of arbitrary unit length u might contain $1 \cdot u \div 0 \cdot u = 1/0$ points. Such extrapolations, from finite cases to an infinite case, are unreliable, but they do not necessarily fail (e.g. they found set theory, see principle *b* of Hallett 1984, p. 7) and that one turns out to be coherent enough, as you will see.

First I should define my terms a little more precisely. Let the *primitive* line be the line that would be *physically* instantiated, e.g. as time, were time infinitely divisible. That idea, of a pre-theoretic geometrical line, makes sense whether or not there actually are any physical continua, and is more fundamental (conceptually) than such formal mathematical models of it as the standard real number line. Let primitive *cardinality* be what collections that may be related by *bijections* (one-to-one correlations) must have in common. That concept is also fundamental, and therefore its nature is also highly debatable, but the following may contribute to such debates, so I shall simply assume that definition. Finally, let k denote the (primitive) cardinality of the continuum. In other words, if lines were composed of points, and if there were physical continua, so that a line of points, say L, would be instantiated, then we would have k(L, being-a-point) in Kessler's notation.

I will begin to clarify what the possibility of k resembling 1/0 amounts to in §2, but an immediate problem is that you may already regard 1/0 as an impossible whole number. You may think, for example, that from $1 \times 0 = 2 \times 0$ we would be able to deduce $1 = 2$, were we to allow arithmetical division by 0, so I shall end this introductory section by challenging that particular reason. For an apposite historical analogy, when Cantor introduced (informally) his transfinite whole numbers he first had to challenge prejudices against their possibility (see Cantor 1883, pp. 892-893) before arguing that they were not just possible, but were actually coherent and useful.

Now, although the $1 = 2$ above does follow from assuming $0/0 = 1$ (together with associativity), why should $0/0 = 1$? One reason might be that $a/a = 1$ whenever a/a is defined at present (for finite a), and additionally defining 0/0 is like allowing a to be 0. But that kind of extrapolation is notoriously unreliable, and note that indeterminate forms within the calculus are often denoted by 0/0, the reason being that all the finite numbers yield 0 upon multiplication by 0. In fact, it need only follow, from dividing $1 \times 0 = 2 \times 0$ by 0, that 0/0 includes x *iff* (if and only if) it contains $2 \cdot x$, and that would allow arithmetical division by 0 (if not as a function) if 0/0 could be a *collection* of numbers (if not a set of numbers). I consider such collections in §2; but incidentally, multifunctions and mereological collections have indeed proved to be mathematically coherent and useful.

Another reason might be that 'division by *x*' means 'multiplication by the multiplicative inverse of *x*' within the number fields, and *x*'s multiplicative inverse is whatever yields 1 when multiplied by *x*. But from 0/0 ≠ 1 it need only follow that division by 0 is not allowed within number fields. You will see (in §5) that it is allowed within number pitches, which contain number fields in an algebraically strong way. Note that I will not be suggesting that functions and fields are not useful. They are extremely useful, but we may certainly extend that repertoire so as to include other structures that nature *might* be instantiating. By analogy, there is an obvious utility to having the cardinality of a collection increase by 1 when a new object is added to it, yet we may consider infinite cardinals whenever we have reason to.

2. A Non-set-theoretic Cardinal.

We may begin to consider the metaphysical possibility of *k* resembling 1/0 by considering the coherence of adjoining an undefined symbol # to the natural numbers (in §3), where the informal properties of # are derived from two heuristic assumptions:

(ha_1) that # is a possibility for *k*; and

(ha_2) that 1/# is the length of a point.

Such an approach is relatively direct because it is analytically metaphysical (i.e. pre-theoretic, not unlike Cantor 1883) rather than axiomatically mathematical (usually set-theoretic, cf. Kitcher 1983, p. 190) and so it avoids prejudging what kinds of numbers are possible (cf. §7). Note that # is not defined to be 1/0, the reason being that # is, if coherent, a cardinal number, which is a more fundamental kind of number than a ratio of magnitudes.

In case, at the outset, your suspicions are roused by # not being one of our numbers already, note that # cannot be a set-theoretic cardinal: were *k* the cardinality of a set, $0 \cdot k$ would equal 0, because the Cartesian product of ∅ (the empty set) with any set is ∅ (note that, for an arbitrary number *x*, $0 \cdot x$ does not necessarily have to equal 0, e.g. it need not within category theory). Furthermore, the major alternative to set theory as a foundation for mathematics has been constructivism, which prefers its lines *not* to be full of points. Consequently, even a coherent # may well have been overlooked. You will be better placed to decide whether or not the metaphysical (and perhaps empirical) hypothesis $k = \#$ coheres with the concept of a line of points after the following essay. But of course, it is hardly unnatural to treat continua mereologically, whether or not they are full of points. And in order to presume as little as possible about what numbers really are, in this pre-theoretic essay, I shall require their sort of collection to be an informal kind of atomic mereological sum (as formally described in Simons 1987, p. 14). A suitable collection has the following 4 informal properties (at least) and I call such a collection a *mere-sum* and denote it by square brackets.

The *first* property of mere-sums of numbers is that the individual numbers are regarded as atoms. That is, mere-sums of numbers are not also mere-sums of whatever comprises those numbers (in a different way) if anything does (e.g. their elements, if numbers are classes). *Secondly*, because the mereological sum of x and y is just x and y, internal brackets can be eliminated (e.g. [[1, 2], 3] = [1, 2, 3]), and the mere-sum of a single number is merely that number (e.g. [1] = 1). *Thirdly*, two mere-sums are naturally defined to be equal *iff* a bijection between them may relate each atom with an equal atom. Consequently, $[x, y] = y$ *iff* each atom of x is also an atom of y, so that x is a part of y (formal mereologies being part-whole theories), which is abbreviated to $x \measuredangle y$ below (e.g. [1, [1, 2]] = [1, 2], so $1 \measuredangle [1, 2]$). Furthermore, if $x \measuredangle y$, and also $z \measuredangle x$ implies $x \measuredangle z$, then x is an atom of y, abbreviated to x @ y below (e.g. 1 @ [1, 2]). And *finally*, arithmetical operations naturally distribute over mere-sums of numbers (e.g. adding 1 to both 1 and 2 yields 2 and 3, and so 1 + [1, 2] = [(1 + 1), (1 + 2)] = [2, 3]).

Mere-sums make very natural pre-theoretic collections (of numbers) and not even that much geometrical mereology will be required, not explicitly (cf. §7). But # does require that primitive lines might be made of points. Points are quite possible and conceivable, of course; e.g., an imaginary black square on a white background has points at its corners, where its edges intersect. Although planes do seem more like, for example, glass panes than sandpaper, that intuition cannot imply that they are not full of points, because points, having size 0, are infinitely smaller than sand grains, which hardly conflicts with planes being infinitely smoother than sandpaper. Furthermore, lines in planes are not like scratches put *onto* glass panes, because the *positions* of such scratches would make much better analogies for primitive lines, and they were clearly there already. In Aristotle's (spatial) line, a point had a potential existence that was actualised only if something happened there, but it is not uncommon to think that if something *could* happen there then an actual position (or point) must have been there already, so that it could happen *there*. While points are not a metaphysical necessity (see Dummett 2000 and Slater 2003), if points do exist then, as there is nowhere in a line where it cannot be intersected by another (0-width) line, lines are clearly *full* of points. A primitive line is effectively thought of as being some infinitude of points with the single thought that such intersections may occur anywhere within it. So the fact that it is also a single operation that yields 1/0 = # (see §3) indicates that k = # is not an intrinsically unreasonable hypothesis.

Nor is $k = 2\hat{}\aleph_0$ of course, where ^ denotes standard cardinal exponentiation (because I use the more familiar superscript notation for a more familiar form of exponentiation in §4) and $\aleph_0$ is the cardinality of the natural numbers (regarded as an actual infinity). That equation for k follows from C-D, so note that the significance of that widespread assumption may be assessed properly only if all the alternatives to it are also considered. For almost a hundred years, mainstream mathematicians have been using numbers that are (isomorphic to) ZF sets, not least because geometry was reduced to analysis following Descartes, and analysis was reduced to set theory following Cantor. But while ZF set theory provides mathematics with a *definite* subject-matter, not incoherently (e.g. see Steinhart 2002), it is hardly a *comprehensive* theory of cardinality, such as would be required for deciding the metaphysical propriety of #. E.g., we ourselves instantiate *the* natural numbers, so we can hardly just define them (see Hamming 1998), and note that it is only *their* emulation by some ZF sets (the finite von Neumann ordinals) that justifies those sets being called natural numbers within ZF. (For various thoughts about *the* natural numbers, see Tieszen 1989, Dehaene 1997 and Heck 2000.) In particular, the totality of the natural numbers might even be potentially infinite, for all we really know. That possibility is already reflected within mathematics by the persistence of constructivism, and it will be accommodated in the last few sections of this relatively platonistic essay by considering lines of # points in two cases, C-I, in which the natural numbers form an actual infinitude, and C-II, in which they do not. At the opposite extreme, for another example, we know that the cardinality of the totality of all the sets, say Ω, is not the size of a set, and yet it is certainly a *primitive* cardinal number because $\aleph_\alpha$ denotes a transfinite cardinal *iff* α denotes an ordinal (so those two classes correlate one-to-one). (Furthermore, it is obviously coherent to regard the totality of the cardinals and the totality of the ordinals as 2 totalities, despite the impossibility of doing so within ZF.)

3. From Notion to Fraction.

First I adjoin # to the *natural* numbers, $N =_{df} [1, 2, 3, \ldots]$, to make $N^{\#} =_{df} [N, \#]$, which I call the *notional* numbers, in order to see how strong the arithmetic of $N^{\#}$ can be, given ha_1 and ha_2. That is a good beginning (especially if C-II is the case) because little is as straightforward as the *informal* arithmetic of the natural numbers. (Incidentally, although a formal extension of an algebraic structure would define new operations upon new objects, with some part of the new structure being isomorphic to the whole of the old domain, I shall call that part and the new operations by their old names as far as possible, for clarity.) The exclusion of 0 from *N* might strike you as odd, because we are used to including it in our ZF set of natural numbers, but it is desirable to exclude it here because the informal properties of # are going to be obtained via the concept of a *line* of # points. In that context, 0 is primarily the length of a point, it is a magnitude (an answer to 'How much?') rather than a multitude (an answer to 'How many?'), and so 0 will be introduced as an abbreviation for 1/# when I consider the ratios of notional numbers, at the end of this section. Of course, 0 is indeed a finite cardinal number (if one of a unique kind) and so beginning with $[N^{\#}, 0]$ would have been a coherent (if less clear) alternative.

To begin with (where *n* is, as usual, a natural number variable) $\# + n$, $\# + \#$, $\#\cdot n$ and $\#\cdot\#$ all equal #, for the following reasons. The first equation, $\# + n = \#$, could hardly be false given the second, which is a special case ($n = 2$) of the third, $\#\cdot n = \#$, which follows (via ha_1) from how the points of a line of length *n* correlate one-to-one with the points of a unit line. Replacing # by *k* in the fourth equation makes it say that planes have the same cardinality as lines, as we would expect nowadays (and it also follows from ha_2 below). All 4 equations should be unsurprising nowadays, as they remain valid if any transfinite cardinal replaces #. And, as we would expect of arithmetical operations that may apply to numbers of points, addition and multiplication may both remain associative and commutative (the consistency of retaining those algebraic strengths is clear enough because any finite expression containing # just equals #). Furthermore, it is trivial to check that multiplication distributes over addition, via a few typical equations such as $\#\cdot(\# + n) = \#\cdot\# = \# = \# + \# = \#\cdot\# + \#\cdot n$.

Note that $N^{\#}$ is clearly (just as *N* is) closed under both operations. I call a mere-sum *S* *closed* under a (binary) commutative operation **o** if *x* **o** *y* @ *S* whenever *x* and *y* are atoms of *S* because that is isomorphic to the familiar definition of closure (a set **S** being closed under **o** if *x* **o** *y* ∈ **S** whenever *x* and *y* are elements of **S**). Isomorphic definitions will not usually be stated explicitly, for brevity, but that definition extends rather naturally to a concept that is more useful with the inverse operations. I say that *S* is *mere-closed* under **o** if *x* **o** *y* ∠ *S* whenever *x* and *y* are atoms (or other parts) of *S*. That concept coheres with the informal meaning of algebraic closure because if a mere-sum is mere-closed then operating within it cannot generate anything that is not there already.

The *inverse* of **o** is usually an operation **i** such that *x* **i** *y* = *z* *iff* *x* = *y* **o** *z*, e.g. 3 – 2 = 1 because 3 = 2 + 1 and nothing else (of current interest) yields 3 when 2 is added to it. But # – # and #/# (which is 0/0) will be collections of numbers (cf. §1), so a more appropriate definition of **i** (in terms of **o**, and within a domain containing atoms *x*, *y* and *z*, and which I call *In*) is *z* @ *x* **i** *y* *iff* *x* @ *y* **o** *z* (which includes the usual definition as a special case). E.g., # – *n* = # follows from *In*, since # + *n* = # and *N* is closed under addition, and #/*n* = # also follows from *In*, since #·*n* = # and *N* is closed under multiplication. Similarly, # – # = $N^{\#}$, since # + # = # = *n* + #, and #/# = $N^{\#}$, since #·# = # = *n*·#. Subtraction and division are not closed in *N*, so they are not mere-closed in $N^{\#}$, and unsurprisingly they are neither associative nor commutative.

For our first surprise, however, multiplication cannot distribute over subtraction, within $N^{\#}$, because (2 – 1)·# = # does not equal 2·# – # = $N^{\#}$. That must seem like bad news for #, but consider the (informal) set **N**∪{0, $\aleph_0$}, where **N** is given by *n* ∈ **N** *iff* *n* @ *N*. Cardinal multiplication cannot distribute over subtraction within that set, lest $\aleph_0 = (2 - 1)\cdot\aleph_0 = 2\cdot\aleph_0 - \aleph_0 = \aleph_0 - \aleph_0 = (1 - 1)\cdot\aleph_0 = 0\cdot\aleph_0 = 0$. Defining $\aleph_0 - \aleph_0$ would be useful, e.g. removing $\aleph_0$ objects from $\aleph_0$ objects would leave *m* objects, where *m* ∈ **N**∪{0, $\aleph_0$}, but $N^{\#}$ would be relatively strong anyway, even were $\aleph_0 - \aleph_0$ undefined, because at least # – # is defined. So it is likely that failures of distributivity are just as natural for infinite cardinals as failures of commutativity are for infinite ordinals (cf. my glance at exponentiation in §4). And although multiplication will stop distributing over *addition* when negative numbers are adjoined (in §4), it is not especially unnatural for a commutative multiplication to fail to distribute over a commutative addition (e.g. it may do so within category theory).

Anyway, consider next the *ratios* of the notional numbers, because within the motivating context of a line of # points we may consider #

points, and n points for any n, and also n line intervals. The continuity of the line makes it possible (in principle) to continue to subdivide intervals endlessly, so it is natural to extend $N^{\#}$ next to a domain that is mere-closed under division. I shall call a ratio of two notional numbers, if it is not #/#, a *fractional* number, an atom of $F^{\#}$. The elementary arithmetic of $F^{\#}$ subsumes that of $N^{\#} \angle F^{\#}$ of course, and includes that of 1/#, and the remaining atoms of $F^{\#}$ are of the form $r = n/m$, where n and m are relatively prime natural numbers with $m > 1$. Addition and multiplication may remain *commutative* and *associative*, with multiplication *distributing* over addition (it is trivial, if tedious, to show the consistency of retaining those algebraic strengths). Note that dividing the notional equation $\#\cdot n = \#$ by m yields $\#\cdot r = \#$ (since $\#/m = \#$), while dividing $\# \pm n = \#$ by m yields $\# \pm r = \#$, and multiplying $\#/n = \#$ by m yields $\#/r = \#$.

It follows from ha_2 that 1/# is the additive identity because, for example, ignoring one of the end-points of a line interval would not affect its length, so $n \pm 1/\# = n$ and $r \pm 1/\# = r$, from which $\# \pm 1/\# = \#$ follows by adding #. And via *In*, $n - n = 1/\#$ and $r - r = 1/\#$. Consequently, $1/\# + 1/\# = 2\cdot n - 2\cdot n = 1/\#$, and $r/\# = r^2 - r^2 = 1/\#$, and $(1/\#)\cdot(1/\#) = 2\cdot n^2 - 2\cdot n^2 = 1/\#$ (which yields the fourth notional equation, $\#\cdot\# = \#$, upon inversion), and so forth. In short, 1/# is isomorphic to the familiar magnitude 0 within the finite part of $F^{\#}$, so 1/# will now (for clarity) be called 0. Also via *In*, $\# - \# = F^{\#}$ and $\#/\# = F^{\#}$, and clearly $\#/\# = \#\cdot 0 = 0/0$, so we may now see (to pick up a point from §1) that dividing $1 \times 0 = 2 \times 0$ by 0 within $F^{\#}$ just yields $F^{\#} = F^{\#}$.

Reiterating those arithmetical operations would be consistent, as a few typical equations (skipped for brevity) would show, so the coherence of # is already indicated (to some extent) by the algebraic strengths of $N^{\#}$ and $F^{\#}$. The most natural way to extend $F^{\#}$ would be by adjoining irrationals (and infinitesimals, see §7), because $\#\cdot 0$ should include all such numbers: if $k = \#$, then lines of arbitrary length are # points, each of length 0; and there are similarly geometrical reasons why # and 0 should be values of $\#\cdot 0$, because a line of # points is 0% of an area, which has $\#\cdot\# = \#$ points. You will see (in §5) that such extensions could retain the algebraic strengths of $F^{\#}$, but for brevity the negative numbers will be adjoined next, because a resulting algebraic structure, the number pitch (defined in §5), extends the other number fields just like it extends the rationals (revealing more of the coherence of #).

4. Dividing by Zero Vectors.

It is useful to assign numerical coordinates to a line's points, relative to two arbitrary points labelled 0 and 1 (see §7), and so it is quite natural (in the current context) to regard the adjunction of negatives as the introduction of two directions, ±1. So consider two signed collections $+F^{\#}$ and $–F^{\#}$ defined by $+x$ @ $+F^{\#}$ *iff* x @ $F^{\#}$ *iff* $–x$ @ $–F^{\#}$, with the familiar properties of signs, e.g. $–x = –y$ *iff* $x = y$ and $(–x)\cdot(–y) = +(x\cdot y)$, following from the natural properties of directions. The equations for +# are those for the fractional #, reading *r* as +*r*, etc., while the equations for –# follow from considering the fractional equations in the direction –1 instead of +1 (e.g. –# ± –*x* = –#, for *x* @ $F^{\#}$). Addition and multiplication may remain commutative and associative (as is easily checked), so the arithmetic of +# and –# follows from that of #. E.g., +# – +# includes all the rationals and +# (via *In*) so, +# being an atom of +# – +# = +# + –#, therefore –# also yields a meresum that includes +# when added to +#, and so +# – +# = [$+F^{\#}$, $–F^{\#}$]. The remainder of the signed arithmetic is mostly that straightforward, but there is one odd-looking result, because (+#)·(+0) = $+F^{\#}$, whereas (+#)·(–0) = –$F^{\#}$, which means that +0 (i.e. +(1/#)) is not quite the same as –0.

Nonetheless the rational equation 0 = (–1)·0 is obtained by replacing $\underline{0} =_{\text{df}} [+0, –0]$ with an individual object (not necessarily a pair-*set*) that relates to the other numbers just like $\underline{0}$ does and which will be called 0 when the positive numbers are called by their previous (unsigned) names. That is not inappropriate because the rational 0 is not an undirected quantity, not in the way that the fractionals are undirected, so it really does not make less sense to think of it as having all the directions (of the domain) rather than none. Furthermore, approaching the rational 0 via $\underline{0}$ coheres with other consequences of ha_1 and ha_2, such as the existence of infinitesimals (see §7), which can have either sign. Of course, if replacing $\underline{0}$ by a single isomorphic object was a particularly unnatural thing to do, then the plausibility of *k* = # would be challenged, but the main thing is that $\underline{0}$ is indeed isomorphic to the rational 0 (and the mainstream approach is less natural, e.g. its integers are equivalence classes of pair-sets of finite ZF ordinals).

That isomorphism follows from how rationals are not changed by the addition or subtraction of $\underline{0}$, and how any rational times $\underline{0}$ equals $\underline{0}$, as follows. From $+F^{\#}$, +0 + +0 = +0, so +0 is an atom of +0 – +0 = +0 + –0, and so –0 is too, and nothing else is, so +0 + –0 = $\underline{0}$, and furthermore (from –$F^{\#}$) –0 + –0 = –0, so $\underline{0} + \underline{0} = \underline{0}$, and so $\underline{0} \pm \underline{0} = \underline{0}$ because $–\underline{0} = \underline{0}$. More briefly now, (from $+F^{\#}$) +*r* ± +0 = +*r*, and (from $–F^{\#}$) –*r* ± –0 = –*r*, and similarly with *n* instead of *r*, and furthermore (+*r*)·(+0) = +0 and (+*r*)·(–0)

= –0, so (–r)·(–0) = +0 and (–r)·(+0) = –0, again with n instead of r. Finally, directly from the properties of signs, $\underline{0}\cdot\underline{0} = \underline{0}$.

So, with $\underline{0}$ replaced by an isomorphic atom called 0, and the positive numbers called by their old names, the new domain consists of #, –# and all of the rationals. Division by # and –# are still multiplication by +0 and (respectively) –0, by definition, so multiplication by 0 is now division by [#, –#], and 1/0 = [#, –#]. Although it was the case within $F^{\#}$ that 1/0 = #, such differences between domains are not too unusual, even within school mathematics, cf. how square numbers (e.g. 1, 4, 9) each have one square root in N, but two in Z (where x @ Z *iff* $x \in \mathbf{Z}$, the informal set of integers that we learnt about at school, and which is arithmetically isomorphic to the ZF set of integers). What is more of a problem is that although # – # now equals the whole domain, #/# is only the non-negative part of it. A more useful structure therefore results from replacing both $\underline{0}$ and $\Theta =_{df}$ [#, –#] by new atoms.

That structure is the rational number pitch, in which 1/0 = Θ_P (defined in §5). But before I define that algebraic structure, note that although Z was bypassed as the number systems were built up via $F^{\#}$, that was not because of any inconsistency between Z and #. In fact, because 1/0 + 1/0 = $\Theta + \Theta$ = # – # = 0/0 (which will become $\Theta_P + \Theta_P = 0\cdot\Theta_P$ below), the familiar rules for adding and multiplying ratios of integers, i.e. $(w/x) + (y/z) = (w\cdot z + x\cdot y)/(x\cdot z)$ and $(w/x)\cdot(y/z) = (w\cdot y)/(x\cdot z)$, may now remain valid when w, x, y and z are *any* integers, and of course, being able to round out the validity of familiar rules indicates coherence. Furthermore, that particular example occurred because multiplication by ±# (below, Θ_P) cannot distribute over addition now that the subtraction of a number is the addition of its negative. So such extensions of validity compensate somewhat for (and thereby indicate the coherence of) that algebraic weakness.

Coherence is similarly indicated by situations that involve to consider exponentiation in any breadth it is apposite to note that $0^{(2-1)} = 0$ and $0^{(1-2)} = \Theta$ (below, Θ_P), whereas $0^2/0 = 0/0 = 0/0^2$, so that the extension of the familiar rule $z^{(x+y)} = z^x\cdot z^y$ to include $z = 0$ is the weaker rule $z^{(x+y)} \underline{\angle}\, z^x\cdot z^y$ (cf. mere-distributivity in §5). But that weakness allows 0^0 to equal 1 instead of 0/0, and it can be useful to stipulate that $0^0 = 1$, e.g. when algebraically manipulating polynomials (cf. Kaplan 1999, p. 169) or when recursively defining exponentiation. Furthermore, a relatively natural (if quasi-multifunctional) way to handle rational powers is via biconditionals such as x @ $y^{½}$ *iff* $x^2 = y$. Then $y^{(½ + ½)} = y \,\underline{\angle}\, [y, –y] = y^{½}\cdot y^{½}$, and the rule $(y^{½})^2 = (y^2)^{½}$ can be kept even when y is negative; whereas the familiar root function, say √, takes only positive values, so $(\surd x)\cdot(\surd y) = \surd(x\cdot y)$ must fail when x and y can be negative (e.g. becoming $-1 \neq 1$ when $x = y = -1$). Note that although $\surd((1)^2) = 1$ is certainly better looking than $((1)^2)^{½} = [1, -1]$, less attractive is $\surd((-1)^2) = 1$.

5. One Pitch, Two Teams.

Some algebraic structures have naturally appeared, so in this section I shall define the *pitch* and *team* structures (to refer to in §7). The substructure of the arithmetic of $[F^{\#}, -F^{\#}]$ within which # and –# only occur in the forms $\underline{0}$ and Θ will be called the *rational number pitch* because (i) it contains the rational number field and (ii) any field may be extended to its corresponding pitch, as follows. A number field **F** is usually a set **F** of numbers together with two arithmetical operations that satisfy the familiar field axioms. But an isomorphic structure is therefore possessed by a mere-sum **Φ** given by x @ **Φ** *iff* $x \in$ **F**, when @ replaces ∈ in those axioms. Adjoining a number Θ_Φ (with the following properties) to the field **Φ** makes the number pitch $\mathbf{\Phi}^\Theta =_{df} [\Theta_\Phi, \mathbf{\Phi}]$. The arithmetical operations are extended by the following 6 equations (where x @ **Φ** and $x \neq 0$).

$\Theta_\Phi + 0 = \Theta_\Phi$ $\qquad\Theta_\Phi + x = \Theta_\Phi$ $\qquad\Theta_\Phi + \Theta_\Phi = \mathbf{\Phi}^\Theta$

$\Theta_\Phi\cdot 0 = \mathbf{\Phi}^\Theta$ $\qquad\Theta_\Phi\cdot x = \Theta_\Phi$ $\qquad\Theta_\Phi\cdot\Theta_\Phi = \Theta_\Phi$

Also, division by 0 is multiplication by Θ_Φ, and *vice versa*, and the subtraction of Θ_Φ is the same as its addition, and addition and multiplication both remain commutative and associative within the pitch, which is therefore rather neat. Pitches are mere-closed under addition, subtraction, multiplication and division. Consistency is easily shown by a few equations such as, for an example of associative multiplication, $0\cdot(0\cdot\Theta_\Phi) = 0\cdot\mathbf{\Phi}^\Theta = [0\cdot\Theta_\Phi, 0\cdot\mathbf{\Phi}] = [\mathbf{\Phi}^\Theta, 0] = \mathbf{\Phi}^\Theta = 0\cdot\Theta_\Phi = (0 \times 0)\cdot\Theta_\Phi$. And the only algebraic cost of extending a field to a pitch is what I call *mere-distributivity*, i.e. if x, y and z are atoms of $\mathbf{\Phi}^\Theta$, then $x\cdot(y + z) \underline{\angle}\, x\cdot y + x\cdot z$, with equality (distributivity) only if $x \neq \Theta_\Phi$.

In particular, when **F** = **Q** (the rational number field), adjoining Θ_P to **Φ** = P (*rho*, for rational, or Pythagoras) yields the pitch P^Θ, which is the same structure that replacing Θ by an isomorphic atom (in §4) would yield, as is easily checked. The previous paragraph therefore provides a summary of the previous sections, whilst being applicable to the other number fields as well. Let the field Δ (*delta*, for Dedekind) be defined by x @ Δ *iff* $x \in$ **R** (the real number field). The adjunction of Θ_Δ to Δ yields the pitch Δ^Θ.

However, lines of # points occur in two possible cases, C-I and C-II (see §6), corresponding to *N* being an actual (completed, finitesque, combinatorial) infinity or (respectively) a potential infinity, and in C-II only reals that could (in principle) be defined by finite laws are legitimate. Denoting such a field by Λ (*lambda*, for legal), the adjunction of Θ_Λ yields what I will call a *legal* real number pitch, Λ^Θ. Similarly, let the field Γ (*gamma*, for Gauss) be defined by *x* @ Γ *iff* $x \in \mathbf{C}$ (the complex number field), which is a Gaussian plane. The adjunction of Θ_Γ yields the pitch Γ^Θ, which is a projection of a Riemann sphere. In C-II, adjoining the imaginary unit i to Λ yields a *legal* complex number field, I (*iota*, for imaginary), with i's adjunction to Λ^Θ yielding I^Θ. Incidentally, had irrational magnitudes been adjoined to $F^\#$ (in §3) both Θ_Δ and Θ_Λ would also have replaced [#, –#], just as Θ_P did, while Θ_Γ and Θ_I would have replaced all the $\#\cdot e^{i\theta}$ for $0 \le \theta < 2\pi$ (legal *θ*, in the case of Θ_I).

A precise description of the increase in *symmetry* caused by the adjunction of Θ_Φ to a field is facilitated by defining the following structure, ⟨*T*, *e*, **a**, **M**⟩, which I call the number *team T*. Teams are so-called because they are commutative generalizations of Abelian (i.e. commutative) *groups*, e.g. ⟨Z, 0, +, ∅⟩ is an improper team, as follows. *T* is any mere-sum of numbers that is mere-closed under an associative and commutative arithmetical operation **a**, with an identity *e* @ *T* such that, for each *x* @ *T*, *e* **a** *x* = *x* and there is a *y* @ *T* such that *e* @ *x* **a** *y*. The finite set **M** contains those *x* for which that last @ *cannot* be replaced by equality, teams being 'proper' if ∅ is a proper subset of **M**, e.g. the proper multiplicative team of the fractionals is ⟨$F^\#$, 1, ×, {0, #}⟩. So, whereas a field **Φ** contains an additive Abelian group ⟨**Φ**, 0, +, ∅⟩ and a multiplicative commutative monoid, a pitch $\mathbf{\Phi}^\Theta$ is relatively symmetrical because it contains two proper teams, ⟨$\mathbf{\Phi}^\Theta$, 0, +, {Θ_Φ}⟩ and ⟨$\mathbf{\Phi}^\Theta$, 1, ×, {0, Θ_Φ}⟩.

6. Another Continuum Problem.

After all that algebra, perhaps a brief recap would be useful. The arithmetic of # was deduced (in §3) from two assumptions, (ha_1) that # is the number of points in a line, and (ha_2) that 1/# is the length of a point. (The symbol # was chosen because it illustrates one intuition for the existence of points within lines.) Directions were then given to the fractional magnitudes (in §4), whence the isomorphism between [+(1/#), –(1/#)] and the rational 0 led to the replacement of [+(#/1), –(#/1)] by a new number Θ_P. (The symbol Θ was chosen because an ideal point at infinity turns an infinite line into an infinite circle, cf. §7.) There were several signs of the coherence of # with our elementary number systems. But, as mentioned in §5, it will make a difference whether the infinitude of *N* is actual or potential, when coordinates are given to a line of # points (in §7), so in this section I will look at those two possibilities. As mentioned in §2, # must be a non-set-theoretic cardinal. Although $0 \cdot \# \neq 0$, whereas $0 \cdot \aleph_\alpha = 0$ for every ordinal α, there are still two possibilities, as follows. Either (C-I) # is *bigger* than every $\aleph_\alpha$, or else (C-II) # is cardinally *incomparable* with every $\aleph_\alpha$. In C-I, lines of # points would contain all transfinite cardinalities of points, which is a lot of points (but then, 0 is *very* small); and in C-II, lines of # points could not even contain $\aleph_0$ points, which is to say that the infinitude of *N* is *potential* (a concept that is usually associated with constructivism, but which has also been associated with proper classes, see Hart 1976).

I call the choice between C-I and C-II *another* continuum problem because of Cantor's famous continuum problem, which concerns the unresolved details of standard cardinal exponentiation (see Feferman *et al.* 2000). So, before looking a little closer at C-I and C-II (although a detailed comparison must await such developments as those mentioned in §7) I will glance at cardinal exponentiation involving #. A simple 'diagonal argument' involving the diagonal of a geometrical square shows that the number of ways in which a 0 or a 1 may be associated arbitrarily with each point of a line of # points is a number bigger than #, say $\#^+$, not $2^\#$ because that notation has already been used for the kind of exponentiation that extends the field operation (e.g. 0^0 in §4) and there is no obvious isomorphism. Now, although $\#^+$ shows that # is increasable (cf. the transfinites), so that within an appropriately extended number system $\#^+$ would be one of the values of 1/0 (cf. –# becoming part of 1/0 in §4), with $\#^+ + \#^+ = \#^+$ and $\#^+ \cdot \# = \#^+$ etc. (cf. transfinite cardinal arithmetic), nonetheless since $\#^+$ does not directly concern continuity such arithmetic is not pursued here.

The first thing to note about C-I is that, within any reasonable theory of (well-ordered) classes, the Cartesian product of the null-class with any class is likely to be the null-class, so # = Ω also seems unlikely. (Given the axiom of choice, we can rule out $\# = 2^\wedge\aleph_0$, of course.) One heuristic principle of Cantorian set theory is that "any potential infinity presupposes a corresponding actual infinity" (principle *a* of Hallett 1984, p. 7), which might imply, intuitively, that # > Ω. But another Cantorian principle is that Ω "cannot be mathematically determined" (principle *c* of Hallett 1984, p. 7). Whilst being actual (in the sense of principle *a*), Ω cannot be as finitesque (in the sense of principle *b*) as the transfinites, which might imply that lines of # points would *not* also contain Ω points. It might therefore be appropriate to call Ω's infinitude *potential*, although Cantor called it *absolute*. In short, proper classes are quite mysterious, and so little can be said pre-theoretically about (e.g. against) C-I at present.

Looking ahead to C-II, note that Ω is often likened to a potential infinity because of ZF's hierarchical nature. ZF's axiom of infinity is just the smallest of several axioms of infinity (see Feferman *et al.* 2000) and without it the natural numbers would form a proper class, just as the totality of all the ZF sets would be a set within a set theory containing a large cardinal axiom. (Assuming that each transfinite is a proper number, one might wish to consider an arbitrary subcollection of Cantorian cardinals, and hence the totality of all such subcollections, which would be cardinally larger than Ω.) In C-II, *N* would effectively be a potentially infinite kind of totality, at least by comparison with a line of points. It is certainly the repeated addition of 1 that yields the names of the finite cardinals in *N*, starting from 1, via $2 =_{df} 1 + 1$, and $3 =_{df} 2 + 1$, and so forth, so their totality (i.e. all of *those* cardinals, the ones with such names) is defined in an endlessly hierarchical kind of way, which is quite different to the way that lines are full of points (cf. §2). So it is logically possible that, although there are *n* points, for any (natural number) *n*, in a line of points, there are not $\aleph_0$ points, just because of the endlessly hierarchical way in which *N* is defined (pre-theoretically).

C-II is a counter-intuitive possibility, but it has therefore been overlooked and has not, in particular, been refuted. In fact, good reasons why *N* should act finitesquely (as **N** does) that are not also reasons why the proper class of all the cardinals should *are* rather elusive (cf. Fletcher forthcoming). Even when *N*'s infinitude is regarded as actual, the concepts of cardinal and ordinal begin to diverge with $\aleph_0$ and ω because of **N**'s endlessness, so that endlessness is certainly able to cause *some* shift away from finitesque behaviour. And the approach of the natural numbers to $\aleph_0$ does resemble the approach of the cardinal numbers to Ω, even though Ω cannot be as actually (or finitesquely) infinite as $\aleph_0$. So note that the possibility of C-II just requires that two infinite collections (the endless sequence of *the* natural numbers and the *primitive* line of points) that are even more different in kind (than $\aleph_0$ and Ω) *might* differ significantly. Furthermore, although # is not necessarily a number at all, it would, were it a number, be primarily a possible number of points, and so arguments that $\aleph_0$ is better at being a number of *numbers* than a number of spatio-temporal *objects* (e.g.

Cooke 2003) amount to an argument for C-II.

7. Infinitesimals.

Admittedly, neither C-I nor C-II appears particularly attractive, intuitively, but that may just be the way with lines of points (cf. the famous Banach-Tarski paradox, and also Freiling 1986). So, neither case having been considered in any detail yet, let alone refuted, the points of an infinite line (satisfying Hilbert's axioms of incidence, order and congruence) will now be given numerical labels, under the assumption of $k = \#$. Calling an arbitrary point 0 gives us our *origin*, and calling any other point 1 defines a *unit* of length and a *positive* direction. Any point, say p, between the points 0 and 1 lies in one of the tenths of that interval (e.g. between the points called 0·0 and 0·1), and in one of the tenths of that tenth (e.g. between 0·00 and 0·01), and so forth, and is therefore associated with an endless decimal expansion, say $d(p)$ (e.g. 0·000…), which is basically a real number.

In C-I, d is a structure-preserving function that maps each point that is a finite distance from 0 to a real number. If just one point mapped to each real, the number of points would be $2\wedge\aleph_0$, and so if $\# \neq 2\wedge\aleph_0$ (e.g. via the axiom of choice) then there are infinitesimals in C-I. Any point, say i, apart from 0 but with $d(i) = 0$, could be used to define an infinitesimal unit of length. The uniformity of the line means that i might have been chosen as the point called 1 (above, following our choice of origin), and so the point that was actually called 1 shows (under the alternative labelling) that there are also points infinitely distant from 0. Such points can be given the numerical label Θ_Δ (see below). Lines are therefore partitioned by the function d from any point q to some $d(q)$ @ Δ^Θ, in C-I, and so the real number line is quite a good mathematical model of the primitive line in that case. (Number lines corresponding more precisely to C-I will not be considered, because this essay is primarily concerned with introducing the possibility of $k = \#$.)

In C-II, the endless decimal expansions would be potentially infinite, so the arbitrary expansions that would be associated with *most* points would be impossible to identify, even in principle, except *via* those points. Consequently, the most analytically useful reals would be those whose expansions could be specified individually by finite laws, which I call the legal ones (e.g., recursive or computable ones; cf. Weyl's line, although it was not composed of points, e.g. see Bell 2000). Legal functions would not include the classical monsters, but could include Dirac's useful delta function because there are also infinitesimals in C-II, as follows. In any infinitely extended line of points, there are points that are n unit lengths from 0 for *every n*, but in C-II there is no sequence of *all* such lengths (since there are not $\aleph_0$ points) and so there are points that are infinitely distant from 0 (relative to any unit of length). Such points are naturally associated with Θ_Λ (see below), and because one of those points might have been called 1 originally, there are also infinitesimals (via the converse of the argument above).

Since the coherence of # is indicated by the use of Θ_Δ or Θ_Λ to label points infinitely distant from 0, I shall briefly justify that use of Θ_Δ (the case of Θ_Λ being analogous). For real $x \neq 0$, all of $\Theta_\Delta + 0$, $\Theta_\Delta + x$, $\Theta_\Delta \cdot x$ and $\Theta_\Delta \cdot \Theta_\Delta$ must equal Θ_Δ (see §5), where the additions correspond to vector additions (e.g. the first corresponds to going an infinitesimal distance from a point infinitely distant from 0, which amounts to remaining infinitely distant from 0) and similarly for the multiplications. So the other three equations, and also the commutative and associative laws, are clearly satisfied. Also required is $\Theta_\Delta + \Theta_\Delta = \Delta^\Theta$, so consider any two points infinitely distant from 0, on either side of 0; going an infinite distance from one point back towards 0 (and possibly beyond it) could amount to being at any other point. And $\Theta_\Delta \cdot 0 = \Delta^\Theta$ is clearly satisfied too because the multiplication of an infinitesimal magnitude with an infinite magnitude may result in any magnitude, of positive (or negative) sign if the signs of the multipliers are the same (respectively different).

Incidentally, # and –# might be used to represent the two infinite regions separately, but using # to label points at infinite distances from 0 is *not* to say that such points are # units from 0, no more than using 0 to label i amounts to saying that i is *not* distinct from 0. Furthermore, coherence may also be indicated by the use of Θ_Δ (or Θ_Λ) and Θ_Γ (respectively Θ_I) within other mathematical structures that, assuming C-I (respectively C-II), resemble division by 0. The ideal point ∞ at infinity in extended lines, for example, could be called Θ_Δ (respectively Θ_Λ), with the planar ∞ becoming Θ_Γ (respectively Θ_I). And 0/0 is already used to denote indeterminate forms in the calculus. In C-II, lines contain n points for any n, and # points, but no intermediate amounts, so Θ_I would be a natural choice for unbounded complex limits, in that case, with ±# denoting unbounded real limits, such as the gradients of vertical lines.

Of course, infinitesimals are certainly counter-intuitive, but it is not, given the *extreme* smallness of 0, *especially* counter-intuitive that infinitesimals should occur in lines of points. One argument against infinitesimals (cf. Grattan-Guinness 2000, p. 236, and Moore 2002, p. 325) concerns a point-object going $I =_{df} (i + i + i + \ldots)$ from 0, and therefore going just as far as one going I from $n \cdot i$, because $n \cdot i + I = I$. Nonetheless, in C-II such counter-intuitive situations cannot arise, while in C-I we may simply deduce that I is undefined. It would be appropriate to regard I as undefined because $(1 + 1 + 1 + \ldots)$ is similarly undefined, and furthermore the Banach-Tarski paradox (which follows from the Banach-Tarski theorem of real analysis via C-D) is usually resolved by deducing that the counter-intuitively decomposed sphere's parts are measureless. Note that the *absence* of infinitesimals (and their reciprocals, see below) is therefore associated with a *similar* counter-intuitiveness.

Consider a rocket going a metre in 1 second, another metre in ½ second, another in ¼ second, etc., along a straight line within an infinite 'flat' space containing no infinitesimals (e.g. $\mathbf{R}^3$). It appears that the rocket should vanish, or at least teleport, after 2 seconds (see Saari and Xia 1995, for similar vanishings from a Newtonian space). But intuitively, the rocket would neither vanish nor teleport, and we might prefer to imagine it reaching instead an infinitely distant part of space (vanishing only from a Euclidean universe of discourse). Were k = #, an infinite 'flat' space would, by containing such infinitely separated places, be more like the space of projective geometry, which is the most symmetrical of the geometries (in their group-theoretic classification). So note that symmetrical structures do seem more likely than asymmetrical ones to be physically instantiated (cf. Penrose 2000, pp. 230-231; see also Castellani 2002).

Infinitesimals are often regarded as unrealistic, but the basic concept cannot be too incoherent because several formal kinds, e.g. 'hyperreal', 'surreal' and 'smooth' ones, already exist (primarily to assist standard analysis). Those in lines of # points, say 'irreal' ones, are not being posited for their utility, but if lines might contain # points then it would obviously be useful to know more about them. So inevitably the issue of axiomatization arises. A good set of axioms for # would require not only more of #'s informal properties than I have been able to mention in this essay, but also a good formal language. That would presumably be a mereological one (judging by the above) but such languages are still in the process of development (see Forrest 2002) and while a mereological approach would probably facilitate the comparison of C-I, C-D, C-II and the cases of lines *not* being full of points, which would clearly be useful, it is precisely because such comparisons would be useful that the choice of a formal language ought to depend upon what the *other* options for *k* are. In short, even more informal metaphysics (such as the above) should precede the formal mathematics of #.

To sum up, there could be physical continua, for all we really know, and lines may well be full of points. One coherent (if occasionally counter-intuitive) metaphysical possibility is of course $k = 2\hat{}\aleph_0$, but another may well be $k = \#$. Since lines of *k* points may have any length, it is hardly counter-intuitive that $0 \cdot k$ should equal 0/0. And continuity is a relatively simple notion, so the informal coherence of the elegant hypothesis $k = \#$ (demonstrated above) makes it relatively plausible that the essence of continuity is captured by $0 \cdot k \neq 0$. Whether or not points exist, there are therefore strong indications that we would be wise to take a mereological approach to the metaphysics of spaces and classes. Furthermore, the introduction of the concept of # may also contribute, indirectly, to the plausibility of the natural numbers forming a (non-constructive kind of) potential infinitude, which is a concept usually associated with lines *not* being full of points.

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