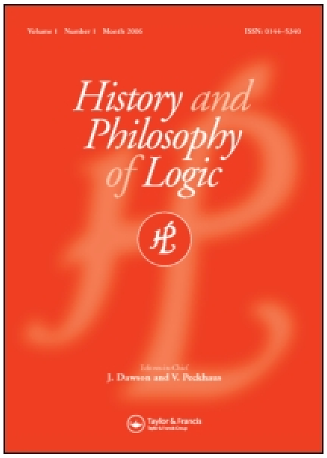

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Categoricity

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Categoricity

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After a short preface, the first of the three sections of this paper is devoted to historical and philosophic aspects of categoricity. The second section is a self-contained exposition, including detailed definitions, of a proof that every mathematical system whose domain is the closure of its set of distinguished individuals under its distinguished functions is categorically characterized by its induction principle together with its true atoms (atomic sentences and negations of atomic sentences). The third section deals with applications especially those involving the distinction between characterizing a system and axiomatizing the truths of a system.

0. PREFACE

Aside from the analysis of the logical structure of mathematical propositions and the formalization of mathematical reasoning, perhaps the most striking achievement of pre-Gödelian mathematical logic was the categorical characterization of traditional mathematical systems (Euclidean geometry, the natural numbers, the rational numbers, etc.) viewed as interpretations of formal languages. Section 1 treats historical and philosophical aspects of the notion of categoricity (and, thus, also isomorphism) within the broader context of a discussion of characterization of a mathematical system by means of a set of sentences which hold in it. Section 2 considers mathematical systems which are categorically characterized by means of one (second order) induction principle supplemented only with atomic sentences and negations of atomic sentences (i.e., using no properly first order sentences). In particular it is shown that a system can be categorically characterized by such means provided only that it is inductive, i.e. that its domain is the closure of a finite number of its individuals under a finite number of its operations (any number of relations may also be present). This theorem leads immediately to a very weak test of categoricity. Section 3 shows that the test has useful applications in axiomatizing inductive systems. The weakness of the test, especially when viewed in relation to examples given, suggests that the importance of categoricity may have been exaggerated and that the relationship between characterizing a model and axiomatizing its truths, is not as close as had been thought. In particular, the test is used to establish a categorical characterization of the natural number system from which it can not be deduced that zero is not a successor. Other examples of deductively very we. k theories which are nevertheless categorical are also given.

1. CATEGORICAL CHARACTERIZATIONS OF MATHEMATICAL SYSTEMS

By the turn of the century mathematicians had distinguished mathematical systems from axiomatizations. A *mathematical system* was thought of, in effect, as a class of mathematical objects together with a finite family of distinguished relations, functions and elements.[1] An *axiomatization* of a system was often thought of as a set of *propositions* about the system. Some mathematicians took the notion of a proposition about a system so literally that they could not conceive of a reinterpretation of a set of axioms (Frege *1906*, 79).

At this time, one must recall, there was no such thing as a formal grammar. Nevertheless, certain mathematicians (e.g. Hilbert *1899*; Veblen *1904*) conceived of the axioms for a mathematical system as propositional forms interpreted in the given system but admitting of other interpretations as well.[2]

Today we *can* speak of mathematical systems without reference to particular formal languages interpreted in them, but we often do not. For example, when we speak of the system of natural numbers we often mean the intended interpretation of one of the formal languages commonly used for number theory. And we have to be reminded of the fact that we can refer to the system of natural numbers in itself, so to speak. We also have to be reminded of the fact that an interpretation of a formal language is not merely a mathematical system but it also involves (among other things) a precise specification of which formal symbols get assigned to which distinguished relations, functions, and elements. In general, a set of formal axioms can be interpreted in a given system in more than one way. Thus, strictly speaking, a

1. The term 'mathematical system' was and is widely used in just this sense (compare Huntington (*1917*, 8) and Birkhoff and MacLane (*1944*, *1953*)). From a philosophical and historical point of view it is unfortunate that the term 'mathematical structure' is coming to be used as a synonym for 'mathematical system'. In the earlier useage, which we follow here, two mathematical system having totally distinct elements can have the same structure. Thus in this sense a structure is not a mathematical system, rather a structure is a 'property' that can be shared by individual mathematical systems. At any rate a structure is a higher order entity. The relation between a given structure and a system having that structure is analogous to the relation between a quality and an object having that quality. For mathematical purposes it would be possible to 'identify' a structure with the class of mathematical systems having that structure, but such 'identification' may tend to distort one's conceptual grasp of the ideas involved.

A referee suggested that some readers could confuse 'mathematical system' in the above sense with 'axiom system' in the sense of an axiomatization. This would be analogous to confusing an event with a description of the event or to confusing the set of solutions to an equation with the equation. All three cases confuse subject-matter with discourse 'about' that subject-matter.

2. Resnik (*1974*, esp. pp. 390–392) discusses this particular aspect of Hilbert's thought as reflected in the so-called 'Frege–Hilbert controversy'. The interpretation of Hilbert advanced here is in full agreement with Resnik.

set of formal axioms does not delimit a class of mathematical systems but rather it delimits a class of interpretations of its language. (An exact definition of 'interpretation' is given in sub-section 2.2 below.)

In the rest of this paper certain confusions can be avoided by keeping the above distinctions in mind. Especially important is the fact that, strictly speaking, a set of formal sentences true in a given interpretation should be regarded as an axiomatization of *the interpretation* rather than an axiomatization of *the underlying mathematical system.*

Let K be a set of non-logical constants and let LK be a formal language having K as its set of primitives. Let i be an interpretation of LK and let $T(i)$ be the set of sentences of LK true in i. A complete axiomatization of i requires the choice of a subset A of $T(i)$ which logically implies the rest.

An axiomatization of a given interpretation provides a *description* of the interpretation. Without meaning to suggest that one can *uniquely* describe an interpretation by means of a set of sentences, Hilbert (e.g. in *1899*) permitted himself remarks to the effect that a set of sentences can '*define*' an interpretation. Frege's criticism of Hilbert shows that Frege *mis*understood Hilbert's remarks as implying the possibility of unique axiomatic characterizations (Frege *1899*, 6–10). But Hilbert's reply shows that Hilbert was fully aware of the impossibility of such characterizations. Hilbert (*1899L*, 13–14) wrote: '... each and every [satisfiable] theory can always be applied to infinitely many systems of basic elements'. (See also footnote 2 above.)

Even today one occasionally finds a passage which admits of the same misconstrual. For example, Kac and Ulam (*1968*, 171) write: 'The axioms are meant to describe simple properties of the objects under consideration; one hopes that in these properties the essence of the objects will be captured completely'.

Nevertheless, by the turn of the century, at least, it had become clear that truth in a formal language has nothing whatever to do with the 'essence' of the objects in an interpretation, but rather depends solely on the form of the interpretation or, as it is sometimes put, on the formal interrelations among the objects. The notion of isomorphism between two interpretations was adopted as a mathematical formulation of the idea of two interpretations having the same form.[3]

3. A mathematically precise definition of isomorphism is given in sub-section 2.3 below. It is important to realize that the concept comes into play in a context where the language is fixed and the interpretations are changed (not when the interpretation is fixed and the language is changed). This confusion is rather widespread in the informal parts of the literature of the recent past. For example, in introducing the proof that any two mathematical systems satisfying the integral domain postulates are isomorphic, Birkhoff and MacLane say that the postulates 'are true of the integers not only as expressed in the usual decimal notation; they are also true of the integers expressed in the binary, ternary or any other scale!' (*1944*, 37). Notice that the Birkhoff and MacLane remark is true and that it would still be true regardless of whether the postulate set in question were categorical. Their remark is totally beside the point and could only be made in this context by persons confusing change of system with change of notation.

One occasionally reads that if *two* interpretations are isomorphic they are 'identical except for the names' of the elements and relations (Fraleigh *1967*, 55) or that isomorphic interpretations 'differ only in the notation for their elements' (Birkhoff and MacLane *1953*, 33). Such remarks can be misleading because generally two isomorphic interpretations have *different* elements and relations. The whole point is that the two have the same 'form', and what sets of objects the two each involve, be they identical or different, is beside the point. (Compare footnote 3 above.)

For example, let *LK* be the algebraic language based on one binary operation symbol *. One familiar interpretation takes as domain the four so-called complex units, 1, −1, *i* and −*i* and as interpretation of * it takes multiplication. Another interpretation takes as its universe the four social classes of the Kariera society and as interpretation of * it takes the function which yields the class of a child when applied to the class of its father and the class of its mother. Levi-Strauss has discovered that such a function exists and, indeed, that the interpretation just mentioned is isomorphic to the interpretation in the complex units (compare Barbut *1966*). Surely one would not want to say in this context that a complex unit and a social class differ only in notation. The form which is common to these two interpretations is, of course, the so-called 'the Klein group'.

The insight that truth in a formal language depends solely on the form of the interpretation (and is independent of content or matter) is partly reflected in the fact that isomorphic interpretations have the same set of truths, i.e. if i and j are isomorphic then $T(i) = T(j)$. Moreover, it has been clear at least since the turn of the century (Hilbert *1899L*, 14) that given any interpretation i, there are other interpretations isomorphic with i but having no content in common with i. The existence of such isomorphic 'images' implies, of course, the impossibility of uniquely characterizing an interpretation by means of a set of sentences in a formal language.[4] Accordingly, it is sometimes said that the best possible characterization of an interpretation would be a 'characterization up to isomorphism', where a set A of sentences is said to *characterize i up to isomorphism* if every interpretation which satisfies A is isomorphic to i.

Thus instead of an ideal of exact characterization, mathematicians adopted the ideal of characterization up to isomorphism, and terminology was introduced to indicate the property of sets of sentences which characterize up to isomorphism the interpretations they characterize (Veblen *1904*, 346). More precisely, a set of sentences A is said to be *categorical* if every two interpreta-

4. As late as 1944 some writers were still not clear about this point. For example, Birkhoff and MacLane *1944* are clear that, within the class of formal languages that they were using, no postulate set could distinguish between isomorphic systems; but they did not see that this is a feature of all classes of formal languages. They wrote: '... no postulate system for the integers (or the type which we have used) could distinguish between two isomorphic systems' (Birkhoff and MacLane *1944*, 37).

tions which satisfy A are isomorphic.[5] (Because of the peculiarity of 'every', a contradictory set A is vacuously categorical.) Incidentally, Veblen noted that the term 'categorical' was suggested by the philosopher John Dewey (*ibid.*).

By the middle of the first quarter of this century categorical characterizations of several important interpretations had been established. Many of these results are reported in Huntington *1905*. It became common to 'identify' the intended interpretation of a formal language used to discuss a standard mathematical system with the system itself. For example, the phrase 'the system of natural numbers' sometimes indicates the intended interpretation of a language LK, where K is a set of 'arithmetic primitive symbols'. Using this terminology it can be said that the following systems had been categorically characterized: the natural numbers, the integers, the rationals, the reals, the complex numbers, and Euclidean space.[6]

Investigation of formal languages led to the further insight that whether a categorical characterization is possible depends not only on the form of the interpretation in question but also on the *logical* devices (variables and logical constants) available in the language chosen. For example, if LK is a first-order

5. Calvin Jongsma (University of Toronto) pointed out that some early mathematicians and logicians understood Veblen to be applying the term 'categorical' to systems that would now be called 'semantically complete' or, to use Church's phrase, 'complete as to consequences', as opposed to deductively complete (Church *1956*, 329; compare Skolem *1928*, 523). Categoricity, of course, implies semantic completeness but the converse does not in general hold, as can be seen from Skolem's paper (*ibid.*). Later I noticed that Birkhoff and MacLane treat the categoricity of the axiom set for integral domains in a section called 'Completeness of the postulates for the integers' (*1944*, 36). It is worth noting that, in that section, there is not a word about completeness in either the semantic or deductive senses. Incidentally, the mathematical use of the term 'categorical' is certainly due to Veblen (*1904*, 346). However, it is not at all clear that Veblen uses it in its modern sense. In fact, he seems to be using 'categorical' to mean 'semantically complete'. The earliest use of 'categorical' in the modern sense is no later than Young (*1911*, 49).

6. The fact that categorical characterizations of the traditional mathematical systems were self-consciously obtained by early mathematical logic suggests that the discovery of such characterizations may have been a stated goal of the field (see footnote 7 below). Developments leading up to research aimed at categoricity results are not well-known. An idea similar to isomorphism is attributed to Galois (1811–1832) and the term is found in an 1870 paper of Camille Jordan (Kline *1972*, 765, 767). Related ideas are found in Cantor's *Grundlagen* of 1883 (Jourdain *1915*, 76, 112) and in Dedekind (*1887*, 93). Cantor and Dedekind each have theorems which can easily be applied to yield categoricity results, but neither seemed to have the idea of characterizing a class of systems by means of sentences (or propositional functions). The earliest genuine categoricity result I know of is due to Huntington (*1902*). Kline finds that 'this notion [categoricity] was first clearly stated and used by . . . Huntington in a paper devoted to the real number system' (*1972*, 1014). Kline is referring to Huntington *1902*, which proves the categoricity of a set of '*axioms*' for 'absolute continuous magnitude'. It was alleged, falsely and without justification, by Young (*1911*, 154) that Hilbert's *Foundations of geometry* (*1899*) contains a proof that Hilbert's axiomatization of geometry is categorical. Ironically, Hilbert *1899* does not even show awareness of semantic completeness, despite Veblen's apparent comment (*1904*, 346) to the contrary.

language (i.e. one having only individual variables) without identity then *no* categorical characterizations are possible, and if *LK* is a first-order language with identity, then only finite interpretations can be categorically characterized.

Kreisel (*1965*, 148) points out that this limitation of first-order languages came as a surprise to logicians,[7] and he also makes the interesting observation that all finite interpretations are categorically characterizable in first order languages so that being finite and being first-order categorically characterizable are equivalent properties of interpretations. Many writers, including Kreisel (*ibid.*) and Montague (*1965*, 136), have noted that the many known categorical characterizations of the familiar classical systems all involve languages of second order, at least.

However, if one moves beyond a first-order language with identity by the smallest possible amount, i.e. by allowing *one* one-place predicate variable, then not only are some infinite interpretations categorically characterizable but many important infinite interpretations are so characterizable. For example, if mathematical induction is written

$$(Po \,\&\, \forall x(Px \supset Psx)) \supset \forall y\, Py$$

then the natural number system (relative to a primitive 0 for zero and *s* for successor) is categorically characterizable. Likewise, the integers, the reals and other important systems are also categorically characterizable using these 'slightly augmented first-order languages' (Montague *1965*).

To be more precise, define the formulas of the *slightly-augmented language with non-logical primitives K*, abbreviated '*saLK*', to be exactly the formulas of the first-order language with identity but based on $K + \{P\}$, where *P* is a one-placed predicate symbol not in *K*. The *sentences* of *saLK* are the formulas which lack free occurrences of the individual variables. The truth conditions for sentences of *saLK* are exactly these of the first-order sentences involving $K + \{P\}$ except that a sentence $S(P)$ involving *P* is true under an interpretation *i* iff it is satisfied by every assignment of a subset of the universe of *i* to *P*. Thus every sentence $S(P)$ is understood to be universally quantified with respect to *P* taken as a variable. Henceforth, *P* is called 'a one-placed predicate variable'. Note that $S(P)$ and $\sim S(P)$ are contraries, *not* contradictories.

Note that *saLK* is not equivalent to the language in which there is universal quantification of *P*, unless one requires (1) that only one occurrence of the universal quantification of *P* is allowed per sentence, and (2) that the single universal quantification must occur at the front. Thus for example, $\forall PS(P)$ is

7. How much of a surprise this was is another matter. Ellentuck says: 'One of the earliest goals of modern logic was to characterize familiar mathematical structures up to isomorphism ... in a first order language' (*1976*, 639). In the opinion of this author it is doubtful whether any logicians held this as a goal, at least for very long. By the time of Skolem *1920*, it was clear that no uncountable systems (e.g., geometry, the reals, or the complex numbers) could be categorically characterized in first order, and there appears to have been very little interest in first order languages before that.

equivalent to the *saLK* sentence $S(P)$, but $\sim\forall PS(P)$ is not in general equivalent to a *saLK* sentence. In particular, the existential predicate-variable quantifier is not definable in *saLK*.

If somehow required to classify a slightly-augmented language as first-order or as second-order, many mathematical logicians would probably not hesitate to call it second-order. However, if slightly augmented languages are so classified it must be noticed that they are weaker than the usual second-order languages (Enderton *1972*, 268–269) in four ways. In the first place, *saLK* has no function variables. Second, it has no n-ary predicate variables for n greater than one. Third, instead of infinitely many one-placed predicate variables *saLK* has but one. Fourth, instead of formulas with arbitrarily many universal predicate-variable quantifications arbitrarily deeply imbedded, *saLK* contains only formulas with at most one such quantification occurring at the front, i.e. not imbedded at all. Thus *saLK* would be an extremely weak second-order language. In fact, Church has implicitly classified *saLK* as an applied first-order language with identity (*1956*, §48).

In the opinion of the author, (1) classification of *saLK* as a first-order language would introduce confusion because there are many properties usually thought of as intrinsically first-order but which do not hold of *saLK*, and (2) classification of it as second-order would tend to mask its expressive weakness and its simplicity. It would seem best simply to refer to *saLK* by the name given above, or by the name 'slightly augmented first-order language'.

The rest of the paper treats categorical characterization in the context of slightly augmented first-order languages.[8] Section 2 establishes an extremely weak sufficient condition for categoricity which is nevertheless useful in constructing categorical sets of sentences. The main theorem is that an interpretation which satisfies an induction principle is categorically characterized by its induction principle together with its true atomic sentences and the negations of its false atomic sentences. In effect, we show that the form of an inductive interpretation is determined by its atoms. The proof of the theorem involves no reference to truth-functional combinations or to quantifications (except, of course, to those involved in induction principles).

Section 3 applies the result of section 2. The paper is intended to be largely self-contained. Moreover, since terminology in logic has not yet been

8. The idea of categoricity is attractive even to logicians who want to avoid quantification over 'higher-order objects'. For example, in order to save categoricity in contexts devoid of such quantificiation Ellentuck *1976* goes to infintary languages, and Grzegorczyk *1962* restricts the interpretations to what he calls 'constructive models'. Other writers 'weaken' the idea of being categorical to being 'categorical in a power'. An axiomatization A is said to be categorical in k, where k is a cardinal number or power, if any two models of A whose universes have cardinality k are isomorphic. For further discussion see Enderton (*1972*, 147). This writer, however, has no interest in avoiding quantification over high-order objects, something he regards as a fundamental aspect of mathematical language. The motivation for considering slightly augmented languages is to isolate an idea which could have served as the core idea in many known categoricity proofs.

standardized, it was thought worthwhile to repeat some rather elementary definitions. However, when this is done it is only done to the extent necessary for the immediate purposes at hand.

2. CATEGORICITY IN saLK

Sub-sections 2.1 and 2.2 below deal with 'grammatical' and semantic preliminaries. Since categoricity is a purely semantic concept having no *intrinsic* dependence on object-language deductions, no system of formal proofs is provided. If the reader wishes to have a system of formal proofs for *saLK* it is sufficient to take any standard system for first-order with identity, e.g. Mendelson (*1964*, 57, 75) and for the monadic predicate variable take the 'rule of substitution' (which amounts to regarding a sentence involving P as a scheme). Sub-section 2.3 repeats the standard, exact notions of isomorphism and categoricity. Sub-section 2.4 associates with each *interpretation* a 'bar interpretation'. Sub-section 2.5 proves the main theorem.

2.1. *Syntax for atoms and induction formulas*

Let K be a finite set of non-logical constants containing at least one individual constant and at least one function symbol. Besides these K can contain any number of individual constants and, for each $n \geqslant 1$, any number of n-ary functions symbols and any number of n-ary relation symbols. For each such K, TK is the set of *constant terms* of K, i.e. TK is the closure of the set of individual constants of K under the operations of attaching an n-ary function symbol f in K to a string $t_1 \ldots t_n$ of n constant terms (forming $ft_1 \ldots t_n$).

An *atomic sentence* of K is an *identity* '$t_1 = t_2$' or a string '$RT_1 \ldots t_n$' where R is an n-ary relation symbol in K and $t_1, \ldots,$ and t_n are all constant terms. The negation of an identity is written '$t_1 \neq t_2$', and the negation of '$Rt_1 \ldots t_n$' is written '$\sim Rt_1 \ldots t_n$'. Atomic sentences and their negations are called *atoms.*

Let P be the monadic predicate variable. If $K = \{0, s\}$ where 0 is an individual constant and s is a monadic function symbol, then the *induction formula for* K is the following:

$$I\{0, s\} \quad (P0 \,\&\, \forall x(Px \supset Psx)) \supset \forall y\, Py.$$

If $K = \{0, 1, s, +\}$ where 0 and s are as above, 1 is an individual constant, and $+$ is a binary function symbol then the *induction formula for* K is as follows:

$$I\{0, 1, s, +\} \quad ((P0 \,\&\, P1) \,\&\, \forall x_1 x_2 ((Px_1 \,\&\, Px_2) \supset (Psx_1 \,\&\, P + x_1 x_2))) \supset \forall y\, Py.$$

In general an induction principle has the form

$$I \quad (B \,\&\, \forall x_1 \ldots x_n (IH \supset IC)) \supset \forall y\, Py,$$

where B is the so-called 'basis', IH is the 'induction hypothesis', and IC is the 'induction conclusion'. Thus in order to define the induction formula *IK* for an arbitrary *K* it is sufficient to define each of the parts, *BK*, *IHK* and *ICK*. The *basis BK* is the conjunction of all the formulas *Pc*, where *c* is an individual constant of *K*. Let *m* be the maximum of the degrees ('arities') of the function symbols in *K*. Then the *induction hypothesis IHK* is the conjunction $Px_1 \,\&\, \ldots \,\&\, Px_m$. For each *n*-ary function symbol *f* in *K*, form $Pfx_1 \ldots x_n$. The conjunction of all such formulas involves only the *m* variables $x_1, \ldots, x_m$ and is called the *induction conclusion, ICK*. Thus *IK*, the *induction formula* for *K*, is the following:

$$IK \quad (BK \,\&\, \forall x_1 \ldots x_m (IHK \supset ICK)) \supset \forall y\, Py.$$

The fact that *IK* is not uniquely determined is not important.

2.2. *Semantics for atoms and induction formulas*

An *interpretation i of LK* is an ordered pair $\langle u, d\rangle$ where *u* is a non-empty set and *d* is a function defined on *K* and such that *dc* is in *u* if *c* is an individual constant, *df* is an *n*-ary function defined on *u* and taking values in *u* if *f* is an *n*-ary function symbol, and *dR* is set of *n*-tuples of members of *u* if *R* is an *n*-ary relation symbol.[9] As usual, the *denotation d^it of a term t under an interpretation i* is defined on *TK* as follows:

$$d^i[c] = dc \quad \text{and} \quad d^i[ft_1 \ldots t_n] = (df)d^i[t_1] \ldots d^i[t_n],$$

i.e. the denotation of an individual constant is its interpretation and the denotation of a function symbol attached to terms is the interpretation of the function symbol applied to the denotations of the terms.

As a result of the way that *TK*, the set of terms, is defined it is obvious that d^i is defined on all of *TK*. The range of d^i is then the set of objects in *u* which are denoted by constant terms. For example, if $K = \{1, s\}$, *u* is taken to be the set of natural numbers including zero, *d*1 is taken to be the number one and *ds* is taken to be the successor function, the range of d^i is only the set of positive numbers; the range of d^i need not be all of *u*.

9. It is true that, in the strict sense of the term 'set', each element "occurs" only once. Thus when one speaks of 'the set of solutions to an algebraic equation' the word 'set' is not being used in the strict sense. Sometimes the word "family" is used to indicate a 'set' wherein an element can have multiple occurrences. If repetitions of the same element are distinguished by indices, one speaks of an 'indexed family' (Halmos *1960*, 34). Thus a system, in the sense of section 1, is a class of mathematical objects together with a finite family of distinguished elements, functions and relations. An interpretation can then be seen as a certain kind of system, namely as a system wherein the family is indexed by a set of symbolic characters (namely, by *K*). It is important, however, that the indexing respect semantico/syntactic distinctions, i.e. that elements are indexed by individual constants, that *n*-ary functions are indexed by *n*-ary function symbols and that *n*-ary relations are indexed by *n*-ary symbols. Bridge (*1977*, 6, 16) treats the topic explicitly in this way. For a treatment of semantic categories, see Tarski *1934*, 215–236.

Let d^iTK be the range of d^i. Notice that because of the way TK is defined, d^iTK is the closure of the set of objects in u denoted by individual constants under the functions denoted by function symbols, i.e. d^iTK is a subset of any subset of u containing the said objects and closed under the said functions. Where no confusion results 'd^i' is sometimes written 'd'.

As usual an identity '$t_2 = t_2$' is *true under i* if dt_1 is the same object as dt_2 (i.e. $dt_1 = dt_2$) and *false under i* if dt_1 and dt_2 are different (i.e. $dt_1 \neq dt_2$). '$Rt_1 \ldots t_n$' is *true under i* if the n-tuple of objects denoted by the terms is in the relation denoted by $R(\langle dt_1, dt_2, \ldots dt_n\rangle \in dR)$, and '$Rt_1t_2 \ldots t_n$' is *false under i*, otherwise. Negation, of course, reverses truth-values.

For a given interpretation $i = \langle u, d\rangle$ an *assignment aP* to the monadic predicate variable P is simply a subset of u. *When aP is assigned to P*, (1) BK is *true under i* if aP contains the objects denoted by individual constants; (2) $\forall x_1 \ldots x_m(IHK \supset ICK)$ is *true under i* if aP is closed under the functions denoted by function symbols; and (3) $\forall yPy$ is true if $aP = u$. More particularly, to say that the induction formula *IK is true of aP under i* is to say that *if aP* both contains the objects denoted by individual constants and is closed under the functions denoted by function symbols, *then aP* is u. And IK is *true under i* if IK is true of *each aP* under i. More particularly, IK is *true under i* iff every subset of u which contains the objects named by individual constants and which is closed under the functions denoted by function symbols *is u*. Since the range of d^i, d^iTK, is such a subset, if IK is true under i then $d^iTK = u$. Conversely, if $d^iTK = u$ then IK is true under i. Thus, to say that IK is true under i is to say nothing but that $d^iTK = u$. In other words, the induction formula 'says' that every object is denoted by some constant term. When induction holds in i, we say that i is *inductive.*

2.3. *Isomorphism and categoricity*

Let $i = \langle u, d\rangle$ and $j = \langle v, e\rangle$ be two interpretations of LK. Then i is said to be *isomorphic* to j if there is a one-one function h from u onto v which 'preserves the structure' in the sense that: if c is an individual constant then $h[dc] = ec$; if f is an n-ary function symbol then for every $b_1, \ldots, b_n$ in u,

$$h[(df)b_1 \ldots b_n] = (ef)[hb_1] \ldots [hb_n];$$

and if R is an n-ary predicte symbol then for all $b_1, \ldots, b_n$ in u,

$$u, \langle b_1, \ldots, b_n\rangle \in dR$$

if and only if

$$\langle [hb_1], \ldots, [hb_n]\rangle \in eR.$$

Let $K = \{0, s\}$ and take i and j as follows. The universes u and v are both the set I of integers, ds and es are both the successor function, $d0$ is zero and $e0$ is one hundred. The function $hx = x + 100$ is an isomorphism between i and j.

Here (and below) we exploit the set theoretic notion of a function being a set of ordered pairs by writing $i = \langle I, \langle 0, \text{zero}\rangle, \langle s, \text{successor}\rangle\rangle$ and $j = \langle I, \langle 0,100\rangle, \langle s, \text{successor}\rangle\rangle$.

If i is isomorphic to j then any sentence of *saLK* true in one is true in the other, i.e. $T(i) = T(j)$. The proof of this is straight-forward but it requires a set of definitions rather more complete than is otherwise required in this paper.

Let A and B be two sets of sentences. Then i is a *model* (or *true interpretation*) of A if $A \subset T(i)$ and A *logically implies* B if every model of A is a model of B. If A logically implies B then, for purposes of smooth expression, B is said to be a *logical consequence* of A. If A has no models, A is said to be *contradictory*. Because of the peculiarity of 'every', a contradictory set implies every set. If A implies B and $B = \{p\}$ then A is said to imply p.

A set A of sentences is *categorical* if all models of A are isomorphic to each other. Because of the peculiarity of 'all', contradictory sets are vacuously categorical. Notice that if A is categorical then for every sentence p not involving P, A implies p or A implies $\sim p$. It fails in the case of sentences involving P only because for them negation does not reverse truth-values: $S(P)$ and $\sim S(P)$ are contraries, not contradictories.

Examples. Let $K = \{0, s\}$. Let $A0$ be the set of sentences true in $iN = \langle N, d\rangle$ where N is the natural numbers, $d0$ is zero, and ds is the successor function. Let S^n0 indicate n occurrences of S followed by 0. Notice that the true atoms are simply the logical identities ($S^n0 = S^n0$) and the negations of the other identities ($S^n0 \neq S^m0$, for $n \neq m$). Let A be the Peano Postulates for zero and successor, i.e.

$$A = \{IK, \forall xy(sx = sy \supset x = y), \forall x(sx \neq 0)\}.$$

Reasoning which shows that A is a categorical characterization of iN is familiar (compare Birkhoff and MacLane *1953*, 54–56). Let

$$A1 = \{IK\}.$$

It is obvious that $A1$ is not categorical because it is satisfied by the unit model $i1 = \langle\{0\}, d\rangle$ where $d0$ is zero and ds is the identity function. More generally it is clear that for any K, IK and any set of positive atoms is satisfied by the unit model, and thus, if IK and a set of atoms is categorical, a negative atom must be present. Let

$$A2 = \{IK, S0 \neq 0, \ldots, S^n0 \neq 0, \ldots\}.$$

It is worth noting that $A2$ implies $\forall x(Sx \neq 0)$. (IK 'says' that every object is named by a term S^n0. The joint effect of the rest of the sentences in $A2$ is to say that if an object is named by a term its successor is not zero.) Nevertheless, it is clear that $A2$ is not categorical because it has as a model $i2 = \langle\{0, 1\}, d\rangle$ where $d0$ is zero and ds is the identity function. Let $A3$ be the result of adding to $A2$ the rest of the negative atoms, $S^n0 \neq S^m0$, for $n \neq m$ *and* $m \neq 0$:

$$A3 = \{IK\} + \{S^n0 \neq S^m0, n \neq m\}.$$

Consider a sentence $SS^n0 = SS^m0 \supset S^n0 = S^m0$. If $m = n$ then the sentence is logically true. If $m \neq n$ then, since the negations of the antecedents are atoms in $A3$, the sentence itself is implied by the atoms in $A3$. By the reasoning of the previous paragraph, then, $A3$ implies $\forall xy(sx = sy \supset x = y)$. Thus $A3$ implies all three of the Peano postulates and is thus categorical.

From the perspective of the next sub-section, the main feature of $A3$ is that it is *atom-complete* in the sense that for every atomic sentence p, either $A3$ implies p or $A3$ implies $\sim p$. The main theorem is that every atom-complete set containing induction is categorical.

2.4. *Bar cointerpretations*

The denotation function d^i maps TK into u. Thus d^i can be used to define an equivalence relation Ei on TK in the usual way, i.e. let $t_1 Eit_2$ iff $d^it_1 = d^it_2$. Let $\bar{t}^i$ be the equivalence class of t, and let $\overline{TK}^i$ be the set of equivalence classes. Let $\bar{d}^i$ be the function from $\overline{TK}^i$ into u such that $\bar{d}^i\bar{t}^i = d^it$. Notice that $\bar{d}^i$ is one–one from $\overline{TK}^i$ onto d^iTK.

Let $i = \langle u, d\rangle$ be inductive. The mapping $\bar{d}^i$ is therefore a one–one onto function from $\overline{TK}^i$, the set of equivalence classes of terms, to the universe of i. Now we define a denotation function ${}^ib$ *on* K in such a way that $\bar{d}^i$ is an isomorphism *from* $\{\overline{TK}^i, {}^ib\}$ to $i = \{u, d\}$. The interpretation $\{\overline{TK}^i, {}^ib\}$ thus induced by $\bar{d}^i$ is called the *bar cointerpretation.*

One *could*, of course, define $\langle\overline{TK}^i, {}^ib\rangle$ as 'the isomorphic image of i under the inverse of $\bar{d}^i$'. *However, the information that we need to highlight is brought out better by defining* $\langle\overline{TK}^i, {}^ib\rangle$ explicitly and *then* verifying that i is its isomorphic image under $\bar{d}^i$.

(1) The universe of $\langle\overline{TK}^i, {}^ib\rangle$ is, of course, $\overline{TK}^i$. The interpreting function ${}^ib$ is defined as follows. (2) ${}^ibc = \bar{c}^i$. Now notice that $\overline{ft_1 \ldots t_n}^i = \overline{ft_{n+1} \ldots t_{2n}}^i$ is implied by the following taken together $\bar{t}_1^i = \bar{t}_{n+1}^i, \ldots, \bar{t}_n^i = \bar{t}_{2n}^i$. This means that the equivalence class of a term $ft_1 \ldots t_n$ is a "function" of the equivalence classes of its components $t_1, \ldots, t_n$. (3) Thus we can define ${}^ibf$ as follows:

$${}^ibf[\bar{t}_1^i, \ldots, \bar{t}_n^i] = \overline{ft_1 \ldots t_n}^i.$$

Finally notice that

$$\bar{t}_1^i = \bar{t}_{n+1}^i, \ldots, \bar{t}_n^i = \bar{t}_{2n}^i,$$

taken together, imply that '$Rt_1 \ldots t_n$' and '$Rt_{n+1} \ldots t_{2n}$' have the same truth-value in i. (4) Thus we can define the relation ${}^ibR$ to hold of $\{\bar{t}_1^i, \ldots, \bar{t}_n^i\}$ iff '$Rt_1 \ldots t_n$' is true in i.

To check that $\bar{d}^i$ is an isomorphism, one need only check:

(1) that $\bar{d}^i$ is one–one and onto from $\overline{TK}^i$ to u,
(2) that $\bar{d}^i\{^ibc\} = dc$,
(3) that $\bar{d}^i\{(^ibf)(\bar{t}_1^i, \ldots, \bar{t}_n^i)\} = (df)(\bar{d}^i\bar{t}_1^i, \ldots, \bar{d}^i\bar{t}_n^i)$, and
(4) that $\langle \bar{t}_1, \ldots, \bar{t}_n \rangle \in {}^ibR$ iff $\langle \bar{d}^i\bar{t}_1^i, \ldots, \bar{d}^i\bar{t}_n^i \rangle \in dR$.

The above reasoning establishes the following:

Lemma. Each inductive interpretation is isomorphic to its bar cointerpretation.

Now we establish the main lemma of the paper.

Lemma. Two inductive interpretations which satisfy the same atoms have the same bar cointerpretation.

Proof: Let $\langle \overline{TK}^i, {}^ib \rangle$ and $\langle \overline{TK}^j, {}^jb \rangle$ be the two bar cointerpretations. To show: (1) $\overline{TK}^i = \overline{TK}^j$, (2) ${}^ibc = {}^jbc$, (3) ${}^ibf = {}^jbf$ and (4) ${}^ibR = {}^jbR$.

To see (1) notice that t_1Eit_2 holds iff '$t_1 = t_2$' is true in i. By hypothesis the latter holds iff '$t_1 = t_2$' is true in j. Again the latter part holds iff t_1Ejt_2. Therefore $\overline{TK}^i = \overline{TK}^j$.

It follows then that $\bar{c}^i = \bar{c}^j$. Thus ${}^ibc = {}^jbc$. (2) is established.

It also follows that for all t,. $\bar{t}^i = \bar{t}^j$, and, in particular, that $\overline{ft_1 \ldots t_n}^i = \overline{ft_1 \ldots t_n}^j$. Thus ${}^ibf = {}^jbf$, (3) is established.

To see (4) notice first that $\langle \bar{t}_1^i, \ldots, \bar{t}_n^i \rangle \in {}^ibR$ iff '$Rt_1 \ldots t_n$' is true in i. By hypothesis the latter holds iff '$Rt_1 \ldots t_n$' is true in j. Again using the definition of bar interpretation, now for j, it follows that '$Rt_1 \ldots t_n$' is true in j iff

$$\langle \bar{t}_1^j, \ldots, \bar{t}_n^j \rangle \in {}^jbR.$$

But since

$$\overline{TK}^i = \overline{TK}^j, \bar{t}^i = \bar{t}^j,$$

so

$${}^ibR = {}^jbR. \qquad \text{Q.E.D.}$$

2.5. *Main theorem*

Any atom-complete set of sentences which includes induction is categorical.

Proof: Let S be such a set of sentences. If S is not satisfiable then S is vacuously categorical.

Assume that S is satisfiable. Let i and j be models of S. Since S is atom-complete, i and j satisfy the same atoms. Since S includes induction, i and j are inductive. By the lemmas, i and j are isomorphic. Q.E.D.

In order to put this theorem in perspective one can note that its import is simply the following. The set of true atoms of an inductive interpretation, taken together with induction, characterize the interpretation up to isomorphism. Thus *if* one is characterizing an inductive interpretation, and *if* one can tell that the axioms already set down are sufficient to imply induction all of the true atomic sentences and the negations of the false ones, *then* the goal of categoricity is achieved.

It is not hard to see that the same theorem holds in all stronger languages, so it holds for second order languages and for higher order languages. Thus one might wonder whether *saLK* can be weakened without losing the theorem. The immediate answer is affirmative because in the entire *proof* not a word was said about any sentences besides (constant) atomic sentences, their negations, and induction. Thus the result holds for *inductive atomic languages which are defined as follows. Let K* be a finite set of non-logical constants as above. The logical constants of the *inductive atomic language based on K, iaLK* are $=$, $\sim$ and a new symbol 'I' to be explained presently. The sentences are all of the atomic sentences of *saLK*, their negations, and I. The interpretations are the same as for *saLK* and the truth conditions for the atoms are the same. But I is true in i iff i is inductive.

The question also arises whether the condition of atom-completeness plus induction can be weakened without losing categoricity. It is clear that one cannot 'throw out' the negations of the atomic sentences because the example A2 in sub-section 2.3 above shows that the true atomic sentences of an interpretation (plus induction) do not characterize the interpretation up to isomorphism. One can also easily show that if S contains induction and is categorical then S is equivalent to an atom-complete set. Thus the above theorem is the strongest possible in the sense that atom completeness is the weakest possible condition sufficient to guarantee that a set of sentences containing induction is categorical.

3. APPLICATIONS

3.1. *Repetition theory*

Imagine that one is dealing with 'sets' of 'repeatable' objects where 'multiplicities' are counted. For example [3, 3] is the 'set of roots of $x^2 - 6x + 9 = 0$' but [3] is 'the set of roots of $x - 3 = 0$'. Such 'sets' are called iterates (or heaps or multiplicities: Hailperin *1976*, 88). At any rate if I is a set of objects being repeated then with each iterate, r, one can associate a unique function f from I into N such that for each repeatable object x, fx is the number of times x is repeated in r. The functions f are called *repetitions* of I.[10]

We consider the case where $I = \{a, b\}$. Let u be the set of repetitions of I, i.e. the set of functions from $\{a, b\}$ into N. Let 0, the null repetition, be the function:

10. Compare footnote 9 above.

$fa = 0, fb = 0$. The a-successor function s_a is the function which 'jacks up' the a-component of f by one, i.e. $s_a f(a) = f(a) + 1$ and $s_a f(b) = f(b)$. Likewise s_b is the b-successor function. Thus we are considering an interpretation $i = \langle u, d \rangle$ of $saLK$ with $K = \{0, s_a, s_b\}$.

A little thought suffices to see that the following axioms are true:

A1 $\forall x(s_a x \neq 0 \;\&\; s_b x \neq 0)$,

A2 $\forall xy((s_a x = s_a y \supset x = y) \;\&\; (s_b x = s_b y \supset x = y))$,

A3 $\forall x(s_a s_b x = s_b s_a x)$,

A4 $\forall x(s_a x \neq s_b x)$,

A5 IK.

By looking at what the constant terms denote one can see that an identity '$t_1 = t_2$' is true in i iff the repetition of $\{s_a, s_b\}$ in t_1 is the same as the repetition of $\{s_a, s_b\}$ in t_2. For example the repetition of $\{s_a, s_b\}$ in the terms

$$s_a s_b s_a s_a s_b 0, \quad s_b s_a s_b s_a s_a 0, \quad s_b s_b s_a s_a s_a 0$$

is

$$\{\langle s_a, 3\rangle, \langle s_b, 2\rangle\}.$$

Using this idea one can show that the above axiom set, $A = \{$A1, A2, A3, A4, A5$\}$, implies each true identity and the negation of each false identity. Thus it implies an atom-complete set including induction, which is categorical by the theorem, and so it is categorical itself.

3.2. *Other possibilities*

It is already clear that various versions of Peano arithmetic (or number theory) admit of this treatment. Kleene (*1952*, 246) has discussed an infinite class of interpretations which he calls 'generalized arithmetics'. It is certainly possible to categorically characterize any one of them and perhaps to give a general formula for treating all of them – of course, using the above method. There is an infinite class of theories of strings based on Tarski *1934*, 172 and categorically axiomatized in Corcoran, Frank and Maloney *1974*. In the same work one finds another infinite class of theories dealing differently with strings based on an idea of Hermes. Both of these classes admit of treatment by this method. In addition it is possible to deal with finitely branching trees and hereditarily finite sets in this way.

For purposes of discussion assume that K has no relation symbols. This restriction does not matter in principle but it holds in all the familiar examples which will come to mind. Define an *equation in K* to be any constant identity (as above) or any sentence of the form $\forall x_1 \ldots x_n t_1 = t_2$ where t_1 and t_2 are terms in K and all variables occurring in t_1 or t_2 or both are among $x_1 \ldots x_n$. Let A be a set of equations. A model of A is called an *A-algebra*. Let $I(A)$ be the set of constant identities implied by A. Let $NI(A)$ be the set of negations of the constant identities *not implied* by A. If i is a model of $A + IK + NI(A)$ then i is what has been called a 'free' A-algebra. For example if $A = \{\forall xyz(x + (y + z) = (x + y) + z)\}$ and $K = \{a_1, \ldots, a_n, +\}$ then the models of $A + IK + NI(A)$ are the free semigroups on n generators.

It is clear that the above methods constitute *one* approach to getting categorical axiomatizations for the theories of free A-algebras. If A or $NI(A)$ is not recursively enumerable then the above approach may not work at all, even given a maximum of ingenuity.

3.3. *Strong induction*

The induction principle IK treated above is the weakest possible induction principle for LK, a language whose set of non-logical constants is K. We saw that IK is true in i iff every object of u is denoted by a term in TK. The weak induction principles IK are essentially unique, but for each K there are many stronger induction principles and, in fact, there are generally several which are maximally strong. To consider the first class of stronger induction principles for LK consider a proper subset $K1$ of K which still contains at least one individual constant and one function symbol. The induction principle $IK1$ used in LK is stronger than IK because it holds in i iff every object in u is denoted by a term in $TK1$ (a proper subset of TK). Clearly, $IK1$ implies IK but not vice versa. Thus $IK1$ is 'stronger' than IK.

For the general definition of a strong induction principle, let T be a proper subset of TK. Let $S(K)$ be a sentence possibly involving the monadic predicate variable. If $S(K)$ has the truth condition that it is true in i *iff* every object in u is denoted by a term in T, then $S(K)$ is a *strong induction principle* for LK. For example take $K = \{1, +\}$. Then IK is

$$(P1 \,\&\, \forall xy(Px \,\&\, Py \supset P(x + y)) \supset \forall zPz.$$

The following, however, is a strong induction principle:

$$(P1 \,\&\, \forall x(Px \supset P(x + 1)) \supset \forall zPz.$$

The truth condition for this sentence is that every object in u is denoted by a term of the following class: 1, (1 + 1), ((1 + 1) + 1), ..., which leaves out (1 + (1 + 1)), ((1 + 1) + (1 + 1)), etc. It is obvious that IK + associativity implies this strong induction principle.[11] At any rate, since every strong induc-

11. One of the referees wanted to know whether there are 'maximally strong' induction principles, i.e. whether there are strong induction principles which are not weaker than any others. One strengthens a strong induction principle by restricting the class T of terms which it 'forces to cover the universe'. For example, the principle cited above can be strengthened by replacing the (non-constant) term '$x + 1$' by '$(x + 1) + 1$'. The most restrictive class of terms is, of course, the null set which corresponds to a contradictory 'induction principle', e.g. $\forall yPy$. Short of this the strongest induction principles would 'force a unit set to cover the universe', e.g. $Pc \supset \forall yPy$.

tion principle implies *IK*, the above results hold when a strong induction principle is substituted for the induction principle.

3.4. *'Negative' applications*

In this sub-section let *LK* be any second-order language with identity and let *DK* be any sound and recursive system of deductions for *LK*. For example, let *DK* be the system of Church *1956* suitably extended with rules and axioms dealing with function symbols.

In this sub-section it is important to recall the distinction between (1) characterizing an interpretation *i* by means of a subset of *T*(*i*) (the truths of *i*) and (2) axiomatizing the truths of *i*. The point of characterizing *i* is descriptive and criterial; one aims at distinguishing *i* from other interpretations. The point of axiomatizing is to form the basis for a deductive development of the truths of *i*. From the standpoint of characterization the best that can be done (when it can be done) is a categorical characterization. With a categorical characterization an interpretation is distinguished from every other interpretation from which it can be distinguished by formal means. From the standpoint of axiomatization it is clear that the best that can be done (when it can be done) is a deductively complete axiomatization, i.e. a recursive subset of *T*(*i*) from which every member of *T*(*i*) is deducible by a (finite) deduction. It is obvious that the set of theorems deducible from a set of axioms is necessarily a recursively enumerable subset of the truths no matter which sound, recursive system of deductions is used but that, in general, the set of theorems is sensitive to choice of deductive system. Below we assume a fixed deductive system *DK*.

Some early postulate theories (e.g., Veblen *1904*, 346) were clear about the *conceptual* distinction between characterization and axiomatization *and* about the possibility of an axiomatically inadequate categorical characterization at least to the extent of explicitly mentioning the possibility that a categorical characterization need not be a (deductively) complete axiomatization. This possibility, of course, entails the possibility of 'logically' incomplete underlying logics (wherein semantic consequences of a given set of axioms are not deducible as theorems).

At that time, however, there was no suspicion of the idea of recursiveness, nor, *a fortiori*, of the relevance of recursiveness and recursive enumerability to problems of axiomatizability. Now we can see that if the set of truths of an interpretation is not recursively enumerable then there is no way to give a complete axiomatization even if the logic is complete. It follows immediately from the Gödel incompleteness result that a (recursive) set of sentences which provides a categorical characterization need not provide a complete axiomatization. Moreover, in such cases, it follows that there are infinitely many other categorical characterizations each of which provides a better axiomatization in the sense of providing the basis for the deduction of additional theorems not deducible from the first characterization.

Separate from the recursiveness considerations which lead to mismatches between characterization and axiomatization are the so-called 'compactness' considerations which lead to additional mismatches (compare Corcoran *1972*, 37*ff.*). Since every deduction is finite and therefore involves only finitely many axioms, no consequence of an infinite axiom set which depends on infinitely many of the axioms can be deducible from those axioms. It is compactness considerations rather than recursiveness considerations which are operative in the rest of this discussion.

It might be thought that *any* categorical characterization of an interpretation provides the basis (given a suitable deductive system) for the deduction of the 'obvious' truths of the interpretation. That is, one might expect that any truth not deducible from a categorical characterization must be a 'pathological' or 'complicated' proposition such as a so-called Gödel sentence or a statement of consistency. Admittedly this point has not been discussed much in the literature (see Paris *1978*). Nevertheless, the test of categoricity given above permits the establishment of categorical characterizations from which the most elementary general truths are not deducible, no matter what sound deductive system is used (regardless of the criterion of recursiveness).

Take $K = \{0, s\}$ and take S as the set of all true arithmetic identities ($s^n 0 = s^n 0$) and the negations of all of the false ones. By the above theorem $S + IK$ is a categorical characterization of $i = \langle N, \langle 0, \text{zero}\rangle, \langle s, \text{successor}\rangle\rangle$. Thus $S + IK$ implies $\forall x(sx \neq 0)$. However, it is impossible to deduce $p = \forall x(sx \neq 0)$ from $S + IK$ using DK or any other sound deductive system $DK1$ because if, say, $p_1 \ldots p_n p$ is a deduction of p from $S + IK$ and $DK1$ is sound then p is implied by the finite number of premises in $p_1 \ldots p_n$. But it is easy to see that no finite number of sentences in $S + IK$ implies $\forall x(sx \neq 0)$.

Examples of this sort can be multiplied. Take $K = \{0, s, +\}$. Take $i = \langle N, \langle 0, \text{zero}\rangle, \langle s, \text{successor}\rangle, \langle +, \text{addition}\rangle\rangle$.

For S take

$$\forall x(sx \neq 0), \forall xy(sx = sy \supset x = y),$$

and the true identities ($s^n 0 + s^m 0 = s^{n+m} 0$) and the negations of the false ones. $S + IK$ is categorical but it is impossible to deduce $\forall xy((x + y) = (y + x))$ from $S + IK$ using a sound deductive system.

These examples are but other illustrations of the vast difference between characterizing an interpretation and axiomatizing its set of truths. The examples point to the conclusion that the connection between the two is weak. In particular, it is now clear that a 'best possible' characterization can be a very poor axiomatization.[12] The class of categorical characterizations of a given inductive system includes many which are virtually useless as axiomatizations.

12. Since semantic completeness is implied by but does not imply categoricity, it follows that semantic completeness is not sufficient for an axiomatization to be 'good'. In fact, as far as axiomatization is concerned, semantic completeness seems to be beside the point unless supplemented by other conditions formulated in accordance with goals arising in particular cases. These observations are due to George Weaver.

It is clear from a survey of the relevant literature not only that the early postulate theorists were unaware of the recursiveness considerations (as mentioned above) but also that they were unaware of compactness considerations as well. One can *not* automatically conclude, however, that they were misguided in using categoricity as an index of worth of an axiomatization. One must realize that the above counterexamples all involve *infinite* sets of axioms whereas the earlier logicians, occasionally explicitly (Veblen *1904*, 343), conceived of an axiomatization as inherently finite. And in the opinion of this writer, the philosophical wisdom of abandoning the finiteness condition should be questioned despite the undeniable advances that came as a result of considering the mathematical consequences of relaxing that condition.

3.5. *Heuristics*

The above test of categoricity requires, for its application to a *given set of axioms* for an inductive system, that one first establish that the axiom set implies each of the true atoms of the system. It is entirely possible that this preliminary step is more demanding in a given case than a straight-forward categoricity proof. However, *if* one is given *the system* (interpretation) alone and the problem is to find a manageable categorical set of axioms *then* the goal of deducibility of the true atoms is often an effective heuristic which leads to the discovery of the required axiom set.

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