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Counterarguments/Counterexamples  
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A *counterargument* for a given argument is an argument having all true premises, a false conclusion, and the same form as the given argument.

Consider the following one-premise argument whose premise and conclusion are about numbers in the sense of the non-negative integers, the so-called natural numbers beginning with zero. Zero is neither positive nor negative, of course, but all other numbers in this sense are positive

Argument 1

If zero is positive, then every number is positive.

Every number that is not zero is positive.

The premise and conclusion are both true: the premise is to be understood as a material conditional with a false antecedent and false consequent. Some people think that this argument is *valid*, i. e., that its conclusion follows logically from its premise, that the conclusion contains no new information, that the conclusion simply repeats some or all of the information in the premise. But they are mistaken.

By putting “one” for “zero” and “even” for “positive”, we get another argument in the same form.

Argument 2

If one is even, then every number is even.

Every number that is not one is even.

The premise is true for the same reason the other one was. But the conclusion is false: three is a *counterexample*. There are infinitely many other counterexamples for the conclusion. Every odd number except one is a counterexample for the proposition “Every number that is not one is even”. Of course, only one counterexample is necessary for a universal proposition to be false.

Since the premise of argument 2 is true but the conclusion false, argument 2 is invalid. The *principle of fact* is that every argument having all true premises and false conclusion is invalid. This is closely related to the fact that no false proposition follows logically from true propositions. Thus, argument 2 is invalid.

Moreover, argument 1 is in the same logical form as argument 2. The *principle of form* is that any two arguments in the same form are both valid or both invalid. It follows that that every argument in the same logical form as an invalid argument is invalid. Thus, argument 1 is invalid.

Argument 1, which seemed to some to be valid, is seen to be invalid by the production of a counterargument, i.e., by exhibition of another argument that is a counterargument for it. This method of proving an argument to be invalid is called the *counterargument method* or the *method of counterargument*.

By reference to the above-stated facts and the above stipulative definition of the word ‘counterargument’, we conclude that argument 2 is a counterargument for argument 1. Moreover, since every argument is in the same logical form as itself, argument 2 is a counterargument for itself. In fact, every argument having all true premises and false conclusion is a counterargument for itself and for every other argument in the same form—regardless of whether anyone ever noticed.

Above ‘counterargument’ was contextually defined and used in the *relational* expression ‘is a counterargument for’. But it is natural to use it in the *predicational* expression ‘is a counterargument’ as in the sentence ‘argument 2 is a counterargument’. Just as *a brother* is a person that is a brother *of someone*, *a counterargument* is an argument that is *a counterargument for some argument*. In fact, the three-word predicational expression ‘is a counterargument’ is often regarded as elliptical for the six-word predicational expression ‘is a counterargument for some argument’.

It is worth emphasizing that there are many invalid arguments not yet known to be invalid and that there are many arguments not yet known to be counterarguments. The method of counterargument requires a *known* counterargument—if it is to be used to produce knowledge of invalidity. Exhibiting an argument *not* known to be a counterargument proves nothing even if the argument exhibited happens to be a counterargument. This is a form of begging the question or *petitio principii*.

As another example, consider the following one-premise argument having a true premise and a true conclusion. This argument has appeared valid to many people.

Argument 3

If two is a prime number, then two is an even prime number.

Two is even.

However, by substituting “one” for “two”, a counterargument is produced.

Argument 4

If one is a prime number, then one is an even prime number.

One is even.

When applying the method of counterargument, it is advisable to follow the *maxim of minimal manipulation*: change the argument as little as necessary.

As a final example, consider the following two-premise argument having all true premises and a true conclusion.

Argument 5.

Every square is a polygon.

Every rectangle is a polygon.

Every square is a rectangle.

This argument is not likely to seem valid to anyone. Yet, someone might not be quite sure that it is invalid. The *method of counterargument* can often be used to achieve certainty that an argument is invalid. By substituting “triangle” for “rectangle”, argument 5 is transformed into argument 6, which therefore has the same form.

Argument 6

Every square is a polygon.

Every triangle is a polygon.

Every square is a triangle.

The two premises are evidently true, but every square is a counterexample for the conclusion, which is therefore false. Thus, argument 6 is a counterargument for argument 5. Argument 5 is thus established to be invalid.

One way of disproving a false *singular* statement that a certain argument is valid is to exhibit an argument *known* to be one of its counterarguments.

One way of disproving a false *universal* proposition is to exhibit an object *known* to be one of its counterexamples. This method of disproving a universal proposition is called the *counterexample method* or the *method of counterexample*. Exhibiting an object *not* known to be a counterexample proves nothing—even if the object exhibited happens to be a counterexample. This is a form of begging the question or *petitio principii*. Despite the fact that every false universal proposition has a counterexample, sometimes it has no known counterexamples and thus the method of counterexample cannot be used—other methods must be tried. For example, it is easy to see that the false universal proposition “every cancer has been detected” has no known counterexamples.

The expressions ‘is a counterargument for’ and ‘has the same logical form as’ are quite similar. Each expresses a *relation* between two arguments, i.e. a relation of an argument to an argument. It is important to resist the temptation to say ‘to another argument’ instead of ‘to an argument’ because both relations have *some* reflexivity: every argument is in the same logical form as itself and every argument with all true premises and a false conclusion is a counterargument for itself. Moreover, both are transitive: every argument in the same logical form as an argument in the same logical form as a given argument is

in the same logical form as the given argument and every argument that is a counterargument for an argument that is a counterargument for a given argument is a counterargument for the given argument. However, although the relation expressed by ‘is in the same logical form as’, *formal identity*, let us say, is symmetrical in the sense that every given argument is in the same logical form as any argument in the same logical form as the given argument, nevertheless the relation expressed by ‘is a counterargument for’, *counterargumentation*, let us say, is not symmetrical: argument 2 is a counterargument for argument 1 but argument 1 is not a counterargument for argument 2. In fact, argument 1 is not a counterargument for any argument: argument 1 does not have a false conclusion. But, counterargumentation is symmetrical with respect to arguments having all true premises and false conclusion: any argument having all true premises and false conclusion that is a counterargument for a given argument having all true premises and false conclusion has the given argument as a counterargument.

The expressions ‘is a counterexample for’ and ‘is a counterargument for’ are quite distinct. Whereas counterargumentation is a *homogeneous* relation in the sense that it relates members of a certain genus to members of *the same* genus—arguments to arguments; the relation expressed by ‘is a counterexample for’, *counterexemplification*, let us say, is a *heterogeneous* relation in the sense that it relates members of a certain genus to members of a usually different genus—numbers to propositions and squares to propositions in the two cases considered above.

Counterexemplification relates an object to a universal proposition whose subject genus includes the object and whose predicate does not apply to the object. In “every swan is white”, swan is the genus serving as subject and being white is the predicate. Every swan that is non-white is a counterexample for the proposition that every swan is white. Conversely, every counterexample for the proposition that every swan is white is a swan that is not white. In order for a universal proposition to be true it is necessary and sufficient for there to be no counterexamples for it. In order for a given object of one given genus to be a *counterexample* for a given proposition it is necessary and sufficient for the proposition to be a universal affirmative proposition having the given genus as its subject and for its predicate to *fail* to apply to the given object.

This stipulative definition of ‘counterexample’ is clear and natural. Moreover it covers all cases that are translated into natural language from symbolic logic where the only two quantifiers are the universal and existential affirmatives. However, it leaves out the universal negative.

It would seem natural to say that zero is a counterexample for the universal negative proposition “No number is square” even though this does not fit the definition above. The choices are to take the sentence ‘no number is square’ to express the universal affirmative proposition “Every number is non-square”, *or* to define an object to be a *virtual* counterexample to a proposition logically equivalent to one it is a real counterexample to, *or* to change the definition. If the term is to be used basically in connection with symbolic logic, the best thing seems to be to keep to the above definition and to admit that,

although the relation of zero to “No number is square” is like counterexemplification, it is not strictly speaking that exact relation.

Moreover, the above definition requires certain rephrasing. Four is a counterexample for “every even number is oblong”. The rephrasing is “every number is oblong if even”: four does not have the property “oblong if even” because it is even but not oblong.

Although counterexemplification is a heterogeneous relation there are cases in which it relates members of a certain genus to members of the same genus—propositions to propositions. “Every proposition is true” is a proposition that is *not* true and thus is a counterexample for “Every proposition is true”. The property of being a proposition that is a counterexample for itself, which belongs to many propositions, is fascinating but so far not important. “Every proposition is false” is a proposition that *is* false and thus is not a counterexample for “Every proposition is false”. However, it *is* a counterexample for “Every proposition is true”.

In certain contexts it is convenient to use alternative terminology. We can define an object to counterexemplify the propositions for which it is a counterexample: zero counterexemplifies “Every number is positive”.

The words ‘counterargument’ and ‘counterexample’ are made by attaching the prefix ‘counter’ to the common nouns ‘argument’ and ‘example’ thus raising the question of whether the opposite prefix ‘pro’ can be similarly used. Although *proargument* has not yet been coined, there is an established use for *proexample*.

Every swan that is black is a proexample for the existential proposition that some swan is black; and every proexample for that proposition is a swan that is black. In order for an existential proposition to be true it is necessary and sufficient for there to be at least one proexample for it. Accordingly, an existential proposition can be proved to be true by one *known* proexample, but it can never be proved to be false by examples. “Some cancer has not been detected” is a true existential that cannot be proved by the method of proexample. Similarly, a universal proposition can be proved to be false by one known counterexample, but it can never be proved to be true by examples.