# Chapter IV: Modal homotopy type theory

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## Contents



We now proceed to take the final step in our journey towards modal homotopy type theory. Analytic philosophers have put modal logic to extensive use in their exploration of so-called alethic, epistemic, doxastic, deontological, temporal and other modalities. These modalities typically qualify ways in which a proposition may be said to be true, as with, for example,

- It is necessarily the case that...
- It is known to be the case that...
- It is obligatory that...
- It will be the case that...

The relevant logical calculus is shaped by, and in turn shapes, reflection on what is imagined to be the philosophical content of these concepts. So philosophers might consider the differences, if any, between *physical*, *metaphysical* and *logical* necessity and possibility. Such discussions will often involve consideration of what has played the role of semantics for these logics, in particular, possible world semantics. For instance, "$P$ is necessarily the case" might be taken to mean "$P$ holds in every metaphysically possible world."

Kripke had given a warning concerning his introduction of possible worlds:

> The apparatus of possible worlds has (I hope) been very useful as far as the set-theoretic model-theory of quantified modal logic is concerned, but has encouraged philosophical pseudoproblems and misleading pictures. (Kripke 1980, p. 48)

All the same, many have decided to overlook his concerns, encouraged by the example of David Lewis (1986). While not every metaphysician has adopted such a concretely realist attitude towards these worlds as Lewis, numerous criticisms even of their instrumental usage have been launched from a range of positions, from the Wittgensteinian to the constructive empiricist. For Hacker (2010), Wittgenstein rightly understood necessary truths as merely the expressions of the normativity of rules, rules of grammar, of mathematics or whatever: "Necessary propositions exhibit neither factual or super-factual ('meta-physical') nor ideational (psychological) truths, but rather conceptual connections." (p. 20). Hacker maintains that there are no possible worlds beyond our actual world. For van Fraassen (2002), philosophers toying with these possible worlds are engaged in a palid imitation of science, where speculative claims are made without any form of check beyond the supposed coherence of the account offered: "The world we live in is a precious thing; the world of the philosophers is well lost for love of it." (p. 18).[1]

But even if one has sympathy for complaints of this kind, it is worth observing that computer scientists have latched onto modal logic and run with it. They do this both by taking up the modalities of the philosophers and putting them to different uses, for instance, temporal logic in model-checking and epistemic logic in multi-agent systems, *and* also by devising new modalities of their own. So, in regard to the latter, modalities have been defined to represent security levels and computational resources, and more generally, what they term *effects* and *coeffects*, relating to features of the context in which programs are executed beyond mere input-output pairings.

Computers scientists are also inventive technically. Where philosophers are still largely working exclusively with propositional modal logics (K, S4, S5, etc.), first-order extensions and Kripke models for semantics, computer scientists employ sub-structural logics, coalgebra, labelled transition systems, descriptive frames and bisimulations, where these topics are often given a category theoretic treatment.

With fewer tools at their disposal, the task of philosophers looking to engineer a useful form of first-order modal logic is not made easy by the lack of clear-cut conceptual constraints placed upon them. Kishida makes just this point in his contribution to *Categories for the working philosopher* (Landry 2017).

> Modal logicians have devoted the overwhelming majority of their inquiries to propositional modal logic and achieved a great advancement. In contrast, the subfield of quantified modal logic has been arguably much less successful. Philosophical logicians–most notably Carnap, Kripke, and David Lewis–have proposed semantics for quantified modal logic; but frameworks seem to keep ramifying rather than to converge. This is probably because building a system and

[1] Elsewhere he writes, "However plausibly the story begins, the golden road to philosophy which possible-world ontologies promise, leads nowhere." (van Fraassen 1989, p. 93)

> semantics of quantified modal logic involves too many choices of technical and conceptual parameters, and perhaps because the field is lacking in a good methodology for tackling these choices in a unifying manner. The remainder of this chapter illustrates how the essential use of category theory helps this situation, both mathematically and philosophically. (Kishida 2017, p. 192)

He then goes on to provide there a rich account of first-order modal logic. What I shall be working towards in this chapter, on the other hand, is the more ambitious target of a modal version of homotopy type theory, naturally still in a category theory-guided way.

We begin by tracing a path which can help us understand why modalities are often conceived in terms of variation over some collection, often construed by philosophers to be a set of worlds. One key finding, however, is that it is not this variation as such that matters, but rather the properties possessed by *adjoint* operators arising from such variation. These properties are encountered in a broader range of situations, and are now forming the basis of a powerful new approach to modal type theory, mentioned in the final section of this chapter. I believe that *computational trinitarianism* is pointing us very strongly in this direction.

As for the expected payoff, modal type theories are already being deployed in computer science. Furthermore, as we will see in the next chapter, they can also make good sense of developments in current geometry. *Linear* dependent type theories are now being developed, and are expected to provide a syntax for the forms of monoidal category used in quantum physics (Schreiber 2014).

As for philosophy, for analytic metaphysics, modal HoTT can help us to think through options on modal counterparts profitably. It also offers a range of novel lines of investigation, such as a way to think beyond modal propositions to elements of modal types, such as '*necessary* steps' and '*possible* outcomes'. But in view of the type theory-inferentialism relationship that has been noted at points through this book, we might also expect to make common cause with inferentialist perspectives on modality, in particular what Robert Brandom denotes as the *Kant-Sellars thesis about modality*:

> ...in being able to use nonmodal, empirical descriptive vocabulary, one already know how to do everything one needs to know how to do in order to deploy modal vocabulary, which according can be understood as making explicit structural features that are always already implicit in what one *does* in describing. (Brandom 2015, p. 143)

The uses of nonmodal vocabulary being made explicit by modal language include those where one describes how the state of something would be under certain kinds of variation of its current situation. As we shall now see, such variation plays a key role in forming modalities for type theory.

# 1 Modalities as monads

First let us consider the *alethic* modalities:

- It is necessarily the case that...
- It is possibly the case that...

Modal operators such as these are often considered to possess certain structural features irrespective of the nature of the associated proposition. For example, a standard reading of possibility admits the following implications:

- $p$ implies possibly $p$.
- Possibly possibly $p$ implies possibly $p$.

When symbolised as entailments $p \vdash \Diamond p$ and $\Diamond\Diamond p \vdash p$, a category theorist will immediately be put in mind of a key construction known as a *monad*. A similar analysis of necessity indicates the dual concept, a *comonad*.

To understand modalities from a type theoretic perspective we will need to make sense of monads and comonads and of how they arise within the context of category theory, but let us ease our way into this material by looking at a simple case. So consider the situation in which there is a type of dogs and a type of people, and where each dog is owned by precisely one person. Then the mapping

$$owner : Dog \to Person,$$

allows any property of people to be transported to a property of dogs, for instance,

$$Being\ French \mapsto Being\ owned\ by\ a\ French\ person.$$

We may say that taking properties of People as elements of $2^{People}$, the 'owner' function induces a mapping

$$owner^* : 2^{Person} \to 2^{Dog},$$

from being a kind of person to being owned by that kind of person.

Now we may order these domains of properties as partially order sets, where the ordering corresponds to inclusion, for instance, 'pug' is included in 'toy dog', and 'being French' is included in 'being European'. However we cannot necessarily invert this mapping to send a property of dogs, say, 'being a pug', to a property of people. Indeed, there is no reason that being a certain kind of dog should correspond to being owned by a certain kind of person.

We may try

$$Pug \mapsto Owning\ some\ pug,$$

but then the composite results in

$$Pug \mapsto Owning\ some\ pug \mapsto Owned\ by\ someone\ who\ owns\ a\ pug.$$

However, people may own more than one breed of dog. A poodle who shares a home with a pug will count as a dog owned by someone who owns a pug.

Since this doesn't work, let's try

$$Pug \mapsto Owning\ only\ pugs,$$

however this leads to

$$\textit{Pug} \mapsto \textit{Owning only pugs} \mapsto \textit{Owned by someone owning only pugs}.$$

Yet again this is not an inverse, not all pugs being owned by single breed owners. The pug sharing its home with a poodle is a case in point.

Despite the failure to find an inverse, in some sense my suggestions were the best approximations to one. Too loose an approximation first time; too constrictive an approximation second time. But even though only approximations to inverses, we can show that these mappings between dog and people properties may be used to make inferences. Say we have two people, $D$ and $P$, trying to establish a relationship between their fields, where $D$, the dog expert, can only think in terms of dogs and their properties, and $P$, the people expert, can only think in terms of people and their properties. Entailments for one expert may be translated to entailments for the other. For instance, when $P$ establishes that being French implies being European, $D$ can know that being owned by a French person entails being owned by a European. And when $D$ establishes that all pugs are toy dogs, then $P$ knows that an owner of only pugs is an owner of only toy dogs, and similarly for 'only' replaced by 'some'. So their inference patterns are in this sense reflected in the other's.

But we might ask for more. Even though there is no process to translate properties of concern to each other faithfully across the divide, the experts are still able to establish jointly a relationship between a dog property and a person property. Let's say that $D$ chooses the property *pug* and $P$ chooses *being French*. $D$ starts listing names of dogs which are pugs. $P$ doesn't understand anything about these matters, but what appears to him are the results of the *owner* function, that is, a series of people's names, which are in fact the owners of a pug. Now $P$ might check to see if each such owner is French. Then they will jointly assert or deny the following claims.

- $D$: All pugs are owned by a French person (whatever such a thing is).
- $P$: Any person who owns a pug (whatever that is) is French.

A similar analysis works for the other mapping:

- $D$: All dogs owned by a French person (whatever such a thing is) are pugs.
- $P$: A French person owns only (if any dog at all) pugs (whatever they are).

What we have here is an *adjoint triple*, $\sum_{owner} \dashv owner^* \dashv \prod_{owner}$, acting between the partially ordered set of dog properties and the partially ordered set of people properties. To make contact with the opening comments of this chapter on the properties of modal operators, we now need to look more closely at the result of composing pairs of adjoints. As we have seen, taking a predicate of dogs and applying the left adjoint followed by the *owner*$^*$ mapping yields an endomap on Dog properties:

$$\textit{Pug} \mapsto \textit{Owning some pug} \mapsto \textit{Owned by someone who owns a pug}.$$

Certainly, in a world where all dogs are owned by some person, if a dog is a pug, then it is owned by someone who owns a pug. However, the opposite condition does not hold since people may own more than one breed of dog. On the other hand, iteration of the construction is *idempotent* in the sense that 'being owned by someone who owns a dog which is owned by someone who owns a pug' is equivalent to 'being owned by someone who owns a pug'. Structurally the resemblance to possibility is clear. Being owned by someone who owns a pug is being construed as though you 'might have been a pug'. So 'owned by someone who owns' resembles the possibility operator, both being monads. This is the best $D$ can do in having $P$ help him to establish a consequence of possession of a dog property. $P$ has information concerning a weaker property. If you learn via $P$ that a particular dog is co-owned with a pug, there's still a chance it may be a pug.

Similarly we can form a *dual* version where we begin with the right adjoint. In our case, for example, we can tell a similar story about the comonad on the category of properties of dogs, generated by the right adjoint of $owner^*$:

$$Pug \mapsto Owning\ only\ pugs \mapsto Owned\ by\ someone\ owning\ only\ pugs.$$

Again, not all pugs are owned by single breed owners, so we have an implication from 'being owned by someone who owns only pugs' to 'being a pug', but not in the other direction. On the other hand, 'being owned by someone who owns only pugs' is equivalent to 'being owned by the owner of a dog owned by someone who owns only pugs.' Evidently, this operation is acting like necessity. If you and all your co-owned fellow dogs are pugs, then you're 'necessarily' a pug. Now, 'owned by someone who owns only' is seen to resemble necessity as a *comonad*. Again, $P$ is doing their best to provide information from which $D$ can look to conclude possession of a property. $D$ may learn via $P$ consequences of a specific dog being one of a group of dogs all of which are pugs. $P$ can provide information about a stronger property.

In sum, the these constructions applied to our pug case are:

- $\Diamond_{owner}$ : $Pug \mapsto Owning\ some\ pug \mapsto Owned\ by\ someone\ who\ owns\ a\ pug.$
- $\Box_{owner}$ : $Pug \mapsto Owning\ only\ pugs \mapsto Owned\ by\ someone\ owning\ only\ pugs$

We have equivalents of

- $P \to \Diamond P$
- $\Box P \to P.$

As we saw above, we also have equivalents of

- $\Diamond\Diamond P \to \Diamond P$
- $\Box P \to \Box\Box P.$

Now recall from the first chapter where I mentioned that partially ordered sets can be considered as categories enriched in truth values. Unsurprisingly then, adjoint triples are commonly encountered operating between ordinary categories, where they also generate monads and comonads. As we should expect with central category theoretic constructions, *monads* appear throughout mathematics. Let's take a look at how these arise. For instance, consider a set, $S$, along with the associated set, $M(S)$, of finite strings of elements of $S$. Then there are two natural mappings:

- $i : S \to M(S)$, which sends an element $s \in S$ to the string of length one, $\langle s \rangle$.
- $m : M(M(S)) \to M(S)$, which sends a string of strings of elements of $S$ to the concatenated string, e.g., $m : \langle \langle pqr \rangle \langle st \rangle \langle uvw \rangle \rangle \mapsto \langle pqrstuvw \rangle$.

Notice that $M$ is behaving very much like $\Diamond$, as is made apparent by representing propositions as objects and entailments as arrows: $p \to \Diamond p$ and $\Diamond\Diamond p \to p$.

This data forms a *monad* since we have a category, here the category of sets, $Set$, an endofunctor from the category to itself, here $M$, with a unit, $i_S : S \to M(S)$, and a multiplication, $m_S : M(M(S)) \to M(S)$, for each $S$ in $Set$, satisfying a number of equations. In this case of strings of elements of a set, some of these equations concern obvious properties of concatenation with singleton strings. Other equations tell us that concatenating strings of strings of strings of elements, it does not matter whether we begin concatenating at the inner or the outer layer.[2]

All monads on a category, $\mathcal{C}$, come from a composition of two adjoint functors between $\mathcal{C}$ and some other category $\mathcal{D}$, the left adjoint followed by the right. In general this composition occurs in several different ways, in the sense that non-equivalent choices of $\mathcal{D}$ are possible. In the case of our strings of elements, $M$ may be taken as the composite of a pair of adjoint functors between the category of sets and the category of monoids, a monoid being a set equipped with an associative binary operation and a unit for this operation. The left adjoint is the free functor which sends a set to the monoid it freely generates, the identity element being the empty string. The right adjoint is the forgetful functor, which sends a monoid to its underlying set.[3]

It seems then as though familiar modal operators such as 'possibly', behaving as they do like a monad, should equally arise from an adjunction. Let's now show this by returning to our dog scenario, but where before I used the *owner* map, I now work with a different map. The constructions I detailed above concerning dog ownership work for any map between sets, and so in particular for the terminal map ($Dog \to \mathbf{1}$). Induced mappings must send dog properties to properties of $\mathbf{1}$. Now, 'properties' of $\mathbf{1}$ are just propositions. The equivalent of the $owner^*$ map is a map which sends a proposition, $P$, to the property of dogs – any dog if $P$ is true, no dog if $P$ is false. Then attempts at inverses to this mapping, left and right adjoints, would send a property of dogs such as 'being a pug' to the familiar constructions of quantified statements: 'Some dog is a pug' and 'All dogs are pugs'. Here we have recovered the dependent sum and dependent product of the chapter 2. Since this is a special case of a construction for general functions, $f$, we often call the left and right adjoints dependent sum, $\sum_f$, and dependent product, $\prod_f$, as well.

Now the step to the possible worlds of modal logic is simple to make. One common philosophical interpretation of *necessarily* and *possibly* is in terms of

[2]We pay less attention to this extra structure in the logical case since we generally take an arrow between propositions to represent entailment, rather than a specific entailment.

[3]So-called *algebras* for this monad are monoids.

a collection of *possible worlds* of which our *actual world* is just one element. So let $W$ be the type of all possible worlds. Any specific choice of $W$ may be taken as specifying what is to be understood as a possible world. Under this interpretation, a proposition that depends on $W$ is necessarily true if it is true in all possible worlds, and possibly true if it is true in some possible world.

Note, however, that these dependent sum and dependent product operations change the dependence from $W$ to non-dependence, or dependence on $\mathbf{1}$ or $*$. In other words, if a proposition $P(w)$ depends on $w : W$, so that it may be true in some worlds and false in others, then $\exists_W P$ and $\forall_W P$ no longer depend on $W$. But the idea of a necessity and a possibility modality is to send a proposition in some context to a proposition in the *same* context so that they may be compared. We should be able to say for instance that $\Box P$ implies $P$ and so on. Thus we need to make $\exists_W P$ and $\forall_W P$ into propositions that again depend on $W$ even if they now depend trivially on $W$.

One of the very pleasant features of a topos, $\mathbf{H}$, is that, if you take any of its objects, say, $A$, then the so-called slice category $\mathbf{H}/A$ is also a topos. The slice category has as objects, maps in the topos $f : B \to A$. A morphism in the slice topos to another object $g : C \to A$ is a map $h : B \to C$, such that $f = g \circ h$, the triangle commutes. A parallel statement also holds true for $(\infty, 1)$-toposes, the categories which model HoTT. Now the equivalent in type theory of the object $f : B \to A$ in $\mathbf{H}/A$ is the dependent type $x : A \vdash B(x) : Type$. Working in a context is equivalent to working in the associated slice of a topos. The empty context corresponds to the object $\mathbf{1}$, and $\mathbf{H}/\mathbf{1} \simeq \mathbf{H}$.

Restricting ourselves still to propositions, we need to make $\exists_W P$ and $\forall_W P$ which belong to the empty context into propositions that again depend on $W$, even if they now depend trivially on $W$. This process is 'context extension' back from the absolute context $\mathbf{1}$ to $W$. The composite monad and comonad are as follows:

$$(\underset{W}{\Diamond} \dashv \underset{W}{\Box}) := \left( \left( W^* \circ \underset{w:W}{\exists} \right) \dashv \left( W^* \circ \underset{w:W}{\forall} \right) \right) \ : \ \mathbf{H}_{/W} \longrightarrow \mathbf{H}_{/W} \, ,$$

taking $\mathbf{H}_{/W}$ for the moment as the category of world-dependent *propositions*, with implications as arrows. With this, if $p \in \mathbf{H}_{/W}$ is a proposition about terms $w$ of $W$ (a $W$-dependent type) then

- $\underset{W}{\Diamond} p(w)$ is true at any $w$ precisely if $\underset{w:W}{\exists} p(w)$ is true, hence if it is the case that $p(w)$ is true for some $w$;
- $\underset{W}{\Box} p(w)$ is true at any $w$ precisely if $\underset{w:W}{\forall} p(w)$ is true, hence if $p(w)$ holds for all $w$.

It may appear to be the case that the further operation provided by applying $W^*$ is unneccesary, but it is crucial for a proper modal HoTT treatment.

Thus we are given one syntactic formalization of the informal meaning of necessity and possibility. The natural semantics for these base change operations is a generalization of the simple traditional possible worlds semantics of propositional necessity and possibility modalities. Notice that we arrived at these constructions without the usual device of using negation. In classical modal logic, the operators are interdefinable as $\neg \Box p = \Diamond \neg p$. Here, however, we defined them independently. Moreover, with this formalization, the modal operator $\Diamond_W$ is left adjoint to $\Box_W$ and hence together form an *adjoint modality*. Indeed, when an adjoint triple is used to form a pair of modalities in this way, they are always in turn adjoint to one another, expressing their 'opposition':

$$\Diamond p(w) \to q(w) \Leftrightarrow p(w) \to \Box q(w).$$

In words, if the possibility of property $p$ entails that $q$ holds at this world, then were $p$ to hold at this world then $q$ would necessarily be the case. Choosing $q$ equal to $p$, we see that a proposition sits between the images of the two operators:

- necessarily true, true, possibly true

following the pattern of

- everywhere, here, somewhere.

The modal adjoints additionally furnish us with an equivalent for Axiom(B) from standard modal logic:

$$Hom(\Diamond p(w), \Diamond p(w)) \cong Hom(p(w), \Box\Diamond p(w)).$$

Also we have (5)

$$Hom(\Diamond\Diamond p(w), \Diamond p(w)) \cong Hom(\Diamond p(w), \Box\Diamond p(w)).$$

This is evidently an S5 form of modal type theory.

A general property of adjoints is that they preserve certain kinds of categorical structure. Left adjoints preserve sums (and colimits in general), while right adjoints preserves products (and limits in general). Examples of this preservation are that

- possibly $p$ or $q \leftrightarrow$ possibly $p$ or possibly $q$
- necessarily $p$ and $q \leftrightarrow$ necessarily $p$ and necessarily $q$

While base change-adjunctions are essentially unique and not free to choose, there is a genuine choice in the above given by the choice of context $W$. This is reflected in the subscripts of $\Diamond_W$ and $\Box_W$ above. It is the choice of this $W$ that gives different kinds of possibility and necessity. More generally there is in fact not just a choice of a context, but of a morphism of contexts, reflecting what is often called *accessibility* of possible worlds.

This construction resembles the dog-owner case better if we consider an equivalence relation on Worlds, represented by a surjection, $W \to V$. Now, necessarily $P$ holds at a world, $w$, if $P$ holds at all worlds related to $w$, that is all worlds with the same image in $V$ as $w$. With our axiomatization via base change, it is immediate to consider the relative case where instead of base change to a unit type $W \to \mathbf{1}$ one considers base change along any surjection $\omega\colon W \longrightarrow V$.

$$(\exists_\omega \dashv \omega^* \dashv \forall_\omega) \;:\; \mathbf{H}_{/W} \underset{\exists_{w:\,\omega^{-1}(-)}}{\overset{\overset{\forall_{w:\,\omega^{-1}(-)}}{\longrightarrow}}{\overset{\overset{\omega^*}{\longleftarrow}}{\longrightarrow}}} \mathbf{H}_{/V}\,.$$

Then we set

$$(\underset{\omega}{\lozenge} \dashv \underset{\omega}{\Box}) := \left(\left(\omega^* \circ \underset{w\,:\,\omega^{-1}(-)}{\exists}\right) \dashv \left(\omega^* \circ \underset{w\,:\,\omega^{-1}(-)}{\forall}\right)\right) \;:\; \mathbf{H}_{/W} \longrightarrow \mathbf{H}_{/W}\,.$$

Since here $\omega$ is a surjection, then it provides an equivalence relation on $W$, where $w_1 \sim w_2$ is given by $\omega(w_1) = \omega(w_2)$. In traditional possible worlds semantics such equivalence relation is called an accessibility relation between possible worlds. Now

- $\lozenge_\omega p$ is true at $w \in W$ iff it is true at at least one $\tilde{w}$ in the same equivalence class of $w$;
- $\Box_\omega p$ is true at $w \in W$ iff it is true at all $\tilde{w}$ in the same equivalence classes of $w$.

We still find ourselves dealing with an S5 kind of modal type theory. In a later section we move away from symmetrically accessible worlds.

Now, even though we have achieved a successful encoding of S5 modalities within a dependent type theory by relying on a type of worlds, there is still much to be taken from this construction by those who refuse to countenance what they take to be such metaphysical fantasies as possible worlds. The fact that I could begin my account with variation over dogs should indicate to us that what is really at stake is variation broadly construed. A type of worlds can be taken to play the role of a generic domain of variation, rather as the probability theorist employs the notion of a sample space, $\Omega$, as a domain for their random variables.

Consider, for $A$ a type and $B$ a property of that type, how we mark the difference between the following:

- This $A$ is $B$.
- This $A$ is necessarily $B$ (by virtue of being $A$).

Using famous examples from Nelson Goodman's 1955 book *Fact, fiction, and forecast*, let us contrast

- This coin in my pocket is silver.
- This emerald is necessarily green.

Where I might look at a gem, $e$, which I know to be an emerald, and observe that it is green, this would only warrant ,

- This emerald is green.
- $\vdash p : Green(e)$

If, on the other hand, I have a witness to a universal statement

- $\vdash f : \prod_{x:Emerald} Green(x)$,

then I can apply this function to my gem, $e$, to construct $f(e) : Green(e)$.

We can see why the language of necessity is invoked from our analysis of modality above. Consider the necessity operator corresponding to the type $A$

through its map to $\mathbf{1}$. When applied to a dependent type $x : A \vdash B(x) : Type$ it produces $x : A \vdash \Box_A B(x) := A^* \prod_{x:A} B(x)$. Now an element of a type is just a map from $\mathbf{1}$ to that type. For instance, a particular element, $a$, of the type $A$ corresponds to a map $a : \mathbf{1} \to A$. This map will generate three maps between all types, $\mathbf{H}$, and all $A$-dependent types, $\mathbf{H}/A$, in particular the map $a^*$ which sends an $A$-dependent type, $B$, to the type in the fibre over $a$, or $B(a)$.

Evaluating our type $\Box_A B(x)$ at $a$ in this way results in $\Box_A B(a) = \prod_{x:A} B(x)$. In other words, in the case of the emerald above, when we found $f$ guaranteeing the greenness of all emeralds, $f : \prod_{x:Emerald} Green(x)$, it transpires that it is at the same time an element of the equivalent type $\Box_{Emerald} Green(e)$, a type which may be said to represent the necessity of $e$'s greenness. We see then that one's entitlement to add '*necessarily*' to a claim about the possession of some property of an individual in a type depends on the derivation one has of the element witnessing its truth, as displayed in the syntactical form of that element.

However, it can't just be a question of a true universal statement playing this role. For example, we might also have the universal truth:

- All the coins in my pocket now are silver.

Of course, in a sense, necessity is present here too. If all the coins in my pocket *are* silver, then a choice of such a coin *must* result in a silver one.[4] On the other hand, there's evidently a difference that Goodman was pointing out with this example in that there seems to be no *modal* element in this case, in the following sense. I've only taken a look at some of the coins that will ever be in my pocket, those which happen to be there now. By tomorrow I may have acquired some copper coins. Indeed, give me a copper coin and I could place such a coin there right now. By contrast, I cannot possibly discover or synthesize a non-green emerald.

But note that we have a necessity operator for every type. When we assess a judgement for its necessity, we must decide what is a relevant 'natural' range of variation. For my coins, a range time-limited to the moment seems unnatural. Then again, I might purchase some trousers and after a time seal up their pockets never to carry any more coins. The coins ever in my pocket may then all happen to be silver. Even so, there seems to be a difference between a type such as *Emerald* and one such as *Coin ever in this pocket*, so that we are unlikely to use the latter as a domain. The former has been of human concern for centuries, certainly since the time of the ancient Egyptians; the latter was cooked up for a philosophical puzzle, and we expect that no language will contain a word for such a type. Furthermore, the truth of '*This emerald is green*' is known to be due to the structural properties it shares with all emeralds. We have discovered that it is the chromium-infused beryl ($Be_3Al_2(SiO_3)_6$) composition which begins to explain its greenness.[5]

Goodman (1955) presented inductive logicians with a range of challenges in making sense of the difference between law-like regularities and ones that happen to occur. The logical empiricists he was addressing were nervous of modal talk,

[4] A point also noted by Brandom (2015, p. 162).

[5] If you're from North America, infusion by vanadium also counts. Note also that we are ignoring the fact that colour is used in counting such Beryl gems as emeralds, making this statement *analytic*.

and so hoped to rely merely on syntactical features. Instead, along with the inferentialist, we can say that much rests on the web of inferential relations to which specific types belong, which dictates what we expect to change and what remain the same as conditions vary over a range. These expectations, naturally enough, change over time. We have a very different understanding today of what kind of measures could lead to a gem changing its colour compared to the expectations of a seventeenth century alchemist. But at any moment, any user of language in its primary function of empirical description will possess "the practical capacity to associate with materially good inferences *ranges of counterfactual robustness*" to speak in Brandom's terms (2015, p. 160). It is this capacity that underpins our use of modal vocabulary.

In sum, our modal vocabularly provides us with the means to make explicit our commitments to the behaviour of entities according to their types. The use of a specific type of *worlds* to construct modal operators is a means to portray a most general form of variation. Let's see now how such variation applies not only to dependent propositions but to dependent types in general.

# 2 Towards modal HoTT

## 2.1 General types

In the section above, I have denoted the expression 'it is necessarily the case that' as $\Box$ and applied it directly to propositions as a form of type. In view of HoTT's understanding that propositions are simply a kind of type, those types at the bottom of the $n$-type hierarchy, we should expect there to be a modal form of the full hierarchy. In other words, we should look to form $\Box A$ for any type $A$.

But before proceeding along these lines, let's reflect on our options. In the philosophical literature it is not uncommon to hear of modalities as being expressions which qualify the truth of judgments (Garson 2018). Since in the constructive type theory tradition, the judgment $\vdash P\ \mathit{true}$ for a proposition $P$ became $\vdash p : P$ in HoTT, what should we make of $\vdash P\ \mathit{necessarily\ true}$? We can't apply an operator to the '$p : P$' part itself, but we might think to modify the nature of the judgment, perhaps to something like $\vdash_{nec} p : P$. Alternatively, as we were led to do earlier, we modify the symbols for the types, $\vdash p : \Box P$.

As we will see, these two choices may be reconciled. Judgment in another stricter domain can be reflected back within the original domain where it will count as having constructed an element of a modified type. For now, however, let us continue with the second of these strategies.

In the previous section we saw propositions depending on a type of worlds. It is perfectly possible then to apply the modal monad and comonad to any such world-dependent types. Indeed, consider the kinds of category taken as models of HoTT, namely, elementary $(\infty, 1)$-toposes. For $\mathbf{H}$ an $(\infty, 1)$-topos and $f : X \to Y$ an arrow in $\mathbf{H}$, then we have seen that base change induces an adjoint triple between slices:

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \overset{\overset{f_!}{\to}}{\underset{\underset{f_*}{\to}}{\overset{f^*}{\leftarrow}}} \mathbf{H}/Y$$

The monad and comonad generated by these maps act on the whole slice $(\infty, 1)$-topos, $\mathbf{H}/X$, that is, on all types dependent on $X$.

Returning to dependence on a type of worlds, so $X = W$, a set, and $Y = V$, the equivalence classes of accessible worlds, now consider a world-dependent type $B(w)$, which we take as a set for clarity. Then $\Diamond_W B(w)$ is the collection of $\langle w', b\rangle$, with $w'$ accessible to $w$ and $b : B(w')$. A possible $B$ at a world is an actual $B$ at some related world. Meanwhile, $\Box_W B(w)$ is the collection of maps which for each world, $w'$, accessible to $w$, select an element of the respective type $B(w')$. So a necessary $B$ at a world is a selection of a $B$ from each accessible world.

There is a natural map $B(w) \to \Diamond_W B(w)$, which sends $b : B(w)$ to $\langle w, b\rangle : \Diamond_W B(w)$, and one from $\Box_W B(w) \to B(w)$, which evaluates the map $w' \mapsto b(w') : B(w')$ at world $w$ as $b(w)$. While still being a monad and comonad, respectively, $\Diamond_W$ and $\Box_W$ as defined above are no longer idempotent. Consider the case where all worlds are mutually accessible, that is, $V = \mathbf{1}$. Then $\Diamond_W \Diamond_W B(w)$ is composed of elements $\langle w', \langle w'', b\rangle\rangle$, with $b \in B(w'')$. Of course there is a projection from this to $\langle w'', b\rangle$, so that we have a natural map $\Diamond_W \Diamond_W B(w) \to \Diamond_W B(w)$. We can similarly find a natural map $\Box_W B(w) \to \Box_W \Box_W B(w)$.

Let us now put these constructions to use to provide a setting in which an old chestnut of a puzzle makes sense. A standard example of the perils of substitution runs as follows:

- It is necessarily the case that $8 > 7$.
- The number of planets is 8.
- It is necessarily the case that the number of planets $> 7$.

For simplicity we keep with the case where all worlds are accessible to one another.

We have, of course, that 7 and 8 are elements of $\mathbb{N}$. But we need a world-dependent version of these numbers. So let us form the trivially world-dependent $W^*\mathbb{N}$, which provides a copy of $\mathbb{N}$ at every world. Versions of our numbers, the constants $7(w)$ and $8(w)$, are then elements of this world-dependent type $W^*\mathbb{N} \simeq W \times \mathbb{N}$. Now we can compare these numbers with 'the number of planets in $w$', also an element of the type $W^*\mathbb{N}$.

Since $8 > 7$ is true, $8(w) > 7(w)$ is a true proposition at all worlds, hence $\Box_W(8(w) > 7(w))(w)$ is constantly true. This is our version of the 'Necessitation rule' that a theorem in the empty context is necessarily true. If we like, we may 'actualise' this result by evaluation at the actual world, $a : 1 \to W$. Such a map induces a map $a^* : \mathbf{H}/W \to \mathbf{H}$, where $a^* \Box_W B(w)$ is equal to $\prod_W B(w)$.

We also have that 'the number of planets$(a) = 8(a) : \mathbb{N}$'. So we may derive the truth of '(the number of planets$(w) > 7(w))(a)$' but not that of $\Box_W$(the number of planets$(w) > 7(w))(a)$. What appears to confuse people in the puzzle is that when $m(w)$ is not a constant function in $W^*\mathbb{N}$ then it is not equal to the constant function $(m(a))(w)$.

Now returning to non-constant world-dependent types, we may wonder whether in general there is any comparability between worlds. If a type is formed by a map from a set $B$ to $W$, that is, as dependent type $w : W \vdash B(w) : \mathit{Type}$, then there are no associated means to identify elements of different worlds. On the other hand, as we saw with the natural numbers, we can form a world-dependent type by base change from a type existing in the empty context. So we might have a general type $A$, then form the $W$-dependent type $W \times A$, the dependency represented by the first projection to $W$. Here, of course, an element $\langle w, a\rangle$ in one world corresponds to the counterpart element in another world, $\langle w', a\rangle$.

We might then form a subtype, $P$, of such a constant type. This will come equipped with its projections to $A$ and to $W$. If for some $a$ in $A$, $W \times \{a\} \subseteq P$, then we have a *section* in $\Box_W P$, represented by $f(w) = a$. To illustrate this, consider that the range of foodstuffs is constant across worlds, or let's say here *situations*. In each situation a recipe for a beef stew is to be given, the ingredients for each recipe coming from that fixed range. Then presumably beef will be an ingredient in each case, whereas potatoes may be left out on occasion. So beef is a *necessary* ingredient of a beef stew, whereas potatoes are not. Elements of $A$ here are acting as *rigid designators* in the sense Kripke gives the term in *Naming and Necessity*:

> Let's call something a *rigid designator* if in every possible world it designates the same object, a *nonrigid* or *accidental designator* if that is not the case. (Kripke 1980, p. 48)

The rigid designator 'Nixon' in Kripke's famous examples is acting like 'beef' to pick out the same entity in each world in which it exists. An element of a type may also be considered as a function to the type from the unit type, $Nixon : \mathbf{1} \to Person$. So then $W^*Nixon(w) : W^*Person(w)$, and may be an element of $P(w)$, our subtype of $W^*Person(w)$. 'The person who won the United States presidential election of $1970(w)$' is also an element of $P(w)$, but it is not in the image of $W^*$. As such it is *nonrigid* in Kripke's sense. We see both transworld identity (Kripke) and counterparts (Lewis) at play.[6]

Of course, one may question such accounts. Does it make any sense to postulate a set of world-transcending people, $Person$, with which to form $W^*Person$ or even the subtype of those people that exist in their worlds? Can we sensibly compare this world to a neighbouring possible world? In a very close world to this one, nobody like me exists. Or the fertilised egg that became me here might have divided to produce monozygotic twins. Nevertheless there's something structurally interesting happening here concerning ways of relating elements between fibres of maps. Let's illustrate this by leaving behind these speculative worlds and confining ourselves to something occurring in our own familiar world.

Let's consider modalities generated by a simple surjective mapping, the map from the type of animals to the type of species, the one which assigns to each animal its species. Then take the dependent type of $x : Animal \vdash Leg(x) : Type$. Then an element of $\bigcirc Leg(x)$ is any leg of a conspecific of $x$, and an element of $\Box Leg(x)$ is a description of a leg possessed by each conspecific of $x$. In terms of a dog called Fido, a 'possible leg' for Fido is any dog's leg, while a 'necessary leg' is an assignment of a leg to each dog. For the latter, we could take, for instance, 'the last leg to have left the ground', or 'the right foreleg'.

Then $Legs(Fido) \to \bigcirc_{spec} Legs(Fido)$ is just the injection of Fido's legs into all dogs' legs. This is relevant to the discussion in philosophy as to whether

[6] These resources should suffice to represent what is studied in so-called *two-dimensional semantics*. This concerns use of actuality or indexicals, as with "It is possible for everything that is actually red to be shiny," to be rendered perhaps as "There is some world, $b$, in which everything that is red in this actual world, $a$, is shiny there in $b$."

possible objects are counted as pertaining to a world. Fido's possible legs pertain to him even if they make reference to other dogs' legs. Similarly, we have a type of possible entities of a kind at a world, members of which make reference to entities of that kind in another world. But we see here that we must maintain type discipline. An element of $\bigcirc_W A(a)$ lives in $a$ in a sense, but as an element of a specific possible type not as an element of $A(a)$. For instance, when $A$ is a type of concrete objects, an element of $\bigcirc_W A(a)$ has no place in the world $a$. It has a *possible* place at $a$, but this makes reference to the place in the world in which the corresponding element actually lives.

In general, there won't be a map from an animal-dependent type to its $\Box$ version. Think of the dependent type $x : Animal \vdash Offspring(x) : Type$. My indicating one of an animal's offspring gives me no means to pick out an offspring of a conspecific. Indeed, this type may well be empty. So which types do allow a map to the $\Box$ version? Which are *necessary*? Well, certainly those types pulled back from ones dependent over species. A *standard* is being provided to allow comparison across conspecifics.

We can see this as follows. Given the map $spec : Animal \to Species$, we have

- $s : Species \vdash BodyPart(s)$
- $\vdash$ front right leg: $BodyPart(Dog)$
- $x : Animal \vdash spec^* BodyPart(x)$

We now have a map from $spec^* BodyPart(x)$ to $\Box_{spec} spec^* BodyPart(x)$. Given an element in $spec^* BodyPart(Fido)$, such as Fido's front right leg, we can name a similar body part for Fido's conspecifics, that is, we can form an element of $\Box_{spec} spec^* BodyPart(Fido)$.[7]

An element generated in such a way might be said to refer to an *essential* characteristic of a dog. If I point to the front right leg of a dog and show you another dog, you will probably choose the same leg. It might be holding this paw in the air, so you could have chosen the left rear leg of the second dog who is cocking this now at a lamp post, but it does seem that the same element of the body plan is the most reasonable choice.

This quality of being able to transfer between the fibres of a map is prevalent throughout mathematics. Indeed, there's a direct route from the simple considerations we have just covered to the topics of connections on fibre bundles and of solutions to (formally integrable) partial differential equations. As I will mention briefly in the following chapter, these latter equations may be seen as recipes which dictate how behaviour carries over to infinitesimally neighbouring points.

As a technical aside: the *algebras* for the possibility monad, types for which there is a $W$-dependent map $\bigcirc_W B \to B$, coincide with the *coalgebras* for the necessity comonad, with corresponding map $B \to \Box_W B$, and are such that there is a natural map $\sum_W B \to \prod_W B$. Given an element $b : B(w')$ from some world, we must generate a function $f : w \mapsto f(w) : B(w)$, such that $f$ picks out

[7]Note that we are assuming that we are dealing with a world in which no animal has lost a leg. Alternatively, we might speak of Patch having lost his front right leg, there being an expectation that such a thing should be present.

the original element, that is, $f(w') = b$. In other words, we must know how to continue the function given its value at a single world.

Before ending this section I should say a word on the *homotopical* aspect of modal HoTT. Until this point in the chapter we have only considered types which are propositions or sets depending on sets. But we may climb the hierarchy and apply the constructions above to the case where our types depend not on a set, but on a group. As in the last chapter, we find that for a type $A$ acted on by a group, $G$, then $\sum_{*:\mathbf{B}G} A(*)$ is a type with structure that of the action groupoid. Now we can *base change* (or *context extend*) back to a $\mathbf{B}G$-dependent type by applying the trivial action of the group. Similarly we can form the dependent *product*, which in this case is composed of the 'fixed points' of the action, those elements of the set left unchanged by all elements of the group. So $\Box_{\mathbf{B}G}$ sends a $G$-action to the trivial action of $G$ on fixed points. Then there's a map from the latter to the original $G$-set, the inclusion, corresponding to the map $\Box A \to A$. The 'necessity' here translates to invariance under the group action.

In the relative case, the map $\mathbf{B}G_1 \to \mathbf{B}G_2$, induced by a group homomorphism $G_1 \to G_2$, provides an adjoint triple between $G_1$-actions and $G_2$-actions, the outer adjoints corresponding to *induced* and *coinduced* actions (see, for instance, Greenlees and May (1995)). It is remarkable to see how the simple ideas we have proposed for presenting modal type theory are intimately related to cutting-edge research in equivariant homotopy theory.

## 2.2 First-order modal logic and Barcan

The form of modal HoTT we have developed should give us a first indication of what to make of one of the thorniest issues in first-order modal logic, the Barcan formulas. These are named after Ruth Barcan who sought to compare formulas of the form $\bigcirc \exists x P(x)$ and $\exists \bigcirc P(x)$, designating claims concerning the possible existence of a $P$ and the existence of a possible $P$. It is perhaps clear why one might wish to resist the forward Barcan inference since it seems to propose that there possibly being something with a property entails that there is something which possibly possesses the property. How can a mere possibility of existence be enough to entail existence?

Here we will need to be careful again with the typing discipline. If the modalities apply to the slice over worlds $W$, then there is a problem in trying to make them interact with quantification over a world-dependent type. Say we have a world-dependent type $w : W \vdash B(w) : Type$, and then some further property $w : W, b : B(w) \vdash P(w, b) : Prop$, then we may quantify over $B$, thereby removing it from the context, and then be able to apply the modal operators. But in this case it won't make sense to apply the modal operators first – they can only apply to the bare context $W$. The only way this reversal could take place is if $B$ is not really $W$ dependent, so that the context is $W, B$. Then we could apply modalities to return to the $W, B$ context, and so apply quantifiers to land in dependency over $W$.

If we have the constant world-dependent type $W \times B$, derived from a plain type $B$, then I might have a dependent proposition

$$w : W, x : B \vdash P(w, x) : Prop$$

Then $w : W \vdash \exists_{x:B} P(w,x) : Prop$, so $w : W \vdash \bigcirc_W \exists_{x:B} P(w,x) : Prop$, in which elements at a world $a$, will be $(w,(b,p))$, $p$ witnessing that $b$ is $P$ in world $w$. Treating the context now as symmetric, we can also form $w : W, x : B \vdash \bigcirc_W B(w,x) : Prop$ and $w : W \vdash \exists_{x:B} \bigcirc_W B(w,x) : Prop$. Here I'm not quantifying over things in my world, but rather over world-independent $B$. I'm saying that some $B$ turning out to have $P$ when it appears in some world is the same as some world containing a $B$ which is $P$, and these are evidently the same claim.

In general, however, the type $B$ may genuinely depend on $World$. Then we haven't a means to exchange as above. But say I have as above $w : W, x : B(w) \vdash P(w,x) : Prop$. Then I can form $\bigcirc_W \sum_{x:B(w)} P(w,x)$, the world-dependent (constant) type containing all possible $B$s which are $P$. Evaluated at my world $a$, this is $\sum_{w:W} \sum_{x:B(w)} P(w,x)$. I can also form $\sum_{(w,x):\sum_{w:W} B(w)} P(w,x)$. Then a version of the Barcan formulas amounts to the equivalence of taking dependent sum in one or two stages.

$$(\bigcirc_W \sum_{x:B(w)} P(w,x))(a) \simeq \sum_{w:W} \sum_{x:B(w)} P(w,x) \simeq \sum_{(w,x):\sum_{w:W} B(w)} P(w,x)$$

The second equivalence is just the rebracketing of the three-part terms in pairs: $(w,(b,p)) \leftrightarrow ((w,b),p)$. We might say:

- At this world, there's possibly a $B$ which is $P$.
- There is a possible $B$ which is $P$.

This solution goes in some respect along the lines of Timothy Williamson (2013) to allow quantification over possible things. However, we arrive at this solution maintaining strict typing discipline, and certainly not allowing untyped quantification over 'everything'.

## 2.3 Contexts and counterfactuals

Of the many philosophers who do not go along with the Barcan formula, let us briefly consider an account by Hayaki (2003), who claims that there is no proper sense in which merely possible entities exist.

> All apparent references to non-actual objects are circumlocutions either for *de dicto* statements about ways the world might have been, or for *de re* statements about ways that actual objects might have been, or for some combination of both. (2003, p. 150)

Since for her possible objects do not exist in our world, she needs to make sense of what appear to be truths we can utter about possible objects. Consider the following pair of propositions relating to the example she discusses:

> I could have had an elder brother who was a banker. He could have been a concert pianist if he had practised harder.

It appears to be the case here that we are referring to a non-existent being, an elder brother, to say counterfactual things about him. If we restrict ourselves to worlds arranged on an equal footing, it certainly sounds as though we are treating this possible man *de re*.

However Hayaki explains the situation through nested trees, where the first sentence presents a level 1 world, and the second a level 2 world. Our world might have gone differently with my parents having another son whose career was in banking. Then in that level 1 world, of the object that is that man it can be said that he might have had a different profession. Hayaki considers this better motivated version of her imagined brother as inhabiting a level 2 world.

What I want to take from this construction is the idea that there's a structure to the variability of possibility that goes beyond variation over a set. Hayaki talks in terms of stories and their continuation. Let's follow her in this by considering winding back the story to a time before her birth but after her parents met, and then winding forward again with their two children and career choices. Leaving aside whether there would be a 'she' if her parents had had a child before her, we can spin out two tales from the assumption of that child's birth and hers, according to the two choices of profession. Then, once we have a branch with a banker brother, we can speak of his possibly being a pianist since we need only wind that branch back to a point where career decisions are being made. But rather than this informal talk of stories and continuations, let's see if they can be included with the formalism of the modal calculus we have been developing.

In this chapter we have been considering the type of worlds as the space of variation for our modal considerations, but given the role of contexts in HoTT as providing the typed variables for terms and further types, we might see whether contexts themselves could act as way to formulate worlds. We can see this idea of winding back through history in terms of deconstructing the type of worlds. Indeed, recall Ranta's idea from Chap. 2 that a context is like a narrative, where we build up a series of assertions, any one of which may depend upon previously introduced terms.

> A man enters a saloon. He is whistling *Yankie Doodle*. A woman enters the same saloon, holding the hand of her child. It is his wife.

As before, think of a work of fiction introducing the reader to things, people and places, describing their features, their actions etc. Then possible worlds relative to what has been stated so far are ways of continuing: new kinds of thing may be introduced, or new examples of existing kinds, and identities may be formed, such as when we find out that Oliver Twist's kindly gentleman is in fact his grandfather. And so on. Necessity then describes what must happen, and possibility what may happen. So for our story we could ask

> Might the child be the man's daughter? Might she call out to him? Must they all be in the same room?

These are all questions about ways the story may or will continue. We may also wonder whether the story could have gone differently

> Might he have been humming rather than whistling? Might he have whistled a symphony? Might the woman have held the child in her arms?

Now formally, recall that a context has the form:

$$\Gamma \;=\; x_0 : A_0,\; x_1 : A_1(x_0),\; x_2 : A_2(x_0, x_1),\; \ldots x_n : A_n(x_0, \ldots, x_{n-1}).$$

In view of the pleasant category-theoretic setting of HoTT, any such context corresponds itself to an object, the iterated dependent sum of the context. Let $W$ represent the iterated sum, and $W_i$ the stages of the construction of $W$, then the maps we considered earlier

$$\mathbf{H}_{/W} \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{H}$$

now factor through the successive stages of the construction of the context:

$$\mathbf{H}_{/W} \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \cdots \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{H}_{/W_2} \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{H}_{/W_1} \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \mathbf{H}$$

Still there are different ways as to how to take this idea further.

There is such a vast store in our shared context that it seems that a story can go almost anywhere its author wishes. Continuing our Western, an elephant escapes from the box car in which it is travels with the circus and tramples all in the saloon underfoot. Or, a tornado rips through the town and takes the child somewhere over the rainbow. Instead, one might imagine a more controlled setting of what can occur next, where paths fan out according to circumscribed choices, as we find with computations paths in computer science or, in a more extreme form, with the collection of real numbers for the intuitionist. Ranta (1991) began the exploration of these themes from the perspective of dependent type theory. Like the real numbers formed from all possible infinite decimal expansions, here we can conceive of the specification of a collection of worlds which are all possible 'complete' extensions of a context $\Gamma$.

> Worlds appear as total infinite *extensions* of finitely representable *approximations* of them. Moreover, all we can say about a world is on the basis of some finite approximation of it, and hence at the same time about indefinitely many worlds extending that approximation. (Ranta 1991, p. 79)

Where the real numbers enjoy the property that we are simply making the choice of a digit from a fixed set at each turn, in the case of narratives we could think to make a circumscribed story by selecting some characters out of a list of possibilities, specifying some of their possible properties, specifying possible relations between them, specifying possible actions that one character does to another and some consequences of these actions, and so on.

This context-based formulation may give us a way to think about the nature of counterfactuals such as

- Had I taken an aspirin this morning, I wouldn't have a headache now.

To make sense of the truth of such counterfactual statements, David Lewis famously invoked the notion of a 'closest' possible world in which the antecedent holds. Here we don't have a metric on our space of worlds, but in that we see the collection of contexts as forming a tree, where an extension shifts along a branch, then one could imagine some kind of minimal stripping back of context to leave out that part which conflicts with the antecedent of the counterfactual, before building up to a context where it holds. Then contexts whose shared initial stage is longer will count as closer in something like a *tree metric*.

But we don't want to count all extensions from the common initial stage as equally close. In the case of my headache, specifying a world by turning back to this morning, asserting that I do take an aspirin, but also that I hit my head on a low ceiling, so that I do in that case have a headache, this should count as being at a further distance than a situation where things continued as similarly as possible. We might then want to think harder about the dependency structure of a context. Although a context is given in general as

$$\Gamma \ = \ x_0 : A_0, \ x_1 : A_1(x_0), \ x_2 : A_2(x_0, x_1), \ \ldots x_n : A_n(x_0, \ldots, x_{n-1}),$$

$A_2$, say, might only really depend on only *one* of its predecessors. The direct dependency graph between types in a context is a *directed acyclic graph*, or a *DAG*. These are famous for expressing the dependency structure of Bayesian networks, a way of representing probability distributions based on causal dependencies. They were developed greatly by Judea Pearl, and described in his book *Causality* (Pearl 2009), where one thing he uses them for is counterfactual reasoning. One minimally modifies the network compatibly with the counterfactual information. We could similarly imagine minimal modifications of context here.

With the idea of branching histories we have come very close to the variants of temporal logic known as computational tree logics. Let us now see if we can develop a temporal type theory.

# 3 Temporal type theory

The philosophical logic literature makes play of the similarity between temporal modalities and those of necessity and possibility, although now with two pairs, oriented according to the time direction. For example, corresponding to possibility, for $\phi$ a proposition, we have

- $F\phi$ is '$\phi$ will be true at some future time';
- $P\phi$ is '$\phi$ was true at some past time'.

Just as with classical treatments of modal logic where possibility and necessity are interdefinable, logicians then look to form dual modalities of $F$ and $P$ as follows:

- The dual of $F$ is $G$, so $G\phi = \neg F \neg \phi$. This means that we read $G\phi$ as 'at no future time is $\phi$ not true', i.e., $\phi$ is always going to be true. ($G$ is for 'Going'.)
- The dual of $P$ is written $H$, whence $H\phi = \neg P \neg \phi$ and $H\phi$ interprets as '$\phi$ has always been true'. ($H$ is for 'Has'.)

In view of the fact that we could construct the dual modalities possibly-necessarily without negation, we should expect a similar treatment to be feasible here.

When in the treatment of possible worlds we used the existence of a map $f : W \to V$, we were understanding worlds in the same preimage of $f$ as related or accessible to one another. But another way to view an equivalence relation, and indeed a binary relation more generally, is as a subcollection of the cartesian product of the collection of worlds with itself. So

$$R \hookrightarrow W^2,$$

is the collection of related pairs of worlds. Naturally from $R$ we have two projections, $p, q$ to $W$, to the first and second members of these pairs. We could then, instead of deploying the map $f$, generate our modal operations on the slice over $W$ using $\sum_p q^*$ for 'possibly' and $\prod_p q^*$ for 'necessarily'. When we are dealing with an equivalence relation $R$, switching $p$ and $q$ in these operations won't lead to anything new. It is worth exploring then what would result from a more general relation.

Temporal logicians have long debated the relevant advantages of instant-based and interval-based approaches. Some have also considered hybrid approaches (Balbiani et al. 2011). As we shall see, the analysis of this section suggests that working with intervals and instants together in the form of something like what is called an *internal category* allows for a natural treatment via adjunctions. Indeed, intervals may be construed as given by pairs of instants marking their boundary, and so as

$$Interval \hookrightarrow Instant^2,$$

where the first instant precedes the second. So consider a category $\mathbf{H}$, and an internal relation given by $b, e : Time_1 \rightrightarrows Time_0$. Here we understand elements of $Time_1$ as time intervals, and $b$ and $e$ as marking their beginning and end points. Now each arrow, $b$ and $e$, generates an adjoint triple, e.g., $\sum_b \dashv b^* \dashv \prod_b$, formed of dependent sum, base change, dependent product, going between the slices $\mathbf{H}/Time_1$ and $\mathbf{H}/Time_0$.

Then along with the monads and comonads generated by composition within a triple, we can also construct some across the triples. Specifically, we find two adjunctions, $\sum_b e^* \dashv \prod_e b^*$ and $\sum_e b^* \dashv \prod_b e^*$. Then we have isomorphisms such as

$$Hom(\sum_b e^* C(t), D(t)) = Hom(e^* C(t), b^* D(t)) = Hom(C(t), \prod_e b^* D(t)).$$

Now consider for the moment that $C$ and $D$ are propositions depending on time instants. Then $\sum_b e^* C(t)$ will contain all instances of intervals beginning at time $t$ where $C$ is true at the end. If this type is inhabited it means "there is some interval beginning now and such that $C$ is true at its end", that is, $FC$, or $C$ will be the case. On the other hand, $\prod_e b^* D(t)$ means "for all intervals ending at $t$, $D$ is true at their beginning", that is, $HD$, or $D$ has always been the case. Hence our adjunction is $F \dashv H$. Similarly, interchanging $b$ and $e$, we find $P \dashv G$. Note that we did not have to assume the classical principle $G\phi = \neg F \neg \phi$ and $H\phi = \neg P \neg \phi$.

Since we have monads and comonads, we can consider the various units and counits

- $\phi \to GP\phi$: "What is, will always have been."
- $PG\phi \to \phi$: "What came to be always so, is."
- $\phi \to HF\phi$: "What is, has always been to come."
- $FH\phi \to \phi$: "What always will have been, is."

As before, in the setting of dependent type theory, we do not need to restrict to propositions, but can treat the temporal operators on general time-dependent types. So if $People(t)$ is the type of people alive at $t$, $FPeople(t)$ is the type of people alive at a point in the future of $t$, and $GPeople(t)$ is a function from future times to people alive at that time. For instance, an element of this latter time is 'The oldest person alive$(t)$', assuming humanity continues.

We can then think of adding other features, such as insisting that $Time$ be an internal category, and so requiring there to be a composition between any two intervals, the end of one matching the beginning of the other. We may also choose to impose additional structure, such as that the internal category be an internal poset, or a linear order. Let's consider here $Time$ as a category, where we have in addition to the two projections from pairs of intervals that adjoin, $p, q : Time_1 \times_{Time_0} Time_1 \to Time_1$, a composition $c : Time_1 \times_{Time_0} Time_1 \to Time_1$. This allows us to express more subtle temporal expressions. We could define a property of time instants that a lightning strike happen at that moment, $t : Time_0 \vdash L(t) : Type$. Then we could characterise the property of an interval that it contains a lightning strike as $\sum_c (ep)^* L(t)$ (note $ep = bq$).

We can also represent *since* and *until*. '$\phi$ has been true since a time when $\psi$ was true', denoted $\phi S \psi$ in the literature, is represented as:

$$\phi S \psi := \Sigma_e (b^* \psi \times \Pi_c (ep)^* \phi).$$

That is, there is an interval ending now such that $\psi$ was true at its beginning and $\phi$ was true at all points inside it. Similarly, '$\phi$ will be true until a time when $\psi$ is true' is

$$\phi U \psi := \Sigma_b (e^* \psi \times \Pi_c (ep)^* \phi).$$

To be precise, this last type is such that any inhabitant of it tells us that there is an interval beginning now such that $\phi$ holds at each of its points, and $\psi$ holds at the end. Of course, $\phi$ may continue to hold after the end of this interval. We could easily express variants where $\phi$ no longer holds after $\psi$ first occurs, or to allow the use of 'until' in the sense where the condition $\psi$ may never happen.

There is also the instant interval map, $i : Time_0 \to Time_1$, which enables us to send a property of intervals, $P(t_1, t_2)$, to a property of times by seeing whether that property holds of the relevant instant interval, $P'(t) := P([t, t])$. Note that this is different from the evaluation of a varying quantity at some moment. Say we have $t : Time_0 \vdash f(t) : \mathbb{R}$, then of course $f(a) : \mathbb{R}$ at some moment, $a : Time_0$, and we may then form a proposition concerning this instantaneous value. So we should agree with the following:

> *Instantaneous events* are represented by time intervals and should be distinguished from *instantaneous holding of fluents*, which are evaluated at time points. Formally, the point $a$ should be distinguished

> from the interval $[a, a]$ and the truths in these should not necessarily imply each other. (Balbiani et al. 2011, p. 32)

Note that one of the consequences of taking $Time$ as an internal category is that the future includes the present, so that $\phi$ could be true now and at no other instant but we would have that $F\phi$ is true when we may imagine that it is supposed to say "$\phi$ will be true at some Future time". Similarly, we would have that $\phi U \psi$ holds now if $\psi$ and $\phi$ both hold now (in general, as defined above it requires $\phi$ to still hold at the instant when $\psi$ becomes true). If we wish to change these consequences, we could let $Time_1$ collect the $<$-intervals instead of the $\leq$-ones. In other words, we could take $Time$ to be a semicategory. While this accords with standard practice, the original alternative has been proposed:

> The most common practice in temporal logic is to regard time as an irreflexive ordering, so that "henceforth", meaning "at all future times", does not refer to the present moment. On the other hand, the Greek philosopher Diodorus proposed that the necessary be identified with that which is now and will always be the case. This suggests a temporal interpretation of $\Box$ that is naturally formalised by using reflexive orderings. (Goldblatt 1992, p. 44)

On the other hand, some temporal logicians look to represent both forms of 'henceforth'.

There are many other decisions to be made in modelling $Time$: linear versus branching, discrete versus continuous, dense or not dense, bounded or unbounded, deterministic or undeterministic, and so on. Computer scientists have formulated various calculi to represent these choices. For instance, branching behaviour is captured in $CLT^*$, a computational tree logic. This calculus allows for quantification between branches as well as along branches, so that one might say of a given state 'It is always going to be that the machine will reach the state', or 'It is possibly going to be that henceforth the machine is in the state.' Such tree logics are used in chip design and verification, as is explained well in Halpern et al. (2001). A type-theoretic version should be easy to formulate, and could very well be useful here.

With some ideas on a temporal type theory in place let us see if we can make sense of one of Bede Rundle's counterexamples to '*and*' being treated as mere conjunction from Chap. 2:

- Alice used to lie in the sun and play cards.

For this proposition to be true, it appears that we need several inhabitants in the dependent sum of past intervals during which Alice lies in the sun and which contain subintervals in which she plays cards. We need the terminal points of the intervals to mark the beginnings and ends of the activities. Individuation of playing and lying-in-the-sun activities as events will then rely on a number of things, including their timing:

> ...a necessary condition for the identity of events is that they take place over exactly the same period of time. (Davidson 2001, p. 124)

So we have a map from $Activity$ to $Time_1$, which generates the dependent type of activities lasting over an interval, $i : Time_1 \vdash Activity(i)$, and we must only have identity of activities as occurring in the same fibre.

It would appear that we are fast approaching the material on *event nuclei* from chapter 2. Recall from there that an event nucleus is composed of a preparatory activity, culminating in an achievement, resulting in a change of state. This neatly matches our set-up. If an event nucleus takes place over an interval, some subdivision of it into two adjoining subintervals and the instant, or, perhaps better, momentary interval, where they abut, corresponds to the timing of its parts. So "He reached for the switch and lit up the room", covers the preparatory motion to the switch, culminating in its being flipped, resulting in light shining there. Such an event nucleus would be an element of the following type:

$$\sum_{i:Time_1} \sum_{(j,k):c^*(i)} \textit{reach-switch}(j) \times \textit{light-room}(k) \times \textit{flip-switch}([ej, bk]),$$

where much remains to be specified about these components as activities, achievements and changes of state.

Variations are possible. Perhaps one would like the achievement to take place in an instant rather than a brief interval. Linguists have wrestled with such choices informally:

> ...theories differ as to whether they take intervals as the basic temporal primitive, and regard events as durative, or whether they take instants as primitive and intervals as composite. Under the first view, a Vendlerian Activity like *running* would be represented as a transition, with a temporal and spatial extent. Under the second view, an Activity would be regarded as a progressive fluent, or property of a state, with the states that it characterizes being accessed via instantaneous incipitative events of *beginning running* and abandoned via terminative events of *stopping running.* (Vendler and his followers seem equivocal between these two interpretations.) Under the latter interpretation, the instantaneous incipitative and terminative events themselves correspond to Vendlerian Achievements, associated with further changes in fluents corresponding to consequent states, such as *running* and *having stopped running.* Vendlerian Accomplishments like *running to the bus stop* are then the composition of an Activity of *running with the goal of being at the bus stop*, the terminative Achievement of *stopping running* and the culminative achievement of *reaching the bus stop*, which in turn initiates its own consequent state of *being at the bus stop.* (Steedman 2012, p. 110)

There is plenty to do here, including making comparisons to other related approaches to temporal types, for instance, the book length treatment (Schultz and Spivak 2017). Something we might do to put their relative power to the test is to take up the challenge to represent complex pieces of script. Balbiani et al. (2011) propose the following examples:

- Ever since he met her for the first time, he could not stop thinking about her and kept calling her several times every night until she would give him a brush-off, and then after being silent for a while he would phone again.

- At the exact moment in which the train passes over the sensor, the rail crossing bar starts to close; the bar will start to open again a while after the train passes over the second sensor.

I think I have shown that the ingredients are available to represent such statements, and others such as the one from Chap. 2

- It took me two days to learn to play the Minute Waltz in 60 seconds for more than an hour.

But now I want to turn to a very recent effort to place modal type theory in a powerful general framework.

# 4 Mode theory

I will end this chapter with a brief discussion of a very interesting body of work which is currently unfolding. Recall the discussion in section 2 above, where I offered two options as to how to modify $\vdash p : P$ with a modality. The one that we went on to use was $\vdash p : \Box P$; the other was to tag the turnstile sign $\vdash_{nec} p : P$. Let us see now if we can make sense of this latter approach.

The idea here is that we have different arenas in which reasoning can take place. In these arenas different rules may apply as to the inferences permissible there. Then even though these inference rules vary, it is possible for communication to take place, or rather representation of another's reasoning in one's own terms, as we saw in the discussion of how the dog and person experts communicate in the first section. Something along these lines was suggested by Haskell Curry, as Fairtlough and Mendler explain:

> Curry's proposal was to take $\bigcirc\phi$ as the statement "in some stronger (outer) theory, $\phi$ holds". As examples of such nested systems of reasoning (with two levels) he suggested Mathematics as the inner and Physics as the outer system, or Physics as the inner system and Biology as the Outer. In both examples the outer system is more encompassing than the inner system where reasoning follows a more rigid notion of truth and deduction. The modality $\bigcirc$, which Curry conceived of as a modality of possibility, is a way of reflecting the relaxed, outer notion of truth within the inner system. (Fairtlough and Mendler 2002, p. 66)

We can illustrate this account in terms of the real suggestions of mathematical physicists Jaffe and Quinn (1993), who, alarmed at the lack of rigour they saw to be intruding into their field, proposed to have less rigorous physicist-style arguments for a mathematical result marked as 'theoretical'. We might say that if physicists have argued something to their own satisfaction, and mathematicians have not decided either way, then the latter should say that it is *possibly* the case. Similarly with roles reversed, if the mathematicians has proved a result for the physicist, or the physicist for the biologist, it should be marked as *necessarily* true.

Fairtlough and Mendler continue in their article by examining whether inference in the outer system might be represented by operators of the form

$$\bigcirc_K^L \phi \equiv K \supset (\phi \wedge L),$$

where we think of $K$ as expressing additional resources for reasoning in that system, and where we do not to establish $\phi$ if we can show that $L$ holds. Reasoners in the outer system have advantages over those in the inner system. They have more resources to deploy and they may find a condition obtains meaning that no further work towards the original conjecture is needed.

Now, of course there's a question of what is meant by speaking of the *same* proposition in different systems. To the extent that one takes the meaning of a proposition to be determined in part by the inference rules present, taking one across verbatim to another setting should alter its meaning. Indeed, this is so – a correct formalism needs to mark this. There may be instances where one category of inference is a subcategory of another, and it is easy there to slip into the practice of coercing the members of the subcategory to count as members of the full category, rather as one coerces a rational number to count as a real number. However, the general situation requires marking of translation between settings.

Very recent work[8] is pursuing this line of thinking in terms of a modal type theory in which a theory of the relevant modes provides one level of syntax on which can be built the reasoning pertaining to that mode. The deepest level of syntax specifies modes, associated to each of which is a class of types. Arrows between modes, say, $\alpha : p \to q$, correspond to adjunctions between these classes of types. Then we may have sequents of the following form, $A_p \vdash_\alpha B_q$.

In a sense we have already seen this mediation between arenas of inference when we took up the triple adjunction between slices,

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \underset{f_*}{\overset{\overset{f_!}{\to}}{\overset{f^*}{\leftarrow}}} \mathbf{H}/Y.$$

The *modes* here are variation over $X$ and variation over $Y$. They generate a pair of left and right adjoint couples, otherwise known as geometric morphisms. Now Licata et al. generalise this situation so as to take as the basic entity a single geometric morphism, that is an adjoint pair, between any two $(\infty, 1)$-toposes, with no requirement that they be slices of a common $(\infty, 1)$-topos. Then there may be multiple such mode morphisms between the various modes. Previous attempts had restricted to a partial order of modes, so at most one adjunction between any pair of modes.

This project has a very expressive scope and is looking to provide a syntactical framework for a wide range of modal type theories, including modal HoTT. The slogan here is that, where HoTT itself is the internal language of $(\infty, 1)$-toposes, modal HoTT is the internal language for collections of $(\infty, 1)$-toposes related by geometric morphisms. This is making sense of a range of previous attempts, and fits smoothly with the relevant mathematics. Harper's *computational* trinitarianism has become *homotopical* trinitarianism (Shulman 2018). As yet, there is no syntactical formulation which picks out those modal type theories which are only to be understood as describing the passage between slices. Modes which involve *variation over a type* make up only a portion of all mode theories, even if we can use such variation in devising models for modal theories, such as when Awodey and Kishida (2012) employ sheaf models to demonstrate

[8]See Licata and Shulman (2016), Licata et al. (2017).

their completeness for first-order modal logic. We will see examples of spatial modalities which are not to be construed as variation over types in the following chapter.

Integrating dependent type theory, and more generally HoTT, with the adjunctions generating the monads of computational effects and comonads of coeffects will allow enormous expressiveness, both in computer science and natural language semantics. Already pragmatic aspects of speech are being represented in terms of extensions of simple type theories by monads:

> Side effects are to programming languages what pragmatics are to natural languages: they both study how expressions interact with the worlds of their users. It might then come as no surprise that phenomena such as anaphora, presupposition, deixis and conventional implicature yield a monadic description. (Marsik and Amblard 2016, p. 259)[9]

If, as I argued in Chap. 2, natural language relies on constructions in dependent type theory, we should expect impressive achievements from its integration with monadic and comonadic adjunctions. We should be able to use this formalism to make common cause with Brandom's *pragmatic expressivism*.

Reading this latest work of Licata et al., there may also be opportunities to revisit Charles Peirce's *gamma* system of *existential graphs*. Peirce thought very highly of his work on these graphs. The *alpha* system corresponds to a propositional logic, while the *beta* system corresponds to a kind of first-order logic. These systems have been translated into a category theoretic framework by Brady and Trimble (2000a, 2000b). We see varieties of this diagrammatic reasoning calculus deployed elsewhere (Melliès and Zeilberger 2016). The gamma system is generally interpreted as Peirce's attempt to formulate a modal logic. The system was far from finished, and had gone through various phases by the time he discussed a late version in (Peirce 1906). Assertions of propositions on coloured sheets, chosen by Peirce in accordance with heraldic tinctures (jules, azure, argent, etc.), four tinctures per class (possibility - Color, intention - Fur, actuality - Metal) correspond to the modes of declaration. Given his success in independently formulating propositional and modal logic in ways that are only recently being recognised, perhaps it would not be such a surprise if Peirce was on the right track with his gamma system.

On that note we end the discussion of what a general modal HoTT should look like, and turn now to put one particular version to use in a formulation of modern geometry.

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[9]See also the very interesting work of Asudeh and Giorgolo (2016) which treats referential opacity using monads.

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