NOTES ON THE FOUNDING OF LOGICS AND METALOGIC: ARISTOTLE, BOOLE, AND TARSKI

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Abstract: This paper is more a series of notes than a scholarly treatise. It focuses on certain achievements of Aristotle, Boole and Tarski. The notes presented here using concepts introduced or formalized by Tarski contribute toward two main goals: comparing Aristotle's system with one Boole constructed intending to broaden and to justify Aristotle's, and giving a modern perspective to both logics. Choice of these three logicians has other advantages. In history of logic, Aristotle is the best representative of the earliest period, Boole the best of the transitional period, and Tarski the best of the most recent period. In philosophy of logic, all three were amazingly successful in having their ideas incorporated into mainstream logical theory. This last fact makes them hard to describe to a modern logician who must be continually reminded that many of the concepts, principles, and methods that are taken to be "natural" or "intuitive" today were all at one time discoveries.

Keywords: Counterargument, countermodel, formal epistemology, formal ontology, many-sorted, metalogic, one-sorted, proof, range-indicator, reinterpretation.

> [U]sing mathematical methods...has led to more knowledge about logic in one century than had been obtained from the death of Aristotle up to... when Boole's masterpiece was published.
>
> Paul Rosenbloom 1950

> [L]ogic ...analyzes the meaning of the concepts common to all sciences and establishes general laws governing these concepts.
>
> Alfred Tarski 1941/1994

> The truth or falsity of the particular premisses and conclusions is of no concern to logicians. They want to know only whether the premisses imply the conclusion.
>
> Elliott Mendelson 1987

Summary of the Article.

The Greek philosopher Aristotle (384-322 BCE), the English mathematician George Boole (1815-1864), and the Polish mathematician Alfred Tarski (1901-1983) are considered to be among the five greatest logicians of history, the other two being the German mathematician Gottlob Frege (1848-1925) and the Austrian mathematician Kurt Gödel (1906-1977). *Prior Analytics* by Aristotle and *Laws of Thought* by Boole are the two most important surviving original logical works before modern logic. Today it is difficult to appreciate the astounding permanence of what they accomplished without seeing their ideas surviving in the work of a modern master. Of the remaining three master logicians, Tarski is the most suitable for this purpose since he was the most interested in articulting the conceptual basis of logic, the most interested in history and philosophy of logic, and the only one to write an introductory book explaining his perspective in accessible terms.

All three would have deserved a place in any history of thought even had they not written a word on logic. All three were intellectual giants: Aristotle a prolific philosopher, Boole an influential mathematical analyst, Tarski an accomplished algebraist, geometer, and set-theorist.

> Aristotle founded logic as organon - as "formal epistemology". He was the first to systematically attempt a theory of demonstrative proof and the first to develop criteria of validity and invalidity of premise-conclusion arguments. He was the first to treat formal deduction and the first to treat independence proofs. Boole founded logic as science - as "formal ontology". He was the first to explicitly recognize the role of tautologies in deduction and to attempt a systematic treatment of "laws of thought" - his expression which was later used in essentially the same sense by Tarski. Tarski founded metalogic - the science explicitly conducted in the metalanguage and focusing among other things on syntax and semantics of idealized languages of sciences including logic. Much of Tarski's theory of metalogic, or "formal methodology", appeared in the 1933 truth-definition monograph, the 1936 consequence-definition paper, and the 1986 logical-notion-definition paper.

The three Tarski papers mentioned might be nearly sufficient to form a modern vantage point from which to survey and appreciate the remarkable accomplishments of

Aristotle and Boole. But it is also useful to consider other works, mainly his *Introduction to Logic and to the Methodology of Deductive Sciences 1941/1994* and his "Truth and Proof" *1969/1993.* This article presents a critically appreciative survey of aspects of the logical works of Aristotle and Boole most important in the development of modern mathematical logic, mainly as represented by the works of Tarski. I use only a small part of Tarski's contribution to logic; even the massive 1986 four-volume *Alfred Tarski: Collected Papers* does not exhaust his enormous output (*Corcoran 1991*). Tarski was probably the most prolific logician of all time.

Prior Analytics presented the world's first extant logical system. Its system, which could be called a logic today, involves three parts: a limited domain of propositions expressed in a formalized canonical notation; a method of deduction for establishing validity of arguments having unlimited numbers of premises; an equally general method of counterarguments or countermodels for establishing invalidity. Roughly speaking, these correspond respectively to the grammar, derivation system, and semantics of a modern logic. Aristotle's logical methodology is at the core of his lasting contribution to modern logic – not his system itself or his results. See my "Logical Methodology: Aristotle and Tarski", *JSL* 57 (1982) 374.

Laws of Thought presented the world's first symbolic logic. Boole's system, which does not fully merit being called a logic in the modern sense, involves a limited domain of propositions expressed in a formalized language as did Aristotle's. Boole intended the class of propositions expressible in his formalized language not only to include but also to be far more comprehensive than that expressible in Aristotle's. However, he was not entirely successful in this. Moreover, where Aristotle had a method of deduction that satisfies the highest modern standards of cogency, soundness and completeness, Boole had a semi-formal method of derivation that is neither sound nor complete. I say 'semi-formal' because Boole was far from clear about the algorithmic specifications of his own symbol manipulations. He did not even attempt a complete characterization of the logic he was developing. More importantly, Aristotle's discussions of goals and his conscientious persistence in their pursuit make soundness and completeness properties that a reader could hope, if not expect, to find his logic to have. In contrast, Boole makes it clear that his primary goal was to generate or derive solutions to sets of equations regarded as conditions on unknowns.[1]

The goal of gaplessly deducing conclusions from sets of propositions regarded as premises, though mentioned by Boole, is not pursued. Contrary to Aristotle, he shows

1 The idea that the premises of a valid argument could be regarded as equational conditions on unknowns while the conclusion could be regarded as a solution was revolutionary, unprecedented and totally original with Boole. The fact that it is not only gratuitous but flatly false is ignored by most modern writers, whether out of politeness or inattention. See my "Boole's Solutions Fallacy", *BSL* 10(2004) 305.

little interest in noting each law and each rule he uses in each step of each derivation. Gaplessness is designated in Aristotle's Greek by *teleios* (complete, fulfilled, finished, perfect, etc.). Tarski uses the German *vollständig* (complete, entire, whole, etc.) or the English 'complete'. But this aspect of logical inquiry is so far from Boole's focus that he has no word for it. Accordingly, the deductive part of Boole's algebraic equation-solving method is far from complete: associative laws are missing for his so-called logical addition and multiplication, to cite especially transparent but typical omissions.

As for a possible third part of Boole's logic, a method of establishing invalidity, nothing answers to this in the realm of equation-solving. Perhaps accordingly, essentially no discussion in Boole's writings concerns independence proofs demonstrating that a given conclusion is not a consequence of given premises: certainly nothing like a method of countermodels (counterinterpretations, or counterarguments) is to be seen. Boole never mentions the ancient problem of showing that the parallel postulate does not follow from Euclid's other premises.

Nevertheless Boole's formalized language went beyond Aristotle's by including tautologies, or – in Boole's phrase also used by Tarski – *laws of thought*, such as Boole's law of non-contradiction. As Boole's title emphasized, *Laws of Thought* brought into a logical system principles formerly thought to belong to philosophy, thereby opening the way to logic in Tarski's sense quoted above. The idea that logic, not metaphysics, establishes general laws involving concepts common to all sciences was not seriously pursued before Boole showed the way.

Boole's contribution widened logic and changed its nature to such an extent that he fully deserves to share with Aristotle the status of being a founding figure in logic. By setting forth in clear and systematic fashion the basic methods for establishing validity and for establishing invalidity, Aristotle became the founder of logic as *formal epistemology*. By making the first unmistakable steps toward opening logic to the study of "laws of thought"— tautologies and metalogical laws such as excluded middle and non-contradiction— Boole became the founder of logic as *formal ontology*.[2]

Neither Aristotle nor Boole had much insight into "logicography", the study of how a logical system is rigorously described, or into the basic methodological premises assumed in descriptions of logical systems. Consequently perhaps, neither made any effort to determine whether their theories of deduction were satisfied by the "proofs", or argumentations, that they themselves offered in support of their claims about the merits or adequacy of their systems. In short, neither applied to his own meta-deductions the

2 Ironically, twenty-five years after Boole presented an expanded view of logic that included concern with logical truth along with the traditional concern with logical consequence, Frege adopted a restricted view of logic focused on logical truth and ignoring, perhaps excluding, logical consequence (*Dummett 1973*, 432-3).

standards they themselves had set forth for deduction of object-language conclusions from object-language premises. In effect neither asked of himself whether he was practicing what he was preaching. And there is no sign in either that their own "proofs" had been deliberately or inadvertently affected by their own theories of proof. This lapse of self-awareness resembles a self-excepting fallacy such as an inconsistent accusation of inconsistency or a communication of a claim that communication is impossible. Even more remarkably, none of their immediate successors, neither supporters nor critics, understood the theories well enough to notice this now-glaring deficiency.

More generally, neither Aristotle nor Boole had much awareness of the details of their own respective frameworks of terminology. Neither had much insight into metalanguage or into what the epistemological-ontological status of its concepts and entities may be, into what we now call metalogic, in the broad sense, or *formal methodology*.

More than any other logician, Tarski identified and addressed the need for a systematic metatheoretic study of logical theories (*Corcoran 1983a*). His 1933 truth-concept monograph made clear the conception of a formal-sentence language as a subset of a universe of strings – concatenations of characters over a finite "alphabet".[3] Without precedent in the history of logic, it established axiomatic foundations of what is now called string theory – the mathematical theory presupposed by any definitions of fundamental concepts such as "truth", "consequence", "logical constant" and "proof". As Tarski said, his axioms were based on "intuitions" involved in such fundamental activities as symbol manipulation in algebra, numerical calculations, and even composing sentences. His monograph, a penetrating and innovative philosophical, mathematical and logical work, clarified, integrated and advanced awareness of basic concepts, principles, and methods of metalogic or formal methodology to such an extent that it has come to occupy a place in the history of logic comparable to *Prior Analytics* and *Laws of Thought*. It contains the world's first axiomatically presented metatheory. Moreover, as pointed out by Gupta (*1980*, 126): "Tarski made truth a scientifically respectable notion; his achievement was recognized by philosophers at the time". Gupta (ibid.) went on to quote Popper: "Thanks to Tarski's work, the idea of objective or absolute truth – ... correspondence with facts – appears to be accepted ... by all who understand it". Tarski deserves to be regarded as the founder of logic as formal methodology.

Moreover, although Aristotle took formal deducibility as a methodological criterion of consequence as did Boole and Tarski, and although Aristotle took presentation of a counterargument as a methodological criterion of independence (non-consequence) as did Tarski (but surprisingly not Boole), neither Aristotle nor Boole attempted a definition of the fundamental concept of consequence as Tarski did. Neither Aristotle nor

3 Tarski *1933* calls characters *signs* and strings *expressions* (*1956/1983*, 166). Church (*1956*, 48f.) calls characters *symbols* and strings *formulas*.

Boole gave more than tantalizing hints as to what they *meant* by phrases such as 'logically implies', 'follows of necessity from', ' is a logical consequence of', and the like. Although Tarski was neither the first person to notice the need for a definition of consequence nor the first to attempt a definition[4], he constructed the conceptual framework within which modern definitions are situated, and his 1936 definition formed the paradigm for later attempts.

Neither Aristotle nor Boole could have achieved much in logic without grasping what it means to say that a given proposition is a consequence of a given set of propositions, but it did not occur to either to attempt a formally and materially adequate definition. Both had methodological criteria without having definitions of the concepts "criterionized". In the 1936 consequence-definition paper Tarski identifies this chasm in pre-modern logic and, in the eyes of many, filled the chasm with a formally and materially adequate model-theoretic definition of consequence. Incidentally, one of the most fundamental achievements of Tarski's 1933 truth-concept paper was to clarify and rigorously exemplify the important but subtle distinction between definitions and criteria (or decision-procedures).

Tarski continued to remind his readers of what is at issue in making this distinction. In his 1969 expository article "Truth and Proof" (*1969/1993*, 116), he wrote:

> Some philosophers and methodologists of science are inclined to reject every definition that does not provide a criterion for deciding whether any given particular object falls under the notion defined or not. In the methodology of empirical sciences such a tendency is represented by the doctrine of operationalism; philosophers of mathematics who belong to the constructivist school seem to exhibit a similar tendency. In both cases, however, the people who hold this opinion appear to be in a small minority. A consistent attempt to carry out the program in practice (that is, to develop a science without using undesirable definitions) has hardly ever been made. It seems clear that under this program much of contemporary mathematics would disappear, and theoretical parts of physics, chemistry, biology, and other empirical sciences would be severely mutilated. The definitions of such notions as atom or gene as well as most definitions in mathematics do not carry with them any criteria for deciding whether or not an object falls under the term that has been defined. Since the definition of truth does not provide us with any such criterion and at the same time the search for truth is rightly considered the essence of scientific activities, it appears as an important problem to find at least partial criteria of truth.

The partial criterion he went on to propose is proof: in order to recognize a proposition as true it is sufficient to deduce it from propositions known to be true. This point is highlighted by his title "Truth and Proof". As Tarski repeatedly emphasized, having a

4 Bernard Bolzano (1781-1848) is often regarded as having preceded Tarski in this regard. See his *1837/1972*, Sect. 155.

definition of truth is not the same as having a criterion of truth; understanding what it means to say that a proposition is true is not the same as having a method for determining whether it is true. This ends the summary.

> Aristotle's *Prior Analytics*
> The principles of logical inference are universally applied in every branch of systematic knowledge. For over two thousand years mathematicians have been making correct inferences of a systematic and intricate sort, and logicians and philosophers have been analyzing ... valid arguments. It is, therefore, somewhat surprising that a fully adequate formal theory of inference has been developed only in the last three or four decades. In the long period extending from Aristotle ..., much of importance ... was discovered about logic by ancient, medieval and post-medieval logicians, but the most important defect in this classical tradition was the failure to relate logic as the theory of inference to the kind of deductive reasonings that are continually used in mathematics.
> *Suppes 1957/1999, xi.*

Early use of deduction in mathematics began long before Aristotle. It has been traced by Immanuel Kant (1724-1804) (*1781/1887*: B, x-xi) as far back as Thales (625?-547? BCE), who is said to have deduced by logical reasoning from intuitively evident propositions the conclusion, far from intuitively evident, that every two triangles, no matter how different in size or shape, nevertheless have the same angle-sum[5]. Thales' result, reported centuries later as Theorem I.32 in Euclid (*c.300 BCE/1908/1925*), was strikingly important at the time and is still fundamental in geometry and trigonometry. Unfortunately, today it is often taken for granted without thought to how stunning it once was, what might have led up to it, how it might have been discovered, how it might have been proved to be true, or even whether there might have been one or more alleged proofs that were found to be fallacious before a genuine proof was discovered[6]. This is one of Aristotle's favorite examples of the power of logical deduction[7] (*Prior Analytics*, 41a26, 46a14, 67a13-30, *Posterior Analytics*, 85b5, 85b11, 85b38, 99a19, *Smith 1989*, 164). Another favorite example of geometrical demonstration is an indirect proof (*reductio ad absurdum*) that the diagonal of a square is incommensurable with the side

5 As is clear from the context, the abstract noun 'deduction' is used for the rational process whereby one proposition, the "conclusion", is *deduced from* a set of one or more propositions, the "premise set", that necessarily logically implies the conclusion. In this sense a person knows that a premise set implies a conclusion by deduction, by deducing the latter from the former, often in many connected, consecutive steps.

6 Nevertheless, there are accomplished mathematicians who have felt the importance of this theorem. Its clarification was one goal of Hilbert's *1899* axiomatization of geometry, as he revealed in a letter to Frege dated 29 December 1899 (*Frege 1980*, 39).

7 The role of deduction in early mathematics is discussed in *Heath 1921/1981*, which goes beyond Kant in its statement that "geometry first becomes a deductive science" with Thales' contributions (Vol. I, 128).

– mentioned explicitly at least four times in *Prior Analytics* (48a33-37, 66a14, 50a37, 65b17) and also elsewhere (e.g. *Sophistical Refutations*, Ch. xvii).

Prior Analytics addressed the two central problems of logic as formal epistemology: how to show that a given conclusion follows from given premises that formally imply it and how to show that a given conclusion does not follow from given premises that do not formally imply it. Aristotle wanted a decisive test or criterion for determining if the conclusion follows and also one for determining if the conclusion does not follow. Using other equally traditional terminology, Aristotle's problems were how to establish validity and how to establish invalidity of an arbitrary argument[8], no matter how many premises or how complicated its propositions.

To understand how Boole changed the field of logic, note that Aristotle's logic was confined to logical epistemology, to concern with determining validity and invalidity of premise-conclusion arguments. Today this is but one concern of logic despite passages such as the above Mendelson quote (*1987*, 1) suggesting that this is still logic's exclusive concern.

Aristotle could not have failed to notice that his two problems are more general than the two he solved in detail (*Rose 1968*, 11, *Sophistical Refutations,* Ch.34). In particular his definition of deduction (*sullogismos*, syllogism) is broad enough to include any chain of reasoning whose conclusion follows logically from its premise-set, regardless of number of premises, complexity of propositions, or subject-matter. Moreover, his geometrical examples show that he did not intend 'syllogism' to be limited to artificially restricted simple cases. According to Rose (loc. cit.), Aristotle's "definition of syllogism at 24b18-20 seems so broad as to include any valid inference: A syllogism is an argument (*logos*) in which, when certain things are assumed, something different from what is assumed follows by necessity from the fact that these things are so".

Aristotle did not solve the problem of formal epistemology in its full generality, nor did he claim to, contrary to what Kant and a few logicians seemed to have thought. He never thought that he had completed logic. The full problem has still not been solved despite occasional statements that it has. Aristotle would never have written

8 An argument or, more fully, a *premise-conclusion, argument* is a two-part system composed of a set of propositions called the premises and a single proposition called the conclusion. An argument is *valid* if the conclusion is logically implied by the premise set, and it is *invalid* otherwise, i.e., if the conclusion contains information beyond that in the premise set. Whether an argument is valid is totally objective having no dependence on any subjective judgment or mental operations. For further discussion of logical terminology see *Corcoran 1989*. In this sense, an argument contains no steps of deductive reasoning – in order to know that an argument is valid a person must produce an *argumentation*, often involving many steps and many intermediate conclusions. Of course, there are other senses of the word 'argument'. In fact mathematicians often use it as a synonym for 'argumentation' and in that sense an "argument" contains a chain of steps of reasoning going beyond the premises set and ending with final conclusion (*Corcoran 1972a*).

what two respected American logicians wrote: "Given premises *and* conclusion, logic can determine whether this conclusion follows" (Lewis and Langford *1932*, 72). Few logicians agree with this statement today.[9]

Moreover, as Boole saw, the case of the problem of formal epistemology that Aristotle did completely solve is a small fragment of what has been accomplished by modern logicians. The class of arguments he treated is just on the threshold of logic. Even though Boole made great progress, the class he treated is still very small by modern standards. Bertrand Russell (1872-1970) was not exaggerating when he wrote (*1945*, 202): "Anyone in the present day who wishes to learn [modern] logic will be wasting his time if he reads Aristotle or any of his disciples".

It is of course impossible to identify or construct a method or criterion for recognizing instances of a relation or property without understanding the relation or property. Moreover, in order to understand a relation or property it is not necessary to have a formally and materially adequate definition of it. But such understanding is necessary for discovering a definition. In order to define "human being" it is necessary to understand the concept, to know what human beings are. Likewise, in order to understand a concept or have a definition of it, it is not necessary, or even usual, to have a criterion for determining that it applies in an arbitrary case. As Tarski noted, an adequate definition need not provide a criterion, and normally does not. In connection with truth definitions he said (*1969/1993*, 116): "Whatever may be achieved by constructing an adequate definition of truth ..., one fact seems to be certain: the definition does not carry with it a workable criterion for deciding whether ... sentences ... are true (... it is not designed for this purpose)." In addition Tarski makes other closely related points (e.g. *1969/1993*, 116-125) that were also unprecedented in the logical writings of Aristotle or even Boole.

Although Aristotle is clear in presenting his two procedural criteria[10] for what amounts to validity and for what amounts to invalidity, nevertheless he gives no hint of

9 The quoted Lewis-Langford sentence is ambiguous. Mathematical logicians will realize that of its two principal readings one contradicts Gödel's 1931 Incompleteness Theorem and one contradicts Church's Theorem. But there are other independent reasons for disagreeing with the Lewis-Langford statement.

10 The word 'criterion' has an ambiguity that is worth attention. Although I have mentioned that Aristotle had a deductive criterion for validity and a radically non-deductive criterion of invalidity, it might be thought that a criterion for a given property, say validity, is automatically also a criterion for its complementary opposite, in this case invalidity. There are two senses of the word 'criterion': a weaker and a stronger. Following Tarski and others, I use the word in the weaker sense in which a criterion for a property is a performable method for determining that a given entity has the property if it does. It is widely recognized that such a method may not determine anything if applied to an entity not having the property. In the stronger sense a criterion is a method which determines whether or not the property applies. In the stronger sense, but not in the weaker, every criterion for a given property is automatically a criterion for the complementary opposite. Hilbert and Ackerman (*1928/1938/1950*, 113) use 'criterion' in the stronger sense.

a *definition* of either concept, or of any functionally equivalent concept. More specifically, Aristotle nowhere recognizes a need to define the metalogical concepts "implies", "consequence", "follows from", "valid", or any of the concepts regarded as interdefineable with them.

This raises the question of whether Aristotle may have subscribed in regard to these concepts to an operationalistic or behavioristic positivism (*Audi 1999*, 632) that "identifies" concepts, or definitions of concepts, with procedures for detecting their instances, e.g. to a view that would take the meaning of 'valid' and that of 'invalid' to be respectively the method by which an argument is recognized as being valid and that by which one is recognized as being invalid. The hypothesis would be that Aristotle held an algorithmic or procedural view of metalogical concepts, a view that would repudiate or render futile any need for an explicit definition of validity or invalidity such as that proposed in the 1936 consequence paper. There is no evidence of this. Aristotle's twin problems invite definitions that he does not supply. The first page of *Prior Analytics* represents Aristotle as a person unusually alert to the importance of defining his basic concepts. His treatment of "term", "premise" "syllogism", "complete" and so on practically promise the modern reader that definitions of the other basic logical concepts such as "argument", "valid", "invalid", "implies" and "consequence" will appear at the proper place. But such explicit definitions have not been found, not in that work, not in the *Organon*, indeed not in the Aristotelian corpus.[11] In view of Aristotle's unusual sensitivity to the importance of definitions, the absence of discussion of definition for any metalogical concept such as "follows of necessity from" (or "is a consequence of") or its negative must be regarded as significant. The issue of whether the absence of this definition was inadvertent or deliberate must remain for future scholars to decide. Moreover, the absence of definitions only increases the significance of the two criteria that he did give us.

For his initial partial solution to the twin problems, Aristotle presented the world's first extant logical system. Aristotle's logic was not a comprehensive "grand logic" such as that of *Frege 1879* or *Whitehead-Russell 1910.* There is no reason to think that it was intended as anything more than a simple exemplification of his logical methodology. His research strategy was like that of Archimedes in dealing with hydrostatics, a carefully delimited part of fluid mechanics that treats the special case of a fluid at rest. Hydrostatics is a necessary preparation for the wider study that includes hydrodynamics, the study of fluid in motion. The strategy is to deal with a simple, perhaps idealized or even fictitious, case first before attacking the problems in their full generality or perhaps before attacking more complicated special cases.

11 There are passages that point to Aristotle's understanding of the consequence or implication relation. In Chapter 17 of *Sophistical Refutations* 176a32, he writes that the consequences of a thesis appear to be parts of the thesis itself.

Archimedes never says that his hydrostatics is a comprehensive theory of fluid mechanics or that it is more than a preparation for the wider study that includes hydrodynamics. Likewise Aristotle nowhere says that his particular system, the so-called assertoric categorical syllogistic, was a comprehensive and exhaustive logic. Kant is frequently criticized, even ridiculed, for making this claim on Aristotle's behalf (*Bolzano 1837/1972,* 9, *Hilbert and Ackermann 1928/38/50,* 55, *Cohen and Nagel 1934/62/93*, 110). Aristotle never said, as some of his less careful but more enthusiastic followers have said, that the syllogistic is adequate for propositional logic. And Aristotle certainly never said anything comparable to the pathetically irresponsible remark of John Stuart Mill (*1843*, Vol. I, 191) that "The whole of Euclid, for example, might be thrown without difficulty into a series of syllogisms, regular in mood and figure."

The above paragraph addresses the question already raised by logicians and philosophers such as da Costa and Santos (per. comm.) of how to explain the gross inadequacy of the theory of propositions (or formal grammar) underlying Aristotle's syllogistic and the apparently simplistic narrowness of the class of arguments it treats. These inadequacies have been widely noticed. They are nearly impossible to miss.[12] They are often not described as virtues, which they clearly are, but as faults to be charged against Aristotle. Regrettably, I did so myself (*1974a,* 123) – ironically, in a paper that was somewhat predicated on the fact that simplifications, even oversimplifications, and unrealistic idealizations play legitimate and important roles in scientific and humanistic progress (ibid. 86).

Addressing this and related issues in a 1989 lecture "Logical methodology: Aristotle and Tarski", I wrote (*1992*, 374):

> Logical methodology is the study of methods in logic, most notably, methods for determining whether a given conclusion is a consequence of or is independent of a given premise-set. Aristotle pioneered logical methodology and Tarski was its most vigorous modern proponent. Aristotle's most original, most influential, and most lasting contribution to logic was his methodological theory, not the logical system known as syllogistic logic, which he considered one of many exemplifications of logical methodology, by no means exhaustive of logical phenomena. Methodology, originally discovered by Aristotle, was independently discovered and expanded by modern logicians including Tarski. The method of deduction for establishing consequence and the method of reinterpretation for establishing independence both originated with Aristotle, who had already distinguished consequence from deducibility – notwithstanding the fact that Aristotle's languages were always fully interpreted. In Tarski's treatment deduction comes to have a syntactic character never envisioned by Aristotle, and consequence comes to have a semantic character hardly conceivable without set theory. In this pa-

12 It is hard to imagine what Russell (*1945*, 199) had in mind in his Chapter 22 "Aristotle's Logic" when he wrote: "Of course it would be possible to re-write mathematical arguments in syllogistic form ..."

per some of Tarski's most philosophical contributions to logic are explained as natural refinements of logical methods whose epistemic purpose has remained constant since Aristotle.

As mentioned above, Aristotle's logical methodology is at the core of his lasting contribution to modern logic. Tarski never mentions details of the so-called Aristotelian syllogistic system, yet he fully embraces the central ideas of Aristotle's logical methodology: recognizing validity by a formal step-by-step deduction and recognizing invalidity by a counterargument, or by what amounts to a counterargument (*1941/1994*: esp. Chs.II and VI). This acceptance of Aristotle's two-part methodology of deduction and counterinterpretation continues in the most recent logic texts, e.g. *Boolos, Burgess, and Jeffrey 2002* and *Goldfarb 2003*. To reiterate: Aristotle's system, which is somewhat similar to a modern logic, involves three parts[13]: a limited domain of propositions expressed in a formalized language[14], a formal method of deduction for establishing validity of arguments having an unlimited number of premises, and an equally general method of countermodel or counterargument for establishing invalidity. The underlying principles for both methods continue to be accepted even today (*Corcoran 1973*, 26-26, *Corcoran 1992, 374*).

Aristotle achieved logical results that were recognized and fully accepted by many subsequent logicians including Boole.[15] The suggestion that Boole rejected Aristotle's logical theory as incorrect is without merit despite the fact that Boole's system may seem to be in conflict with Aristotle's. Interpretations of *Prior Analytics* established the paradigm within which Boole's predecessors worked, a paradigm which was unchallenged until the last quarter of the 1800s after Boole's revolutionary insights had taken hold. The origin of logic is better marked than that of perhaps any other field of study — *Prior Analytics* marks the origin of logic (*Smith 1989*, vii and *Sophistical Refutations*: Ch. 34). The system is presented almost entirely in the space of about 15 pages in a recent trans-

13 The tripartite character of modern logics is so well established that it is more often presupposed than specifically noted or discussed. But see *Corcoran 1973*, 27-30, *Corcoran 1974*, 86-87, and *Shapiro 2001*.
Aristotle's text does not mention the tripartite nature of his own system nor does it formulate its aims in the way stated here. For a discussion of aims a logical theory may have, see *Corcoran 1969*.

14 The locus classicus for the expression 'formalized language' is probably the truth-definition paper by Alfred Tarski (*1933/1956/1983*). Today, however the expression carries a somewhat more restrictive and more natural sense of a system of meaningful sentences (to use Tarski's locution) in which the logical forms of the propositions expressed is represented by the grammatical forms of the sentences (*Church 1956*, 2, 3). The adjective 'formalized' carries no pejorative connotations such as is often attached to the adjective 'rationalized' and always attached to 'formalistic', 'moralistic', 'puristic' and 'scientistic'. Mary Mulhern pointed out the need for this clarification (per. comm.).

15 Its full acceptance today is contingent on proper translation into standard logics. This issue is treated on the last page of this article.

lation, Chapters 1, 2, and 4 through 6 of Book A, and it is discussed throughout the rest of A, especially Chapters 7, 23-30, 42 and 45. Chapters 8-22 are mainly passages interpolated at a later time (*Corcoran 1974*:88).

Prior Analytics presupposes no previous logic on the part of the reader. However, it does require knowledge of basic plane geometry, including ability and experience in deducing non-evident theorems from intuitively evident premises such as those taken as axioms and postulates a generation or so later by Euclid (fl. 300 BCE). Especially important is familiarity with *reductio ad absurdum* or indirect deduction. Aristotle repeatedly refers to geometrical proofs, both direct and indirect[16]: many specific examples are cited below. It also requires readers to ask themselves what is demonstrative knowledge, how do humans acquire it, what is a proof, and how is a proof made.

Proof Produces Knowledge: The Knowledge-Through-Proof Thesis.

For Aristotle a proof proves (to those for whom it is a proof) that its conclusion is true. For him, and for many modern logicians, in the strict sense of 'prove', there is no way to prove a false proposition.[17] Since 'prove' and it cognates are interchangeable respectively with 'demonstrates' and its cognates, this means that for Aristotle a demonstration demonstrates (to those for whom it is a demonstration) that its conclusion is true. Every proof produces demonstrative (apodictic) knowledge of its conclusion. The Socratic proof/persuasion distinction reappears in Aristotle along with the knowledge/belief distinction. A proof does not merely persuade; it produces knowledge, not merely belief. After all, fallacious argumentations produce belief.

The fundamental *knowledge-through-proof* thesis, that proof establishes knowledge, was never doubted by Boole although he said little worth repeating on the topic. In full awareness that the thesis had been challenged in the years after Boole, Tarski reaffirmed and endorsed it in some of his writings. For example, in his article "Truth and Proof", he wrote (1969/1993, 117):

> The notion of proof ... refers just to a procedure of ascertaining the truth ... This procedure is an essential element of ... the axiomatic method, the only method now used to develop mathematical disciplines.

16 The importance of the axiomatic geometry for understanding Aristotle's logic has been overlooked by historians of logic. Lukasiewicz *1951* simply overlooks it (*Corcoran 1974*, 96, 125) and Bochenski takes our breath away by stating (*1956/1961*, 283) that "Euclid was the first to carry out the idea of an axiomatic system in mathematics" and later to call Euclid "the Father of Geometry" (*1956/1961*, 536), apparently forgetting Thales, Pythagoras and countless other pre-Euclidean geometers. One hypothesis is that Lukasiewicz and Bochenski were so focused on regarding logic as formal ontology as opposed to formal epistemology that they ignored contrary facts.

17 Cohen and Nagel (*1934/1962/1983*, xix, 6f, 19, 22, passim) are exceptions. And, through use of the unfortunate expression 'proof from hypotheses', Church (*1956*, 87) is an apparent exception.

To be sure, Tarski realized that this thesis was not universally accepted, but he was not inclined to stir controversy gratuitously. Consequently many of his writings do not broach the issue, and some of them seem to evade it. Nevertheless, some of his best known articles could hardly have been written by a person who doubted that knowledge is achieved through genuine proof. Examples that readily come to mind include the truth-definition paper, the consequence definition paper, and "Truth and Proof". A person skeptical about the possibility of knowledge could hardly be expected to take the existence of paradoxes as challenges. On the contrary, such a person should embrace paradox as confirmation or even "proof" of skeptical suspicions.[18]

Aristotle's Truth-and-Consequence Conception of Proof.

> Deduction should be discussed before proof. Deduction is more general. Every proof is a deduction, but not every deduction is a proof.
> *Aristotle, Prior Analytics.*

For Aristotle a proof begins with premises that are known to be true and shows by means of chaining of evident steps that its conclusion is a logical consequence of its premises. Thus a proof is a step-by-step deduction whose premises are known to be true. In Aristotle's words: "Every proof is a deduction, but not every deduction is a proof" (*Prior Analytics*, 24a7, Gasser *1991*, 232). For him, one of the main problems of logic (as opposed to, say, geometry) is to describe in detail the nature of the deductions and to say how the deductions come about.

Thus, at the very beginning of logic we find what has come to be known as *the truth-and-consequence conception of proof*: a proof is a discourse or extended argumentation which begins with premises known to be *truths* and which involves a chain of reasoning showing by evident steps that its conclusion is a *consequence* of its premises. The adjectival phrase 'truth-and-consequence' is elliptical for the more informative but awkward '*established*-truth-and-*deduced*-consequence'. Nevertheless, we still find otherwise competent logicians seeming to espouse the absurdity that a proposition that is a consequence of truths is proved. This is similar to Quine's inadvertent statement (1950/1959, xv) that "If one statement is held to be true, each statement implied by it must also be held to be true" – which could be improved by changing 'implied' to 'held to be implied'.

18 Church also accepted the knowledge-through-proof thesis, but sometimes he was not as clear as he could be – he did not explicitly distinguish between admitting a premise and knowing it to be true nor did he explicitly distinguish between subjective conviction and genuine knowledge. He once wrote (*1956*, 53-4): "Indeed it is essential to the notion of proof that, to any one who admits the presuppositions on which it is based, a proof carries final conviction."

Over and above the premises and conclusion, every proof has a chain-of-reasoning that shows that the (final) conclusion fc follows logically from the premises. An Aristotelian direct proof based on three premises p1, p2, and p3, and having a chain-of-reasoning with three intermediate conclusions ic1, ic2, and ic3, can be pictured as below.

p1
p2
p3
?fc
ic1
ic2
ic3
fc
QED

Note that in such an Aristotelian proof the final conclusion occurs twice: once as a goal to be achieved and once as it has been inferred.[19] This picture represents only a direct proof. The picture is significantly different for indirect proofs, for *reductio-ad-impossibile.* To represent a simple indirect proof, ~fc (a contradictory opposite of the final conclusion) is added as a new assumption and the X indicates that the last intermediate conclusion ic3 contradicts one of the previous intermediate conclusions or one of the premises.

p1
p2
p3
?fc
~fc
ic1
ic2
ic3
X
QED

19 If we use the word 'argumentations' as in footnote 8 for the genus of which proofs form a species, i.e., for a three-part system composed of a premise-set, a conclusion to be reached, and a chain-of-reasoning, then the Aristotelian truth-and-consequence proof admits of easy definition in the traditional genus-et-differentia form. In order for an argumentation to be a *proof* (of its conclusion to a given group of people) it is necessary and sufficient for the premises to be known to be true (by the group) and for its chain-of-reasoning to show (to the group) that its conclusion is a logical consequence of its premises. Every proof has established premises and cogent reasoning. In order for an argumentation to be *fallacious* (to a given group of people) it is necessary and sufficient for the premises to be not known to be true (by the group) or for its chain-of-reasoning to not show (to the group) that its conclusion is a logical consequence of its premises. Every fallacious "proof" has faulty premises or faulty reasoning.

As suggested above, there is no way to understand Aristotle's logic without being aware of his rigorous training and deep interest in geometry. Aristotle spent seventeen years in Plato's Academy, whose entrance carried the motto: *Let no one unversed in geometry enter here*. The fact that axiomatic presentations of geometry were available to the Academy two generations before Euclid's has been noted often. Ross points out that "there were already in Aristotle's time *Elements of Geometry*" (*1923/1959*, 47). According to Heath (*1908/1956*, Vol. I, 116-7), "The geometrical textbook of the Academy was written by Theudius of Magnesia ... [who] must be taken to be the immediate precursor of Euclid, and no doubt Euclid made full use of Theudius ... and other available material".

In the article "Euclid", *Encyclopedia Britannica 2004* reports: "The latest compiler before Euclid was Theudius, whose text was used in the Academy and was probably the one used by Aristotle." Hippocrates of Chios (fl. 440 BCE) might have compiled the first axiomatic geometry text (*Encyclopedia Britannica 2004*). Alexander of Aphrodisias (*c.200*, Preface) says that geometry is an application of Aristotle's logic.

It would be an exaggeration to suggest that an understanding of Aristotle's goals and achievements in *Prior Analytics* would be impossible without attention to his many geometrical examples. Nevertheless, it is fair to criticize the many works on the history of logic which do not mention these examples or which do not mention the axiomatic treatments of geometry available to Aristotle. Even more deserving of criticism are those works on the history of logic, or on logic itself for that matter, which date the origin of the axiomatic method after Aristotle rather than before his time (e.g., *Fraenkel 1958/1991*, 3, *Goldfarb 2003*, 79). Although Tarski does not make the above errors, he still fails to trace the axiomatic method back any farther than Euclid (*1946/1995*, 120), and he mentions Aristotle as the creator of logic without indicating Aristotle's interests in geometrical proof or in the axiomatic method (*1946/1995*, 19). It would be difficult to improve on Robin Smith's succinct description of the *Analytics* (*1989*, xiii):

> The Content of the Prior Analytics: From Aristotle's viewpoint, the *Prior Analytics* is simply the first part of the *Analytics:* the second part is the work known to us as the *Posterior Analytics.* The subject of the latter is proof or demonstration (*apodeixis*), that is, argumentation which produces scientific understanding (*epistēmē*). Aristotle makes it clear from the start that this is also the subject of the entire *Analytics*, and thus of its first part, the *Prior*. Aristotle conceives of a demonstrative science as a system of demonstrations, which in turn are a type of deduction (*sullogismos*). Accordingly, the *Prior Analytics* gives an account of deductions in general and the *Posterior* discusses the specific character of those deductions which are demonstrations.

Smith's interpretation echoes that of Alexander, who first explained that Aristotle's *Prior Analytics* and *Posterior Analytics* are about deductions and proofs, respectively, and then adds that the first is called *Prior* because its subject, deduction, "is by nature" prior to the subject, demonstration (*c.200*, Preface). Likewise, Smith's view is consistent with the opinion expressed by Kneale and Kneale (*1962*, 113) that "…Aristotle's logical theorizing appears to have been prompted mainly by thought about demonstration as it occurs, for example, in geometry…".

The *Analytics* as a whole forms a treatise on scientific knowledge (24a, 25b28-31). On Aristotle's view each item of scientific knowledge is initial, known directly in itself by experience in the broad sense (*epagoge* sometimes translated 'induction'), or else it is derivative, deduced ultimately from initial items known in themselves[20] (*Posterior Analytics*, *passim*, esp. II, 19). The *Posterior Analytics* deals with the acquisition and deductive organization of scientific knowledge. It is the earliest general treatise on axiomatic method in sciences. The *Prior Analytics*, on the other hand, develops the underlying logic used in the inference of deductively known scientific propositions from those known in themselves (*Corcoran 1974*, 91).

Aristotle's word for a proof or demonstration was *apodeixis*, from which we get our word 'apodictic'. He called an ultimate premise of a demonstration an *archê apodeixeôs*, literally "demonstration beginning", "beginning of a demonstration", and less literally, "demonstrative or apodictic beginning" (*Posterior Analytics* Article I. Ch.2.72a7). According to D. Hitchcock (per. comm.), Aristotle seemed to have no special word for the conclusion of a proof per se; he used the same word *sumperasma*, literally "end", "conclusion", for the conclusion of a deduction whether or not it was a proof. Later writers who accepted Aristotle's insights adopted other terminology for the distinction between the original premises and the conclusions of the demonstrations – some called the former axioms and called the latter theorems (*Boole 1847*, 13, *Tarski 1956/1983*, 465-6, 497); some called the former primary or ultimate propositions, or laws, and the latter secondary (*Boole 1847*, 18, *1848*, 187, *1854*, 240). The fact that so many different and even contrasting pairs have been used to mark variations on this conceptual contrast may signal both its importance and the discomfort that has been felt regarding it.

As far as can be determined from his published writings, Boole subscribed to the truth-and-consequence conception of proof. We do know that he never disputed it. In the late 1970s and early 1980s toward the end of his life, Tarski also accepted it (per. comm.), although he does not seem to explicitly endorse it in his published writings. However, he was clear that the truth-and-consequence conception of proof is not implied by the knowledge-through-proof thesis. On at least two occasions separated by

20 For Aristotle's thoughts on how axioms are known, see Hintikka's masterful *1980* essay "Aristotelian induction" that I summarized in 1982 in *Mathematical Reviews*, where I wrote: "The fact that Aristotle traced all knowledge of axioms to experience is already clear and well known to scholars despite contrary information persistently appearing in popularizations".

years, he said that he thought Euclid was actually "proving theorems" in some sense but not deducing them from premises and definitions.

Tarski agreed with modern scholars that Euclid "smuggled premises", i.e., used propositions not among his stated premises and definitions (Euclid I.1), and he was aware of occasions where Euclid proved something other than what is claimed (Euclid IX.20), thereby appearing to commit the fallacy of "ignoratio elenchi" or wrong-conclusion. But Tarski went on to point out that Euclid never said that his goal was to show that the theorems were logical consequences of his basic propositions (per. comm.). If Euclid accepted the knowledge-through-proof thesis while rejecting the truth-and-consequence conception, he was not the only important figure in the history of logic to do so.

This view of Euclid might seem to conflict with what Tarski wrote in the 1960s about "the science of geometry as it was known to ancient Egyptians and Greeks in its early pre-Euclidean stage". He wrote of the pre-Euclidean period: "A sentence was accepted as true either because it seemed intuitively evident or else because it was proved on the basis of some intuitively evident sentences, and thus was shown by means of intuitively certain argument to be a consequence of these other sentences" (1969/1993, 117). Thus, Tarski seems to be attributing to pre-Euclidean geometers approximations of both knowledge-through-proof thesis and the truth-and-consequence conception of proof. However, as Tarski goes on to explain, he thinks that these views were not rigorously applied.

Aristotle's Logic is Not Limited to Proof.

But the logic of the *Prior Analytics* is not designed solely for such demonstrative use (cf., e.g., 53b4-11; *Kneale and Kneale 1962*, 24). On the contrary, Aristotle subscribes to what may be called the *epistemic-neutrality thesis*, the thesis that the process of deduction is the same whether the premises are known to be true or whether the premise set contains a mere hypothesis not known to be true and not known to be false, or even if the premises are all known to be false. In the beginning of *Prior Analytics* (24a25-30, *Smith 1989*, 1), Aristotle says that whether a person is trying to demonstrate a conclusion from premises known to be true or trying to deduce it from premises merely assumed for purposes of reasoning "it will make no difference as to whether a deduction comes about …"

The three points made in the last few paragraphs are closely related, but, since the problems and tensions they bring reverberate throughout history, they should be amplified to make their independence evident.

One point is that scientific knowledge has an initial-plus-derived structure (Cf. *Corcoran 1994*, 16). This is not to say that initially known truths are all obvious to everyone, or even to everyone sufficiently acquainted with the subject matter. Likewise, this is not to say that derivations from initially known truths are all obviously cogent to everyone, or even to everyone sufficiently acquainted with the subject matter.

A second point is that the process of deduction is separate from the process of experiential acquisition of knowledge of the initial truths and, moreover, it is the same regardless of the nature of the subject-matter – logic is *topic neutral*. In other words, and more generally, methods for determining validity and invalidity of premise-conclusion arguments stand apart from the sciences – some say logic *underlies* or is under or is foundational for the sciences; others say it *transcends* or is over or governs the sciences. From an Aristotelian point of view, which troubles some people, a person does not need to know science before learning logic, but a person must have inner knowledge of logic before making much progress in learning science. In the terms used by followers of Aristotle, logic is not a science per se; rather it is an instrument of science, an *organon*. In that logic stands apart from any science's subject-matter, it may be considered to be a *formal* discipline.

A related but third point is that deduction per se produces consequences of whatever it is applied to, deduction makes explicit the implicit content of premise sets; it does not produce new information about the subject-matter of the premises. Deduction per se is merely information processing and not knowledge producing –except in the sense of producing knowledge that a conclusion follows from premises.

This third point, which may be called *the non-creativity of deduction*, is dramatically illustrated by the contrast between the classical formal conception of proof due largely to Aristotle and the contentual, or cognitivist, conception vigorously articulated by some of Kant's more independent followers as well as by Frege. To make this contrast it is convenient to use the word *deduction* as above for the process of coming to know that a given conclusion follows from given premises (without regard to whether the premises are known or not) in contrast with using *inference* for the process of coming to know that a conclusion is true based on previous knowledge of premises[21]. For the classical formalist a proof involves a deduction, a series of deductive steps, which in principle could be grasped and checked by someone who did not even understand much less have knowledge of the premises. In fact, in connection with checking an alleged "proof" to determine whether

21 The distinction between deduction and inference, often but not always using these very words, has been recognized, even emphasized by most major logicians. The truth of the premises is totally irrelevant to the cogency of a deduction, but there is no such thing as an inference based on false premises, or even on premises not known to be true. A person who deduces a conclusion from premises need not believe the conclusion, but a person who infers a conclusion necessarily believes it, according to the usage of this book. Frege had several occasions to make reference to aspects of this crucial but sometimes overlooked point. E.g., on the third page of his classic 1919 article "Negation" he writes: "Of course we cannot infer anything from a false proposition; but the false proposition may be part of a true proposition from which something can be inferred" (*Geach-Black 1952/1966*, 119). Also see *Frege 1879* §11 and my 2006 abstract.

it is a [genuine] proof, classical formalists want to first determine whether it is even a deduction, and to do this they often recommend bracketing or disinterpreting the sentences expressing the premises, i.e. disregarding their meanings except for the purely formal aspects. For the cognitivist a proof is a series of inferences which are totally inaccessible to a person who did not have full knowledge of the truth of the premises.

Misunderstanding of this third point, which involves the formality of deduction, alternates with understanding of it down through the ages. People who understand it tend to think that it is so obvious that it need not be mentioned. Boole could not have missed this point, but he never makes it. However, a few years after Boole's 1854 *Laws of Thought*, Jevons – an English logician who continued the development of Boole's work – seems very close when he wrote (*1870*, 149): "The very purpose of syllogism is to deduce a conclusion which will be true when the premises are true. The syllogism enables us to restate in a new form the *information ... contained* in the premises, just as a machine may deliver to us in a new form the material ... put into it"

Boole fully accepted Aristotle's truth-and-consequence conception of proof. For example, he wrote (*1997*, 32): "The demonstrations of geometry are a particular application of logical argument to questions of space, magnitude and figure. As all reasoning consists in the deduction of conclusions from premises, so geometrical reasoning consists in deducing geometrical relations from geometrical premises, which premises are either truths established by previous reasoning or they are the axioms and definitions upon which all geometry rests....Axioms are self-evident truths."

Kazarinoff's Convincing-argument Conception of Proof.

> A proof is an argument that *has convinced* and *now convinces*. ... A mathematical proof is a temporal, communicable phenomenon in the minds of living men.
> *Kazarinoff 1970*

As obvious as Aristotle's theory of proof may seem, there are many competent, knowledgeable people who have not accepted it. Moreover, despite the fact that Aristotle conceived of his view through the study of mathematics in Plato's Academy, some of the most articulate people who do not accept it are mathematicians. In fact, there is no evidence that Euclid or any other Ancient Greek mathematicians coming after Aristotle accepted it. Nicholas Kazarinoff summarizes his contrary view in the following paragraph taken from the section titled "Proof" in his influential and instructive book[22] on geometrical constructions *Ruler and the Round: Classical Problems in Geometric Constructions.*

22 This thoroughly readable book for a general audience was written by an accomplished and respected mathematician, former Professor and Chair at the Departments of Mathematics, the University of Michigan and the University of Buffalo.

> The standard by which practically all the world's mathematicians judge a proof is this; a proof is that which has convinced and now convinces the intelligent reader. Of course, one asks, who are the intelligent readers? The best answer I can give to that question is that within a given culture the intelligent readers of mathematical proofs are those people who are generally accepted to be mathematicians. Moreover, proof is relative: What is good mathematics in this culture in this age may not be considered good mathematics in this or another culture in a future age, just as today we consider much mathematics of past cultures and ages to be incomplete or incorrect. Next, attention should be focused on the point that a proof is an argument that *has convinced* and *now convinces*. The use of past and present tenses is deliberate. I maintain that an argument is not a proof until it has been articulated, heard or read, and, finally, found to be convincing, so convincing that there exist live men who are presently convinced of it. A mathematical proof is a temporal, communicable phenomenon in the minds of living men. Mathematical proofs are not arguments written on tablets of gold in Heaven (or on Earth); they are certain collections of thoughts that many people, intelligent readers, hold in common.

Kant's Constructive Conception of Proof.

> The true method [of proof] ... was ... to bring out what was necessarily implied in the concepts ... formed *a priori* and ... put into the figure in the construction.
> *Immanuel Kant 1781.*

For Aristotle, once the basic premises are set forth in a proof, the fact that they are known to be true is irrelevant to what is done by the thinker thereafter – in the proof itself. Deduction of the conclusion from the premises is simply information processing which would be the same regardless of the cognitive status of the information. In fact, for Aristotle it is irrelevant whether the conclusion is deduced from the premises before the premises are established to be true or whether the premises are established to be true first and then the conclusion is deduced from them. Aristotle's truth-and-consequence conception of proof could just as accurately be called the consequence-and-truth conception of proof. Moreover, any appeal to the nature of the subject-matter is also irrelevant. As will be discussed in detail below, in order to deduce a conclusion from premises it is not necessary to be acquainted with the subject-matter of the premises and conclusion of the argument, but only with the logical form of the argument.

As commonplace as this may be, Kazarinoff gives no recognition of it – perhaps he has never thought of it; perhaps he has rejected it. And Kazarinoff is not alone in not accepting the truth-and-consequence conception of proof. Kant has an amazing passage about the origin of proof in the preface to the second edition of *The Critique of Pure Rea-*

son (*Kant 1781/1887*: B, x-xi). I quote it in full, but broken into two parts so that I can point out some important aspects of it that bear on the issues under discussion here.

> In the earliest times to which the history of human reason extends, *mathematics*, among that wonderful people, the Greeks, had already entered upon the sure path of science. But it must not be supposed that it was as easy for mathematics as it was for logic—in which reason has to deal with itself alone—to light upon, or rather to construct for itself, that royal road. On the contrary, I believe that it long remained, especially among the Egyptians, in the groping stage, and that the transformation must have been due to a *revolution* brought about by the happy thought of a single man, the experiment which he devised marking out the path upon which the science must enter, and by following which, secure progress throughout all time and in endless expansion is infallibly secured. The history of this intellectual revolution—far more important than the discovery of the passage round the celebrated Cape of Good Hope—and of its fortunate author, has not been preserved. But the fact that Diogenes Laertius, in handing down an account of these matters, names the reputed author of even the least important among the geometrical demonstrations, even of those which, for ordinary consciousness, stand in need of no such proof, does at least show that the memory of the revolution, brought about by the first glimpse of this new path, must have seemed to mathematicians of such outstanding importance as to cause it to survive the tide of oblivion. A new light flashed upon the mind of the first man (be he Thales or some other) who demonstrated the properties of the isosceles triangle. The true method, so he found, was....

What Kant has said above indicates deep awareness of the importance of proof and of how difficult it must have been to create the first proof. Now he is going to say how the first proposition was proved. A person who accepts the truth-and-consequence conception of proof expects Kant to say that the true method is to logically deduce the proposition from premises that have been established intuitively or experientially. But that is not what Kant says. I quote the remainder of the passage.

> The true method, so he found, was not to inspect what he discerned either in the figure, or in the bare concept of it, and from this, as it were, to read off its properties; but to bring out what was necessarily implied in the concepts that he had himself formed *a priori*, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with *a priori* certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept."

Kant is saying nothing about starting with premises known to be true, nor about applying information-processing procedures to reveal the consequences implicit in those premises. On the contrary, he is talking about doing intuitive geometrical constructions – something totally irrelevant to Aristotle's conception of proof. Kant is basing apodictic judgment on processing concepts, not propositions. Kant starts with concepts put into a figure previously constructed; Aristotle starts with propositions previously established. For Kant the diagram is essential; for Aristotle it is irrelevant.

Boole's Laws of Thought.

> It has been said that Galileo's greatest achievement was to persuade the world's scientists that physical reality is mathematical, or at least that science should be pursued mathematically. In his words, 'The Book of Nature is written in mathematical characters'.[23] In a strikingly similar spirit, Boole (1854, p. 12) stated 'it is certain that [logic's] ultimate forms and processes are mathematical'. Perhaps Boole's greatest achievement was to persuade the world's logicians that logical reality is mathematical, or at least that logic should be pursued mathematically.
> *John Corcoran 2003*

The publication of *Laws of Thought* in 1854 launched mathematical logic. Tarski noted that the continuous development of mathematical logic began about this time, and he says that *Laws of Thought* is Boole's principal work (*1941/1946*, 19). The influential Hilbert-Ackermann *Principles of Mathematical Logic* (*1928/1938/1950*, 1) is even more specific in saying: "The entire later development [of mathematical logic] goes back to Boole. Lewis and Langford, in agreement with Hilbert-Ackermann, write that Boole's work "is the basis of the whole development [of mathematical logic] ..." (*1932/1959*, 9). Decades of historical research only confirm these conclusions about the origin of the continuous development of mathematical logic (*Peckhaus 1999*, 433).

If, as Aristotle tells us, we do not understand a thing until we see it growing from its beginning, then those who want to understand logic should study *Prior Analytics* and those who want to understand mathematical logic should also study *Laws of Thought*. There are many wonderful things about *Laws of Thought* besides its historical importance. Of all of the foundational writings concerning mathematical logic, this one is the most accessible[24].

Boole had written on logic previously, but his earlier work did not attract much attention until after his reputation as a logician was established. When he wrote this book he was already a celebrated mathematician specializing in the branch known as analysis. Today he is known for his logic; his work in analysis is largely forgotten although his books are still in print. The earlier works on logic are read almost exclusively by people who have

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23 This Galileo quote and its importance were emphasized by Herbert Hauptman in his 1982 presentation to the Buffalo Logic Colloquium on the nature of applied mathematics.

24 The secondary literature on Boole is lively and growing, as can be seen from an excellent recent anthology (*Gasser 2000*) and a nearly complete bibliography that is now available (*Nambiar 2000(1)*). Boole's manuscripts on logic and philosophy, once nearly inaccessible, are now in print (*Grattan-Guinness and Bornet 1997*).

read *Laws of Thought* and are curious concerning Boole's earlier thinking on logic. In 1848 he published a short paper "The Calculus of Logic" (*Boole 1848*) and in 1847 his booklet "The Mathematical Analysis of Logic" (*Boole 1847*) was printed at his own expense[25].

Contrary to appearances, by the expression 'mathematical analysis of logic', Boole did not mean that he was analyzing logic mathematically or using mathematics to analyze logic. Rather his meaning was that he had found logic to be a new form of mathematics, a new form of analysis, and that it was not a form of metaphysics or philosophy as had been thought previously. More specifically, his point was that he had found logic to be a form of the branch of mathematics known as mathematical analysis, which included algebra and calculus[26]. In his 1847 booklet, addressing "the few" who think that analysis is worth study for its own sake, Boole refers to logic as a form of analysis "in which every equation can be solved" (*1847*, 7). Since his time the scope of what is meant by '[mathematical] analysis' has shifted by incorporating fields that were then unknown and by separating off subfields that were then thought of as analysis but which have since come to exhibit radically different features. The 1992 book by Gelbaum and Olmstead, *Counterexamples in Analysis,* gives a clear impression of the nature of [mathematical] analysis in the 1900s. For a somewhat abstract view of the subject see the 1930 classic by E. Landau with the revealing title *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational and Complex Numbers.*

In his Introduction (*1847*, 4) Boole wrote: "...I propose to establish the Calculus of Logic, and....claim for it a place among the acknowledged forms of *Mathematical Analysis*..." [Italics added]. Accordingly, in his 1848 paper we read: "In a work lately published I have exhibited the application of a new and peculiar form of Mathematics to the expression of the operations of the mind in reasoning". Thus, more generally, Boole's use of the word 'analysis' did not relate to the activities of analyzing, whether analyzing concepts, propositions, deductions, thought processes or anything else. It relates to a discipline called "Mathematical Analysis". Whitehead and Russell were referring to mathematicians working in this discipline by the word 'analysts' in the second sentence of *Principia Mathematica*, where they credit "the work of analysts and geometers"(*1910*, v). George Berkeley's famous 1734 essay *The Analyst* concerns mathematical analysis.

25 Without even mentioning the 1848 paper, Boolos (*1998*, 244), in crediting Boole with key ideas relating to the disjunctive normal form and the truth-table method, notes that the 1847 work is much less well-known than the *Laws of Thought*.

26 Books on this subject typically have the word 'analysis' or the words 'real analysis' in the title, e.g., Gelbaum and Olmstead (1964/1992), *Counterexamples in Analysis.* For a short description of this branch of mathematics, see the article "Mathematical Analysis" in the 1999 *Cambridge Dictionary of Philosophy* (Audi *1999*, 540-541). A recent, and in many respects revealing, axiomatic formalization of analysis can be found in *Boolos, Burgess, and Jeffrey (2002*, 312-318).

Long before Boole there had been "acknowledged forms of Mathematical Analysis" dealing separately with numbers, lengths, areas, volumes, time intervals, weights, and other quantities. But Boole thought of himself as advancing a new non-quantitative form of analysis that dealt with classes. And he somehow took it to be logic. It was this new form of "Mathematical Analysis" that Boole applied "to the expression of the operations of the mind in reasoning" (*1848*, 183).

Bertrand Russell (*1903*, 10) recognized the pivotal nature of this book when he wrote: "Since the publication of Boole's *Laws of Thought* (1854), the subject [mathematical logic] has been pursued with a certain vigour, and has attained to a very considerable technical development". A few years later Russell (*1914*, 49-51) was even more definite saying: "The modern development of mathematical logic dates from Boole's *Laws of Thought* (1854)". Grattan-Guinness (*2004*) noted that Boole's system received its definitive form in this book.[27] Although this work begins mathematical logic, it does not begin logical theory. The construction of logical theory began, of course, with Aristotle, whose logical writings were known and admired by Boole. In fact, Boole explicitly accepted Aristotle's logic as "a collection of scientific truths" (*1854*, 241) and regarded himself as following in Aristotle's footsteps. He thought that he was supplying a unifying foundation for Aristotle's logic and at the same time expanding the ranges of propositions and of deductions formally treatable in logic.

Boole thought that Aristotle's logic was "not a science but a collection of scientific truths, too incomplete to form a system of themselves, and not sufficiently fundamental to serve as the foundation upon which a perfect system may rest" (*1854*, 241). He was one of many readers of *Prior Analytics* who failed to discern the intricate and fully developed logical system that Aristotle had devised. What Kretzmann (*1974*, 4) said of Aristotle's *On Interpretation* applies with equal force to *Prior Analytics*: "In the long history of this text even what is obvious has often been overlooked". Boole was not the first or the last in a long series of scholars who wrote about Aristotle's logical works without mentioning even the presence of Aristotle's references to geometrical demonstrations. Not until the early 1970s did philosophically and mathematically informed logicians, taking Aristotle at his word in the first line of *Prior Analytics*, finally discover Aristotle's system (*Corcoran 1972* and *Smiley 1973*). Gasser, in his 1989 historical and philosophical study of the concept of proof writes (*1989*, 1): "...from the very beginning, logic claimed to be the study of proof and sought to develop the science of proof. This

27 It remains a puzzle to this day that despite significant changes Boole says in the preface that *1854* "begins by establishing the same system" as was presented in *1847*.

new understanding of Aristotle's logic is fully reflected in the 1989 translation of *Prior Analytics* by Robin Smith[28].

As has been pointed out by Grattan-Guinness (*2003*, and *Grattan-Guinness and Bornet 1997*), in 1854 Boole was less impressed with Aristotle's achievement than he was earlier in 1847. In *1847* Aristotle's logic plays the leading role, but in *1854* it occupies only one chapter of the fifteen on logic. Although his esteem for Aristotle's achievement waned as Boole's own achievement evolved, Boole never found fault with anything that Aristotle produced in logic, with Aristotle's positive doctrine. Boole's criticisms were all directed at what Aristotle did not produce, with what he omitted. Interestingly, Aristotle was already fully aware that later logicians would criticize his omissions; unfortunately he did not reveal what he thought those omissions might be (*Sophistical Refutations*, Ch. 34).

Likewise, philosophical concern with problems of understanding the nature of logical reasoning also predates Aristotle's time. In a way, concern with understanding the nature of logical reasoning was brought to a climax by Socrates (469? -399 BCE), who challenged people to devise a criterion, or test, for recognizing proofs, a method for determining of a given alleged proof whether it indeed is a proof, i.e., whether it proves to its intended audience that its conclusion is true, or whether, to the contrary, it is fallacious despite any persuasiveness the audience might find it to have (*Phaedo* 90b-90e). Despite Frege's impetuous claim (*1879*, Preface), the Socratic challenge has still not been answered. Perhaps the identification of logic as a potential field of study, or as a possible branch of learning, should be taken as the time when humans, having discovered the existence of logical deduction, were able to perceive a difference between objective proof and subjective persuasion. As Gasser (*1989*, 117) puts it: "One of the main objectives of logic is to distinguish between persuasion and proof, and belief and knowledge". See *Corcoran 1994*.

Boole's Philosophy of Logic.

> Perhaps the most extraordinary view of logic which has ever been developed with success is that of the late Professor Boole… His book...*Laws of Thought*… is destined to mark a great epoch in logic; for it contains a conception which in point of fruitfulness will rival that of Aristotle's *Organon*".
> *Peirce 1865*

28 Smith's scholarship combines knowledge of modern mathematical logic with an appreciation of Aristotle's thought gained through reading Aristotle's writings in the ancient Greek language. For a useful discussion of some of the linguistic and interpretational problems that Smith confronted see the critically appreciative essay-review by Gasser (*1991*).

Boole subscribed to Aristotle's initial-plus-derivative conception of the structure of each science without reservation. In his own words (1854, 5): "All sciences consist of general truths, but of those truths some only are primary and fundamental, others are secondary and derived." It is also evident, from the above and many other passages, that he also agreed with Aristotle's pluralistic conception of the realm of scientific knowledge as composed of many separate sciences. What is not evident from the above quotation, but is nevertheless also true, is that Boole agreed with Aristotle on the fundamental ontological principle that each science had its own domain (Sagüillo, *1999*, 268-75).

Aristotelian, Boolean, and Modern Logics.

The common view is that Aristotle's logic conflicts with modern logic whereas Boole's is in agreement with it. This view could not be further from the truth. First, as noted above, Boole accepted as valid absolutely every argument accepted as valid in Aristotle's system. Thus any conflict with modern logic found in Aristotle's logic would be found in Boole's to the extent that Boole's logic is faithful to his own philosophy. Second, as first noted by Smiley (*1962*, 72), if Aristotle's logic is correctly translated into modern logic the fit is exact. If categorical sentences are translated into many-sorted symbolic logic according to Smiley's method or either of the methods given below, an argument with arbitrarily many premises is valid according to Aristotle's system if and only if its translation is valid according to modern standard many-sorted logic. As Parry showed me in 1973 (per. comm.), this result can be proved from the combination of Parry's insights (*1965*, 343) with my proof of the completeness of Aristotle's categorical logic (*1972*, 696-700).

Aristotelian Logic as Many-Sorted Symbolic Logic Using Sortal Variables: The ranges of the sortal variables are all non-empty as with ordinary one-sorted logic. Each range is assigned independently of all others (*Church 1956*, 340, *Parry 1965*, 342). In my example, ess ranges over the spheres and pee over the polygons. *Hilbert 1899* uses this style.

Every sphere is a polygon.	$\forall s \exists p\ s = p$
Some sphere is a polygon.	$\exists s \exists p\ s = p$
No sphere is a polygon.	$\forall s \forall p\ s \neq p$
Some sphere isn't a polygon.	$\exists s \forall p\ s \neq p$

Using Range-Indicators (Common Nouns) with General (Non-Sortal) Variables: Each initial variable occurrence follows an occurrence of a range-indicator or "common noun" that determines the range of the variable in each of its occurrences bound by the quantifier preceding the range-indicator. This is reflected in the practice of Tarski and others of using variables as pronouns having as common nouns as antecedents as in "For every number x, x precedes x+1". The word 'number' in 'number x' determines the range of x in all three occurrences. To each range-indicator a non-empty set is assigned as its "extension". In my example, the extension of ess is the class of spheres and pee the class of polygons.

Every sphere *x* is a polygon *y*.	
For every sphere *x* there exists a polygon *y* such that $x = y$.	$\forall Sx \exists Py\ x = y$
Some sphere *x* is a polygon *y*.	
For some sphere *x* there exists a polygon *y* such that $x = y$.	$\exists Sx \exists Py\ x = y$
No sphere *x* is a polygon *y*.	
For no sphere *x* is there a polygon *y* such that *x* is *y*.	
For every sphere *x,* for every polygon *y*, *x* isn't *y*.	$\forall Sx \forall Py\ x \neq y$
Some sphere *x* is not any polygon *y*.	
For some sphere *x,* every polygon *y* is such that *x* isn't *y*.	
There exists a sphere *x* such that, for every polygon *y*, *x* isn't *y*.	$\exists Sx \forall Py\ x \neq y$

Many-sorted logic with sortal variables is prominent in Hilbert (*1899/1971*) and merely described in Hilbert and Ackermann (*1938/1950*, 102, *Church 1956*, 339). Many-sorted logic with range-indicators and non-sortal variables was pioneered by Anil Gupta in his *1980* book. Also see my *1999*.

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