NOTES ON A SEMANTIC ANALYSIS OF
VARIABLE BINDING TERM OPERATORS

John Corcoran and John Herring

A *variable binding term operator* (vbto) is a non-logical constant, say v, which combines with a variable y and a formula F containing y free to form a term (vy:F) whose free variables are exactly those of F, excluding y. The expression (vy:F) is called a *variable bound term* (vbt). In case F has only y free, (vy:F) has the syntactic propreties of an individual constant; and under a suitable interpretation of the language vy:F) denotes an individual. By a semantic analysis of vbtos we mean a proposal for amending the standard notions of (1) "an interpretation of a first-order language" and (2) "the denotation of a term under an interpretation and an assignment", such that (1') an interpretation of a first-order language associates a set-theoretic structure with each vbto and (2') under any interpretation and assignment each vbt denotes an individual.

In this paper we consider a recent semantic analysis ([1]) of vbtos ([2], pp. 65-69) which interprets each vbto as a function mapping the class of finite sequences of objects and/or subclasses of the universe into the universe, and according to which the denotation of a vbt (vy:F) depends not only on the "truth set" of the formula F (see below), but also on both (1) the objects mentioned by the free variables of the term and (2) the order in which they are mentioned in the formula.

This is spelled out in detail as follows. Let L be a first order language with identity and such that (1) V is the infinite set of variables, (2) C is the set of non-logical constants including vbtos, and (3) no vacuous variable binding occurs (neither with quantifiers nor with vbtos). An interpretation, i, of L is a

---

pair (D,m) where D is a non-empty set (the universe or do-
main) and m is a function defined on C and assigning to each
non-logical constant an appropriate "extensional meaning"
relative to D. As usual, mb is in D where b is an individual
constant, mf is a function from D^n to D where f is an n-ary
function symbol, and mP is a subset of D^n where P is an n-ary
predicate. The question of the nature of mv, v a vbto, is settled
as follows: mv is function from $M(D \cup PD)$ to D, where P is the
power set operation and where M forms the class of finite
sequences of objects in the class to which it is applied. Let a
be a mapping from V to D, assigning a member of the domain
to each variable. We use the notation ia ambiguously: to in-
dicate an interpretation i together with an assignment a and
also to indicate the function which assigns to each formula its
truth-value in ia and to each term its denotation in ia. The
function ia is defined as usual with the following amendment
for vbts. First define, for each variable y and formula F con-
taining y free, *the truth-set of F relative to y under* ia to be
the set of objects d in D that satisfy F when the constants in
F have the meanings given by i and the free variables in F
other than y are taken as denoting in accordance with a. Where
ia is understood, we use the notation Ty(F) to indicate the truth
set of F relative to y under ia. Now, let F be a formula whose
free variables are exactly $x_1, x_2, ..., x_n, y, z_1, z_2, ..., z_m$ (all
distinct) in order of first occurrence. According to this seman-
tic analysis, ia(vy:F), the denotation under ia of the vbt (vy.F),
is

$$mv(ax_1, ax_2, ..., ax_n, Ty(F), az_1, az_2, ..., az_m).$$

1. A Minor Point of Convention

From the nature of the last clause it is clear that mv is never
actually applied to any sequences except those of the form
sSs' where s and s' are both finite (perhaps null) sequences
of objects from D and S is a subset of D. Thus, except when
D is a singleton, it is always possible to construct from a given
interpretation i another interpretation j such that j is (elemen-

tarily) equivalent to i but not isomorphic with i simply by changing what mv does to a sequence not of the above form, for example to DDD. This phenomenon rules out at the outset the possibility of getting categoricity in any power except unity, and it is also-inelegant. In the balance of the paper we regard the analysis as amended so that mv is always a function from MDPDMD to D, i.e., that mv be defined on all and only sequences of the form sSs', as above.

2. *Failure of Certain Rules of Inference*

In order to see how Universal Instantiation (UI) and Substitutivity of Identicals (SI) fail consider a language which contains two non-logical constants, an individual constant b and a vbto v. Let j be the interpretation with universe $\{0, 1\}$ and such that (1) mb$=0$ and (2) mv assigns 0 to the unit sequence $\{0\}$, and 1 to all other sequences. Under this interpretation (vy:y $=$ b) denotes 0 but (vy:y $=$ x) denotes 1, for all assignments. Thus the following is true:

(2.1) $\forall x((vy{:}y{=}b) \neq (vy{:}y{=}x))$.

But application of universal instantiation yields:

(2.2) $(vy{:}y{=}b) \neq (vy{:}y{=}b)$,

which is not only false but contradictory (unsatisfiable).

One way to locate the source of the above phenomenon is to point out that this semantic analysis permits the denotations of two terms (under an assignment and interpretation) to differ merely on the ground that an individual is mentioned by a constant in one but by a variable in the other. Individuals mentioned by constants never get "looked at" by mv, whereas individuals mentioned by variables always do. This feature also accounts for failure of what is here called Substitutivity of Identicals. Here the point is simply that the following sentence is not logically true (universally valid) because it is false in the above interpretation:

(2.3) $\forall x(b{=}x \supset ((vy{:}y{=}b){=}(vy{:}y{=}x)))$,

A final oddity traceable to the above feature is failure of Existential Generalization (or ∃ Introduction, abbreviated EG). in j we have the truth of:

(2.4) $(vy:y=b)=(vy:y=b)$,

whereas the following result of existential generalization is false in j:

(2.5) $\exists x((vy:y=x)=(vy:y=b))$.

There is another feature, unrelated to individual constants, which also leads to failure of rules: mv "looks at" the order in which individuals are mentioned in a term. In particular, consider the interpretation j′ (of the above language) with domain $\{0, 1\}$, where mb=0 and mv assigns 0 to all sequences of the form sSs′ with s′ null and 1 to all sequences of the form sSs′ with s′ non-null. In this case mv assigns different values to d{d} and {d}d for all d in $\{0, 1\}$. The following is true in j′:

(2.6) $\forall x((vy:x=y)\neq(vy:y=x))$,

because the first term will always denote 0 while the second always denotes 1. Again UI fails.

The fact that mv "looks at" the order in which individuals are mentioned by variables in a term also leads to additional failure of SI. For example the following is true in j′:

(2.7) $\forall xz(x=z\supset((vy:z=z\&x=y\&y=z)\neq(vy:x=x\&x=y\&y=z)))$.

Perhaps more serious is that the dependence on order leads to failure of Substitutivity of Equivalents (SE). The point here is that the following is not valid:

(2.8) $\forall xy(F\equiv G)\supset\forall x((vy:F)=(vy:G))$, where F and G have exactly x and y free.

SE fails because, e.g. the following is false in j′:

(2.9) $\forall xy((x=y)\equiv(y=x))\supset\forall x((vy:x=y)=(vy:y=x))$.

In the context of a logic without identity some of the above objections do not hold. However UI, EG and other (desirable) versions of SE fail even in that case. The semantic analysis in question was given in a context of logic without identity.

3. *A Suggested Amendment*

Common sense seems to dictate that speculation concerning a general semantic analysis of vbtos should be preceded by an examination of "standard" vbtos. The hope is, of course, that an acceptable general semantic analysis can be gotten by generalizing on the semantic features shared by the few vbtos in common usage.

Let us first confine our attention to the four "standard" vbtos: (1) set abstraction in set theory, (2) minimalization in number theory, (3) selection (or Hilbert epsilon) and (4) description. (Use of the last two is not confined to any "special science" and, therefore, they may be treated as logical constants. The only effect such treatment would have would be to shift specification of the analysis of these two completely to the definition of denotation i.e. to eliminate these two vbtos from the domain of m (where (D, m) is an interpretation.)

For each of these vbtos one can find (or devise) three equally rational but non-equivalent conventions of usage: the contextual, the mathematical and the classical, as we here call them. The contextual convention amounts to treating the vbts as though, in a sense, they had no definite meaning in themselves but rather that in context they contribute to abbreviation of a longer expression (cf. [11], p. 51). The usual rules of reasoning are not applied directly to expressions involving vbts; such expressions must be "deabbreviated" before reasoning can take place. This convention, besides being awkward, does not require semantic analysis. The mathematical convention presupposes that some vbts have denotations: only those for which certain ancillary conditions hold. For brevity of expression below, we assume that F(y) is a formula with y free in the "interpreted" language in question and such that

denotations have been assigned to whatever other variables are free in F(y). When the truth-set of F(y) is in the universe of sets then the abstraction operator can meaningfully be applied. When the truth-set of F(y) is non-empty then (in number theory) minimalization and (generally) selection can be applied. When the truth-set of F(y) is a singleton then the description operator can be applied. In the other cases the resulting vbt is regarded either (1) as not well-formed or (2) as meaningless. The first alternative involves allowing grammaticality to depend on material considerations in such a way that knowledge of non-grammatical fact is a prerequisite to determining well-formedness (cf. [6], p. 595). The second alternative is widely used (for this use of abstraction cf. [2], p. 101; of minimalization cf. [2], p. 214; of selection cf. [7], p. 101; and of description cf. [12], p. 22) but unfortunately it involves some rather important revisions of standard logic (cf. [12]. In any case our attention is focused on the classical convention which was the topic of the semantic analysis outlined above. The classical usage has two distinguishing features. In the first place each vbt is regarded as a term, in the second place each vbt has a denotation. Where no "natural" denotation is available a denotation is assigned by convention.

In particular, the above four operators are used as follows. (1) Where the truth-set of F(y) is in the universe of sets the vbt formed by the abstraction operator denotes the truth-set and otherwise it denotes the null-set (cf. [13], p. 34). (2) Where the truth-set of F(y) is non-empty in number theory then the vbt formed by the minimalization operator denotes the minimal member and otherwise it denotes zero ([3], p. 74). (3) The denotation of terms formed by the selection operator also is determined by considering the truth-set. One neat way of doing this (cf. [4], p. 466) is to use a "choice function" which has its value in any non-empty set it is applied to and is arbitrarily defined to have the same value at the null set as at the whole domain. Then the denotation of a vbt formed by a selection operator is taken to be the value of the "choice function" applied to the truth set. (4) The denotation of a vbt formed by the description operator is the member of the truth-set when the

truth-set is a singleton and otherwise it is arbitrarily assigned.

In all of these cases the extensional meaning (denotation) of a vbt depends only on the truth-set, just as the extensional meaning (truth-value) of a quantified sentence depends only on the truth-set. The universal quantification is true if and only if the truth-set is the whole domain; the existential quantification is true if and only if the truth-set is non-null. The overall conclusion to be "gathered" from examination of the classical cases is that for each vbto v and each interpretation (D, m), mv is a function from PD to D and that for each ia:

(3.1) $ia(vy:F) = mv(Ty(F))$,

where (as above) F is a formula with at least y free, ia(vy:F) is the denotation of the vbt under the interpretation i and assignment a and Ty(F) is the truth-set of F relative to y i.e. the set of objects d in D which satisfy F when the constants in F are interpreted according to i and the free variables (except y) in F are interpreted according to a.

This conclusion is obviously valid for the so-called "n-bounded minimalization" operators of recursion theory. For each natural number n there is an n-bounded minimalization operator whose vbts denote the least member less than n in the truth-set, if such there be; if not they denote n. In addition, one can manufacture vbtos *ad libitum* and see that they all follow the pattern of depending for their denotations on the value of a function mv (from PD to D) applied to their truth-sets. One apparent exception is an operator which binds two or more variables at once and here the difference is only apparent because, since the truth-sets are now subsets of Cartesian products of the universe, the function mv will naturally take as its domain the power set of such a Cartesian product. Another apparent exception is an operator which, like "bounded minimalization", binds one variable while introducing another variable as free ([3], p. 75). Here mv must "look at" both the truth-set and the value of the "new" free variable. The first apparent exception is mentioned only to indicate that as long as variables are only being bound the above noted principle generalizes. However, neither apparent exception is

properly within the class of vbtos as defined above — we have confined our attention to term forming operators which bind one variable and which do not introduce any "new" variables free.

Our view is that the denotation of a vbt should depend only on the truth-set. This means that whenever F is a formula containing at least y (but not y') free and F' is a formula containing at least y' (but not y) free then every universal closure of the following is logically true:

$$(3.2) \quad \forall yy'(y=y' \supset F \equiv F') \supset ((vy{:}F)=(vy'{:}F')).$$

We call 3.2 the *truth-set principle* (TSP). TSP implies the following versions of SI and SE. Let $\&(x_i=z_i)$ indicate a conjunction of the n identities $(x_i=z_i)$, i between 1 and n. Let F' result from F by properly replacing zero or more occurrences of x_1 by z_i and/or vice versa and similarly for the other identities. SI contains every universal closure of the following:

$$(3.3) \quad \&(x_i=z_i) \supset ((vy{:}F)=(vy'{:}F')).$$

SE contains every universal closure of the following where F and G both contain y free:

$$(3.4) \quad \forall y(F \equiv G) \supset ((vy{:}F)=(vy{:}G)).$$

We propose that in every interpretation (D, m), for each vbto v, mv be taken as a function from PD to D and also that "denotation" be defined as in 3.1 above. It should be clear that "our" proposal is really a mere amendment of the analysis originally considered. Each of our interpretations (D, m) is easily "identifiable with" one of these proposed in the original analysis. Aside from mathematical elegance, our proposal can be seen as placing an additional condition on the functions mv, viz., that their values be determined by the set in the sequence to which they are applied.

Our analysis was "gathered" from a study of vbtos in actual use, as, no doubt, was the analysis considered in the first two sections of this paper. The difference is that the latter allows for possible interpretations prohibited by ours. Since ours covers all of the usual cases and has additional merits

besides, we feel that some argumentation is needed to establish the usefulness of the more liberal analysis.

Incidentally, although no rules of inference were proposed along with the first semantic analysis, we propose (^{2}) that the truth-set principle (3.2 above) be added as an axiom to any already sound and complete system of first-order reasoning. We conjecture that the resulting system will also be sound and complete. Proof of this conjecture is beyond the scope of the present study.

4. *The First Analysis Revisited*

Because we regard the amended analysis as "the correct" rendering of the semantic presuppositions involved in use of variable binding term operators, and because the first analysis seemed *prima facie* plausible, we feel that a useful purpose is served by a reconsideration of the first analysis with a view toward understanding the reasons for its plausibility and how those reasons go wrong.

For the sake of discussion let us imagine that no semantic analysis of vbtos has been proposed. Now let us consider the following situation. An "interpreted" language with vbtos is given and the problem is to determine how the vbtos should be semantically analysed. Let F be a formula containing exactly the distinct variables $x_1, x_2, \ldots x_n, y, z_1, z_2, \ldots,$ and z_m free in the given order of first occurrence. Within the classical framework (vy:F) denotes unambiguously, and obviously the denotation of (vy:F) depends on the values assigned to the free variables. Thus, in accordance with standard definitional practice it is possible to use the vbt to define a function as follows:

$$(4.1) \quad \forall x_1 x_2 \ldots x_n z_1 z_2 \ldots z_m (f x_1 x_2 \ldots x_n z_1 z_2 \ldots z_m = (vy:F)).$$

Variants of this observation occur naturally in the most elementary discussions of vbtos (e.g. cf. [7], p. 101). Notice that

_____

(^{2}) Cf. [5], pp. 251, 252, especially rule 4.

the function f so defined is not in general symmetric — i.e., changing the order of the arguments cannot be expected to leave the value unchanged. Notice also that the above definitional process produces functions of every (finite) degree.

At this point we have noticed that, given an interpreted language, one naturally associates a function with each vbt and indeed that the value of the vbt depends not only on the objects mentioned by its free variables but also on the order in which they are mentioned.

Now we need to ask: what else could the value of a vbt depend on ? By considering the standard examples it becomes clear that the truth-set is a "relevant variable". Thus we are lead to postulate for each vbto v, the existence of a function g_v of "varying" degrees $n+1+m$ such that under any assignment of $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_m$ to the variables free in a term (vy:F) by an assignment a (as above) we will have:

$$(4.2) \quad g_v(a_1a_2...a_nYb_1b_2...b_m) = ia(vy:F),$$

where Y is the truth-set of F relative to y and ia(vy:F) is the denotation of the vbt under the given interpretation and assignment. From 4.2 above, it is obvious that *given an assignment,* the value of (vy:F) depends only on the truth-set of F relative to y . More precisely, let F and G be any two formulas having only the indicated variables free and in the given order of first occurrence. Then, *under a given assignment,* if the truth-set of F relative to y is the same as that of G *then* (vy:F) and (vy:G) have the same denotation. This means that the following principle must hold:

$$(4.3) \quad \forall x_1x_2...x_nz_1z_2...z_m(\forall y(F \equiv G) \supset (vy:F) = (vy:G)).$$

The "naturalness" of this conclusion gives some (weak) evidence that we are on the right track. Moreover, it also implies that g_v can always be defined as a function from MDPDMD to D as follows.

(4.4) Let c be arbitrarily chosen from D and let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m$ be members of D and Y a subset of D.

(a) If Y is a truth-set relative to y for some formula F as

above where the a_i and b_i are taken as assigned by a to the x_i and z_i respectively then set:

$g_v(a_1a_2...a_nYb_1b_2...b_m) = ia(vy:F)$.

(Note that 4.3 guarantees that every such formula F will give the same value).

(b) If Y is not a truth-set for any such F then set:

$g_v(a_1a_2...a_nYb_1b_2...b_m) = c$.

We hope that the above discussion convinces the reader that by means of "plausible reasoning" (in the sense of Polya [9]) an analysis of vbtos *in use* produces a function of the kind required in the unamended semantic analysis of vbtos. Thus we have accounted, it seems, for the *prima facie* plausibility of the first analysis. Now we have to indicate where it went wrong.

Recall from elementary geometry that the method of Pappus is a strategy for discovering constructions with a certain property, say P, and that its proper application always involves two steps, a step of discovery and a step of justification. In the step of discovery one "works backward", i.e., one imagines that the desired construction has been carried out and one applies known (and/or conjectured) theorems to deduce what it must "look like". In case the first step is fruitful it issues in a new theorem (or conjecture) of the following kind: *If a construction with property P has been accomplished then situation A results.* At this point one conjectures the converse and, then, the step of justification involves deducing the converse. In many cases the converse is seen to be false but even in such cases it may be easy to see how to amend it. Moreover, in pursuing the second step one often discovers deficiences in what was taken as plausible in the first step.

Our view is that the failure of the unamended analysis can be thought of as attributable to omission of the second step — but pursuit of the second step will also make it clear that the first step can be improved.

From the first step (above) of application of the Method of Pappus we "discover" that any interpreted language involving

a vbto v presupposes that a function g_v with certain properties is associated with v. But we have not yet asked whether every (or indeed any) such function, when used as an interpretation of a vbto, issues in an "intuitively correct" interpretation of a language. The first section of this paper indicates a negative answer to this question. Now we must ask (to pursue the method) what (possibly additional) conditions (if any) can be placed on functions in order that they will issue in "intuitively correct" interpretations.

To do the latter we must specify as well as possible the exact conditions characterizing "intuitively correct" interpretations. We list three. First, that under an assignment of values to the free variables the denotation of the vbt is *this function* of (at most) the truth-set and the individuals mentioned by variables; more precisely, that (in the notation of 4.4):

(4.5) $ia(vy{:}F)=g_v(a_1a_2...a_nYb_1b_2...b_m)$.

Second, substitutivity of identicals holds. The third condition is that an arbitrary expansion to individual constants is always possible. Allowing for arbitrary expansions to individual constants is simply to leave open the possibility of adding new individual constants to the language and amending the interpretations accordingly without changing the structures assigned to any of the "old" constants (especially to vbtos). In particular:

> (4.6) Let L and L' be two languages with constants C and C' with C a subset of C' and with C'-C a set of individual constants. Let (D, m) be an interpretation of L and let m' be any function from C'-C into D. Then (D, m+m') is an interpretation of L' where m+m' is a function defined on C' such that, for d in C, (m+m') (d)=m(d) and for d in C'-C, (m+m')(d)=m'(d).

Given these three conditions what must hold of the functions assigned to vbtos ?

It is almost immediate that the functions g_v must be subject to the condition that "all projections are identical", i.e., that one does not change the value of the function by dropping in-

dividual arguments. For example, we must have that:

$$g_v(a_1a_2Yb_1b_2) = g_v(a_2Yb_1b_2) = \quad g_v(Yb_1b_2) = g_v(Yb_1) = g_v(Y).$$

The reason for this is that replacing a variable by a "new" constant which names the same individual cannot change the denotation of a vbt. For example the following must hold for every ia:

$$(4.7) \quad ((a=x \ \& \ c=z) \supset ((vy{:}Fxyz) = (vy{:}Fayc)).$$

"Identity of projections" implies that the value of g_v depends only on the value of the set in the sequence to which it is applied.

In a loose way, therefore, we have indicated how continued pursuit of a method which accounts for the initial plausibility of the first analysis would, by itself, issue in an analysis equivalent to the amended version.

The reader is left to puzzle out how complete symmetry of g_v (implied by identity of projections) is compatible with possible non-symmetry of the f's above. The following observations, which are made for more general reasons, should help.

The first step in the above application of the Method of Pappus started out with the trivial observation that under an interpretation i the value of a vbt depends on the values assigned to the variables. This leads one to focus on a fixed interpretation and a fixed vbt while "varying" the values of the variables. However semantic analysis of vbtos requires a different perspective. For this one takes a fixed interpretation and assignment ia and thinks of "varying" F and y. That is one thinks of the *metalinguistic* expression 'ia(vy:F)' as defining (for fixed i, a and v) a function from (essentially) VXL into D:

$$(4.8) \quad ia(vy{:}F) = h(y,F).$$

State University of New York   John CORCORAN
at Buffalo   John HERRING

Abstract: We consider a recent semantic analysis of variable binding term operators which interprets each such operator as a function assigning individuals to sequences of individuals and set and according to which the denotation of each variable bound terms depends only on the "truth-set" of the formula operated on but also on the individuals mentioned by the free variables (under an assignment) and on the order in which they are mentioned in the term. It is easily seen that this semantic analysis has two sequences which would be regarded as undesirable. First, it implies excessively large equivalence classes of interpretations. Second it implies unsoundness of various rules of inference including unrestricted universal instantiation, existential generalization, and certain substitution principles. A possible emendation is suggested. Finally, reasons for the initial plausibility of the original analysis are examined in detail and found to be largely correct.

REFERENCES

[1] ENGELER, Erwin, Review of HATCHER [2], *American Mathematical Monthly*, (January 1970), p. 80.

[2] HATCHER, William, *Foundations of Mathematics*, Philadelphia (1969).

[3] HERMES, Hans, *Enumerability, Decidability and Computability*, translated by G. T. HERMAN and O. PLASSMAN, New York (1965).

[4] HILBERT, David, "The Foundations of Mathematics" in VAN HEIJENOORT, J. (ed.), *From Frege to Gödel*, Cambridge, 1967, pp. 464-469.

[5] KALISH, David and MONTAGUE, R., *Logic Techniques of Formal Reasoning*, New York (1964).

[6] KNEALE, Wm. and KNEALE, Mary, *The Development of Logic*, corrected edition, Oxford (1964).

[7] KNEEBONE, G. T., *Mathematical Logic and the Foundations of Mathematics*, Princeton (1963).

[8] NELSON, David, Review of HATCHER [2], *Mathematical Reviews,*, V. 38 (1969), p. 1002.

[9] POLYA, George, *Mathematics and Plausible Reasoning*, Princeton (1954).

[10] ROBINSON, Abraham, Review of HATCHER [2], *American Scientist* (Summer 1969), p. 123A.

[11] RUSSELL, Bertrand, "On Denoting" in MARSH, Robert (ed.) *Logic and Knowledge*, London (1956).

[12] SCHOCK, Rolf, *Logics Without Existence Assumptions*, Stockholm (19668).

[13] SUPPES, Patrick, *Axiomatic Set Theory*, Princeton (1960).

Added in proofs: The completeness result conjectured at the end of section 3 has been obtained. See: CORCORAN, HATCHER and HERRING, "Variable Binding Term Operators", *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, forthcoming.