Tarski's Convention T: condition beta.

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Tarski's Convention T—presenting his notion of *adequate definition of truth* (sic)—contains two conditions: alpha and beta. Alpha requires that all instances of a certain T Schema be provable. Beta requires in effect the provability of 'every truth is a sentence'. Beta formally recognizes the fact, repeatedly emphasized by Tarski, that sentences (devoid of free variable occurrences)—as opposed to *pre-sentences* (having free occurrences of variables)—exhaust the range of significance of *is true*. In Tarski's preferred usage, it is part of the meaning of *true* that attribution of being true to a given thing presupposes the thing is a sentence. Beta's importance is further highlighted by the fact that alpha can be satisfied using the recursively definable concept of being *satisfied by every infinite sequence*, which Tarski explicitly rejects. Moreover, in Definition 23, the famous truth-definition, Tarski supplements "being satisfied by every infinite sequence" by adding the condition "being a sentence". Even where truth is undefinable and treated by Tarski axiomatically, he adds as an explicit axiom a sentence to the effect that every truth is a sentence.

Surprisingly, the sentence just before the presentation of Convention T seems to imply that alpha alone might be sufficient. Even more surprising is the sentence just after Convention T saying beta "is not essential". Why include a condition if it is not essential? Tarski says nothing about this dissonance.

Considering the broader context, the Polish original, the German translation from which the English was derived, and other sources, we attempt to determine what Tarski might have intended by the two troubling sentences which, as they stand, are contrary to the spirit, if not the letter, of several other passages in Tarski's corpus.

Contents

1. Introduction

> Philosophical logic … includes … epistemological problems, e. g. the problem
> of what is truth and whether there is any criterion of truth. Yet these matters …
> do not belong to logic … in which no reference whatever will be made to truth

> and falsehood. The problem of truth belongs to those epistemological issues which are extremely difficult … and probably will not be solved soon.—*Łukasiewicz 1929/1963*, pp.7f

This paper concerns Alfred Tarski's English monograph "The concept of truth in formalized languages" (*Tarski 1933/1956*) first published in his collection (*Tarski 1956*). Central to the monograph is presentation and discussion of a certain one of the infinitely many definitions of the infinite class of true sentences of a certain interpreted first-order object language for the "calculus of classes"—specifically Definition 23 (*1956*, p.195).[1]

He takes his construction of one particular truth definition for this one formalized language to exemplify a general abstract method applicable to many other formalized languages. His very words are instructive.

> I shall construct a definition of this kind in connexion with a particular concrete language and show some of its most important consequences. The indications which I shall then give in §4 of this article will, I hope, be sufficient to show how the method illustrated by this example can be applied to other languages of similar logical construction.—*Tarski 1956*, p.168

Surprisingly, Tarski did not coin an abbreviated name for the language; thus necessitating repeated awkward locutions. We call it the *Object Language of Classes*, OLC. Among OLC's expressions are certain strings[2]—devoid of free-variable occurrences—that have truth-values[3] and are called *sentences*. Besides the variables, OLC contains negation, disjunction, the universal quantifier, and—as its only relation symbol—the "inclusion sign". OLC has no other characters and in particular no expressions to serve as proper class names such as Boole's '0' and '1'. OLC uses Polish parenthesis free notation as opposed to *Principia* notation and thus, strictly speaking is not what had been called 'the language of the calculus of classes' or 'the language of the logic of classes'. Moreover, it is not an algebraic language using function symbols such as plus and minus.

Every sentence contains at least one occurrence of what Tarski calls *inclusions*: a string of three signs—an inclusion sign occurrence followed by two variable occurrences. In OLC's "intuitive interpretation", the variables range over the classes of individuals and

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[1] As will become clear below, this is not—and could not have been—how Tarski stated one of his goals. His main goals are stated in several different ways and it is difficult to determine whether some or all are restatements of others.

[2] In order to formalize his treatment of strings needed even for defining 'sentence', Tarski axiomatized string theory in *1933/1956* (*Corcoran-Frank-Maloney 1974*). Tarski's 1956 word for "string" was the misleading word 'expression' (*Tarski 1956*, pp. 153, 166, 169, 172, 173). Of course, for Tarski an expression need not express anything; sentences do express but Tarski has no word for what is expressed by sentences—whether true or false. In later writings he seems to suggest that true sentences express "states of affairs". Following Frege and Church (*Corcoran 2009*), we would say sentences express what are called 'propositions'—a word Tarski disliked and did not use in *1956*. He preferred 'sentential logic' to 'propositional logic', 'sentential function' to 'propositional function', etc.See *Tarski 1944*, p. 342. The now ubiquitous word 'string', meaning "concatenation of characters", does not occur in *1933/1956* nor for that matter do 'concatenation' or 'character' in the syntactical sense (a point brought to our attention by Dr. Mary Mulhern, then of the University of Buffalo). However, in later writings Tarski used 'string' and 'concatenation' in the required senses, e.g. *1983*, p.174.

[3] The expression 'have truth-values' must be understood as abbreviating 'are either true or false': Tarski studiously avoids using the expression 'truth-values' and he never discusses the ontological status of what other people call truth-values.

the inclusion sign is "equivalent in meaning" with the expression 'is included in'. The universe of discourse contains all classes of "individuals" (*1956*, 168f). Every sentence is either true or false and no sentence is both. But Tarski does not use this current terminology.[4]

The definition presented was discovered by Tarski; his priority claims have never been challenged. His task of presenting such a definition presupposes that the readers know what is meant by saying that a given expression of OLC is a true sentence.[5] Tarski does not discuss how those sentences came to be, how they acquired truth-values, how the reader came to know of those sentences, or how the reader came to know what it means to say that such a sentence is true. Thus, Tarski's task is neither stipulational nor instructional: his goal is not to stipulate a meaning for a previously meaningless expression nor is it to instruct anyone in the meaning of the expression to be defined.

Tarski does not discuss how he came to select the language of the calculus of classes, or even which other languages he considered. However, his readers would have known that prominent logicians regarded it as one of the three branches of mathematical logic: e.g. "Considered as a formal calculus, mathematical logic has three analogous branches, namely (1) the calculus of propositions, (2) the calculus of classes, (3) the calculus of relations."(*Whitehead-Russell 1910*, p. 88); also: "The subject of Symbolic Logic consists of three parts, the calculus of propositions, the calculus of classes, and the calculus of relations." (*Russell 1903*, p. 11).[6]

Tarski's discussion uses, not in OLC of course, but in an unnamed metalanguage, we call the *Metalanguage of Classes*, MLC (*1956*, p.169ff). MLC contains a certain symbol '*Tr*' in the grammatical category of class names. He also uses a third language: a metalanguage of MLC containing the two-character symbol '*Tr*' but also containing the four-character quotes-name of '*Tr*', namely ' '*Tr*' '. Today's readers must continually remind themselves that MLC and the formalized languages Tarski discusses are all thoroughly interpreted.

It remains perhaps to add that we are not interested here in formal languages and sciences in one special sense of the word 'formal': namely sciences to the signs and expressions of which no material sense is attached. For such sciences the problem here discussed has no relevance, it is not even meaningful.

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[4] Although Tarski never says so in, even in *1983*, a modern reader would notice that OLC can be reinterpreted by assigning a new universe of discourse and a new meaning for the relation symbol. Thus, it can be transformed into a language for set theory, for number theory, for geometry [e.g. the geometry of line segments in Tarski *1941*], or even for string theory. Thus, OLC can be taken as a paradigm example of a relational language. However, in *1933/1956* Tarski takes OLC to be a fragment of a higher-order language and in particular where the variables are dedicated class variables.

[5] Today we would say that negation, disjunction, and the universal quantifier are logical constants and that the relation symbol—the "inclusion sign"—is non-logical or descriptive. However, it would be awkward, to say the least, for Tarski to use this terminology because he regards the calculus of classes as a "fragment of mathematical logic" (1956, 168). Moreover, we would refer to the variables as individual variables but this would be awkward for Tarski as they are dedicated class variables. Alternative interpretations are never under consideration by Tarski. By the way, Tarski *1956* never says that the variables range over classes; his unvarying locution is "the variables represent names of classes of individuals" (*1956*, 169, 249). This is despite the fact that OLC has no such names—a fact Tarski conveniently does not mention.

[6] It is beyond the scope of this paper to discuss the important and often puzzling role in Tarski's thought over the years of "the calculus of classes" and similar systems. Sometimes they are logics in which truth coincides with logical truth, but sometimes they are in effect first-order theories where the logical truths (tautologies, in the broad sense) form a proper subset of the truths.

—*Tarski 1933/1956*, p.166.

Moreover, the possibility of reinterpreting them is out of the question—not only in the truth-definition paper (*Tarski 1933/1956*) but also, perhaps surprisingly, in the famous consequence-definition paper (*Tarski 1936/1956b* and *1936/2002)* as documented in (*Corcoran-Sagüillo 2011*).[7] The symbol '*Tr*' denotes the set of true sentences of OLC. Perhaps, to explicitly recognize language-relativity, it would be better to say something like the following: The symbol '*Tr*' denotes the set of expressions of OLC that are true-in-OLC. In particular, MLC contains infinitely many sentences having a certain definitional form with the symbol '*Tr*' as the definiendum.

For every expression x, $x \in Tr$ if and only if […].

In accordance with Tarski's terminology, as misleading as it maybe to those not accustomed to it, we call such a sentence—with an MLC formula devoid of occurrences of the symbol '*Tr*' (to avoid circularity[8]) and having only x free on the right as definiens—a *definition of the symbol 'Tr'*.[9] Such a sentence is thus a definition of the symbol '*Tr*' regardless of what formula is on the right provided the formula on the right does not contain any free occurrence of a variable other than x and is devoid of '*Tr*'. We considered putting cautionary quotes around 'definition' in such contexts but thought that such a precaution would create other difficulties. It is instructive to see the example Tarski gives, viz. Definition 23 (*Tarski 1933/1956*, p.195). Following standard practice in mathematics, Tarski leaves it to the reader to supply 'for every expression x': the initial universal-quantifier phrase.[10] The symbol 'S' denotes the class of sentences of OLC.

Definition 23. x is a true sentence—in symbols $x \in Tr$—if and only if $x \in S$ and every infinite sequence of classes satisfies x.

Restoring the omission and deleting the clutter would produce something like the following.

Definition 23*. For every expression x, $x \in Tr$ if and only if $x \in S$ and every infinite sequence of classes satisfies x.

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[7] The currently popular idea—that the difference between logical and non-logical constants is that the former have "fixed meanings" but the latter have "variable meanings"—has no place in pre-WWII logic. It seems to be based on failures to distinguish artifacts of modern formulations of mathematical logic from features of logical phenomena perceived by logicians as far back as Aristotle and the Stoics.

[8] Of course, such a pre-sentence would also have to be devoid of 'true sentence', which Tarski takes to be an exact synonym of '*Tr*'.

[9] As exasperating as it may be to find this out, we must tell our readers that Tarski never defines *definition of the symbol 'Tr'*. Incidentally any string of that form having a variable other than '*x*' free on the right would not be a sentence and thus would not be a definition of '*Tr*' in this sense.

[10] Tarski's introductory logic book *1941/1994* repeatedly emphasizes this point, but we have not found it in *1933/1956*. Incidentally, such omissions routinely made by mathematicians tend to annoy logicians. Martin Davis, a mathematician who is also a logician, compromised by using the semi-colon to abbreviate the string of universal quantifiers omitted in the order they would normally occur. Thus, ': $xy = yx$' could abbreviate 'for every number x, for every number y: $xy = yx$'. Thus, the semi-colon between the quantifier phrase and the formula it governs is made to stand for itself preceded by the quantifier phrase.

To see why we take the range indicator to be 'expression'—and not say 'object'—see the first paragraph of page 173 in *1933/1956*. But since Tarski does not consistently keep to the convention stated there, we cannot be sure of our choice: e. g. see *1933/1956*, pp.190 where the variable *a*—allegedly restricted to classes—is used with a wider or at least different range.[11]

Tarski's Definition 23 is a definition of the symbol '*Tr*' that is true. Perhaps we should say *true-in-MLC* as opposed to *true-in-OLC*. Definition 23 is a true sentence of MLC—a *true definition* of the symbol '*Tr*'. The following are *false definitions* of the symbol '*Tr*', definitions of the symbol '*Tr*' that are false, thus, false sentences of MLC.

For every expression *x*, *x* ϵ *Tr* if and only if *x* is identical with *x*.
For every expression *x*, *x* ϵ *Tr* if and only if *x* is not identical with *x*.

This terminology almost violates normal standards of English usage: an English speaker not informed of the strange usage would be inclined to say that neither of the displayed sentences is a definition of the symbol '*Tr*'.[12]

It is worth noting that 1933/1956 has two definitions of the symbol '*Tr*', the semantical Definition 23 and a structural definition called Theorem 28 on page 208.

Theorem 28: In order that *x* ϵ *Tr*, it is necessary and sufficient that *x* is a […].

But at first Tarski does not call this a definition of the symbol '*Tr*'; he says that because of its form it could be regarded as such a definition. On the next page he does explicitly call it a definition.

Even more confusing is the fact that sometimes the expression *definition of the symbol* is redundantly lengthened by Tarski to *formally correct definition of the symbol*, *methodologically correct definition of the symbol* , or just *correct definition of the symbol* (*1956*, pp. 188, 406, and 246 respectively].[13] It is difficult to become accustomed to the fact that these extending expressions are mere fillers or expletives not used to attribute any material significance (*Corcoran 2003*, p. 266). However, other similar expressions *are* used to restrict the class of definitions to those which "work", in senses to be discussed below.

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[11] Even more troubling is the last full sentence on p. 187 of *1933/1956*, where Tarski begins to discuss strengthening condition alpha. There he refers to "[…] sentences of the type '*x is not a true sentence*', in which in the place of '*x*' we have the name of an arbitrary expression (or any other object)[…]". Taken literally, this is astounding, perhaps incoherent. We have here one more indication that more research is need before *1933/1956* can be regarded as understood.

[12] This ignores the strange fact that logicians tend to use 'is identical with [to]' for "is the same thing as" in contexts where normal speakers would say 'is'. In normal English, 'is identical with' most often means "different but very similar": Corcoran's copy of *Tarski 1956* is identical to Weber's. Tarski almost always says 'is identical with [to]' for "is the same thing as", but the simple 'is' does appear in the identity sense in *1956*, p.194, footnote 1. Of course, the logicians are concerned—often uselessly—to avoid the identity/predication ambiguity of 'is', but their misguided attempts to avoid one ambiguity lands them in another. Incidentally, it is to Tarski's credit that in *1933/1956*, he never says 'equal' for 'is' of identity, except of course in connection with numbers, which is so entrenched it is hard to avoid without appearing pedantic, or worse.

[13] As an indication that *formally correct*, *methodologically correct*, and *correct* are empty redundant rhetoric or filler preceding *definition of the symbol*, one can observe that Tarski never mentions definitions of the symbol '*Tr*' that are not *formally correct*, *methodologically correct*, or *correct*.

In the case that the symbol '*Tr*' were to be regarded as meaningless, each definition of the symbol '*Tr*' could be said to *assign to* '*Tr*' a class of expressions, viz. the class of expressions that satisfy the formula on the right […]. For example, the above two assign respectively the universal class and the null class.

A definition of the symbol '*Tr*' can be called a *definition of the class X* if it assigns *X* to the symbol '*Tr*'. However, it would be absurd to try to assign a class to the symbol '*Tr*' if it already had a class assigned to it—as in Tarski's case.

Tarski does not use the terminology italicized in the last paragraph; but he does imply, e.g. that the expression '*x* is not identical with *x*' defines the null *property* and the expression '*x* is *x*' defines the universal *property* (*1956*, 194). However, he never says why 'property' is used instead of 'class' or what he takes to be the relation of classes to properties. Tarski does not reserve 'class' for the entities referred to in the language of classes, OLC. He uses 'class' routinely for aggregates of objects of various sorts such as expressions and numbers. In this context, it would be natural to say that the expression '*x* is not identical with *x*' is a definition of the null class and the expression '*x* is identical with *x*' is a definition of the universal class but Tarski does not say such things in 1933/1956.

Of course, in the sense introduced above, no definition of the symbol '*Tr*' endows '*Tr*' with meaning. Unless '*Tr*' has previously been endowed with meaning, none of the definitions is a meaningful sentence in Tarski's sense. If '*Tr*' is considered an uninterpreted symbol, then the definitions of '*Tr*' must be considered as formal sentences lacking truth-values. However, anticipating this situation in the preamble to Convention T, Tarski endowed '*Tr*' with meaning. Tarski explicitly took "the symbol '*Tr*' to denote the class of all true sentences" (*1933/1956*, p.187). By 'true sentence', Tarski meant "true sentence of OLC". Notice that in *defining* the symbol '*Tr*', i.e. in endowing the symbol '*Tr*' with meaning, Tarski was not giving a definition of the symbol '*Tr*' in the above sense, although using normal English it would be perverse to deny that Tarski gave a definition of the symbol '*Tr*'.

Once the symbol '*Tr*' is meaningful, every definition of the symbol '*Tr*' becomes a meaningful sentence in MLC and thus has a truth-value. One might be inclined to say that many, even *most*, of them are false. However, since there is a countable infinity of truths among them, 'most' cannot be used literally here. Anyway, all of those that are true are definitions of the class of true sentences (of OLC): each of them would assign to the symbol '*Tr*' the class of true sentences (of OLC)—which is exactly what Tarski took '*Tr*' to denote. The definitions of the symbol '*Tr*' that "work" are those that are true. But Tarski never says this—perhaps because he does not want to use the word 'true' in the meta-metalanguage. It would be natural to use the expression

materially adequate definition of the symbol '*Tr*'

to refer to a formally correct definition that happens to be true. Tarski does not seem to grasp this point: he never warns the reader that being true is not sufficient for being materially adequate in his sense.[14]

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[14] Corcoran—in *1999* and elsewhere—mistakenly thought that 'materially adequate' meant "true" and that convention T was establishing a test for determining the truth of a definition of the symbol '*Tr*'.

Notice that whether such a definition is true has nothing to do with the deductive apparatus of the metatheory—the non-logical axioms and the logical axioms and rules of inference. This point, which is obvious enough in itself, follows from the discussion of definability-in-a-model in the Editor's Introduction to *Tarski 1956/1983*.[15] Why doesn't Tarski mention this fact? Does he have something to lose were this point clearly stated? Would it raise problems he wanted to avoid treating? Perhaps the fact that the truth of a definition is independent even of the existence of the deductive apparatus of the metatheory is so obvious to Tarski that he does not see fit to mention it. If this is so, he is insensitive to the needs of his readers.

It would be natural to call a formally correct definition of the symbol '*Tr*' *materially correct*, *adequate*, or *materially adequate* if it is true, i.e. if it assigns to '*Tr*' the class of true sentences. Indeed, this usage occurs often in the literature (*Audi 1999*, p. 540). However, one should not jump to the conclusion that a usage that would be natural today is what Tarski would have used over 75 years ago. In this regard, it is important to note that if Tarski had meant to call true definitions of symbols materially adequate he could easily have done so. We will see reasons below why that meaning was not used by *Tarski 1933/ 1956*. Anyway it becomes clear that by calling a definition materially adequate, Tarski was not merely saying that it was true, i.e. that the extension it would assign to the symbol '*Tr*' would be the class of true sentences of OLC. Similar or closely related points appear in recent literature, e.g. *Patterson 2012*, p. 110.

Tarski's Convention T—which presents his notion of *adequate definition of truth* (sic)[16]—contains two conditions, alpha and beta, that are separately necessary and jointly sufficient (*Tarski 1933/1956*, pp. 187–8). The two-condition notion presented is also called *material adequacy* (loc. cit., pp. 209, 265, and 273) and *material correctness* (loc. cit., pp. 195, 197, 230, and 246) elsewhere.

Alpha requires in effect that all instances of a T Schema[17] be provable—a condition elsewhere also called *material adequacy* (*Tarski 1936/1956a*, p. 404; *Tarski 1944,* pp. 343ff., p. 353).

Beta requires in effect the provability of 'every truth is a sentence'. Beta formally recognizes the fact repeatedly emphasized by Tarski that sentences—as opposed to pre-sentences having free occurrences of variables—exhaust the range of significance of the predicate *is true*.[18] Figuratively speaking, we might put this by saying that for Tarski it is

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[15] Tarski read and commented on several drafts of Corcoran's Editor's Introduction. He finally approved what was printed. Tarski would never have permitted a technical inaccuracy to be printed in the new edition of *Tarski 1956*. He did write that he did "not always agree with Corcoran's judgments"—referring to Corcoran's interpretations of the historical development of Tarski's thinking (*1956/1983*, p. xiv).

[16] Is Tarski stipulating what is henceforth to be meant by *adequate definition of truth*? That is, is he regarding this expression as a previously meaningless unitary formula better written as one hyphenated word *adequate-definition-of-truth* so that it can be *given* a meaning? Or is he asserting ("postulating") a necessary and sufficient condition for a definition of truth to be adequate, assuming that the reader knows what *adequate definition of truth* means? See also e.g. *Tarski, A. 1931/1956*, p. 118; *Tarski, 1933/1956*, p. 194n; *Corcoran 1999*. But we set these problems to the side for the moment.

[17] Specifically, '*x* ε *Tr* if and only if *p*' is the schema template and the side condition requires '*x*' to be replaced by a structural-descriptive name of a sentence of the object-language while '*p*' is replaced by a translation of that sentence into the metalanguage. See *Corcoran 2006*. The instances of the schema are metalanguage sentences. Convention T is an italicized meta-metalanguage sentence.

[18] What we call pre-sentences Tarski calls sentential functions containing "free variables". In Tarski's framework, there is no such thing as a free variable, strictly speaking. He means to refer to free *occurrences*

part of the meaning of *true* that attribution of true to a given thing presupposes the thing is a sentence (*Tarski 1933/1956*, pp. 186–7–8, 195–7; *Tarski 1941/1946*, pp. 4 and 11, *Tarski 1969/1993*, pp. 101ff.).

The essential necessity of beta—and that alpha alone is not sufficient—is highlighted by several other facts. Perhaps the simplest is that Tarski's stated problem is "the construction of the (sic) definition of *true sentence*", which uses 'sentence' (*Tarski 1933/1956*, p. 186).[19] Another is the fact that Convention T's preamble introduces the definiendum's symbol as denoting the class of true sentences. His exact words are: "Using the symbol '*Tr*' to denote the class of all true sentences, the above postulate can be expressed in the following convention" (loc. cit.). Still another is the fact that alpha is satisfied by the recursively definable concept of being *satisfied by every infinite sequence*, which Tarski emphatically rejects as inadequate (*Tarski 1933/1956*, p.189) and which he supplements adding the condition of being a sentence in the famous truth-definition (*Tarski 1933/1956*, p.195). He stresses the point that "composite sentences are in no way compounds of simple sentences" and that therefore no recursive definition of *true sentence* is possible (*Tarski 1933/1956*, p.189; *Tarski 1944*, p. 353). It is worth quoting the *1933/1956* passage.

In general, composite sentences are in no way compounds of simple sentences. Sentential functions do in fact arise in this way from elementary functions, i.e. from inclusions; sentences on the contrary are certain special cases of sentential functions. In view of this fact, no method can be given which would enable us to define the required concept directly by recursive means.

The final reason we cite underlining the essential nature of condition beta is that even where truth is undefinable and treated by Tarski axiomatically, he adds as an explicit axiom a sentence to the effect that every truth is a sentence. This point is made in italics in Theorem III (*Tarski 1933/1956*, p.256).

It should also be mentioned that the sentences alpha requires to be provable are provable simply by adding to the metatheory, not a definition of the symbol *Tr*, but a variant of sentence (5) on page 159. However, that would not satisfy beta.

In our introductory remarks, which avoid symbols, we said that beta requires *in effect* the provability of 'every truth is a sentence' in order to avoid making beta appear to be requiring provability of a tautology—which it does not. However, in Tarski's metalanguage—which includes part of ordinary English—two ways of expressing beta's sentence using Tarski's own locutions are as follows.[20]

every true sentence is a sentence

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of variables. Sentences are sentential functions not containing "free variables". This point is especially important in *Tarski 1933/1956*, p. 194. See also e.g. *Tarski 1944*, pp. 353-4. However, In Tarski *1941/1946*, on p. 5 sentential functions must contain free variable occurrences and hence no sentence is a sentential function.

[19] The italics are Tarski's. Incidentally, it is extremely unlikely in our opinion that by 'the definition […]' Tarski meant to imply that there is only one, or even only one that is true. Since Polish has no definite article, the occurrence of 'the' traces back to a translator. We leave open the further question of whether Tarski ever explicitly stated or even implied that there are infinitely many true sentences of MLC that are definitions of the symbol '*Tr*'.

[20] Following Tarski's meticulous style, we put sentences in *clausal form—*no initial uppercase and no period. See *Corcoran 2006*, pp. 226f and *Corcoran 2009*.

every object that is a true sentence is a sentence

Each is an apparent tautology. In Tarski's variable-enhanced English, the range of the variable 'x' includes the class of strings of characters used in the object language. Another Tarskian way of expressing beta's sentence, using the variable, is

for every object x, if x is a true sentence, then x is a sentence

—another apparent tautology. The reason none of these are tautologies for Tarski is that he takes 'true sentence' to be a unitary non-compound expression without separately significant parts, a point that might be brought out by writing 'true-sentence' or better 'truesentence'. Taking 'true sentence' and 'sentence' to be class names abbreviated respectively 'Tr' and 'S', Tarski gives two ways of expressing beta's sentence in his presentation of Convention T.[21]

for any x, if $x \, \varepsilon \, Tr$, then $x \, \varepsilon \, S$
$$Tr \subseteq S$$

The fact that Convention T has two conditions and not just one is mentioned several times, twice on page 246 of *Tarski 1933/1956*.[22]

Paradoxically, in the English translation, the sentence just before the presentation of Convention T seems to imply that alpha might be sufficient. Even more paradoxical is the sentence just after Convention T: it says beta "is not essential". Why include a condition if it is not essential? He says nothing about this apparent dissonance. We look at the broader context, the Polish original, and the German translation from which the English was derived. We also consider other sources to determine what Tarski might have intended by the two troubling sentences which, as they stand, are contrary to the spirit, if not the letter, of several other passages in Tarski's corpus (*Tarski 1933/1956*, *Tarski 1941/1946*, *Tarski 1969/1993*, and elsewhere).

Some current writers hold that beta is unnecessary. *Patterson 2012*, on p. 124, while not explicitly dismissing beta, states that alpha is sufficient. However, *Horsten 2011* not only says in effect that beta is not essential, but on p. xx he cites the Tarski passage as authority. Horsten gives absolutely no reason why he regards beta as "not essential"; he does not explain what *he* means by saying beta is not essential; nor does he say how omitting beta contributes to his own project.[23]

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[21] Tarski *1933/1956* does not comment on the fact that he regards 'true sentence' as a single word with no independent meaning given to the separate parts 'true' and 'sentence'. However, he does comment on using quote-names as unitaries not containing the expressions named on page 159. He also discusses taking 'it is not true that' as a single unitary expression wherein 'true' has no independent meaning on page 169. Other expressions regarded as unitary include the following: 'for all', 'is a', 'is identical with', 'is included in', 'provable sentence', 'free variable', 'universal quantification', and, arguably, 'adequate definition of truth'.

[22] As mentioned above, the property set forth in Convention T is not always referred to using 'adequate definition of truth': on the just mentioned page, 'correct definition of truth' is used and so is 'material correctness'.

[23] These points were not mentioned in any of the several reviews of *Horsten 2011* published before 2014. They are not even mentioned in Corcoran's critical notice (*Corcoran 2012*) nor are they are found in the scathing *Corcoran-Masoud 2014*.

2. Range of significance of 'is true'

> To say that "snow is white" is, for the ends of logic, equivalent to saying that "snow is a white thing".—George Boole 1854/2003, p. 52.
> To say that "snow is white" is true is, for Alfred Tarski, equivalent to saying that "snow is white" is a true sentence.—Frango Nabrasa 2014, personal communication.

The notion of range of significance belongs to semantics in Tarski's sense and it is intimately connected to his notion of semantic category. The range of significance of a predicate is the class of objects of which the predicate can be coherently affirmed or denied. This means that a predicate can be truly or falsely affirmed or denied only of the objects in its range of significance. If it is affirmed or denied of something else the resulting sentence is *incoherent*. In terms borrowed from the wider literature, each sentential function having one free variable determines three sets of objects: 1) those satisfying it, its *truth set*; 2) those satisfying its negation, its *falsity set*; 3) those satisfying it or its negation, its *range of significance* (*Whitehead-Russell 1910*, p. 199) or *range of values* (*Whitehead-Russell 1910*, p. 15). The notion is introduced in Appendix B of Russell's *The Principles of Mathematics*, §497 (*Russell 1903*).

For example, the predicate 'exceeds seven' can be truly affirmed of any number exceeding seven and truly denied of any other number—but affirming or denying it of , say, the sun or Tarski, results in incoherence. The following two sentences express incoherent meanings.

The sun exceeds seven.
It is not the case that the sun exceeds seven.

The sun is not the sort of thing that can coherently be said to exceed seven or not exceed seven.

In order to determine that a sentence expresses a coherent message it is not necessary to known that it is true or that it is false: understanding its meaning is necessary and often sufficient. Likewise, to determine that a sentence expresses an incoherent message understanding its meaning is necessary and often sufficient. In one broad sense of 'meaningful"—i.e. "having a meaning"—incoherent sentences are meaningful as are coherent sentences of course: it is by dint of their meanings that we determine that they are incoherent. Using a handy variant of the Frege-Church terminology, we can say that the two sentences above containing 'sun' have meanings but neither meaning is a proposition (*Corcoran 2009*).

When Tarski says in effect that the sentences of a given language form the range of significance of the languages truth predicate, he means coherent sentences (*1944*, p. 342). He often says "meaningful sentence" where it would be preferable to say coherent sentence. There are obvious senses of *meaningful* in which 'the sun exceeds seven' is perfectly meaningful and through grasping its pathological meaning we see that it is incoherent.

When a predicate is used in an atomic sentential function or pre-sentence such as '*x* exceeds seven', its range of significance is the range of values of the variable. This ties into the fact that the range of significance of the negation of a predicate—here 'it is

not the case that *x* exceeds seven' or more idiomatically '*x* doesn't exceed seven'—is the same as that of the predicate itself. Whatever can be coherently affirmed of a given thing can coherently be denied of it, and conversely.

It is evident from Definition 23 that for Tarski the range of significance of the predicate 'is a true sentence' is the entire universe of discourse of the intended interpretation of the metalanguage, i.e., the range of values of the individual variable '*x*'. See *Tarski 1933/1956*, pp. 173, 195.

The extension of a predicate is the subset of objects in its range of significance that it is predicable truly of. The extension of a class name is the class it names. In Tarski 1933/1956, the extension of 'is a true sentence' is the class of true object language sentences.

As said above, in Tarski's preferred usage, it is part of the meaning of *true* that attribution of being true to a given thing presupposes the thing is a sentence. In terms of this paper, Tarski took the set of sentences of OLC to be the range of significance of 'true', or 'is true'. To see this preference in other contexts see Tarski 1936, Tarski 1944, and Tarski 1969 where the analogues to the T schema use 'true' as a predicate adjective as in 'is true' and not as an attributive adjective modifying 'sentence' as in 'is a true sentence'. In these three places 'is true' amounts to 'is a true sentence'.

3. Convention T in context

> In the long history of this text, even what is obvious has often been overlooked.
> —Norman Kretzmann on a passage in the *Organon*, Buffalo, 1972.

This section contains verbatim Tarski's presentation of Convention T together with the *preceding two sentences* and the *following sentence*, which will be referred to as such in the commentary to follow in the next section. The rest of this section contains Tarski's exact words (*Tarski 1933/1956*, pp. 186–187).

Not much more in principle is to be demanded of a general definition of true sentence than that it should satisfy the usual conditions of methodological correctness and include all partial definitions of this type as special cases; that it should be, so to speak, their logical product. At most we can also require that only sentences are to belong to the extension of the defined concept, so that, on the basis of the definition constructed, all sentences of the type '*x is not a true sentence*', in which in the place of '*x*' we have the name of an arbitrary expression (or of any other object) which is not a sentence, can be proved.

Using the symbol '*Tr*' to denote the class of all true sentences, the above postulate can be expressed in the following convention:

CONVENTION **T**. *A formally correct definition of the symbol 'Tr', formulated in the metalanguage, will be called an* adequate definition of truth *if it has the following consequences:*

(α) *all the sentences which are obtained from the expression 'x ϵ Tr if and only if p' by substituting for the symbol 'x' a structural-descriptive name of any sentence of the language in question and for the symbol 'p' the expression which forms the translation of this sentence into the metalanguage;*

(β) *the sentence 'for any x, if x ϵ Tr then x ϵ S' (in other words 'Tr ⊆ S').*

It should be noted that the second part of the above convention is not essential; so long as the metalanguage already has the symbol '*Tr*' which satisfies the condition (α), it is easy to define a new

symbol 'Tr'' which also satisfies the condition (β). It suffices for this purpose to agree that Tr' is the common part of the classes Tr and S.

4. Commentary on the English edition

> In […] the case of the monograph on the concept of truth […], the translation had to be based not upon the original, […] Polish, but upon the […] German version. This made it even harder for the translator to give a fully adequate rendering of the original intentions and ideas of the author. In addition, […] the translator was deprived of the benefit of extensively discussing with the author even the major difficulties […], and so achieving a meeting of minds before the text as set up in type.—Alfred Tarski *1956*, p. x

To understand the immediately relevant implications of Tarski's "preceding two sentences", it is convenient to omit certain temporarily extraneous words.

> Not much more […] is to be demanded of a […] definition of true sentence than that it should […] include all partial definitions of this type […]. At most we can also require that only sentences are to belong to the extension of the defined concept […].

The first sentence says "not much more […] than" condition alpha is needed. This implies that *something* more is needed. The second sentence implies that condition beta is at least as strong as that something more, i.e., that beta would be a *sufficient* supplement to alpha. Note that it does not strictly speaking imply that the full strength of beta is necessary. Moreover, the expression "at most we can also require" implies awareness that the adding of requirements can be overdone: an exact balance is being sought.

Now with this conclusion in mind, we consider the immediately relevant implications of Tarski's "following sentence", again omitting certain temporarily extraneous words.

> […] the second part of the above convention is not essential; so long as the metalanguage […] satisfies the condition (α), it is easy to define a new symbol 'Tr'' which also satisfies the condition (β). […] to agree that Tr' is the common part of the classes Tr and S.

If the first clause is taken to say that beta is not essential, i.e. that it can be dropped without interfering with the concept of adequacy being defined, then the second clause is a blatant non sequitur. There is no way that the defining of a new symbol satisfying beta could possibly show that beta is not essential—in the relevant sense.

Something is wrong here.[24] Either there must be an error in our interpretation of 'essential', or there was a mistranslation somewhere, or Tarski's attention lapsed momentarily. The last possibility will be explored in a later section. The second possibility will be considered in the next section. Other problems with this passage have been discussed above in Section 1.

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[24] Corcoran, the senior author of this paper, must have read these two pages (pp. 186–187) of *Tarski 1933/1956* dozens of times. But he never noticed that anything was amiss until reading the *Horsten 2011* citation on page 49, where these passages are alleged to "justify" Horsten's erroneous application of truth to pre-sentences. For more on *Horsten 2011*, see *Corcoran 2012*.

4.1 Tarski's postulate

Immediately following the below quote, Tarski wrote "the above postulate can be expressed in the following convention". We seem justified, therefore, in calling the proposition expressed by this two-sentence paragraph *Tarski's Postulate.* How a postulate can be a convention is a problem demanding attention.

Not much more in principle is to be demanded of a general definition of true sentence than that it should satisfy the usual conditions of methodological correctness and include all partial definitions of this type as special cases; that it should be, so to speak, their logical product. At most we can also require that only sentences are to belong to the extension of the defined concept, so that, on the basis of the definition constructed, all sentences of the type '*x is not a true sentence*', in which in the place of '*x*' we have the name of an arbitrary expression (or of any other object) which is not a sentence, can be proved.

Tarski (*1933/1956*, p.187) is saying that this paragraph "can be expressed in" Convention T. We can wonder what relation Tarski has in mind by "can be expressed in". We can also wonder how we are meant to understand 'postulate'. How does a postulate differ from an axiom? Incidentally, contrary to certain recent writers, Tarski takes 'axiom' pretty much in a traditional sense: when he calls a sentence an axiom, it is clear that he thinks that it is true, that he thinks he knows that it is true, and that it is suitable to be used in proofs of other sentences.

This being the case, it goes without saying that all of the expressions in an axiom are understood. If a postulate is a kind of axiom, or if 'postulate' is a stylistic variant of 'axiom', then 'adequate' must be understood before a sentence containing it could be postulated, i.e. taken as a postulate.

In this connection it is important to remember that, for Tarski, simply having all objective properties of an axiom is not sufficient for a sentence to be an axiom: it must be subjectively selected for axiomatic service. Moreover, that selection is subjective, as Tarski emphasizes in 1933/1956 and thus it is entirely appropriate for him to use the word 'convention' in this connection. Patterson *2012* has some discussion of Tarski's use of 'convention'.

5. Commentary on the German and Polish editions

> If you by your rules would measure what with your rules doth not agree, forgetting all your learning, seek ye first what its rules may be.
> —Richard Wagner, *Die Meistersinger*.

5.1 Preliminary remarks: the "Tarski-Blaustein-Woodger triangle"

The widely studied Woodger *1956* English edition of Tarski's monograph on truth is a translation of Leopold Blaustein's *1935* German rendering of the Polish original text, which appeared as a book in *1933*. It is, thus, a translation of a translation. Not, to be sure, a mere translation, because, as is well known, Tarski himself thoroughly revised Woodger's version (in both the first, Woodger's, and second, Corcoran's, editions),

rewrote and added some footnotes, and endorsed the final text we are nowadays acquainted with.[25] Nevertheless, in spite of his personal collaboration in preparing the English edition, the fact still remains that, in the end, *The Concept of Truth in Formalized Languages* is a translation—a translation of a translation—and this obviously introduces some intricacies when interpretive issues come to the fore.[26]

Actually, one might argue that Tarski's rewriting/revising work on Woodger's version, although valuable in many respects, had the unintended effect of making things somewhat convoluted from a historical and exegetical point of view. To further complicate matters, just as the English edition is not an exact rendition of the source paper in German, so also Blaustein's version is not an exact translation of the Polish original dissertation. Tarski revised, expanded and edited it too, making things a little bit more complicated for the historically-minded reader. Strictly speaking, what is usually referred to as "Tarski's truth-definition paper" is, when one looks into the details, a moderately intricate object of study consisting of three non (exactly) overlapping texts— an object that might be pictorially displayed as the figure below:
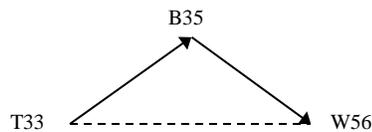
This figure might be called the "Tarski-Blaustein-Woodger triangle" (TBW, for short). [Notation for vertices is self-explanatory. Arrows indicate translation from one vertex to the other, and the dotted line indicates absence of direct translation.][27]

There are some differences in content between the vertices of TBW.[28] The main, well-known difference between T33 and B35 is an eleven pages postscript added to the latter

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[25] See Corcoran, John. "Editor's preface to the revised edition", Woodger, J. H., "Translator's preface", and Tarski, "Author's acknowledgments", *Tarski 1956/1983*. For more details, especially related to errors and misprints, see *Corcoran 2013*: "Errors in Tarski's 1983 truth-definition paper".

[26] Tarski's book—*Pojęcie prawdy w językach nauk dedukcyjnych*—appeared as number 34 of the *Travaux de la Société des Sciences et de Lettres de Varsovie (Classe III – Sciences Mathématiques et Physiques)*. Blaustein's translation—*Der Wahrheitsbegriff in den formalisierten Sprachen*—appeared in the first number of the Polish international journal *Studia Philosophica*. In order to avoid unnecessary complication, we did not mention the early, unpublished 1931 version of Tarski's book—*Pojęcie prawdy w naukach dedukcyjnych*—which was scheduled to appear in the volume XXIV of the *Comptes Rendus des Séances de la Société des Sciences et de Lettres de Varsovie (Classe III – Sciences Mathématiques et Physiques)*. The unpublished version is mentioned twice in *Tarski 1933/1956* (p. 152 [Bibliographical Note] and p. 247 [footnote 1]). As for its title and scheduled publication, see *Sur les ensembles définissables des nombres réels* (footnote 2, pp. 211-2).

[27] We are deliberately ignoring here translations into languages other than English, since most of them derive from Woodger's edition, and also because of the pervasive utilization of that edition by students and scholars. But we are aware that Blaustein's version has also been directly translated into other tongues—for instance, there is the Italian 1961 rendition of B36 by Francesca Rivetti Barbò, *Il concetta di verità nei linguaggi formalizzati*. Even so, W56 has a special place among all the translations, in virtue of Tarski's direct involvement in preparing corrections and additions for it although he was unable to make extensive changes in the final proof sheet before publication (*Tarski 1956, Translator's Preface*).

text, the *Nachwort*, derogating the principles (formerly assumed) of the so-called "theory of semantical categories", but there are further, although minor, non-substantial discrepancies between both editions. Besides a slight divergence in the titles (the Polish one means literally the same as 'the concept of truth in the languages of the deductive sciences'), here are three further examples of differences, chosen at random: (1) footnote 59 of B35 does not preserve the original ordering of the opening sentences of its counterpart in T33: the Polish note begins with the phrase '*Przy tej interpretacji terminu "funktor"* ', but in the German note the corresponding phrase—'*Bei derjenigen Interpretation des Terminus "Funktor"* '—occurs only in the fifth line; (2) the sentence '*eine solche Funktion wollen wir als* Anführungsfunktion *bezeichnen*', which occurs in §1 of B35 (and which Woodger renders as 'we shall call such functions *quotation-functions*') has no counterpart in T33; and (3) a textbook on set theory by Sierpiński which is listed in the bibliography of T33, and mentioned in footnote 77, does not occur in the bibliography of B35; in the corresponding footnote 77 of the German edition, reference is made instead to the third edition of Fraenkel's *Einleitung in die Mengenlehre*.[29]

As to the differences between B35 and W56, they are peripheral and mainly concern some footnotes. Perhaps the most remarkable and noteworthy example is footnote 56, which was drastically shortened by Tarski in the English edition.[30] It is a very special footnote: it contains a striking statement, *viz.* that the language of Leśniewski's "complete system of mathematical logic" is not to be counted among the formalized languages for which Tarski's method of defining the concept of truth and other semantical concepts is given. Tarski wrote that Leśniewski's system is "the only complete system of mathematical logic, as far as I know, whose formalization—in contrast to, for instance, the system of Whitehead-Russell—is unobjectionable and exhibits perfect precision."[31] This far, so good: Tarski is praising his relatively unknown teacher's work while criticizing the world renowned *Principia Mathematica*.

But then he goes on to say "unfortunately, that system [sc. Leśniewski's] seems to me, in virtue of certain specific characteristics, an especially unrewarding object for methodological and semantical investigations."[32] A few lines after, he excludes Leśniewski's system from the family of "formalized" sciences, in the technical sense he calls 'formalized': "From a formal point of view, it would be actually hard to fit that system under the general characterization of the formalized deductive sciences given at the beginning of §2."[33] Eventually, Tarski concludes: "In order to adapt Leśniewski's system, under these circumstances, to the requirements of the present investigations, it

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[28] Kalinowski's 1972 French translation of Tarski's monograph provides the reader with a systematic guide to the differences between T33, B35, and W56, by means of a system of comparison used in the text itself.

[29] Sierpiński's book is *Zarys teorji mnogości—Część pierwsza: Liczby pozaskończone* ('An outline of set theory—First part: transfinite numbers', 1928). The *1933/1956* corresponding footnote is footnote 1 (p.234).

[30] 39 lines in B35 down to only 14 lines in *1933/1956* [footnote 2, p. 210].

[31] "… das einzige mir bekannte vollständige System der mathematischen Logik, dessen Formalisierung—im Gegensatz z.B. zum System Whitehead-Russell—keine Einwände zulässt und vollkommene Präzision aufweist."

[32] "… leider scheint mir dieses System wegen gewisser spezifischer Eigentümlichkeiten ein überaus undankbares Objekt für methodologische und semantische Untersuchungen zu sein."

[33] "Formal gennomen würde es sogar schwer fallen, dieses System der allgemeinen, am Anfang des §2 gegebenen Charackterisierung der formalisierten deduktiven Wissenschaften unterzuordnen."

would have to be subjected to a quite thorough reworking, which, however, would fall entirely outside the framework of this work." [34] This is a striking statement, which almost contradicts the praise initially made. Unfortunately, however, reference to Leśniewski's system has been entirely removed from the corresponding *1933/1956* footnote.[35]

   Given this state of affairs (existence of some minor textual differences *plus* double possibility of mistranslation), the reader of the English edition of Tarski's truth-definition paper should be cautious. When confronted with an intractable or awkward passage in *1933/1956*, we should always ask two natural questions, before attributing to Tarski himself some inadequacy:

Is Woodger's passage P faithful to Blaustein's corresponding passage P' (if there is such)? By its turn, is Blaustein's passage P' faithful to Tarski's original passage P" (if there is such)?

This trace-back strategy may appear as a hermeneutic platitude, but it is worth being explicitly stressed here. For the intriguing claim we have been considering in *Tarski 1933/1956*—that condition beta "is not essential"—might well be the result of a mistranslation, either from B35 to W56, or else from T33 to B35; still, the claim itself might as well have no counterpart in B35 or T33. So let us look more closely to the German and Polish editions in order to see whether one of these possibilities is indeed the case.[36]

5.2 The German and Polish editions

   To begin with, let us observe that Tarski's claim is not the result of adding to the English edition a passage which has no counterpart in the German or Polish editions. There is such a counterpart in both the *1933* and *1935* texts, as we will see below. But before checking this point, let us further remark that, at least with respect to Tarski's "preceding two sentences", Woodger's translation seems to be faithful to Blaustein's text. And the same holds for Blaustein's version: it also seems to be faithful to Tarski's original Polish.

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[34] "Um unter diesen Umständen das System Leśniewski's den Bedürfnissen der vorliegenden Untersuchungen anzupassen, müsste es einer recht gründlichen Umarbeitung unterzogen werden, was jedoch den Rahmen dieser Arbeit vollständig sprengen würde."

[35] The Polish/German footnote 56 did not go unnoticed by some scholars. See *Sundholm 1993*, *Sundholm 2003* and also Philippe de Rouilhan, 'Tarski et l'universalité de la logique', in Frédéric Nef/Denis Vernant (éd.), *Le formalism en question—le tournant des anées 30*, J. Vrin, 1998. Some readers might think at first that one cannot define the concept of truth for the language of Leśniewski's "complete system of mathematical logic" for reasons quite analogous to the reasons given in the informal proof of Theorem I in §5 (which concerns the language of what Tarski there calls "the general theory of classes"). But the reasons are in a deeper level. Not even an *axiomatic* semantics as delineated by Tarski's method (neither in the form of an axiomatic theory of truth nor of any other semantical theory) is available for Leśniewski's language—or any of its "fragments". Leśniewski's language belongs to a wholly distinct kind of regimented languages. As odd as it may seem, we should repeat that Tarski explicitly says that Leśniewski's language is not a "formalized" language.

[36] In order to avoid complication we are disregarding here some changes that took place in the second edition of LSM (there are some differences between W56 in *Tarski 1956* and W56 in *Tarski 1983*. For two examples, compare footnotes 1 and 2 on page 174 in *Tarski 1956*, with the same footnotes on the same page in *Tarski 1983*. The reader should be aware of these differences.

As in section 4, let us omit for convenience certain temporarily extraneous words from the German and Polish passages corresponding to Tarski's "preceding two sentences". Then, from Blaustein's edition, we get:

Von einer […] Definition der wahren Aussage soll man […] nicht viel mehr verlangen, als dass sie […] alle Teildefinitionen von diesem Typus […] umfasse […]; höchstens kann man noch verlangen, dass zum Umfang des definierten Begriffs ausschliesslich Aussagen gehören […].

And, from the Polish original edition, we get the following:

Od […] definicji zdania prawdziwego nie należy […] o wiele więcej wymagać ponad to, by […] obejmowała wszystkie cząstkowe definicje tego typu […]; conajwyżej można jeszcze żądać, by do zakresu zdefinjowanego pojęcia należały wyłącznie zdania [...].[37]

The German phrase '*nicht viel mehr* […] *als*' and the Polish phrase '*nie* […] *o wiele więcej* […] *ponad to*' contribute to the meaning of their contexts exactly as the English phrase 'not much more […] than' contribute to the meaning of the first of Tarski's "preceding two sentences". Thus, the German and Polish counterparts of the *first* of the "preceding two sentences" similarly imply that *something* more is needed than what is stated in condition alpha. Now, the German '*höchstens*' and the Polish '*conajwyżej*' (today spelled '*co najwyżej*') mean essentially the same as the English adverbial phrase 'at most'. Therefore, the German and Polish counterparts of the *second* of the "preceding two sentences" similarly imply that what is stated in condition beta would be a *sufficient* supplement to alpha.[38]

The critical passage is Tarski's "following sentence", the context of the claim that condition beta "is not essential". What we find in the German and Polish corresponding excerpts? Again, let us omit for convenience certain temporarily extraneous words from them. Making some minor adjustments for the sake of readability, we get the following:

[…] der zweite Teil obiger Konvention [**besitzt**] keine wesentliche Bedeutung […]: sobald die Metasprache […] die Bedingung (α) befriedigt, kann man leicht ein neues Symbol „*Wr'* " definieren, welches ausserdem die Bedingung (β) erfüllt; […] [**nehmen wir an**], dass *Wr'* der gemeinsame Teil der Klassen *Wr* und *As* ist.

In the same fashion, from the Polish original edition, we get:

[…] druga część powyższej umowy nie posiada istotnego znaczenia: z chwilą gdy […] [**metajęzik**] […] [**czyni**] zadość warunkowi (α), można z łatwością

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[37] We are especially indebted here to Magda Stroińska, David Hitchcock, James Smith, and Andrew McFarland, for kindly helping us with the abbreviation and adjustments in the Polish passages quoted in this section. We also benefitted from the research reported in *Hitchcock and Stroińska 2002*.

[38] In the entry for '*najwyżej*' in Stanisławski's *Wielki Słownik polsko-angielski*, one reads 'at (the very) most' as the English phrase corresponding to the Polish '*co najwyżej*'. This is stronger than 'at most', but does not alter significantly the content of the passage. Rather, it seems to reinforce our affirmation that the second of Tarski's "preceding two sentences" does not strictly speaking imply that the full strength of beta is necessary.

> zdefinjować nowy symbol „*Vr'* ", który ponadto spełnia warunek (*β*); […]
> [**przyjmijmy**], że *Vr'* jest to część wspólna klas *Vr* i *S*.

As said above, the statement that condition beta "is not essential" has a counterpart in both the *1933* and *1935* editions of Tarski's truth-definition paper. The German corresponding phrase is '*besitzt keine wesentliche Bedeutung*', and Tarski's original phrase is '*nie posiada istotnego znaczenia*'.

First of all, it is interesting to note that Woodger's 'is not essential' is not a literal translation of Blaustein's version. In particular, it leaves out the word '*Bedeutung*'. Translating literally the German phrase would give us something like 'has no essential meaning'. Blaustein carefully preserves the structure of Tarski's original phrase—putting, in particular, '*Bedeutung*' for the Polish '*znaczenia*' (nominative: '*znaczenie*'). A direct translation from the 1933 passage into English would also give us something like 'has no essential meaning' (maybe 'has no essential significance'). Be that as it may, leaving out '*Bedeutung*' (and, indirectly, '*znaczenia*')—does it have any serious interpretive effect at all? Whether one says that condition beta "is not essential", or whether one says that it "has no essential *meaning*" (or "significance"), our problem seems to remain untouched. We still do not know what this claim amounts to.[39]

Incidentally, a word of caution is in order here: do not take the terms '*Bedeutung*' and '*znaczenie*', as used by Tarski here and elsewhere in his monograph, with any technical and precise philosophical content. Blaustein seems to follow the implicit rule of translating in most cases '*znaczenie*' as '*Bedeutung*', and '*sens*' as '*Sinn*'. But there is absolutely no technical distinction presupposed here—let alone, as one could think at first, a tacit adoption of a theory of meaning like Frege's. If one reads the Polish original, one notes the fact that Tarski uses interchangeably '*sens*' and '*znaczenie*' in quite similar contexts, a fact which by itself indicates that the intended distinction is a merely stylistic one.

The focus of the problem relates to the word 'essential'. In what sense of 'essential' did Tarski intend to say that condition beta "is not *essential*" or, if one prefers a literal translation, "has no *essential* meaning"? Again, checking the German and Polish passages does not shed much light on the issue. For 'essential' is a faithful version of '*wesentlich*', which by its turn is a faithful version of '*istotny*' ('*istotnego*' is the genitive singular form). There is no mistranslation here with respect to these words. But remind that Tarski's "following sentence" has two clauses, the second of which seems to be an explanation of (or a sketchy argument for) the first clause. Perhaps Woodger mistranslated Blaustein's second clause, or else Blaustein mistranslated Tarski's original second clause.

Once again, checking the source passages shows that there are no substantive translational problems. Blaustein's German second clause translates almost literally the Polish corresponding passage. Some puristic criticism might target '*sobald*' ("as soon as") and point out that it is not literal for '*z chwilą gdy*' ("at the moment [when]"), but this would be irrelevant to our problem. As to Woodger's translation, it is not as literal as the

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[39] In a personal communication, Magda Stroińska and David Hitchcock remarked to us that "in the phrase '*istotnego znaczenia*', which literally means 'essential meaning' and is in the genitive case, the word '*znaczenia*' adds very little semantic content". They remarked further that the phrase is common Polish for 'essential'. In conclusion, they said that "Woodger's translation without the word 'meaning' seems quite faithful to the Polish original, and Blaustein's use of the word '*Bedeutung*' seems a bit over-literal". We are again indebted to them for this valuable and kind assistance.

German rendition. One remarkable feature of the English sentence is Woodger's choice of uniformly using the verb 'satisfy', which translates two German verbs used in the source text, '*befriedigen*' (Blaustein's choice for '*czynić zadość*') and '*erfüllen*' (Blaustein's rendering of '*spełniać*'). But in the German version, as in the original Polish, '*befriedigen*' and '*erfüllen*' ('*czynić zadość*' and '*spełniać*') are used as mere stylistic variants, as would be, for instance, 'satisfy', 'fulfill', and 'comply with' in English. No material question is raised by Woodger's choice of 'satisfy' to translate two interchangeable verbs in the source paper.[40]

We finish our somewhat lengthy discussion in this section by suggesting the following conclusion: the awkwardness of the claim that condition beta "is not essential" (or "has no essential meaning") is also present in the German and Polish editions. If we are not wrong, the strangeness of Tarski's statement is not the result of a mistranslation. We should approach the problem from a different viewpoint.

6. Observations about *Tarski 1933/1956*, pp. 186-189.

> The extension of the two concepts [sc. true and provable] is thus not identical.
> —Tarski 1933/1956, p. 186.
> In order to predicate 'not identical' a plural is needed. 'Socrates is not identical' is a favorite example of incoherency. Tarski wants to say that the two concepts are not coextensive; he needs 'the extensions …are': The *extensions* of the two concepts *are* thus not identical. Did Corcoran miss this?
> —John Michael Herring 1985, personal communication.

Alfred Tarski's writings have been justly praised for their meticulous precision, clarity, and rigor. Indeed, many passages in *Tarski 1933/1956* are paradigms of such virtues.[41] Unfortunately however, these properties are not routinely exemplified in the

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[40] The metacalculus developed in MLC contains *explicitly* 5 axioms, 12 lemmas, 32 definitions, and 28 theorems. It is a rather remarkable fact, easily rectifiable, that the verb "to satisfy" is used as an undefined semantical term belonging to MLC's vocabulary in the statement of two axioms, several definitions, and one lemma in *Tarski 1933/1956*. In order to undo the *appearance* of circularity (in fact, there would be none), Woodger carefully avoids using "to satisfy" in Def. 22 as an undefined MLC term. Inadvertently however, he uses this same verb in the alternative definition of satisfaction in footnote 1 on page 193 and in Def. 24. Woodger's conscious manoeuver has no counterpart in the German or Polish. Blaustein uses the verb "*erfüllen*" twice in Def. 22 as part of the defined expression '*die Folge f erfüllt die Aussagefunktion x*' and as an informal metalanguage semantical verb in the phrase '*welche eine von den vier folgenden Bedingungen erfüllen*', which immediately follows it. The same slip is in Def. 24, but not in the alternative definition of satisfaction, where he uses "*erfüllen*" in the defined expression and "*genügen*" in the immediately following phrase. In the Polish, Tarski is more meticulous: he finds a way out different from Woodger's. Instead of using the verb "*spełniać*" twice, he resorts to "*spełniać*" in the defined expression, '*ciąg f spełnia funkcję zdaniową x*', and to the informally synonymous semantical verb, "*czynić zadość*" in the phrase Blaustein translates '*welche eine von den vier folgenden Bedingungen erfüllen*'. The same manoeuver is used by Tarski in the alternative definition of the notion of satisfaction, and in Def. 24. By the way, although OLC remains the same (apart from typographic differences) throughout T33, B35, and W56, its corresponding metalanguages change. MLC is *not* OLC's metalanguage in the German and Polish monographs.

[41] One famous example is the last paragraph on page 192 giving the truth-condition for universal quantification. Another, perhaps almost as important but much less well known, is the rarely mentioned fact

crucially important Section 3 of *Tarski 1933/1956.* Our unhappy task in this part of our paper—begun in the above epigraph—is to justify our sadly negative accusation.

In the first sentence, Tarski uses the expression 'the construction of the definition of *true sentence'*—italicizing the expression 'true sentence' in the context following 'the definition of*'*. It requires some study and interpretation to determine what role the italics are intended to play: whether he is referring to a definition of the concept of true sentence, a definition of the extension (or truth-set) of the concept of true sentence, a definition of the expression 'true sentence', definition of the extension of the expression 'true sentence', definition of the sense or meaning of the expression 'true sentence', a definition of the property of being a true sentence, or something else entirely.[42] He seems to be implying that it would be improper to use the expression 'definition of true sentence' without italics—an expression having marginal status as non-elliptical grammatical English. After all, Tarski has already sensitized the reader to use-mention confusions and to other similar confusions. Moreover, as already mentioned, English does not allow using a common noun phrase as a proper name of its extension: 'dog' cannot be used to denote the class of dogs—without introducing a nonce convention.[43]

Nevertheless the expression 'definition of true sentence' without italics occurs twice on this page: lines 11 and 20. In this section, the expression 'definition of' followed by an italicized noun or noun phrase never occurs again—but followed by an unitalicized noun or noun phrase it occurs several time.

The second sentence is the following.

It might appear at first sight that at the present stage of our discussion this problem can be solved without further difficulty, that 'true sentence' with respect to the language of a formalized deductive science means nothing other than 'provable theorem', and that consequently Def. 17 is already a, definition of truth and moreover a purely structural one.

Where do we begin? In the first place, there is no justification for saying that one common noun phrase *means* another. Perhaps by 'means nothing other than', Tarski intends 'means the same as'. Perhaps by 'means nothing other than "provable theorem"', Tarski intends 'means nothing other than what "provable theorem" means'. Anyway, some change is needed. In the second place, the expression 'provable theorem' is a glaring pleonasm: comparable to 'female women' or 'canine dog'. That expression had never before occurred in Tarski's writings and it never occurs in later writings. When Tarski introduced 'provable' (*1956*, p. 182), he used the expression '*x* is a provable (accepted) sentence or a theorem'. Perhaps 'provable theorem' was intended to be - 'provable sentence' or perhaps 'theorem'.

Farther down the same page Tarski writes the following.

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that Tarski meticulously uses the clausal form as opposed to the assertoric form of sentences that are used as components of compound sentence. The sentence 'it is snowing' does not begin with an uppercase nor does it end with a period (*Corcoran 2006*, pp. 226f and *Corcoran 2009*).

[42] By a definition of the concept of true sentence one might understand a sentence such as 'the concept of true sentence is […]' or maybe '*x* is the concept of true sentence iff […]'.

[43] This is reminiscent of some confusion rife in Boolean era logic, e.g. thinking that 'Every dog is an animal' is not about dogs and animals but about classes.

The extension of the two concepts is thus not identical. From the intuitive standpoint all provable sentences are without doubt true sentences (the Defs. 13-17 of § 2 were formulated with that in mind). Thus the definition of true sentence which we are seeking must also cover sentences which are not provable.VIII.3 186.16

The first of these three sentences was dealt with in the epigraph. The second could be shortened to read simply 'all provable sentences are true sentences'. The third sentence uses roman type where italic is required, it uses the previously undefined expression 'the definition … must also cover sentences which are not provable', and, most importantly, it suggests—contrary to Tarski's own previous disclaimer—that a definition of truth tells us which sentences are true. What he should say is simply that we cannot get a definition of true sentence simply by replacing 'provable sentence' by 'true sentence' in Definition 17.[44]

On the next page, p.187, we meet Convention T:

CONVENTION **T**. *A formally correct definition of the symbol 'Tr', formulated in the metalanguage, will be called an* adequate definition of truth *if it has the following consequences:*

As mentioned above, 'formally correct' is meaningless filler. Not only does it detract from the clarity of the statement, it throws the reader off balance by suggesting, if only for a moment, that the reader should turn back a few pages to find out the difference between a *formally correct definition of the symbol 'Tr'* and a *definition of the symbol 'Tr'*. Next, what is *formulated in the metalanguage* doing here? All such definitions are metalanguage sentences. And even if there is some useful purpose for *in the metalanguage* , surely *formulated* is filler. Moreover, the repetition of 'definition' makes the reader wonder whether 'adequate definition of truth' could just as well be reduced to 'adequate'. Making these changes would result in the following.

CONVENTION **T***. *A definition of the symbol 'Tr' will be called* adequate *if it has the following consequences:*

Further, after meticulously using 'if and only if' in definition after definition, the reader is perplexed to find 'if' standing alone. Why not 'if and only if'? Lastly, the *will be called* is hardly the verb for a *postulate*. It is not even the verb for a definition: *is called* is the usual verb. Incidentally, if Convention T is intended as a definition in the meta-language of the meta-language MLC, then why is it not called a definition? Making these new changes produces the following.

CONVENTION **T****. *A definition of the symbol 'Tr' is called* adequate *if and only if it has the following consequences:*

Another alternative would be the following.

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[44] Tarski introduced 'provable' (*1956*, p. 182) using expression '*x* is a *provable (accepted) sentence* or a *theorem*'. According to accepted conventions, this should be replaced by '*x* is a *provable sentence*'. Tarski has no interest in defining the expression '*provable (accepted) sentence* or in defining 'accepted *sentence*', an expression not used in this monograph. Besides, there are infinitely many provable sentences only finitely many of which have been accepted: 'accepted' cannot be a synonym for 'provable' and it is a mistake to stipulate that it is.

DEFINITION **T*****. *A definition of the symbol 'Tr' is* adequate *if and only it has the following consequences:*

Why did Tarski select the form printed and not some other?

Two pages later (*Tarski 1933/1956*, p.188) we read the following two-sentence paragraph that immediately follows Convention T and that has already been mentioned previously in our discussion.

It should be noted that the second part of the above convention is not essential; so long as the metalanguage already has the symbol 'Tr' which satisfies the condition (α), it is easy to define a new symbol '*Tr′*' which also satisfies the condition (β). It suffices for this purpose to agree that *Tr*' is the common part of the classes *Tr* and *S*.

There are several independent criticisms to be made of the first sentence following Convention T; we explicitly refrain from making all of them, even all that we know of. Note the "new symbol" '*Tr′*' [sc. accenting the symbol '*Tr*'].

Taken literally, Tarski's statement is what is known colloquially as "a flaming non-sequitur". There is no sense of 'essential' known to us—or to any of the dozens of native English speakers we consulted—that makes sense here. The fact, if such it is—that a definition satisfying beta *also* is easy to construct from one satisfying alpha—has no bearing on whether beta is essential for any given purpose. Were the expression 'not essential' changed to something like 'not difficult to satisfy once alpha has been satisfied', we could begin to deal with Tarski's statement.

Notice that Tarski's expression 'to define a new symbol '*Tr′*' which also satisfies the condition (β)' cannot be taken literally: it is not a *symbol* satisfying beta that is at issue but a *definition* satisfying beta. Perhaps Tarski wanted something like 'to construct a new definition which uses a new symbol '*Tr′*' and which also satisfies the condition (β)'.

However, the new expression Tarski suggests is evidently the following.

Tr' is the common part of the classes *Tr* and *S*.

But this is not a definition of the symbol '*Tr*'. In fact it is not even a sentence of MLC. It is an expression in another language: one gotten from MLC by adding a new symbol '*Tr′*'—accenting '*Tr*'. Moreover, since no meaning has been attached to the accented '*Tr′*', the new expression has no truth-value.

Incidentally, Tarski's expression 'It suffices for this purpose to agree …' is unnecessarily inexact and misleading to beginners and others not accustomed to how mathematicians write informally. No agreeing or disagreeing is relevant: we are talking about manipulating strings of characters.

What Tarski might have had in mind is taking a definition of the symbol '*Tr*' satisfying alpha and changing the right side into a conjunction of the old right side with '*x* is a sentence'. This would yield a definition of the symbol '*Tr*' in MLC—an expression having the required form and devoid of any new symbol. Further, if the old definition were true, the new one would be as well.

But, it is not clear—to us at least—whether beta would be satisfied by the new definition or even whether alpha would be satisfied.

To conclude this section we can say that we have demonstrated a pattern of careless, inexact, and overly elliptical writing in this section. Compare *Corcoran 2013*.

7. Conclusions

> Do not forget that philosophical writings are artifacts; they are not the reality that philosophy is about. Do not become like the archeologists who inadvertently spend their time studying the debris left by previous archeologists.
> — Frango Nabrasa, 2001, personal communication.

We have demonstrated beyond a reasonable doubt that there is no way to conclude from Tarski's statements apparently retracting his inclusion of condition beta that he indeed intended to limit Convention T's conditions to alpha alone. In fact, evidence points to no intentions to eliminate beta. On the contrary, retraction of condition beta would call into question the sincerity of Tarski's repeated statements that the range of applicability of truth in a given context is the set of sentences of a fixed formalized language given in advance. We can only purpose that Tarski misspoke when he apparently retracted condition beta. Anyway, no responsible person could justify omitting condition beta in their own definition of adequacy by citing Tarski as an authority—at least not the Tarski of *1933/1956*.

Our studies took us into unexplored corners of Tarski's thought and uncovered many questions to explore in the future. We raised more questions than we answered. Many of our answers were hedged or labeled as speculation. Contrary to our hopes and expectations, our investigations revealed incoherencies, non sequiturs, obscurities, redundancies, misleading passages, and other flaws not previously identified in print. However, we did not and would not question the importance and general rigor widely attributed to Tarski's *1933/1956* work.

Tarski's writings are portrayed as a rich source of historical, philosophical, linguistic, and sociological problems. We hope that our paper will help to make interpretational study of Tarski's logical work to become as extensive, rich, and rewarding as interpretational study of Aristotle's logical work has become since Łukasiewicz brought it to the attention of the logically informed world.

Acknowledgements

> We call in others to aid us in deliberating on important questions—distrusting ourselves as not being equal to deciding.—Aristotle 1112b10.

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