Amphi-ZF : axioms for Conway games

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Abstract

A theory of two-sided containers, denoted ZF_2, is introduced. This theory is then shown to be synonymous to ZF in the sense of Visser [8], via an interpretation involving Quine pairs. Several subtheories of ZF_2, and their relationships with ZF, are also examined. We include a short discussion of permutation models (in the sense of Rieger–Bernays) over ZF_2. We close with highlighting some areas for future research, mostly motivated by the need to understand non-wellfounded games.

1 Introduction

In a passage on pages 64–67 of *On Numbers and Games* [1, 2] (hereafter abbreviated in the usual way as ONAG, and unless specified otherwise all page references and quotations apply to both editions), Conway discusses options for the formalisation of his theory of combinatorial games formed as two-sided sets of *options* for the players Left and Right, these options themselves being two-sided sets of options. He points out that, although his theory could be formalised in ZF, it would be more natural to formulate a theory of two-sided sets and formalise the theory of games in this. More radically, he proposes a 'Mathematicians' Liberation Movement' [ONAG, p66] in which some general foundational principle that should allow all reasonable ('permissible') constructions and inductions to be considered grounded on a foundation essentially equivalent to that of ZF without requiring further investigation.

In this passage, it is clear that Conway has the idea of a two-sided set theory based on a principle of induction, and also the idea for an interpretation of it in usual ZF. Indeed he refers to Kuratowsi ordered pairs and using the Scott trick of equivalence classes of sets of minimal rank to hint at how this interpretation might be carried out [ONAG, p65]. Completing this programme of devising the two-sided set theory and interpreting it in ZF as indicated is straightforward. We believe however that it is much more natural to use Quine's ordered pairs [7]—not so much because of the 'typing' advantages of the pairing that was Quine's original motivation, but because every set can be considered as a pair of sets using Quine's pairing. This raises the possibility that ZF and its two-sided version are actually essentially the same theory, a conjecture that is verified here.

In this paper we formalise an axiomatic theory ZF_2 of Conway games, called Amphi-ZF . Then we show this theory to be equiconsistent with ZF in a 'strong' sense (i.e. *synonymous*) using relative interpretations. The axioms we give are intentionally an obvious generalisation of ZF to a language with two membership relations $\in_L$ and $\in_R$. Moreover, the axioms we give, as well as the proofs of interpretations, generalise to any number of membership relations without difficulty.

By 'interpretation' we shall mean a *relative interpretation*, as defined by Visser [8]. We briefly describe such objects and their category-theoretic framework, which makes discussion of the interpretations much simpler, here. The category INT has as objects logical theories. All theories are assumed to have only relations as non-logical symbols; among these relations included we assume there is a unary relation δ, indicating the domain. (Note that equality is also included as a logical symbol). We assume full first-order logic, including the equality rules and the logical axiom $\forall x\ \delta(x)$. By a *relative translation* $\mathfrak{f}\colon T_2 \to T_1$ of an $\mathscr{L}_2$-theory T_2 into an $\mathscr{L}_1$-theory T_1 we mean a mapping $\mathfrak{f}$ of atomic formulas $R(x_0, \dots, x_{n-1})$ of $\mathscr{L}_2$ to formulas $R(x_0, \dots, x_{n-1})^{\mathfrak{f}}$ of $\mathscr{L}_1$, in the same free variables. In particular $\delta(x)^{\mathfrak{f}}$ is some domain formula $\delta^{\mathfrak{f}}(x)$, and $(x = y)^{\mathfrak{f}}$ is in our case required to be simply $(x = y)$. This mapping is extended to all $\mathscr{L}_2$-formulas by taking $(\neg\theta(\overline{x}))^{\mathfrak{f}}$ to be $\neg\, \theta(\overline{x})^{\mathfrak{f}}$, $(\phi(\overline{x}) \to \psi(\overline{x}))^{\mathfrak{f}}$ to be $\phi(\overline{x})^{\mathfrak{f}} \to \psi(\overline{x})^{\mathfrak{f}}$, and $(\forall\overline{x}\,\phi(\overline{x}))^{\mathfrak{f}}$ to be $\forall\overline{x}\,(\bigwedge_i \delta^{\mathfrak{f}}(x_i) \to \phi(\overline{x})^{\mathfrak{f}})$. A *relative interpretation* $\mathfrak{f}\colon T_2 \to T_1$ is a relative translation satisfying $T_1 \vdash \exists x\,\delta(x)^{\mathfrak{f}}$ and also $T_2 \vdash \phi \Rightarrow T_1 \vdash \phi^{\mathfrak{f}}$ for all statements ϕ in the language of T_2. Following Visser we require that, for theories U, V, W in INT:

- the interpretation $\mathrm{id}_U\colon U \to U$ leaves relations unchanged;
- if $\mathfrak{f}\colon U \to V$ and $\mathfrak{g}\colon V \to W$ then $R(\overline{x})^{\mathfrak{f}\mathfrak{g}}$ is $(R(\overline{x})^{\mathfrak{f}})^{\mathfrak{g}}$.

Further, two interpretations $\mathfrak{f}, \mathfrak{g}\colon U \to V$ are considered *equivalent* if

- $V \vdash \forall x(\delta(x)^{\mathfrak{f}} \leftrightarrow \delta(x)^{\mathfrak{g}})$, and
- $V \vdash \forall\overline{x}\left(\bigwedge_i \delta(x_i)^{\mathfrak{f}} \to (\phi(\overline{x})^{\mathfrak{f}} \leftrightarrow \phi(\overline{x})^{\mathfrak{g}})\right)$ for all formulas ϕ in the language of U.

Finally, the morphisms in INT are these interpretations, modulo equivalence, though generally we shall refer here to specific interpretations. The interpretations $\mathfrak{f}\colon U \to V$ and $\mathfrak{g}\colon V \to U$ are said to be *inverse* to each other if the corresponding morphisms are; that is, if $\mathfrak{f}\mathfrak{g} = \mathrm{id}_U$ and $\mathfrak{g}\mathfrak{f} = \mathrm{id}_V$ in this category.

Having introduced ZF_2, we shall show that there are interpretations $\mathfrak{f}\colon \mathrm{ZF}_2 \to \mathrm{ZF}$ and $\mathfrak{g}\colon \mathrm{ZF} \to \mathrm{ZF}_2$ which are inverse to each other. In Visser's terminology, ZF_2 and ZF are *synonymous*; more informally, they are essentially the same theory.

We also briefly look at the state of von Neumann–Bernays–Gödel set theory (NBG) and its two sided analogue, NBG_2, and report the expected result that these are also synonymous.

Even though ZF and ZF_2 (or NBG and NBG_2) are equal, there are sometimes strong psychological reasons to prefer the two-sided theory. Conway's games provide one example: there is an elegance in studying combinatorial games and their 'one line proofs' in a system which shows that not only are these games a generalisation of number, but they are also the only notion of 'set' that is required. It seems plausable that other such situations may arise, and in such examples the more natural two-sided structure would be more suggestive and helpful for the development of the subject. Another possibility is that a construction (for example a model theoretic construction of a new model of two-sided set theory from a model of ZF_2) is straightforward, but when 'translated' to normal set theory via our interpretations becomes powerful and interesting. We close by offering a couple of speculations along these lines using the idea of two-sided Rieger–Bernays permutation models.

Most of the results presented here appeared in Cox's Master's Thesis at the University of Birmingham [3] and this thesis can be consulted for additional information and other views on combinatorial games in general.

2 Amphi-ZF

In this section we shall present the axioms for Amphi-ZF or ZF_2. Let $\mathscr{L}_2$ denote the first-order language with non-logical symbols $\in_{\mathrm{L}}, \in_{\mathrm{R}}$, both denoting binary relations.

Throughout we shall use '=' to denote the usual logical *identity* in the first-order language $\mathscr{L}_2$. So for example, for us $\{0, 1 \mid\} \neq \{1 \mid\}$. The axiom of extensionality (below) will identify = with the notion of two games having the same Left and Right options or members. In other words our = is the notion which Conway calls identity and notates as $\equiv$ [ONAG, p15]. We believe adhering to standard usage in first-order logic is more important here than adhering to Conway's usage. We will hardly need Conway's notion of equality here; when we do need it we use the symbol $\approx$ for it.

An object of an $\mathscr{L}_2$-structure will be called a two-sided set or a *game*. The relations $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$ are the Left and Right membership relations. There is also a symmetric membership relation defined by

- $x \mathrel{\sqsubset\!\!\!-} y \Leftrightarrow (x \in_{\mathrm{L}} y \vee x \in_{\mathrm{R}} y)$.

In general, we will use square versions of familiar set theoretic notation such as $\mathrel{\sqsubset\!\!\!-}, \sqcup, \sqsubseteq$ to indicate a variation of the familiar notion that is symmetric in Left and Right.

The subset relations are defined as follows. Let x, y be games. Then

- $x \subseteq_{\mathrm{L}} y \Leftrightarrow \forall z \in_{\mathrm{L}} x\, (z \in_{\mathrm{L}} y)$;
- $x \subseteq_{\mathrm{R}} y \Leftrightarrow \forall z \in_{\mathrm{R}} x\, (z \in_{\mathrm{R}} y)$;
- $x \sqsubseteq y \Leftrightarrow (x \subseteq_{\mathrm{L}} y \wedge x \subseteq_{\mathrm{R}} y)$.

Due to the symmetric nature of the system of axioms to be presented, it is useful to adopt the following notation. We will write a sub or superscript P (as in, for example, $\in_{\mathrm{P}}$) to indicate that one of L (Left) or R (Right) versions of the symbol is to be used (e.g. $\in_{\mathrm{L}}$ or $\in_{\mathrm{R}}$); in any expression a P will only represent a particular player (L or R) at any one time, however many times it is used. For example, the string $\forall x \subseteq_{\mathrm{P}} z\ \exists y\, (y \in_{\mathrm{P}} x)$ refers to either $\forall x \subseteq_{\mathrm{L}} z\ \exists y\, (y \in_{\mathrm{L}} x)$ or $\forall x \subseteq_{\mathrm{R}} z\ \exists y\, (y \in_{\mathrm{R}} x)$. Further, if $\phi_{\mathrm{L}}, \phi_{\mathrm{R}}$ are first-order sentences we define $\bigwedge_{\mathrm{P}} \phi_{\mathrm{P}}$ to be $\phi_{\mathrm{L}} \wedge \phi_{\mathrm{R}}$.

A0 (Zero Game Axiom)**.** There exists a zero game, i.e.

$$\exists x\ \forall y\, (y \notin_{\mathrm{L}} x \wedge y \notin_{\mathrm{R}} x).$$

A1 (Axiom of Extensionality)**.** Two games are equal if and only if their respective options are equal;

$$\forall x\ \forall y \Big(\bigwedge_{\mathrm{P}} (\forall z\, (z \in_{\mathrm{P}} x \leftrightarrow z \in_{\mathrm{P}} y)) \to x = y \Big).$$

Extensionality justifies the use of the familiar notation from ONAG, for example using $\{u, v \mid x, y\}$ to denote the game with Left and Right options u, v and x, y repsectively.

A2 (Pair-game Axiom)**.** If x, y are games there is a game with these games as left options;

$$\forall x\ \forall y\ \exists z\, (x \in_{\mathrm{L}} z \wedge y \in_{\mathrm{L}} z).$$

The replacement axiom will imply that there is a game with only these options, and will also guarantee that a similar game with right options x, y and no left options exists.

A3 (Replacement)**.** If $\phi_{\mathrm{L}}(x, y, \bar{a})$ and $\phi_{\mathrm{R}}(x, y, \bar{a})$ are first-order formulas in the free variables shown, then

$$\forall \bar{a}\ \forall I \Big(\bigwedge_{\mathrm{P}} (\forall x \mathrel{\mathsf{E}} I\ \exists! y\, \phi_{\mathrm{P}}(x, y, \bar{a})) \to \exists A\ \forall z \bigwedge_{\mathrm{P}} (z \in_{\mathrm{P}} A \leftrightarrow \exists x \mathrel{\mathsf{E}} I\, \phi_{\mathrm{P}}(x, z, \bar{a}))\Big).$$

The next axiom is not really required; as in ordinary ZF it can be deduced from the Replacement axiom. Still, we shall refer to this theorem as Separation when working in Amphi-ZF.

A4 (Separation)**.** For all first-order formulas $\phi_{\mathrm{L}}(\bar{u}, v)$, $\phi_{\mathrm{R}}(\bar{u}, v)$ in free variables shown we have

$$\forall \bar{x}\ \exists y\ \forall z \bigwedge_{\mathrm{P}} (z \in_{\mathrm{P}} y \leftrightarrow z \in_{\mathrm{P}} x \wedge \phi_{\mathrm{P}}(\bar{x}, z)).$$

If x is a game for which $\bigwedge_{\mathrm{P}}(y \in_{\mathrm{P}} x \leftrightarrow \phi_{\mathrm{P}}(y))$, we write

$$x = \{\, y^{\mathrm{L}} : \phi_{\mathrm{L}}(y) \mid y^{\mathrm{R}} : \phi_{\mathrm{R}}(y) \,\}$$

or even (extending our useful shorthand notation using the P) $x = \{\, y^{\mathrm{P}} : \phi_{\mathrm{P}}(y^{\mathrm{P}}) \,\}$. Separation guarantees the existence of any set of the form $\{\, y^{\mathrm{P}} : y^{\mathrm{P}} \in_{\mathrm{P}} a \wedge \phi_{\mathrm{P}}(y^{\mathrm{P}}) \,\}$. If x is a game, i.e. an object of our theory, Conway uses x^{L} as a variable ranging over Left-elements of x, i.e. over z such that $z \in_{\mathrm{L}} x$. Similarly x^{R} ranges over Right-elements of x, i.e. z such that $x \in_{\mathrm{R}} x$. This is chiefly used as an abbreviation in an implicit use of Separation. For example $\{\, x^{\mathrm{L}} \mid x^{\mathrm{R}} \,\}$ is an abbreviation for $\{\, z : z \in_{\mathrm{L}} x \mid z : z \in_{\mathrm{R}} x \,\}$, which is of course x itself. We shall occasionally use such notation too.

We still require an axiom of Union. There are several different types of union we may wish to use, and so for clarity we designate a symbol for each, as follows.

$$\begin{aligned}
\mathop{\boxplus} x &= \{\, z : \exists y \mathrel{\mathsf{E}} x\, (z \mathrel{\mathsf{E}} y) \mid z : \exists y \mathrel{\mathsf{E}} x\, (z \mathrel{\mathsf{E}} y) \,\};\\
\bigsqcup x &= \{\, z : \exists y \in_{\mathrm{L}} x\, (z \mathrel{\mathsf{E}} y) \mid z : \exists y \in_{\mathrm{R}} x\, (z \mathrel{\mathsf{E}} y) \,\};\\
\biguplus x &= \{\, z : \exists y \mathrel{\mathsf{E}} x\, (z \in_{\mathrm{L}} y) \mid z : \exists y \mathrel{\mathsf{E}} x\, (z \in_{\mathrm{R}} y) \,\};\\
\bigcup x &= \{\, z : \exists y \in_{\mathrm{L}} x\, (z \in_{\mathrm{L}} y) \mid z : \exists y \in_{\mathrm{R}} x\, (z \in_{\mathrm{R}} y) \,\}.
\end{aligned}$$

Our Union axiom simply states that the largest of these, $\mathop{\boxplus} x$, exists for all games x. From this the existence of each other type follows simply by Separation.

A5 (Axiom of Union)**.**

$$\forall x\ \exists y\ \forall z \bigwedge_{\mathrm{P}} (z \in_{\mathrm{P}} y \leftrightarrow \exists w \mathrel{\mathsf{E}} x\, (z \mathrel{\mathsf{E}} w)).$$

We can also define binary operations of unions and intersections of games, along these lines. If x, y are games then $\{\, x, y \mid \,\}$ exists, and so the game $\{\, x^{\mathrm{L}}, y^{\mathrm{L}} \mid x^{\mathrm{R}}, y^{\mathrm{R}} \,\}$ (which is the Conway notation for $\{\, z : z \in_{\mathrm{L}} x \vee z \in_{\mathrm{L}} y \mid z : z \in_{\mathrm{R}} x \vee z \in_{\mathrm{R}} y \,\}$) does by Union and Separation. We call this game $x \cup y$. Analogously we define

$$x \cap y = \{\, z : z \in_{\mathrm{L}} x \wedge z \in_{\mathrm{L}} y \mid z : z \in_{\mathrm{R}} x \wedge z \in_{\mathrm{R}} y \,\},$$

and $x \setminus y = \{\, z : z \in_{\mathrm{L}} x \wedge z \notin_{\mathrm{L}} y \mid z : z \in_{\mathrm{R}} x \wedge z \notin_{\mathrm{R}} y \,\}$. Notice that we may also form successor games $s_{\mathrm{L}}(x) = \{\, x^{\mathrm{L}}, x \mid x^{\mathrm{R}} \,\} = x \cup \{\, x \mid \,\}$ and $s_{\mathrm{R}}(x) = x \cup \{\, \mid x \,\}$, and define $1 = s_{\mathrm{L}}(0)$, $2 = s_{\mathrm{L}}(1)$ and $-1 = s_{\mathrm{R}}(0)$, etc. From this we can state that there exists a Left- and Right-inductive game.

A6 (Infinity)**.**

$$\exists x \bigwedge_{\mathrm{P}} (0 \in_{\mathrm{P}} x \wedge \forall y \in_{\mathrm{P}} x(s_{\mathrm{P}}(y) \in_{\mathrm{P}} x)).$$

We choose the following—the E-induction principle [ONAG, p64] is derived from it in the usual way.

A7 (Foundation)**.**

$$\forall x \neq 0 \ \exists y \mathrel{\mathsf{E}} x \ \forall z \mathrel{\mathsf{E}} x \, (z \not\mathrel{\mathsf{E}} y).$$

Remark 2.1. There are other natural choices for a foundation axiom. Let $\mathrm{wf}(R)$ be the statement that R is a wellfounded relation, i.e.

$$\forall x \, (\exists y \, (y \, R \, x) \to \exists y \, R \, x \ \forall z \, R \, x \, (z \not R \, y)).$$

Then our foundation axiom is simply $\mathrm{wf}(\mathsf{E})$. We might instead choose to posit that each membership is wellfounded; that is, $\bigwedge_{\mathrm{P}} \mathrm{wf}(\in_{\mathrm{P}})$. This is a clear consequence of $\mathrm{wf}(\mathsf{E})$, but the converse is not obvious. In fact we will see in Section 6 that this second statement is strictly weaker than the first.

For the final axiom, we use the symmetric subset relation $\sqsubseteq$ defined above.

A8 (Power Game)**.**

$$\forall x \ \exists y \ \forall z \, (\bigwedge_{\mathrm{P}} z \in_{\mathrm{P}} y \leftrightarrow z \sqsubseteq x).$$

By the Power Game and Separation axioms

$$y = \{ \, u : u \sqsubseteq x \text{ and } u \text{ is inductive} \mid u : u \sqsubseteq x \text{ and } u \text{ is inductive} \, \}$$

is a game, as are $\{ \, y^{\mathrm{L}} \mid \, \}$ and $\{ \, \mid y^{\mathrm{R}} \, \}$. Defining the operator $\bigcap$ in the obvious way, i.e.

$$\bigcap u = \{ \, w : \forall v \in_{\mathrm{L}} u \, (w \in_{\mathrm{L}} v) \mid w : \forall v \in_{\mathrm{R}} u \, (w \in_{\mathrm{R}} v) \, \},$$

we may define the game $\omega = \bigcap \{ \, y^{\mathrm{L}} \mid \, \}$. Finally here, we issue a word of warning. Taking '$\approx$' to be equality as defined by Conway and $+$ to be Conway's addition of games, while is is the case that $s_{\mathrm{L}}(n) \approx n + 1$ for all $n \in_{\mathrm{L}} \omega$, we do not in general have that they are *identical*; for example,

$$2 = s_{\mathrm{L}}(1) = \{ \, 0, 1 \mid \, \} \neq \{ \, 1 \mid \, \} = 1 + 1.$$

Nor, in fact, do we have the equality $s_{\mathrm{L}}(x) \approx x + 1$ for all games.

There are many ways to define ordered pairs, but it is convenient to say that an ordered pair (u, v) is simply the game $\{ \, u \mid v \, \}$; a function is a game f with no Right options and only ordered pairs for Left options, subject to the condition that $\forall x \ \forall y \ \forall z \, ((x, y) \in_{\mathrm{L}} f \wedge (x, z) \in_{\mathrm{L}} f \to y = z)$. We write $f(x) = y$ if $(x, y) \in_{\mathrm{L}} f$, and remark that $\biguplus x$ is the game whose left elements are those in the domain of f, and whose right members are those objects in the image of f. Note that $\bigsqcup f$ is the game whose Left options are the elements of the domain of f and whose right options are the elements of the image of f.

It is easily seen that ω is Left-inductive. Using this we can define a transitive closure of a set, and go on to prove an appropriate E-induction principle, i.e.

$$\forall x \, (\forall y \mathrel{\mathsf{E}} x \, \phi(y) \to \phi(x)) \to \forall x \phi(x)$$

for all formulas $\phi(u)$. In fact, for each union type $\mathbb{U}$ of $\bigsqcup, \boxplus, \bigcup, \uplus$ the transitive closure $\mathrm{TC}(x, \mathbb{U})$ of a game x is defined recursively by setting $\mathrm{TC}(x, 0, \mathbb{U}) = x$, and $\mathrm{TC}(x, s_{\mathrm{L}}(n), \mathbb{U}) = \mathbb{U}\,\mathrm{TC}(x, n, \mathbb{U})$ for $n \in_{\mathrm{L}} \omega$; then $\mathrm{TC}(x, \mathbb{U})$ is the game

$$\{\, z\colon\ \exists n \in_{\mathrm{L}} \omega\ \exists y \in_{\mathrm{L}} \mathrm{TC}(x, \mathbb{U}, n)\ (z \in_{\mathrm{L}} y) \,|\, z\colon\ \exists n \in_{\mathrm{L}} \omega\ \exists y \in_{\mathrm{R}} \mathrm{TC}(x, \mathbb{U}, n)\ (z \in_{\mathrm{R}} y) \,\},$$

i.e. $\mathrm{TC}(x, \mathbb{U}) = \bigcup A$, where $A = \{\, \mathrm{TC}(x, \mathbb{U}, n) : n \in_{\mathrm{L}} \omega \,|\, \mathrm{TC}(x, \mathbb{U}, n) : n \in_{\mathrm{L}} \omega \,\}$. In particular $\mathrm{TC}(x, \boxplus)$ is transitive in the relations $\in_{\mathrm{L}}, \in_{\mathrm{R}}, \mathsf{E}$, while $\mathrm{TC}(x, \bigcup)$ is transitive in $\in_{\mathrm{L}}, \in_{\mathrm{R}}$, but not necessarily in E.

By E-recursion we can define the operations of negation, addition, multiplication, etc. We can also repeat Conway's definitions of what it means for a game to be less than, greater than, equal to, etc., another game, exactly as in ONAG.

3 Interpreting Amphi-ZF in ZF

Working in ordinary ZF now, and basing the following on Quine's notion of ordered pairs [7], we make the following definitions.

Definition 3.1. For all sets x we define

$$f_{\mathrm{L}}(x) = \{\, s(u) : u \in x \cap \omega \,\} \cup (x \setminus \omega),$$
$$f_{\mathrm{R}}(x) = \{\, 0 \,\} \cup \{\, s(u) : u \in x \cap \omega \,\} \cup (x \setminus \omega).$$

(Here $s(u)$ denotes the set successor of u, $\{\, u \,\} \cup u$.)

It is useful to note that $f_{\mathrm{L}}, f_{\mathrm{R}}$ have a mutual left-inverse, defined by

$$g(x) = \{\, u \in \omega : s(u) \in x \,\} \cup (x \setminus \omega).$$

We can use f_{L} and f_{R} to define relations $\in_{\mathrm{L}}, \in_{\mathrm{R}}$ by $x \in_{\mathrm{P}} y \Leftrightarrow f_{\mathrm{P}}(x) \in y$. Formally, this is as follows.

Definition 3.2. We define a translation $\mathfrak{v}\colon \mathrm{ZF}_2 \to \mathrm{ZF}$ by setting $(x \in_{\mathrm{P}} y)^{\mathfrak{v}}$ to be equivalent to $(f_{\mathrm{P}}(x) \in y)$.

We argue that $\mathfrak{v}$ is an interpretation, i.e. that for all axioms A of Amphi-ZF , $\mathrm{ZF} \vdash A^{\mathfrak{v}}$. Most of these are easy to see. We shall prove here the most difficult case that, if Found_2 denotes the amphi-foundation axiom, then $\mathrm{ZF} \vdash \mathrm{Found}_2^{\mathfrak{v}}$; the other axioms are left to the reader. First define a cumulative hierarchy of games in ZF as follows.

$$\mathscr{G}_0 = 0;$$
$$\mathscr{G}_{\alpha+1} = \{\, f_{\mathrm{L}}(z) : z \subseteq \mathscr{G}_\alpha \,\} \cup \{\, f_{\mathrm{R}}(z) : z \subseteq \mathscr{G}_\alpha \,\};\ \text{and}$$
$$\mathscr{G}_\lambda = \bigcup_{\delta<\lambda} \mathscr{G}_\lambda \ \text{for limit ordinals } \lambda.$$

Notice that this is exactly the interpretation in ZF of an obvious cumulative hierarchy of games, since for all sets x, z we have $z \subseteq x \Leftrightarrow z \sqsubseteq^{\mathfrak{v}} x$. By showing that every set is a member of this hierarchy, we can deduce the translation of Amphi-foundation in ZF.

Proposition 3.3. In ZF, for ordinals α, β we have the following.

- If $\alpha < \beta$ then $\mathscr{G}_\alpha \subseteq \mathscr{G}_\beta$.

- $\mathscr{G}_\alpha$ is $\in$-transitive.
- Every set is in some $\mathscr{G}_\alpha$.

Proof. Each claim is proved by induction. Assume that whenever $\gamma < \alpha < \beta$ we have $\mathscr{G}_\gamma \subseteq \mathscr{G}_\alpha$. If $\beta = \alpha + 1$, say, and $x \in \mathscr{G}_\alpha$ then for some $\gamma < \alpha$ there is $z \subseteq \mathscr{G}_\gamma$ such that $x = f_{\mathrm{P}}(z)$. As $z \subseteq \mathscr{G}_\gamma$ we have $z \subseteq \mathscr{G}_\alpha$ and so $x \in \mathscr{G}_{\alpha+1}$; thus $\mathscr{G}_\alpha \subseteq \mathscr{G}_{\alpha+1}$. The claim is clear when β is a limit.

To see the second claim suppose $\mathscr{G}_\alpha$ is transitive (in $\in$) for all $\alpha < \beta$. If $\beta = \alpha + 1$ and $y \in x \in \mathscr{G}_{\alpha+1}$ then $x = f_{\mathrm{P}}(z)$ for some $z \subseteq \mathscr{G}_\alpha$, and $y \in f_{\mathrm{P}}(z)$. If $y \notin \omega$ then $y \in z \subseteq \mathscr{G}_\alpha$, so by monotonicity $y \in \mathscr{G}_\beta$. If instead $y \in \omega$ then $y = 0$ (in which case $0 = f_{\mathrm{L}}(0) \in \mathscr{G}_1 \subseteq \mathscr{G}_\beta$) or $y = s(u)$ for some $u \in z \cap \omega$. Assuming the second case, $u \in \mathscr{G}_\alpha$, and so by transitivity $u \subseteq \mathscr{G}_\alpha$. Since the successor map and f_{R} coincide on ω, $y = f_{\mathrm{R}}(u) \in \mathscr{G}_\beta$. If instead β is a limit ordinal then $\mathscr{G}_\beta$ is a union of transitive sets, and hence the claim.

For the final claim, suppose $x \subseteq \mathscr{G}_\alpha$, but $x \notin \mathscr{G}_{\alpha+1}$. Then $g(x) \nsubseteq \mathscr{G}_\alpha$. Let $y \in g(x) \backslash \mathscr{G}_\alpha$. If $y \notin \omega$ then $y \in x \subseteq \mathscr{G}_\alpha$, a contradiction. Therefore $y \in \omega$, so $y \in s(y) \in x \subseteq \mathscr{G}_\alpha$. By transitivity of $\mathscr{G}_\alpha$ we have $y \in \mathscr{G}_\alpha$, a contradiction. Therefore $x \subseteq \mathscr{G}_\alpha \Rightarrow x \in \mathscr{G}_{\alpha+1}$, i.e. $\mathscr{P}(\mathscr{G}_\alpha) \subseteq \mathscr{G}_{\alpha+1}$ for all α. In particular as $V_0 = \mathscr{G}_0$ we have that $V_\alpha \subseteq \mathscr{G}_\alpha$ for all α. The claim follows. □

Theorem 3.4. $\mathfrak{v}\colon \mathrm{ZF}_2 \to \mathrm{ZF}$.

Proof. We show here that $\mathrm{Found}_2^{\mathfrak{v}}$ follows from ZF; the remaining axioms are easily verified. Define the 'game rank' $\mathrm{gr}(y)$ of a set y to be the least ordinal α such that $y \subseteq \mathscr{G}_\alpha$. Let x be an arbitrary set, and pick $y \mathrel{\mathsf{E}} x$ of minimal game rank α. Supposing $z \mathrel{\mathsf{E}} x \wedge z \mathrel{\mathsf{E}} y$, some $f_{\mathrm{P}}(z) \in y$ and so $z \subseteq \mathscr{G}_\beta$ for some $\beta < \alpha$. Hence $\mathrm{gr}(z) \leq \beta < \mathrm{gr}(y)$, contradicting our choice of y. □

Everything we have done here works in a similar way for weaker theories. In particular we consider the 1-sided set theory EST, with axioms of extensionality, empty set, pair set, sum set, separation and replacement.

It is important to note that we can define ω in EST and use the principle of induction and recursion on it. This is because either the axiom of infinity holds and ω is a set as usual, or else ω is a definable class—the class of all ordinals. The functions $f_{\mathrm{L}}, f_{\mathrm{R}}$ and g defined above may then be defined in EST.

EST_2 is the two-sided version of EST with the two-sided versions of all these axioms as described above.

The following follows by the techniques just given.

Theorem 3.5. $\mathfrak{v}\colon \mathrm{EST}_2 \to \mathrm{EST}$.

4 Interpreting ZF in Amphi-ZF

Firstly we observe that ZF can be interpreted easily as a class in any model $\mathscr{G}$ of Amphi-ZF . This allows us to define a set-membership relation E in $\mathscr{G}$, essentially copying $\in$. More precisely, we may define $\mathscr{G}_{\mathrm{L}}$ to be the subclass of games which are hereditarily Right-empty; formally this is an interpretation $\mathfrak{l}\colon \mathrm{ZF} \to \mathrm{ZF}_2$ where we let $\delta^{\mathfrak{l}}(x)$ be the formula $\mathrm{TC}(x, \uplus) \subseteq_{\mathrm{R}} 0$, and $(x \in y)^{\mathfrak{l}}$ is $(x \in_{\mathrm{L}} y)$.

Proposition 4.1. The subclass $(\mathscr{G}_{\mathrm{L}}, \in_{\mathrm{L}})$ satisfies the axioms of ZF; hence $\mathfrak{l}\colon \mathrm{ZF} \to \mathrm{ZF}_2$.

In order to find an interpretation $\mathfrak{g}\colon \mathrm{ZF} \to \mathrm{ZF}_2$ whose domain contains *all* games we may construct a (definable) bijection $F\colon \mathscr{G}_{\mathrm{L}} \to \mathscr{G}$, and mimic the behaviour of $\in$ in $\mathscr{G}$. This bijection can be defined in such a way that $\mathfrak{g}$ is inverse to $\mathfrak{v}$.

Working in Amphi-ZF, functions $f^{\mathrm{I}}_{\mathrm{L}}, f^{\mathrm{I}}_{\mathrm{R}}$ are determined (uniquely) by

$$f^{\mathrm{I}}_{\mathrm{L}}(x) = \{\, s^{\mathrm{I}}(u) : u \in_{\mathrm{L}} x \cap \omega \,|\, \} \cup (x \setminus \omega),$$
$$f^{\mathrm{I}}_{\mathrm{R}}(x) = \{\, 0 \,\} \cup \{\, s^{\mathrm{I}}(u) : u \in_{\mathrm{L}} x \cap \omega \,|\, \} \cup (x \setminus \omega).$$

(Here we use ω to denote the game $\{\, 0, 1, \dots \,|\, \}$ in $\mathscr{G}_{\mathrm{L}}$). The appropriate definition of F is then rather straightforward: if x is a set then we interpret it as the game $\{\, y : f_{\mathrm{L}}(y) \in x \,|\, y : f_{\mathrm{R}}(y) \in x \,\}$, where each set y is already interpreted as a game. Thus we define

$$F(x) = \{\, F(y) : f^{\mathrm{I}}_{\mathrm{L}}(y) \in_{\mathrm{L}} x \,|\, F(y) : f^{\mathrm{I}}_{\mathrm{R}}(y) \in_{\mathrm{L}} x \,\},$$

and notice that F has an inverse according to the rule

$$F^{-1}(x) = \{\, f^{\mathrm{I}}_{\mathrm{P}}(y) : F(y) \in_{\mathrm{P}} x \,|\, \}.$$

We define $\in$ by the rule $\forall x\, \forall y (y \in x \leftrightarrow F^{-1}(y) \in_{\mathrm{L}} F^{-1}(x))$, i.e. we define a mapping of $\mathscr{L}_{\in}$-formulas by taking $(y \in x)^{\mathfrak{g}}$ to be $(F^{-1}(y) \in_{\mathrm{L}} F^{-1}(x))$. Then the following is not difficult to prove.

Theorem 4.2. $\mathfrak{g}\colon \mathrm{ZF} \to \mathrm{ZF}_2$.

Theorem 4.3. The morphisms $\mathfrak{v}$ and $\mathfrak{g}$ are inverse to one another in INT. That is, ZF and ZF_2 are synonymous.

The problem with this approach, natural as it is, is that it appeals to induction and recursion in a strong way, so is not available in models of EST_2. We must attempt to define $\mathfrak{g}$ in EST_2 directly. As the definitions are technical and perhaps not obvious we will spend a little time motivating them.

Start by considering $0, 1, 2, \dots$ in $\mathscr{V} \vDash \mathrm{EST}$. These are of course given by $n = \{0, 1, \dots, n-1\}$ but each corresponds to a set $n^{\mathfrak{v}}$ in $\mathscr{G} = \mathscr{V}^{\mathfrak{v}}$. We calculate its notation in $\mathscr{G}$. As 0 is empty it has no members, so no $f_{\mathrm{P}}(x)$ is in 0 so $0^{\mathfrak{v}} = \{\,|\,\}$. Similarly $1 = \{0\}$ has single member $0 = f_{\mathrm{L}}(0)$, so $0^{\mathfrak{v}} = \{0^{\mathfrak{v}}\,|\,\} = \{0\,|\,\}$. $2 = \{0, 1\}$ has members $0 = f_{\mathrm{L}}(0)$ and $1 = \{0\} = f_{\mathrm{R}}(0)$ so $2^{\mathfrak{v}} = \{0^{\mathfrak{v}}\,|\,0^{\mathfrak{v}}\} = \{0\,|\,0\}$. Similarly, $3 = \{0, 1, 2\} = \{f_{\mathrm{L}}(0), f_{\mathrm{R}}(0), f_{\mathrm{R}}(1)\}$ so $3^{\mathfrak{v}} = \{0\,|\,0, \{0\,|\,\}\}$, $4 = \{0, 1, 2, 3\} = \{f_{\mathrm{L}}(0), f_{\mathrm{R}}(0), f_{\mathrm{R}}(1), f_{\mathrm{R}}(2)\}$ so $4^{\mathfrak{v}} = \{0\,|\,0, \{0\,|\,\}, \{0\,|\,0\}\}$, and so on.

This motivates the following curious notation for the integers.

Definition 4.4 (EST_2)**.** For $n \in \omega$, define $\nu(n)$ as follows. Let $\nu(0) = \{\,|\,\}$ and $\nu(1) = \{0\,|\,\}$. For $n \in \mathbb{N}$, we define

$$\begin{aligned} \nu(n+2) &= \nu(n+1) \cup \{\,|\,\nu(n)\} \\ &= \{\nu(0)\,|\,\nu(0), \dots, \nu(n)\}. \end{aligned}$$

We also define $\nu(\omega) = \{\nu(0)\,|\,\nu(0), \nu(1), \dots\}$. Notice that $\nu(\omega)$ may not exist as a set in a model of EST_2 (its existence requires an infinity axiom), but is a definable ‘amphiclass’.

Now, using the new notions, we may repeat Definition 3.1.

Definition 4.5 (EST_2)**.** Define

$$\begin{aligned}
\tilde{f}_{\mathrm{L}}(x) &= \{\,|\,\nu(n+1) : \nu(n) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{if } 0 \notin_{\mathrm{L}} x \\
&= \{\,|\,0\} \cup \{\,|\,\nu(n+1) : \nu(n) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{otherwise} \\
\tilde{f}_{\mathrm{R}}(x) &= \{0\,|\,\} \cup f_{\mathrm{L}}(x)
\end{aligned}$$

Also,

$$\begin{aligned}
\tilde{g}(x) &= \{\,|\,\nu(n) : \nu(n+1) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{if } 0 \notin_{\mathrm{R}} x \\
&= \{0\,|\,\} \cup \{\,|\,\nu(n) : \nu(n+1) \in_{\mathrm{R}} x\} \cup (x \setminus \nu(\omega)) && \text{otherwise.}
\end{aligned}$$

Now we can define our interpretation.

Definition 4.6 (EST_2)**.** For all x, y,

$$(x \in y)^{\mathfrak{g}} \leftrightarrow (0 \notin_{\mathrm{L}} x \wedge \tilde{g}(x) \in_{\mathrm{L}} y) \vee (0 \in_{\mathrm{L}} x \wedge \tilde{g}(x) \in_{\mathrm{R}} y).$$

The following are now straightforward.

Lemma 4.7 (EST_2)**.** For all $n \in \omega$ and all x, $(\nu(n) \in x)^{\mathfrak{g}}$ if and only if $(\nu(n+1) \in \tilde{f}_{\mathrm{L}}(x))^{\mathfrak{g}}$. Consequently, $\tilde{f}_{\mathrm{P}}(x) = f_{\mathrm{P}}^{\mathfrak{g}}(x)$ and $\tilde{g}(x) = g^{\mathfrak{g}}(x)$.

Proposition 4.8. Let $\mathscr{G} \vDash \mathrm{EST}_2$. Then $\mathscr{G} \vDash \mathrm{EST}^{\mathfrak{g}}$.

Finally we show that the interpretations are inverse to one another, i.e. $\mathfrak{g}\mathfrak{v} = 1$ and $\mathfrak{v}\mathfrak{g} = 1$. First, we require a preparatory lemma within a model of EST.

Lemma 4.9. In any model of EST, $\tilde{f}_{\mathrm{P}}^{\mathfrak{v}} = f_{\mathrm{P}}$; $\tilde{g}^{\mathfrak{v}} = g$.

Proof. Induction on ω. □

This immediately gives the following .

Proposition 4.10. (a) In EST_2, for all x, y, $x \in_{\mathrm{P}} y$ if and only if $(x \in_{\mathrm{P}} y)^{\mathfrak{g}\mathfrak{v}}$.

(b) In EST, for all x, y, $x \in y$ if and only if $(x \in y)^{\mathfrak{v}\mathfrak{g}}$.

Corollary 4.11. The theories EST and EST_2 are synonymous in the sense of Visser.

It may be of interest to develop a catalogue of equivalent subtheories of ZF and ZF_2 exending EST and EST_2 respectively, equivalent via the interpretations just defined. This involves showing $\mathrm{EST} + A \vdash A_2^{\mathfrak{v}}$ and $\mathrm{EST}_2 + A_2 \vdash A^{\mathfrak{g}}$ for various axioms A where A_2 is the two-sided version of A. Of course a great many such results may be given, and we give a very small sample here.

The proof of the following is straightforward.

Proposition 4.12. For sentences $A \in \{\mathrm{Inf}, \mathrm{Pow}, \mathrm{Inf} \wedge \mathrm{Pow}\}$, $\mathrm{EST} + A \cong \mathrm{EST}_2 + A_2$.

The case for foundation is less straightforward. It is not immediately obvious whether $\mathrm{EST} + \mathrm{Found} \cong \mathrm{EST}_2 + \mathrm{Found}_2$, although we can say that, by constructing a cumulative hierarchy for each theory within the other (as in Proposition 3.3), we can show that EST+Pow+Found is synonymous with its amphi-equivalent. However there is good reason to avoid considering foundation alone. Foundation's role is essentially to provide us with $\in$-induction (denoted Ind($\in$)), or its amphi-equivalent, Ind(E). However this does not follow from EST + Found (or $\mathrm{EST}_2 + \mathrm{Found}_2$) alone: we require the additional axiom that every object has a transitive closure, which is normally provided by the infinity axiom (see Kaye and Wong's article [5] or Mancini and Zambella [6]).

Proposition 4.13. The theories EST + Ind($\in$) and EST_2 + Ind(E) are synonymous, via the interpretations $\mathfrak{g}$ and $\mathfrak{v}$.

Proof. Working in EST + Ind($\in$), define a 'game rank' inductively, by

$$\mathrm{gr}(x) = \sup\{\mathrm{gr}(u) \colon u \in g(x)\} + 1.$$

If $\phi(x, \bar{a})$ is any formula such that

$$\forall x(\forall y \mathrel{\mathsf{E}^{\mathfrak{g}}} x \phi(y, \bar{a}) \to \phi(x, \bar{a})).$$

Then $\forall x\, \phi(x)$ can be proved by considering some x of minimal game rank for which $\neg\phi(x)$.

Analogously, if we work within EST_2 + Ind(E) we can define a set rank by

$$\mathrm{rank}(x) = \sup\{\mathrm{rank}(u) \colon \tilde{g}(u) \mathrel{\mathsf{E}} x\} + 1,$$

and proceed as above. □

Of particular interest in combinatorial game theory are the so-called short (hereditarily finite) games. We obtain a suitable theory for such games by negating our infinity axiom, Inf, and ensuring that full induction is available as in the last proposition. By ZF − Inf and ZF_2 − Inf we denote the theories of ZF and ZF_2 minus their respective infinity axioms. By ZF − Inf^* and ZF_2 − Inf^* we denote these theories plus an appropriate axiom, TC, of transitive containment. (In ZF we take $\forall x\, \exists y(x \subseteq y \wedge \forall u\, \forall v(u \in v \wedge v \in y \to u \in y))$; in amphi-ZF we take the same, but with $\in$ replaced by E.) Notice that ZF − Inf^* is equivalent to the theory EST + Ind($\in$) + Pow (and analogously for the appropriate amphi-variants). This immediately gives us the following.

Theorem 4.14. $\mathfrak{g}\colon \mathrm{ZF} - \mathrm{Inf}^* \to \mathrm{ZF}_2 - \mathrm{Inf}^*$ and $\mathfrak{v}\colon \mathrm{ZF}_2 - \mathrm{Inf}^* \to \mathrm{ZF} - \mathrm{Inf}^*$ are inverse to one another in INT.

Notice that by a result of Kaye and Wong [5] this implies $\mathrm{ZF}_2 - \mathrm{Inf}^* \cong \mathrm{PA}$ in INT, where PA is the theory of Peano Arithmetic.

5 Amphi-NBG

For the sake of completeness, in particular as Conway [ONAG, p67] mentions it, we briefly describe the two-sided version NBG_2 of von-Neumann–Bernays–Gödel set theory, NBG.

Following one of the popular formulations of NBG without choice, we take a two-sorted language with variables for Class-like Games $A, B, C, \ldots$ and (set-like) games $a, b, c, \ldots$. The well-formed atomic formulas are of the form $A = B$ (identity of Class-like Games), $a = b$ (identity of games), $a \in_{\mathrm{L}} B$, $a \in_{\mathrm{R}} B$, $a \in_{\mathrm{L}} b$, and $a \in_{\mathrm{R}} b$ (membership). We use $a = B$ as an abbreviation for $\bigwedge_{\mathrm{P}} \forall x\, (x \in_{\mathrm{P}} a \leftrightarrow x \in_{\mathrm{P}} B)$.

B1 (Extensionality)**.**

$$\forall x\ \forall y \Big(\bigwedge_{\mathrm{P}} (\forall z\, (z \in_{\mathrm{P}} x \leftrightarrow z \in_{\mathrm{P}} y)) \to x = y \Big)$$

and

$$\forall X\ \forall Y \Big(\bigwedge_{\mathrm{P}} (\forall z\, (z \in_{\mathrm{P}} X \leftrightarrow z \in_{\mathrm{P}} Y)) \to X = Y \Big).$$

B2 (Pair).

$$\forall x\ \forall y\ \exists z\,(x \in_{\mathrm{L}} z \wedge y \in_{\mathrm{L}} z).$$

B3 (Union).

$$\forall x\ \exists y\ \forall z\ \bigwedge_{\mathrm{P}} (z \in_{\mathrm{P}} y \leftrightarrow \exists w \mathrel{\mathsf{E}} x\,(z \mathrel{\mathsf{E}} w)).$$

B4 (Power).

$$\forall x\ \exists y\ \forall z\,(\bigwedge_{\mathrm{P}} z \in_{\mathrm{P}} y \leftrightarrow z \sqsubseteq x).$$

B5 (Infinity).

$$\exists x \bigwedge_{\mathrm{P}} (0 \in_{\mathrm{P}} x \wedge \forall y \in_{\mathrm{P}} x(s_{\mathrm{P}}(y) \in_{\mathrm{P}} x)).$$

B6 (Foundation).

$$\forall X \neq 0\ \exists y \mathrel{\mathsf{E}} X\ \forall z \mathrel{\mathsf{E}} X\,(z \not\mathrel{\mathsf{E}} y).$$

B7 (Comprehension). For all first-order formulas $\phi_{\mathrm{L}}(\bar{U}, \bar{u}, v)$, $\phi_{\mathrm{R}}(\bar{U}, \bar{u}, v)$ in free variables shown we have

$$\forall X\ \forall \bar{x}\ \exists Y\ \forall z\ \bigwedge_{\mathrm{P}} (z \in_{\mathrm{P}} Y \leftrightarrow \phi_{\mathrm{P}}(\bar{X}, \bar{x}, z)).$$

B8 (Replacement).

$$\begin{aligned}\forall F_{\mathrm{L}}, F_{\mathrm{R}}\ \forall x\,(\bigwedge_{\mathrm{P}}(\forall u\ \forall v\ \forall w\ ((u, v) \in_{\mathrm{L}} F_{\mathrm{P}} \wedge (u, w) \in_{\mathrm{L}} F_{\mathrm{P}} \to v = w)\\ \to \exists y \bigwedge_{\mathrm{P}} \forall z\,(z \in_{\mathrm{P}} y \leftrightarrow \exists u\,(u \in_{\mathrm{P}} x \wedge (u, z) \in_{\mathrm{L}} F_{\mathrm{P}}))\end{aligned}$$

Our formulation of NBG is the obvious one-sided version of these axioms. One could if one wished add to either NBG or NBG_2 any of the usual forms of the axiom of global choice.

Working in NBG, Definition 3.1 applies to both sets and classes, sending sets to sets and classes to classes. This gives an interpretation $\mathfrak{v}\colon \mathrm{NBG}_2 \to \mathrm{NBG}$ formally similar to the one in Section 3 where $(x \in_{\mathrm{P}} y)^{\mathfrak{v}}$ is $(f_{\mathrm{P}}(x) \in y)$, equality is preserved and the property of being set-like is preserved. Without any difficulty the arguments above show this is indeed an interpretation. Working in the other direction, the interpretation $\mathfrak{l}$ of Section 4 is extended to preserve the 'set-like' predicate on objects, and gives $\mathfrak{l}\colon \mathrm{NBG} \to \mathrm{NBG}_2$, restricting to hereditarily right-empty sets and right-empty classes of such sets. The class $\mathscr{G}_{\mathrm{L}}$ of hereditarily right-empty sets is in 1–1 correspondence with the class of all set-like games, for the same reason as in Section 4, and this yields also a bijection between the subclasses of $\mathscr{G}_{\mathrm{L}}$ and the class-like games. Thus the technique of mimicing $\in$ in the whole collection of games goes through too, giving an interpretation $\mathfrak{g}\colon \mathrm{NBG} \to \mathrm{NBG}_2$ which is inverse to $\mathfrak{v}$. The straightforward details are omitted.

6 Rieger–Bernays permutation models

In defining ZF_2 we have chosen certain axioms almost arbitrarily, where other obvious axioms might have been equally intuitive. For instance, our choice of a Pair-set axiom is simple, though it could be argued that a symmetric version would be more fitting. More interestingly, various different types of union are available—none of which is obviously more appropriate than the rest—and any axiom positing the existence of one such union would suffice. In these cases each choice of axiom is equivalent, modulo the other axioms of ZF_2, to the axioms it has been chosen above. In the case of union axioms it may be interesting to consider much smaller fragments of ZF_2 (perhaps obtained by weakening the replacement scheme) which do not necessarily provide this equivalence, though we will not do so here.

Of more interest to us is the weakening of foundation. Foundation is of particular interest in the theory of games through potential applications of non-wellfounded games in the semantics of computer processes.

The Rieger–Bernays permutation construction (see for example Forster [4]) can be used to obtain models of ZF_2 in which the full foundation axiom fails but some form of foundation remains, perhaps just enough to preserve certain structure present in Conway games, while also allowing one to consider illfounded games. (See also the questions at the end of this paper.)

As usual, we let $\mathrm{Sym}(\mathscr{G})$ denote the collection of permutations of $\mathscr{G}$.

Definition 6.1. Let $\mathscr{G} \vDash \mathrm{ZF}_2$, and suppose $\pi \in \mathrm{Sym}(\mathscr{G})$. For $x, y \in \mathscr{G}$ we write $x \in^{\pi}_{\mathrm{L}} y$ for $x \in_{\mathrm{L}} \pi y$ and $x \in^{\pi}_{\mathrm{R}} y$ for $x \in_{\mathrm{R}} \pi y$. By $\mathscr{G}^{\pi}$ we denote the first-order structure $(\mathscr{G}, \in^{\pi}_{\mathrm{L}}, \in^{\pi}_{\mathrm{R}})$.

It will be useful to use ZF_2^- to refer to the theory with all the axioms given above for ZF_2 except for the foundation axiom A7.

The following is an easy translation of the usual result in ZF.

Theorem 6.2. Suppose $\mathscr{G} \vDash \mathrm{ZF}_2$. If $\pi \in \mathrm{Sym}(\mathscr{G})$ is definable, then $\mathscr{G}^{\pi} \vDash \mathrm{ZF}_2^-$.

In Remark 2.1 we discussed a variant (denoted $\bigwedge_{\mathrm{P}} \mathrm{wf}(\in_{\mathrm{P}})$) of our amphi-foundation axiom ($\mathrm{wf}(\mathrel{\varepsilon})$). We can use Rieger-Bernays permutation models to prove that the variant is strictly weaker.

Theorem 6.3. Let $\mathscr{G} \vDash \mathrm{ZF}_2$, with as usual $1 = \{0\,|\,\}$, $-1 = \{\,|\,0\}$, and (for this theorem only[1]) use Conway's definition of $2 = \{1\,|\,\}$ and $-2 = \{\,|\,-1\}$. Let π be the permutation

$$(1\ \ -2) \cdot (-1\ 2).$$

Then $\mathscr{G}^{\pi}$ is a model of

$$\mathrm{ZF}_2^- + \bigwedge_{\mathrm{P}} \mathrm{wf}(\in_{\mathrm{P}}) + \neg\, \mathrm{wf}(\mathrel{\varepsilon}).$$

Proof. Let $\mathrm{wf}_{\mathrm{P}}(x)$ denote the formula

$$\exists y\,(y \in_{\mathrm{P}} x) \rightarrow \exists y \in_{\mathrm{P}} x\ \forall z \in_{\mathrm{P}} x\,(z \notin_{\mathrm{P}} y), \tag{1}$$

the statement that x is wellfounded. Note that if we can prove $\forall x\ \mathrm{wf}_{\mathrm{L}}(x)$ then $\forall x\ \mathrm{wf}_{\mathrm{R}}(x)$ follows by the symmetry of π and ZF_2.

Fix an amphiset x and assume that $\mathscr{G}^{\pi} \vDash \neg\, \mathrm{wf}_{\mathrm{L}}(x)$. Suppose first that $x \in X = \{\pm 1, \pm 2\}$. Since 1 and 2 are $\in^{\pi}_{\mathrm{L}}$-empty, they satisfy wf_{L}; therefore $x = -1$ or $x = -2$. These contain sole $\in^{\pi}_{\mathrm{L}}$-members 1 and 0 respectively, which are $\in^{\pi}_{\mathrm{L}}$-empty. Hence $\mathrm{wf}_{\mathrm{L}}(x)$, a contradiction.

Now suppose $x \notin X$. Let

$$x' = \{\,u \in_{\mathrm{L}} x \colon u \notin X \,|\, u \in_{\mathrm{R}} x\,\}.$$

As $\mathscr{G} \vDash \mathrm{wf}_{\mathrm{L}}(x')$, we can pick $u \in_{\mathrm{L}} x'$ such that $\forall v \in_{\mathrm{L}} x'\,(v \notin_{\mathrm{L}} u)$. As $x' \subseteq_{\mathrm{L}} x$, $u \in_{\mathrm{L}} x$; since $x \notin X$, $u \in^{\pi}_{\mathrm{L}} x$. Since x is not $\in^{\pi}_{\mathrm{L}}$-wellfounded, there is $v \in^{\pi}_{\mathrm{L}} x$ such that $v \in^{\pi}_{\mathrm{L}} u$.

[1] See the discussion towards the end of Section 2.

Since $v \in_{\mathrm{L}}^{\pi} x$, $v \in_{\mathrm{L}} x$, so that $v \notin_{\mathrm{L}} u$. As $v \in_{\mathrm{L}}^{\pi} u \wedge u \notin_{\mathrm{L}} u$, $u \in X$. This contradicts the choice of u.

Finally, $1 \in_{\mathrm{L}}{}^{\pi} -1 \in_{\mathrm{R}}{}^{\pi} 1$ showing $\neg\, \mathrm{wf}(\in)$. □

Given $\mathscr{G} \vDash \mathrm{ZF}_2$ there are two 'obvious' permutations to look at. The first swaps the left and right members, $x \mapsto x^* = \{u : u \in_{\mathrm{R}} x \,|\, v : v \in_{\mathrm{L}} x\}$. Then it is easy to see that $x \in_{\mathrm{L}}{}^* y$ iff $x \in_{\mathrm{L}} y^*$ iff $x \in_{\mathrm{R}} y$ and similarly for R, so the permutation model $(\mathscr{G}^*, \in_{\mathrm{L}}{}^*, \in_{\mathrm{R}}{}^*)$ is just $(\mathscr{G}, \in_{\mathrm{R}}, \in_{\mathrm{L}})$ with $\in_{\mathrm{L}}$ and $\in_{\mathrm{R}}$ swapped. Another way of saying this is that the map $x \mapsto -x$ given as usual by

$$-x = \{-u : u \in_{\mathrm{R}} x \,|\, -v : v \in_{\mathrm{L}} x\}$$

is an isomorphism $(\mathscr{G}, \in_{\mathrm{L}}, \in_{\mathrm{R}}) \to (\mathscr{G}^*, \in_{\mathrm{L}}{}^*, \in_{\mathrm{R}}{}^*)$.

The second 'obvious' permutation is the additive inverse $x \mapsto -x$ itself. It is easy to check that $x \in_{\mathrm{L}}{}^- y$ iff $x \in_{\mathrm{L}} -y$ iff $-x \in_{\mathrm{R}} y$ and similarly for R. Since the rank of $-x$ is the same as that of x it follows that $\mathscr{G}^-$ satisfies full foundation, i.e. $\mathscr{G}^- \vDash \mathrm{ZF}_2$. In fact, $(\mathscr{G}^-, \in_{\mathrm{L}}{}^-, \in_{\mathrm{R}}{}^-)$ is actually isomorphic to $(\mathscr{G}, \in_{\mathrm{L}}, \in_{\mathrm{R}})$ via the isomorphism defined recursively in $\mathscr{G}$ by

$$\phi(y) = \{\phi(u) : -u \in_{\mathrm{R}} y \,|\, \phi(v) : -v \in_{\mathrm{L}} (y)\}.$$

It is unclear what this ϕ operation is, except that it too is a permutation and may be used to give a further permutation model also isomorphic to the original, via yet another somewhat obscure map. We are not sure if this is an interesting or profitable line of enquiry and have left it here.

This concludes our outline of what might be called the 'traditional' permutation model construction. But since we are working in a two-sided set theory, we may consider a two-sided analogue of these permutations.

In the following we use relations $\equiv_{\mathrm{L}}$ and $\equiv_{\mathrm{R}}$ defined by by $x \equiv_{\mathrm{P}} y \leftrightarrow x \subseteq_{\mathrm{P}} y \subseteq_{\mathrm{P}} x$.

Definition 6.4. Asssume $\mathscr{G} \vDash \mathrm{ZF}_2$. Suppose $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ is a pair of permutations from $\mathscr{G}$. For $x, y \in \mathscr{G}$, we write

$$x \in_{\mathrm{L}}^{\pi} y \leftrightarrow x \in_{\mathrm{L}} \pi_{\mathrm{L}} y;$$
$$x \in_{\mathrm{R}}^{\pi} y \leftrightarrow x \in_{\mathrm{R}} \pi_{\mathrm{R}} y.$$

By $\mathscr{G}^{\pi}$ we denote the structure $(\mathscr{G}, \in_{\mathrm{L}}^{\pi}, \in_{\mathrm{R}}^{\pi})$. If, in addition,

$$\forall x, y\, ((\bigwedge_{\mathrm{P}} \pi_{\mathrm{P}} x \equiv_{\mathrm{P}} \pi_{\mathrm{P}} y) \to x = y), \tag{2}$$

then we call π an *amphi-permutation*. We say that π is a *proper* amphi-permutation if additionally it is not the case that

$$\forall x\, \exists y\, (x \equiv_{\mathrm{L}} \pi_{\mathrm{L}} y \wedge x \equiv_{\mathrm{R}} \pi_{\mathrm{R}} y).$$

Remark 6.5. It is interesting to note that our interpretation in Section 3, $(x \in_{\mathrm{P}} y)^{\mathfrak{v}}$, takes a form that is 'dual' to the amphi-permutation model, with $x \in_{\mathrm{P}} y$ if and only if $(f_{\mathrm{P}}(x) \in y)$, for 1–1 (but not bijective) functions $f_{\mathrm{L}}, f_{\mathrm{R}}$. In the same way, amphi-permutations may be used to build models with two-sided membership from single-sided models: $x \in_{\mathrm{P}} y$ if and only if $x \in \pi_{\mathrm{P}}(y)$.

Remark 6.6. The reason for the condition (2) is that we would like $\mathscr{G}^\pi$ to satisfy extensionality. It is easily checked that this condition is equivalent to extensionality in $\mathscr{G}^\pi$. In the case of a definable amphi-permutation π (meaning, of course, that each π_{P} is definable) in a model of ZF_2, condition (2) is also equivalent to the assertion that the map

$$\hat{\pi}(x) = \{u : u \in_{\mathrm{L}} \pi_{\mathrm{L}} x \mid v : v \in_{\mathrm{R}} \pi_{\mathrm{R}} x\}$$

is 1–1. If this map were also onto our $\mathscr{G}^\pi$ would be the same as $\mathscr{G}^{\hat{\pi}}$ and this reduces to the case of the single permutation. The condition that the map $\hat{\pi}$ is onto is equivalent to the assertion that π is improper; this explains the choice of terminology.

It follows also, by a pigeonhole argument, that if $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ is an amphi-permutation and $\hat{\pi}$ has finite support then π is improper. For if $S = \mathrm{Supp}(\hat{\pi})$ then $\hat{\pi}$ maps S into S since it is 1–1. Note too that $\mathrm{Supp}(\hat{\pi}) \subseteq \mathrm{Supp}\,\pi_{\mathrm{L}} \cup \mathrm{Supp}\,\pi_{\mathrm{R}}$.

Example 6.7. We give an example of an definable amphi-permutation $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ of $\mathscr{G} \models \mathrm{ZF}_2$ such that $\mathscr{G}^\pi$ satisfies extensionality but does not contain an empty set. This shows that proper amphi-permutations exist and that the amphi-permutation construction does not preserve stratified formulas. For simplicity, our π will have $\pi_{\mathrm{R}} = \pi_{\mathrm{L}}^{-1}$.

As usual, let $0 = \{\,|\,\}$, $n+1 = \{0, 1, \ldots, n\,|\,\}$ and $-(n+1) = \{\,|\,-n, \ldots, -1, 0\}$ for $n \in \omega$. Now define copies of these amphisets by

$$n_k = \{0, 1, \ldots, (n+k-1) \,|\, 0, -1, \ldots, -(k-1)\}$$

and

$$(-n)_k = \{0, 1, \ldots, (k-1) \,|\, 0, -1, \ldots, -(n+k-1)\}$$

for all $n, k \in \omega$. (By the only sensible convention for the meaning of '$0, 1, \ldots, -1$' we have $n = n_0$ for all n.) Note that the amphisets n_i are all distinct.

We define

$$\pi_L \colon z_k \mapsto (z+1)_k$$

for $z = n$ or $-n$, and $n, k \in \omega$, and $\pi_R = \pi_L^{-1}$.

To check the properties of $\mathscr{G}^\pi$ it suffices to check that $\hat{\pi} \restriction S$ maps into S and is 1–1, where $S = \mathrm{Supp}\,\pi_L = \mathrm{Supp}\,\pi_R = \{n_k : n, k \in \omega\}$. A simple calculation shows that $\hat{\pi}(n_k) = (n+1)_k$, $\hat{\pi}((-n)_k) = -(n+1)_k$ for $k \in \omega$ and $n > 0$ and that $\hat{\pi}(0_k) = 0_{k+1}$ for $k \in \omega$. So $\hat{\pi}$ is 1–1 and $0 = 0_0$ is the only amphiset not in its image.

We know of no easily stated conditions on an improper amphi-permutation that ensures $\mathscr{G}^\pi \models \mathrm{ZF}_2^-$. The following proposition is the unsatisfactory result of our investigation into this question.

Proposition 6.8. Suppose $\mathscr{G} \models \mathrm{ZF}_2$ and $\pi = (\pi_{\mathrm{L}}, \pi_{\mathrm{R}})$ is a definable amphi-permutation with the stronger property that

$$\bigwedge_{\mathrm{P}} \forall x, y\, (x \equiv_{\mathrm{P}} y \leftrightarrow \pi_{\mathrm{P}} x \equiv_{\mathrm{P}} \pi_{\mathrm{P}} y).$$

Then π is improper.

Proof. Given x, let u, v satisfy $x = \pi_{\mathrm{L}} u$ and $x = \pi_{\mathrm{R}} v$, and define y so that $y \equiv_{\mathrm{L}} u$ and $y \equiv_{\mathrm{R}} v$. Then by the condition $\pi_{\mathrm{L}} y \equiv_{\mathrm{L}} \pi_{\mathrm{L}} u \equiv_{\mathrm{L}} x$ and similarly for R, so π is improper. □

7 Open questions and suggestions for future research

Our intuition about ZF_2 is based on that of ZF but the theory ZF_2 is finer-structured with respect to its subtheories. A full investigation of subtheories of ZF_2 and in particular the effect of weakening the union game, power game and foundation axioms should be given. Theorem 6.3 presents a small start in this direction.

We have shown that ZF and its amphi version ZF_2 are the same theory via two inverse interpretations $\mathfrak{v}$ and $\mathfrak{g}$. These interpretations are natural but may not be the only possibilities. The main question concerns exactly what axioms are required to define these interpretations and to give a catalogue of equivalent subtheories of ZF and ZF_2. A start was made to this programme in Section 4.

Given a model $\mathscr{G} \vDash ZF_2$, which we regard as a collection of combinatorial games, the operations of addition and additive inverse are definable using recursion as usual,

$$x + y = \{u + y : u \in_{\mathrm{L}} x \,|\, v + y : v \in_{\mathrm{R}} x\} \cup \{x + u : u \in_{\mathrm{L}} y \,|\, x + v : v \in_{\mathrm{R}} y\}$$

and

$$-x = \{-u : u \in_{\mathrm{R}} x \,|\, -v : v \in_{\mathrm{L}} x\}.$$

These operations are central to the theory of games, as are

$$0 \leqslant x \leftrightarrow \forall u \in_{\mathrm{R}} x \;\exists v \in_{\mathrm{L}} u\, (0 \leqslant v)$$

and

$$0 \lhd\!\mid x \leftrightarrow \exists u \in_{\mathrm{L}} x \;\forall v \in_{\mathrm{R}} u\, (0 \lhd\!\mid v)$$

with $x \leqslant y \leftrightarrow 0 \leqslant y - x$ and $x \lhd\!\mid y \leftrightarrow 0 \lhd\!\mid y - x$. Obviously foundation is required for all these definitions, but how much? Is the full axiom of foundation required to make sense of these notions?

Given $\mathscr{G} \vDash ZF_2^-$, we might not be able to define $+, -, \leqslant, \lhd\!\mid$ internally, but we can at least regard each $x \in \mathscr{G}$ as an (external) game (in the metatheory) with three outcomes: either L or R wins, or there is a draw—meaning that after standardly many (i.e. $\mathbb{N}$ in the metatheory) turns there is no winner. Then $+, -, \leqslant, \lhd\!\mid$ can all be defined on these games in the metatheory (with the proviso that we need to account for games that go on for infinitely many turns). It would seem to make sense to study these notions for various models $\mathscr{G} \vDash ZF_2^-$.

The amphi-permutation construction for ZF_2 is richer than that of single-sided membership, and this needs a thorough investigation. What sentences are necessarily preserved by such a construction? Also, the amphi-permutation construction (and its 'dual', see Remark 6.5) also enables other models of two-sided theories to be obtained from one-sided memberships. This should also be investigated.

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