Translation of: P. Ehrenfest,‘Energy fluctuations in the radiation field or crystal lattice through superposition of quantized normal modes’*

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Abstract

It is shown that and why smaller fluctuations result from the superposition of quantized normal modes than those claimed by Einstein: Equation (II) and (III) compared to (IV).

§1. In a discussion about the Einsteinian light quanta held about a year ago, Frau T. Ehrenfest-Afanassjewa made a remark that can be formulated as follows:

Einstein had derived his equation for energy fluctuations of a volume element in the black radiation field from Planck's radiation formula and got this result: the magnitude of these fluctuations is incompatible with the conception of the radiation field as a superposition of light waves.[1]

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*‘Energieschwankungen im Strahlungsfeld oder Kristallgitter bei Superposition quantisierter Eigenschwingungen’, *Zeitschrift für Physik* 34 (1925), 362–373. I have numbered footnotes in this translation continuously, whereas in the original Ehrenfest resets footnote numbering with each new section.

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[1] A. Einstein, Phys. ZS. **10**, 185, 817, 1909, Solvay Congress 1911, Rapp. p. 419. — In addition to the papers referred to in footnotes, see the following papers about light fluctuations: M. v. Laue, Verh. d. D. Phys. Ges. **17**, 198, 1915; H. A. Lorentz, Théories statist., Note IX, Teubner 1916. — W. Bothe, Räuml. Energieverteilung in der Hohlraumstrahlung [Spatial energy dispersion in blackbody radiation], ZS. f. Phys. **20**, 145, 1923; M. Planck, Ann. d. Phys. **73**, 272, 1924.

But on the other hand, it is well known that one can derive the Planck radiation formula for blackbody radiation from wave concepts if one quantizes only the Rayleigh–Jeans normal modes of the blackbody governed by Planck's energy ensemble statistics.

Thus the following internal contradiction arises — not so different from the question of the 'true' nature of radiation fields: starting from wave concepts one attains, via Planck's equation, light fluctuations that are incompatible with wave concepts. — How do we resolve this contradiction?

The answer to this question was discovered and proved by Ornstein and Zernike already in 1919, and in essence is contained in their note.[2] In his fluctuation calculations, Einstein makes the traditional assumption that the entropies of different volume elements are purely additive, so the fluctuations in them are independent of one another from the standpoint of probabilistic calculations. But this is not a good assumption in the case of the radiation field comprised of a superposition of normal modes of the blackbody.[3]

§2. Though the root of this contradiction is entirely clear and you have probably already decided to forsake the representation of blackbody radiation through quantized normal modes, allow me to show you two fluctuation equations that result from holding on to this hypothesis. — To some degree this is justified by the following remarks:

1. Energy fluctuations must also appear in volume elements of a crystal lattice, e.g. a cold diamond, and according to Einstein's equation[4] one can be sure that the specific heat of a solid body – excluding the gap where the velocity of light is $\gg$ the velocity of elastic waves – must obey entirely analogous laws as the Einsteinian light fluctuations. For energy fluctuations in crystal lattices, however, one is for the time being still entirely reliant on the method of normal modes.[5]

[2]Ornstein and Zernike, Energiewisselingen der zwarte straling er licht-atomen, Akad. v. Wetensch. Amsterdam **28**, 280, 1919/20.

[3]Ornstein and Zernike use this opportunity to cite Laue's investigation of the non-additivity of entropy that persists through coherence. M. v. Laue, Ann. d. Phys. **20**, 365, 1906; **23**, 1, 795, 1907.

[4]Einstein Solvay Congress Report, p. 419 (1911). — The translational motion of dilute He molecules also seem to posses temperatures of fluid Helium, in essence the kT–value, thus also in principle through thermal contact with e.g. a diamond whose thermal motion is already completely degenerate here. Without high concentrations of fluctuation in the energy of the diamond lattice, such a thermal equilibrium would surely arguably be entirely inconceivable.

[5]Except where the application of Bose–Einstein statistics to the atoms of a crystal opens up a new path. — See A. Einstein, Quantentheorie einatomiger idealer Gase; Berl.

2. Also if, following Einstein, one bypasses interpreting light fluctuations with the help of light corpuscles, then one of the terms of the fluctuation formula – derived from the Planck radiation law – indeed works splendidly: the "Wien term"; but the other term, the "Rayleigh–Jeans", seems to indicate, as is known, a peculiar affinity with light corpuscles – and in fact if one doesn't at all wish to ignore interference phenomena, of light corpuscles with a wave-like affinity.[6] But this also makes it possible that in the future theory of light fluctuations, some features of wave superpositions may likely linger.

3. The comparison of the equation of the fluctuation formula (II) derived in §4 with that derived by Einstein (IV) in §6 illuminates in more detail the contradiction between the following two statements:

   a) The fluctuations arise from the superposition of quantized blackbody oscillations (based on II)

   b) The entropy and fluctuations of various parts of the radiation field are independent from one another (based on IV).

4. Moreover, the comparison of "Problem II" (§4) with "Problem III" (§5) shows that one obtains entirely different values for fluctuations in a volume element depending on whether one considers time-like (beat) fluctuations or fluctuations caused by changing excitation of the blackbody normal modes. This crucial distinction is not treated in the literature.

§3. "Problem I." Assume there is a "cavity" of volume V bounded by a shell of mirrors. In the frequency range $\nu, \nu + d\nu$ we have the normal modes

Akademiebar. 1924.

[6]The text of the paper by S. N. Bose – Plancks Gesetz und Lichtquenten-hypothese, ZS. f. Phys. **26**, 178, 1924 – shows whether Planck's radiation law can be derived from the representation of independent light-corpuscles. But this is not the case. Independent light corpuscles would correspond to the Wien radiation law. See A. Einstein, Ann. d. Phys. **17**, 132,1905; P. Ehrenfest, Ann. d. Phys. **36**, 91, 1911; G. Krutkow, Phys. ZS. **15**, 133, 363, 1914; P. Ehrenfest and H. Kamerlingh Onnes, Ann. d. Phys. **46**, 1023, 1915. — Along those lines, M. Wolfke, Phys. ZS. **22**, 375, 1921, and W. Bothe, ZS. f. Phys. **20**, 145, 1923, have worked with the hypothesis that complex-picture (spatially contiguous) light corpuscles suffice to obtain the Planck law. This would not, however, account for the appearance of interference phenomena.

$$Z = V \cdot \frac{8\pi\nu^2 d\nu}{c^3} \tag{1}$$

They are quantized, which means the energy bound of a single normal mode can only take on the values 0, $h\nu$, $2h\nu, \ldots$. Once every day at 0 o'clock let them be subject to a "Planck lottery," by connecting them for a short time to an infinitely large body of temperature T. Their phases then play independently of one another, and the *lottery mean* $[\![\varepsilon]\!]$,[7] taken over many days, of the energy content ε of each single one of these Z normal modes is

$$[\![\varepsilon]\!] = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} \ , \tag{2}$$

thus the lottery mean E_0 of their total energy content E is

$$E_0 \equiv [\![E]\!] = Z[\![\varepsilon]\!] \ . \tag{3}$$

One further obtains for the lottery mean the quadratic fluctuation of E[8]:

$$\frac{[\![(E - E_0)^2]\!]}{E_0^2} = \frac{1}{Z} + \frac{1}{Q} \tag{I}$$

where

$$Q = \frac{E_0}{h\nu} \tag{4}$$

(The number of energy bits $h\nu$ contained in E_0).

§4. "Problem II." Consider an arbitrary subregion v[9] of the blackbody V, and indeed only imaginarily demarcated within V and not, for example, bounded by partitions. — Let $e(t)$ represent its energy content falling within the frequency range $\nu, \nu + d\nu$ at t o'clock . Thanks to interference beats, the Z normal modes of V – in contrast to ε and E – fluctuate within the same day about a mean value $\langle e(t)\rangle = \eta$[10] specific to that day, and which we will

[7] [Ehrenfest originally indicated lottery mean by bolding the square brackets. For clarity I have used double brackets instead. –EC]

[8] See §4 equation II for the special case where $v = V$, thus $z = Z$, $q = Q$ are constant.

[9] Consisting of one or more parts. But every part must be very large in wavelength compared to the frequency range $\nu, \nu + d\nu$.

[10] [Ehrenfest originally used bolded parenthesis; for clarity I use angle brackets instead. –EC]

return to in §5. We consider for the moment the lottery average $[\![e(t)]\!]$ over very many days. Before time t, its independent value e_0 is given by[11]

$$[\![e(t)]\!] = e_0 = v\frac{[\![E]\!]}{V} = z[\![\varepsilon]\!] \ , \tag{5}$$

where[12]

$$z = \frac{v}{V}Z = v \cdot \frac{8\pi\nu^2 d\nu}{c^3} \ . \tag{6}$$

For the lottery mean, the quadratic fluctuation of $e(t)$ about e_0 gives – again the value is independent before hour t[13]:

$$\frac{[\![[e(t) - e_0]^2]\!]}{e_0^2} = \frac{1}{z} + \frac{1}{Q} \ . \tag{II}$$

§5. "Problem III." Now let's consider the time-like fluctuation that $e(t)$ in the course of a day, i.e. when the blackbody remains isolated, exhibits about the daily mean

$$\langle e(t)\rangle = \eta \ . \tag{7}$$

The mean quadratic fluctuation within such a day

$$\langle [e(t) - \eta]^2\rangle \tag{8}$$

for different days still possesses wholly different varying values according to chance. Thus we ask about the 'mathematical expectation' of the quantity (8), i.e. about its lottery mean after many days. For its relation to e_0^2 we obtain[14]:

$$\frac{[\![\langle [e(t) - \eta]^2\rangle]\!]}{e_0^2} = \frac{1}{z} - \frac{1}{Z} \ . \tag{III}$$

[11]See §10 (38) (39).

[12]Were ν to be bounded by mirror partitions, then z would be the number of its normal modes falling within the frequency range $\nu, \nu + d\nu$.

[13]See the proof in §12 (52).

[14]See §11 (45) for proof. — It is worth noting that the fluctuation III maintains its value instead of Planck's in the case where one subjects the normal modes of the blackbody to a mostly arbitrary lottery – (see remarks in the conclusion of §11) –, e.g. were the value exactly the same also for classical statistics ($h = 0$).

§6. The contrast of fluctuation formula II to the Einstein formula. — For the quadratic fluctuation of the energy contents of v, Einstein obtains the expression[15]:

$$\frac{\overline{(e-e_0)^2}}{e_0^2} = \frac{1}{z} + \frac{1}{q} \;, \tag{IV}$$

where

$$q = \frac{e_0}{h\nu} \tag{9}$$

(The number of energy bits $h\nu$ contained in e_0).

In cases where $v \ll V$, thus $q \ll Q$, the Einstein formula (IV) requires many more fluctuations than (II).

If one lets V, thus also Z and Q, grow to infinity, then (IV) remains unaffected; however, the right sides of (II) and (III) converge towards

$$\frac{1}{z} \;, \tag{10}$$

i.e. the value that follows directly also from the interference of non-quantized waves.[16]

Only when v coincides with V, so $z = Z$ and $q = Q$, do (IV) and (II) agree with one another [and with (I)]. In this case (III) yields a null value, as one would expect.

§7. Dependence or independence of fluctuations in different volume elements of the radiation field (and crystal lattice)? — If the subregion v is increased by a factor of n while V is kept constant, then z, e_0 and q likewise are multiplied by n while of course Z, E_0 and Q remain unchanged. From (IV) one sees that $\overline{(e-e_0)^2}$ is exactly n-times larger, which together with the entropy and so also the fluctuations of different volume elements, Einstein considered to be independent in the derivation of (IV). — The analogous quadractic fluctuations in (II) and (III) definitely do not increase proportionally with n, which plainly shows that in the superposition of blackbody oscillations [Holhraumschwingungen], the energy fluctuations in individual parts of v are dependent upon one another. One should also

[15] Well, we would naturally obtain this value instead of (II) if v were not merely imaginarily but instead physically bounded by walls and one could directly subject its normal modes – through contact with a blackbody – to a Planck lottery. (Cf. "Problem I" for the entire V.)

[16] Lorentz, *Théories statistiques*, Note IX.

expressly be reminded that the subregions v may be made up of spatially separated pieces.

Which of the two hypotheses better suits reality – that of the independents of the entropies or that of the construction of normal modes [Eigenschwingungen] through superpositions? — It seems that the second cannot be maintained over and above the first. But if one abandons it [the second] for the radiation field, then one must also let it drop for energy distributions in a crystal lattice!

Mathematical Addenda

§8. Without sacrificing any of the essential features in our problem, for simplicity we may consider the electromagnetic blackbody oscillations as oscillations of a string.

$$\varrho \frac{\partial^2 s}{\partial t^2} = \varkappa \frac{\partial^2 s}{\partial x^2}\,, \tag{1}$$

$$s\,(0,t) \;=\; s\,(L,t) \;=\; 0 \tag{2}$$

$$s(t,x) \;=\; \sum_h C_h \;\sin(h\omega t + \tau_h)\; \sin h\gamma x\,, \tag{3}$$

$$\omega \;=\; \pi \frac{e}{L}\,, \tag{4}$$

$$\gamma \;=\; \frac{\pi}{L}\,, \tag{5}$$

$$\frac{\varrho\omega^2}{2} \;=\; \frac{\varkappa}{2}\gamma^2\,. \tag{6}$$

The energy of the segments l[17](possibly consisting of multiple parts):

$$u(t) \;=\; \int_{(l)} dx \left\{ \frac{\varrho}{2}\left(\frac{\partial s}{\partial t}\right)^2 + \frac{\varkappa}{2}\left(\frac{\partial s}{\partial x}\right)^2 \right\}\,. \tag{7}$$

After inserting (3) into (7), for the part of $u(t)$ falling within the frequency range $\nu, \nu + d\nu$ one obtains:

[17]Corresponding to the sub-volume v in "Problems II and III".

$$e(t) = \sum_h \sum_k B_h B_k [\cos(h\omega t + \tau_h) \cos(k\omega t + \tau_k) \Phi_{hk} + \sin(h\omega t + \tau_h) \sin(k\omega t + \tau_k) \Psi_{hk}] , \quad (8)$$

where

$$\left.\begin{array}{l} \nu \leqq h\omega \leqq \nu + d\nu \\ \nu \leqq k\omega \leqq \nu + d\nu \end{array}\right\} \quad (9)$$

$$h, k \gg 1 , \quad (10)$$

$$\frac{h-k}{h+k} \ll 1 , \quad (11)$$

$$B_h = \sqrt{\frac{\varrho \omega^2}{2} h C_h} = \sqrt{\frac{\varkappa \gamma^2}{2} h C_h} , \quad (12)$$

$$\begin{aligned} \Phi_{hk} &= \int_{(l)} dx \; \sin h\gamma x \; \sin k\gamma x \\ &= \frac{1}{2} \int_{(l)} dx \; \cos(h-k)\gamma x - \frac{1}{2} \int_{(l)} dx \; \cos(h+k)\gamma x , \end{aligned} \quad (13)$$

$$\begin{aligned} \Psi_{hk} &= \int_{(l)} dx \; \cos h\gamma x \; \cos k\gamma x \\ &= \frac{1}{2} \int_{(l)} dx \; \cos(h-k)\gamma x + \frac{1}{2} \int_{(l)} dx \; \cos(h+k)\gamma x . \end{aligned} \quad (14)$$

If only the parts comprising l are large compared to the frequency range (9) with respect to their wavelengths, then one can easily see in light of (10), (11) that in (13) and (14) the second integral is completely negligible compared to the first integral:

$$\frac{1}{2} \int_{(l)} dx \; \cos(h-k)\gamma x = \Omega_{hx} , \quad (15)$$

thus is

$$e(t) = \sum \sum B_h B_k \; \cos\{(h-k)\omega t + \tau_h - \tau_k\} \cdot \Omega_{hk} . \quad (16)$$

§9. The B_h and τ_h may now be subjected to any lottery[18] of which initially only is known that all B_h and all τ_h perform independently of each other, so that for the lottery mean $[\![\,]\!]$ is valid:

$$[\![\cos(\tau_h - \tau_k)]\!] = [\![\sin(\tau_h - \tau_k)]\!] = 0\,, \tag{17}$$

$$[\![B_h\, B_k]\!] \;=\; [\![B_h]\!]\,[\![B_k]\!] \tag{18}$$

for $h \neq k$. Furthermore:

$$[\![B_h^2]\!] \;=\; [\![B_k^2]\!] \;=\; [\![B_l^2]\!] \;= \ldots = \mathbf{II}, \tag{19}$$

$$[\![B_h^4]\!] \;=\; [\![B_k^4]\!] \;=\; [\![B_l^4]\!] \;= \ldots = \mathbf{IV} \tag{20}$$

for all $h, k, l \ldots$, are within the frequency range (9). Then is valid[19]:

$$\begin{aligned}[\![e(t)]\!] = &\sum_h [\![B_h^2]\!]\,\Omega_{hh} \\ &+ \sum_h {\sum_k}' [\![B_h]\!]\,[\![B_k]\!]\,[\![\cos[(h-k)\omega t + \tau_h - \tau_k]]\!]\,,\end{aligned} \tag{21}$$

so for (17) (18):

$$e_0 \;\equiv [\![e(t)]\!] = \mathbf{II} \sum_h \Omega_{hh} \tag{22}$$

(independent of t). Furthermore:

$$[\![[e(t)]^2]\!] \;=\; \sum_h \sum_k \sum_l \sum_m [\![B_h\, B_k\, B_l\, B_m]\!]\,[\![\varphi(h,k)\varphi(l,m)]\!]\,\Omega_{hk}\;\Omega_{lm} \tag{23}$$

where

$$\begin{aligned}&[\![\varphi(h,k)\cdot\varphi(l,m)]\!] \\ &\qquad = [\![\cos[(h-k)\omega t + \tau_h - \tau_k]\cdot\cos[(l-m)\omega t + \tau_l + \tau_m]]\!]\,;\end{aligned} \tag{24}$$

[18] Limitation on the Planck Lottery not until §12. — Consider the comment at the end of §11.

[19] $\sum_h {\sum_x}'$ connotes exclusion of the element $k = h$, which was kept separate because it has unusual statistical properties.

for fixed t the action/gamble of $(\tau_h - \tau_k)$ and $(\tau_l - \tau_m)$ is decisive/critical here, and regarding (17) all quantities in (24) are null except for the following:

$$
\begin{aligned}
[\![\varphi(h,k)\varphi(h,k)]\!] &= \frac{1}{2}\,, && (25)\\
[\![\varphi(h,k)\varphi(k,h)]\!] &= \frac{1}{2}\,, && (26)\\
[\![\varphi(h,h)\varphi(l,l)]\!] &= 1\,, && (27)\\
[\![\varphi(h,h)\varphi(h,h)]\!] &= 1\,. && (28)
\end{aligned}
$$

Consequently,

$$
\left.\begin{aligned}
[\![[e(t)]^2]\!] = &\sum_h \sum_k{}' [\![B_h^2]\!][\![B_k^2]\!]\cdot\frac{1}{2}\Omega_{hk}^2 && \text{(see 25)}\\
&+\sum_h \sum_k{}' [\![B_h^2]\!][\![B_k^2]\!]\frac{1}{2}\Omega_{hk}^2 && \text{(see 26)}\\
&+\sum_h \sum_l{}' [\![B_h^2]\!][\![B_l^2]\!]\cdot 1\cdot\Omega_{hh}\Omega_{ll} && \text{(see 27)}\\
&+\sum_h [\![B_h^4]\!]\cdot 1\cdot\Omega_{hh}^2 && \text{(see 28)}
\end{aligned}\right\} \qquad (29)
$$

or for (19), (20)

$$
[\![[e(t)]^2]\!] = \mathbf{II}^2 \sum_h \sum_k{}' \Omega_{hk}^2 + \mathbf{II}^2 \sum_h \sum_l{}' \Omega_{hh}\Omega_{ll} + \mathbf{IV} \sum_h \Omega_{hh}^2\,. \qquad (30)
$$

For the time average $\langle\,\rangle$ one obtains:

$$
\eta \equiv \langle e(t)\rangle = \sum_h \sum_k B_h B_k \Omega_{hk} \langle\varphi(h,k)\rangle = \sum_h B_h^2 \Omega_{hh}\,, \qquad (31)
$$

$$
\begin{aligned}
[\![\langle e(t)\rangle^2]\!] &= \sum_h \sum_k [\![B_h^2\, B_k^2]\!]\Omega_{hh}\Omega_{kk}\\
&= \mathbf{II}^2 \sum_h \sum_k{}' \Omega_{hh}\Omega_{kk} + \mathbf{IV}\sum_h \Omega_{hh}^2\,.
\end{aligned} \qquad (32)
$$

§10. But now according to (15):

$$
\Omega_{hh} = \Omega_{kk} = \frac{l}{2}\,, \qquad (33)
$$

so

$$\sum_h \Omega_{hh} = Z\frac{l}{2}\,, \tag{34}$$

$$\sum_h \Omega_{hh}^2 = \frac{l^2}{4}Z\,, \tag{35}$$

$$\sum_h {\sum_k}' \Omega_{hh}\Omega_{kk} = \frac{l^2}{4}Z(Z-1)\,, \tag{36}$$

where Z is the number of normal modes that fall within the frequency region (9). Furthermore[20]:

$$\sum_h {\sum_k}' \Omega_{hk}^2 = \frac{l^2}{4}Z^2\left(\frac{1}{z}-\frac{1}{Z}\right)\,, \tag{37}$$

where[21]

$$z = \frac{l}{L}Z\,, \tag{38}$$

so:

$$e_0 \equiv [\![e(t)]\!] = \frac{l}{2}\,\mathbf{II}\,Z\,, \tag{39}$$

$$[\![e^2]\!] = \frac{l^2}{4}\mathbf{II}^2\left[Z^2\left(\frac{1}{z}-\frac{1}{Z}\right)+Z^2\left(1-\frac{1}{Z}\right)\right]+\frac{l^2}{4}\,\mathbf{IV}\,Z\,, \tag{40}$$

$$[\![\eta^2]\!] \equiv [\![\langle e(t)\rangle^2]\!] = \frac{l^2}{4}\mathbf{II}^2Z^2\left(1-\frac{1}{Z}\right)+\frac{l^2}{4}\mathbf{IV}\,Z\,. \tag{41}$$

§11.[22] For the derivation of the quadratic mean fluctuation from these formulae, one notes that:

$$[\![(e-e_0)^2]\!] = [\![e^2]\!] - e_0^2\,, \tag{42}$$

$$\langle [e(t)-\eta]^2\rangle = \langle e^2\rangle - \eta^2\,, \tag{43}$$

$$[\![\langle [e(t)-\eta]^2\rangle]\!] = [\![\langle e^2\rangle]\!] - [\![\eta^2]\!] = [\![\langle e^2\rangle]\!] - [\![\eta^2]\!] = [\![e^2]\!] - [\![\eta^2]\!]\,. \tag{44}$$

Then by inserting (39, 40, 41) into (42, 44) one finds:

$$\frac{[\![\langle (e-\eta)^2\rangle]\!]}{e_0^2} = \frac{1}{z}-\frac{1}{Z}\,, \tag{45}$$

[20]Proved in §14.
[21]Cf. the z in Eq. (6) §4.
[22]From here onward e will be shorthand for writing $e(t)$

$$\frac{[\![(e-e_0)^2]\!]}{e_0^2} = \left(\frac{1}{z} - \frac{2}{Z}\right) + \frac{\mathbf{IV}}{\mathbf{II}^2}\frac{1}{Z}\,. \tag{46}$$

One sees: that the fluctuation (45) corresponding to 'Problem IIII' (§5) has the same value for all lotteries satisfying only the conditions (17 to 20) in §9. — That the fluctuation corresponding to 'Problem II' (§4) still depends on the value that the lottery yields for $\mathrm{IV}/\mathrm{II}^2$.

§12. Now notice (19, 20) in §9, and recall that

$$\frac{\mathbf{IV}}{\mathbf{II}^2} = \frac{[\![B^4]\!]}{[\![B^2]\!]^2} = \frac{[\![\varepsilon^2]\!]}{[\![\varepsilon]\!]^2}\,, \tag{47}$$

where ε represents the energy content of a normal mode located in the frequency range (9), §8,[23] then ε_h and ε_h^2 are of course proportional to B_h^2 and B_h^4. However, for a Planck lottery (see following paragraphs):

$$\frac{[\![\varepsilon^2]\!]}{[\![\varepsilon]\!]^2} = e^{h\nu/kT} + 1\,, \tag{48}$$

so according to (46), (47):

$$\frac{[\![(e-e_0)^2)]\!]}{e_0^2} = \frac{1}{z} + \frac{e^{h\nu/kT} - 1}{Z}\,. \tag{49}$$

However, for the energy content E of the whole string, as long as it falls within the frequency range (9) of \$ 8, [one] obtains:

$$[\![E]\!] = Z[\![\varepsilon]\!] = Z\frac{h\nu}{e^{h\nu/kT} - 1}\,. \tag{50}$$

Thus for the number of energy amounts [quanta?] $h\nu$ contained therein:[24]

$$Q \equiv \frac{[\![E]\!]}{h\nu} = \frac{Z}{h\nu}e^{h\nu/kT} - 1\,, \tag{51}$$

so according to (49)

$$\frac{[\![(e-e_0)^2]\!]}{e_0^2} = \frac{1}{z} + \frac{1}{Q}\,. \tag{52}$$

§13. Derivation of equation (48). — It is for a Planck lottery:

$$[\![\varepsilon]\!] = h\nu\frac{Q}{P}\,, \tag{53}$$

[23]Cf. the ε of §3.
[24]Cf. the Q of §3.

$$[\![\varepsilon^2]\!] = (h\nu)^2 \frac{R}{P}\,, \tag{54}$$

where with the use of $\dfrac{h\nu}{kT} = \omega$

$$P = \sum_0^\infty e^{-s\omega} = \frac{1}{1 - e^{-\omega}}\,, \tag{55}$$

$$Q = \sum_0^\infty s e^{-s\omega} = -\frac{dP}{d\omega}\,, \tag{56}$$

$$R = \sum_0^\infty s^2 e^{-s\omega} = \frac{d^2 P}{d\omega^2}\,, \tag{57}$$

which after a quick intermediate calculation yields:

$$\frac{[\![\varepsilon^2]\!]}{[\![\varepsilon]\!]^2} = e^\omega + 1 = e^{h\nu/kT} + 1\,. \tag{58}$$

§14. Derivation of equation (37). — According to equation (15) in §8 it was:

$$\Omega_{hx} = \frac{1}{2}\int_{(l)} dx \cos(h-k)\gamma x\,. \tag{59}$$

If for the purpose of approximation one makes use of the inequalities (10, 11) in §8, one sees[25] that

$$\begin{aligned} \sum_h \sum_k{}' \Omega_{hk}^2 &\cong \sum_h \sum_{-\infty}^{+\infty}{}' \Omega_{h,h+s}^2 \\ &= \sum_h 2 \sum_1^\infty \left(\frac{1}{2}\int_{(l)} dx \cos s\gamma x\right)^2 = \frac{Z}{2}\sum_1^\infty A_s^2\,, \end{aligned} \tag{60}$$

where

$$A_s = \int_{(l)} dx \cos s\gamma x\,. \tag{61}$$

For evaluating the infinite sum in (60), the following trick: consider a function $f(x)$ that in the subregion (l) is equal to 1, elsewhere in the interval $0 \leqq x \leqq L$ equal to 0. By Fourier expansion of this $f(x)$ one has:

$$f(x) = \sum_0^\infty a_s \cos\, s\gamma x\,, \tag{62}$$

[25] $\sum_{-\infty}^{+\infty}{}'$ implies that $s = 0$ should be dropped. For justifying the approximation, one in addition has to consider that $\Omega_{h,h+s}^2$ falls off sharply as s increases.

$$\int_0^L dx[f(x)]^2 = La_0^2 + \frac{L}{2}\sum_1^\infty a_s^2 \tag{63}$$

and it is

$$\int_0^L dx[f(x)]^2 = \int_{(l)} dx 1 = l\,, \tag{64}$$

$$a_0 = \frac{1}{L}\int_0^L dx f(x)\cos 0\gamma x = \frac{l}{L}\,, \tag{65}$$

$$a_s = \frac{2}{L}\int_0^L dx f(x)\cos s\gamma x = \frac{2}{L}\int_{(l)} dx \cos s\gamma x = \frac{2}{L}A_S\,. \tag{66}$$

Introducing (64), (65), (66) into (63) yields

$$\sum_1^\infty A_s^2 = \frac{l^2}{2}\left(\frac{L}{l} - 1\right)\,, \tag{67}$$

and as a consequence, by applying (38), (60) goes over into:

$$\sum_h \sum_l{}' \Omega_{hk} \cong \frac{l^2}{4} Z^2 \left(\frac{1}{z} - \frac{1}{Z}\right)\,. \tag{68}$$

Leiden, August 1925.