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Second-Order Wave Equation for Spin-$\frac{1}{2}$ Fields: 8-Spinors and Canonical Formulation

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The algebraic structure of the 8-spinor formalism is discussed, and the general form of the 8-component wave equation, equivalent to the second-order 4-component one, is presented. This allows a canonical formulation that will be the first stage of the future Clebsch parametrization, i.e., a relativistic generalization of the Bohm–Schiller–Tiomno pioneering work on the Pauli equation.

1. INTRODUCTION

In a recent series of papers,[1] we started from the remark that a relativistic, second-order wave equation constitutes the most natural generalization of Schrödinger's wave mechanics. From this standpoint, despite the impressive quantity of results directly obtained from the Dirac form of the wave equation for spin-$\frac{1}{2}$ particles, it is a bit unsatisfactory that this linearization cuts off a substantial part of the solutions of the second-order equation. Nothing similar, indeed, is usually done on the Klein–Gordon or on the Proca equation. Moreover, beyond a series of well-known remarks[2] about the use of this second-order formalism, we should take into account the fact that

(a) If we are looking for a causal interpretation of the quantum equations, a classical analogy can be found only starting from a second-order differential equation.

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(b) If we want, subsequently, to interpret the appearance of a quantum potential[3] as the global effect of a stochastic process induced by a subquantum medium,[4] we must use second-order differential equations.

(c) Both of the preceding steps are essential in a causal physical interpretation of the quantum mechanical nonlocal effects[5] now suggested by experiments.[6]

However, we should also remark that, for differential equations containing second-order derivatives in time, the usual form of the conserved current density has a zero component that is not positive-definite. This has two main consequences: This zero component cannot be directly interpreted as a probability density, and the scalar product, defined by means of this current, cannot be utilized to define a positive norm on the vector space of the states. In our preceding papers,[1] we solved this problem by showing that, for spin-$\frac{1}{2}$ fields ruled by a second-order, relativistic wave equation, it is possible to define a conserved current density whose zero component is always positive-definite. That will allow us to define coherently the statistical interpretation of the theory and the Hilbert space of the states, without restricting ourselves to the solutions of the Dirac equation. Moreover, in doing so, we arrived at the conclusion that it is possible to build a coherent theory of the second-order equation by means of states defined as 8-component spinors obeying a first-order equation.

The aim of the present paper is, first of all, to gain a deeper insight in this 8-spinor formulation and, subsequently, to define in the right way all of the quantities (Hamiltonians, Lagrangians, spin densities, and so on) needed for the next step of this work, i.e., to give a relativistic generalization of the Clebsch parameter analysis carried by Bohm *et al.*,[7] in the nonrelativistic case, for a two-component spinor field.

We briefly recall here the notation and some results of the preceding papers.[1]

For the spinors of different rank, we will adopt the following notations:

$$\text{2-spinor:} \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \qquad u_a,\ a \equiv (1, 2)$$

$$\text{4-spinor:} \qquad \psi = \begin{bmatrix} u \\ v \end{bmatrix}; \qquad \psi_\alpha,\ \alpha \equiv (1, 2, 3, 4)$$

$$\text{8-spinor:} \qquad \Psi = \begin{Bmatrix} \psi \\ \varphi \end{Bmatrix}; \qquad \Psi_{\mathrm{A}},\ \mathrm{A} \equiv (1, 2, 3, 4, 5, 6, 7, 8)$$

The spinor representations will always be chosen so that the upper and lower parts of a higher-rank spinor will always be themselves lower-rank spinors: u, v are 2-spinors in ψ; ψ, φ are 4-spinors in Ψ, and so on. For the 4×4 Dirac matrices γ_μ, $\mu\equiv 0, 1, 2, 3$, and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ obeying the anticommutation rules

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \qquad \{\gamma_\mu, \gamma_5\} = 0,$$
$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{1}$$

we adopt the representation

$$\gamma_0 = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}, \qquad \gamma_\kappa = \begin{bmatrix} 0 & -\sigma_\kappa \\ \sigma_\kappa & 0 \end{bmatrix}, \qquad \gamma_5 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix} \tag{2}$$

where the Pauli matrices are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

Moreover, for each 4-spinor ψ and 4×4 matrix Γ, we define $\bar\psi = \psi^+\gamma_0$ and $\bar\Gamma = \gamma_0\Gamma^+\gamma_0$. It can be shown that, in this representation, a Lorentz transformation takes a quasi-diagonal form, so that, if $\psi = [{}^u_v]$ is a 4-spinor, u comes out to be a first-type 2-spinor and v a second-type 2-spinor. In fact, it should be remarked that u and v transform themselves in an opposite way under the same Lorentz transformation.

Now, if we write

$$D_\mu = \frac{1}{mc}\left(i\hbar\,\partial_\mu - \frac{e}{c}A_\mu\right) \tag{4}$$

the second-order, relativistic wave equation that we are talking about is the 4-spinor equation

$$(I - \not{D}^2)\,\psi(x) = 0 \tag{5}$$

with $\not{D} = \gamma_\mu D^\mu$. In this notation, the Dirac equation is written in the form

$$(I - \not{D})\,\psi(x) = 0 \tag{6}$$

We have shown[1] that

$$j_\mu(x) = \tfrac{1}{2}(\overline{\not{D}\psi}\,\gamma_\mu\,\not{D}\psi + \bar\psi\gamma_\mu\psi) \tag{7}$$

is a conserved current density, with

$$j_0(x) = \tfrac{1}{2}[(\not{D}\psi)^+ \not{D}\psi + \psi^+\psi] \geqslant 0 \tag{8}$$

This positive conserved density j_0 can now be considered as a probability density, for a statistical interpretation of (5), and it can be used as a basis for the definition of a scalar product for the state vectors.

Moreover, since (5) contains second-order derivatives in time, the state of the system at a given time x^0 determines the time evolution of the spinor only if we know, as initial conditions, both $\psi(x^0)$ and $\not{D}\psi(x^0)$. This leads to the idea that the state of the system at any fixed time is specified by means of a couple of 4-spinors, and hence by means of an 8-spinor $\Psi(x)$ which is in correspondence with $\psi(x)$ and $\not{D}\psi(x)$. It can be seen that these 8-spinors should obey to a first-order wave equation. In the subsequent section, we will analyze the algebraic structure of this theory in its 8-component representation.

2. CLIFFORD ALGEBRA FOR 8-SPINORS

In general,[8] given n anticommuting "numbers," we can generate, by multiplication, a Clifford algebra C_n of 2^n linearly independent elements that can be used as a basis for a 2^n-dimensional vector space either on the field of the real or of the complex numbers. If $n = 2k$ or $n = 2k + 1$, the lowest-order representation of C_n has 2^k components. This amounts to saying that, for a representation of order 2^k, we can find a maximum of $n = 2k + 1$ anticommuting matrices. However, also the case of $n = 2k$ anticommuting matrices needs the same order of representation. Hence, for representations of order 2^k, there is room for the Clifford algebras C_{2k}, C_{2k+1}, the difference lying in the fact that C_{2k} is the basis for a real vector space whereas C_{2k+1} spans a complex vector space, obtained by complexification of the coefficients of C_{2k}. Moreover, in general, starting with a C_{2k} algebra, we can determine the $2k+1$th anticommuting matrix by multiplying all of the given $2k$-generating elements of C_{2k}. We finally remark that the anticommutators of the n generating elements of C_n are usually connected to the metric tensor of the corresponding vector representations.

In our 8-spinor representation, we can find up to 7 linearly independent anticommuting matrices leading to a Clifford algebra C_7. Take, for instance, the 6 anticommuting matrices

$$\mathbb{G}_\mu = \begin{Bmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{Bmatrix}, \qquad \mathbb{G}_4 = \begin{Bmatrix} 0 & I \\ I & 0 \end{Bmatrix}, \qquad \mathbb{G}_5 = \begin{Bmatrix} 0 & -iI \\ iI & 0 \end{Bmatrix} \tag{9}$$

and build, by multiplication, the 7th anticommuting element

$$\mathbb{G}_6 = \mathbb{G}_0 \mathbb{G}_1 \mathbb{G}_2 \mathbb{G}_3 \mathbb{G}_4 \mathbb{G}_5 = \begin{Bmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{Bmatrix} \tag{10}$$

For these $\mathbb{G}_A$, $A \equiv (\lambda; L) \equiv (0, 1, 2, 3; 4, 5, 6)$, we find

$$\begin{aligned} \mathbb{G}_\lambda^2 &= g_{\lambda\lambda} \mathbb{I}, & \mathbb{G}_L^2 &= \mathbb{I} \\ \mathbb{G}_\lambda^+ &= g_{\lambda\lambda} \mathbb{G}_\lambda, & \mathbb{G}_L^+ &= \mathbb{G}_L \\ \bar{\mathbb{G}}_\lambda &= \mathbb{G}_\lambda, & \bar{\mathbb{G}}_L &= -\mathbb{G}_L \end{aligned} \tag{11}$$

or, by introducing the metric tensor $g_{A\Sigma}$ of the corresponding 7-vector representation, which is

$$\{\mathbb{G}_A, \mathbb{G}_\Sigma\} = 2g_{A\Sigma} = 2 \left(\begin{array}{c|c} g_{\lambda\sigma} & 0 \\ \hline 0 & \delta_{LS} \end{array}\right) \tag{12}$$

we have

$$\mathbb{G}_A^2 = g_{AA} \mathbb{I}, \qquad \mathbb{G}_A^+ = g_{AA} \mathbb{G}_A \tag{13}$$

that summarize all the essential properties of the $\mathbb{G}$'s matrices.

In the 7-dimensional space with metric tensor $g_{A\Sigma}$ we must separate a 4-dimensional, "external," Minkowski space, described by means of the first four dimensions of indices $\lambda \equiv (0, 1, 2, 3)$; and a 3-dimensional "internal" space, constituted by the remaining three dimensions of indices $L \equiv (4, 5, 6)$. The names "external" and "internal" space have been chosen because we would keep the Minkowski coordinates unaffected by rotations on the other three coordinates, and vice versa, so that no manifestation of the existence of this surplus of dimensions can be found in the ordinary space-time. This means that, in our 7-dimensional total space, we will not consider the complete group of rotations SO(3, 4), but only its subgroup $SO_E(3, 1) \otimes SO_I(3)$, which is the tensor product of a Lorentz group $SO_E(3, 1)$ acting on the external Minkowski space, times a rotation group $SO_I(3)$ acting on the internal space. Hence, in the corresponding 8-spinor representation, among the 21 generators of the complete group

$$\mathbb{S}_{A\Sigma} = \frac{i}{2} [\mathbb{G}_A, \mathbb{G}_\Sigma] \tag{14}$$

we select only the 9 generators

$$\mathbb{S}_{\mu\nu}=\frac{i}{2}[\mathbb{G}_\mu, \mathbb{G}_\nu]$$
$$\mathbb{S}_{MN}=\frac{i}{2}[\mathbb{G}_M, \mathbb{G}_N] \tag{15}$$

namely, the 6 generators of the Lorentz transformations and the 3 generators of the internal rotations. It is straightforward to see that, from the anticommutation rules, we have

$$[\mathbb{S}_{\mu\nu}, \mathbb{G}_M]=[\mathbb{S}_{MN}, \mathbb{G}_\mu]=[\mathbb{S}_{MN}, \mathbb{S}_{\mu\nu}]=0 \tag{16}$$

i.e., the external and internal transformations do commute. Hence, every covariant, external, bilinear quantity is invariant under internal rotations and every tensor, internal, bilinear quantity is invariant under proper Lorentz transformations.

The explicit form of the generators of the proper Lorentz transformations is, in our representation,

$$\mathbb{S}_{\mu\nu}=\begin{Bmatrix}\sigma_{\mu\nu} & 0\\ 0 & \sigma_{\mu\nu}\end{Bmatrix} \tag{17}$$

so that an arbitrary Lorentz transformation has the form

$$\mathbb{L}=\begin{Bmatrix}L & 0\\ 0 & L\end{Bmatrix} \tag{18}$$

where L is the matrix representing the transformation for the 4-spinors ψ.

To determine the form of the generators of the internal rotations, it is useful to define the matrix

$$\mathbb{G}=i\mathbb{G}_0\mathbb{G}_1\mathbb{G}_2\mathbb{G}_3=\begin{Bmatrix}\gamma_5 & 0\\ 0 & \gamma_5\end{Bmatrix} \tag{19}$$

with the following properties

$$\{\mathbb{G}, \mathbb{G}_\mu\}=[\mathbb{G}, \mathbb{G}_M]=0$$
$$\mathbb{G}^2=\mathbb{I}, \qquad \mathbb{G}^+=\mathbb{G}, \qquad \bar{\mathbb{G}}=-\mathbb{G} \tag{20}$$

which imply also

$$[\mathbb{G}, \mathbb{S}_{\mu\nu}]=[\mathbb{G}, \mathbb{S}_{MN}]=0 \tag{21}$$

It is clear that, in some sense, in the internal space $\mathbb{G}$ behaves like $\mathbb{I}$, the difference between them lying in their behavior with relation to the external space. It is easy to verify now that the internal generators are

$$\mathbb{S}_{MN} = -\varepsilon_{MNL}\mathbb{G}\mathbb{G}_L \tag{22}$$

where ε_{MNL} is the usual completely antisymmetric, three-indices symbol. We can also define three more matrices

$$\mathbb{S}_L = -\tfrac{1}{2}\varepsilon_{LMN}\mathbb{S}_{MN} = \mathbb{G}\mathbb{G}_L \tag{23}$$

so that

$$\mathbb{S}_{MN} = -\varepsilon_{MNL}\mathbb{S}_L \tag{24}$$

The properties of these matrices are the following:

$$\mathbb{S}_M^2 = \mathbb{I}, \qquad \mathbb{S}_M^+ = \mathbb{S}_M, \qquad \bar{\mathbb{S}}_M = \mathbb{S}_M \tag{25}$$

the anticommutation rules are

$$\begin{aligned} \{\mathbb{S}_M, \mathbb{S}_N\} &= 2\mathbb{I}\delta_{MN} \\ \{\mathbb{S}_M, \mathbb{G}_N\} &= 2\mathbb{G}\delta_{MN} \end{aligned} \tag{26}$$

and the commutation rules with external matrices are

$$[\mathbb{S}_M, \mathbb{G}_\mu] = [\mathbb{S}_M, \mathbb{S}_{\mu\nu}] = 0 \tag{27}$$

In our representation, their explicit form is

$$\mathbb{S}_4 = \begin{Bmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{Bmatrix}, \qquad \mathbb{S}_5 = \begin{Bmatrix} 0 & -i\gamma_5 \\ i\gamma_5 & 0 \end{Bmatrix}, \qquad \mathbb{S}_6 = \begin{Bmatrix} I & 0 \\ 0 & -I \end{Bmatrix} \tag{28}$$

The matrices $\mathbb{S}_M$ will be used in the following as the generators of the internal rotations $\mathbb{R}$, which will commute with all of the proper Lorentz transformations. Of course, the required invariance of $\bar{\Psi}\Psi = \Psi^+\mathbb{G}_0\Psi$ under such an internal rotation will impose the relation

$$\bar{\mathbb{R}} = \mathbb{R}^{-1} \tag{29}$$

We remark here that the matrices $\mathbb{G}_M$ and $\mathbb{S}_M$ behave, in many respects, exactly like the Pauli matrices σ_κ. One of the main differences, however, is in the existence, in the 8-spinor case, of the γ_5 matrix, which, while behaving like an identity in the internal commutation relations, acts in a different way with respect to the external matrices. This shows why the $\mathbb{S}_M$ matrices are essentially different from the $\mathbb{G}_M$ ones.

The tensor character of the bilinear quantities is determined by the relations

$$\begin{aligned} \bar{\mathbb{L}}\mathbb{G}_\mu\mathbb{L} &= \Lambda_\mu{}^\nu\mathbb{G}_\nu \\ \bar{\mathbb{R}}\mathbb{G}_M\mathbb{R} &= R_{MN}\mathbb{G}_N \end{aligned} \tag{30}$$

where $\Lambda_\mu{}^\nu$ is the Lorentz transformation in the usual 4-vector representation and R_{MN} is the matrix representing an internal rotation for internal 3-vectors. We can easily deduce, in fact, that under proper transformations $\bar{\Psi}\Psi$ and $\bar{\Psi}\mathbb{G}\Psi$ behave like scalars, $\bar{\Psi}\mathbb{G}_\mu\Psi$ and $\bar{\Psi}\mathbb{G}\mathbb{G}_\mu\Psi$ like external 4-vectors, $\bar{\Psi}\mathbb{G}_M\Psi$ and $\bar{\Psi}\mathbb{S}_M\Psi$ like internal 3-vectors, and so on.

As for the pseudocharacter of these quantities, provided that the 4-spinors ψ constitute a representation of the Lorentz group including space inversions and time reversals, we will require that the improper transformations for 8-spinors will be described by means of the following operations

$$\begin{aligned} \Psi_P(x_P) &= \mathbb{P}\Psi(x), \qquad x_P \equiv (g_{\mu\mu}x_\mu) \\ \Psi_T(x_T) &= \mathbb{T}\Psi^*(x), \qquad x_T \equiv (-g_{\mu\mu}x_\mu) \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mathbb{P} &= \mathbb{S}_6\mathbb{G}_0 = \begin{Bmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{Bmatrix} = \bar{\mathbb{P}} \\ \mathbb{T} &= i\mathbb{G}_1\mathbb{G}_3 = \begin{Bmatrix} i\gamma_1\gamma_3 & 0 \\ 0 & i\gamma_1\gamma_3 \end{Bmatrix} = \bar{\mathbb{T}} \end{aligned} \tag{32}$$

In this way, the complete Lorentz group (rotations plus space-time inversions) will assume a quasi-diagonal form for 8-spinors, in which each upper and lower 4-spinor transforms itself in the correct way. It is very easy to verify, for example, the relations

$$\begin{aligned} \bar{\mathbb{P}}\mathbb{P} &= \mathbb{I}, & \bar{\mathbb{T}}\mathbb{T} &= \mathbb{I} \\ \bar{\mathbb{P}}\mathbb{G}_\mu\mathbb{P} &= g_{\mu\mu}\mathbb{G}_\mu, & \bar{\mathbb{T}}\mathbb{G}_\mu\mathbb{T} &= g_{\mu\mu}\mathbb{G}_\mu^* \\ \bar{\mathbb{P}}\mathbb{S}_{\mu\nu}\mathbb{P} &= g_{\mu\mu}g_{\nu\nu}\mathbb{S}_{\mu\nu}, & \bar{\mathbb{T}}\mathbb{S}_{\mu\nu}\mathbb{T} &= -g_{\mu\mu}g_{\nu\nu}\mathbb{S}_{\mu\nu}^* \\ \bar{\mathbb{P}}\mathbb{G}\mathbb{G}_\mu\mathbb{P} &= -g_{\mu\mu}\mathbb{G}\mathbb{G}_\mu, & \bar{\mathbb{T}}\mathbb{G}\mathbb{G}_\mu\mathbb{T} &= g_{\mu\mu}\mathbb{G}^*\mathbb{G}_\mu^* \\ \bar{\mathbb{P}}\mathbb{G}\mathbb{P} &= -\mathbb{G}; & \bar{\mathbb{T}}\mathbb{G}\mathbb{T} &= \mathbb{G}^* \end{aligned} \tag{33}$$

that guarantee the covariance of the corresponding external quantities. However, differently from the case of the proper Lorentz transformations $\mathbb{L}$, the internal tensor quantities undergo some transformations under the space-time inversions. In fact, in terms of upper and lower 4-spinors, they are defined by means of Lorentz scalars and/or pseudoscalars, which are not invariant under space and time inversions. We have now, from the definitions of $\mathbb{P}$ and $\mathbb{T}$:

$$\bar{\Psi}_P(x_P)\,\mathbb{G}_M\,\Psi_P(x_P) = \begin{cases} \bar{\Psi}(x)\,\mathbb{G}_4\,\Psi(x) \\ \bar{\Psi}(x)\,\mathbb{G}_5\,\Psi(x) \\ -\bar{\Psi}(x)\,\mathbb{G}_6\,\Psi(x) \end{cases}$$

$$\bar{\Psi}_P(x_P)\,\mathbb{S}_M\,\Psi_P(x_P) = \begin{cases} -\bar{\Psi}(x)\,\mathbb{S}_4\,\Psi(x) \\ -\bar{\Psi}(x)\,\mathbb{S}_5\,\Psi(x) \\ \bar{\Psi}(x)\,\mathbb{S}_6\,\Psi(x) \end{cases}$$

$$\bar{\Psi}_T(x_T)\,\mathbb{G}_M\,\Psi_T(x_T) = -\bar{\Psi}(x)\,\mathbb{G}_M^*\,\Psi(x) = \begin{cases} -\bar{\Psi}(x)\,\mathbb{G}_4\,\Psi(x) \\ \bar{\Psi}(x)\,\mathbb{G}_5\,\Psi(x) \\ -\bar{\Psi}(x)\,\mathrm{G}_6\,\Psi(x) \end{cases} \tag{34}$$

$$\bar{\Psi}_T(x_T)\,\mathbb{S}_M\,\Psi_T(x_T) = \bar{\Psi}(x)\,\mathbb{S}_M^*\,\Psi(x) = \begin{cases} \bar{\Psi}(x)\,\mathbb{S}_4\,\Psi(x) \\ -\bar{\Psi}(x)\,\mathbb{S}_5\,\Psi(x) \\ \bar{\Psi}(x)\,\mathbb{S}_6\,\Psi(x) \end{cases}$$

The preceding transformation properties for internal tensor quantities can be also derived by direct inspection of their definition by means of scalars and pseudoscalars:

$$\begin{aligned}
\bar{\Psi}\mathbb{G}_4\Psi &= \bar{\psi}\varphi - \bar{\varphi}\psi \\
\bar{\Psi}\mathbb{G}_5\Psi &= -i(\bar{\psi}\varphi + \bar{\varphi}\psi) \\
\bar{\Psi}\mathbb{G}_6\Psi &= \bar{\psi}\gamma_5\psi - \bar{\varphi}\gamma_5\varphi \\
\bar{\Psi}\mathbb{S}_4\Psi &= \bar{\psi}\gamma_5\varphi - \bar{\varphi}\gamma_5\psi \\
\bar{\Psi}\mathbb{S}_5\Psi &= -i(\bar{\psi}\gamma_5\varphi + \bar{\varphi}\gamma_5\psi) \\
\bar{\Psi}\mathbb{S}_6\Psi &= \bar{\psi}\psi + \bar{\varphi}\varphi
\end{aligned} \tag{35}$$

and of the behavior of these scalar and pseudoscalars under space-time reflections. As for the charge conjugation operation, we will propose as generalization

$$\Psi_C(x) = \mathbb{C}\,\Psi^*(x) \tag{36}$$

where

$$\mathbb{C} = i\mathbb{G}_2 = \begin{Bmatrix} i\gamma_2 & 0 \\ 0 & -i\gamma_2 \end{Bmatrix} = -\bar{\mathbb{C}} \tag{37}$$

which, of course, reproduces the transformation properties of the external, covariant, bilinear quantities

$$\begin{aligned} \bar{\mathbb{C}}\mathbb{C} &= \mathbb{I}, & \bar{\mathbb{C}}\mathbb{G}\mathbb{G}_\mu\mathbb{C} &= -\mathbb{G}^*\mathbb{G}_\mu^* \\ \bar{\mathbb{C}}\mathbb{G}_\mu\mathbb{C} &= \mathbb{G}_\mu^*, & \bar{\mathbb{C}}\mathbb{G}\mathbb{C} &= \mathbb{G}^* \\ \bar{\mathbb{C}}\mathbb{S}_{\mu\nu}\mathbb{C} &= \mathbb{S}_{\mu\nu}^*, \end{aligned} \tag{38}$$

The internal tensor quantities transform as follows under charge conjugation:

$$\begin{aligned} \bar{\Psi}_C\mathbb{G}_M\Psi_C &= -\bar{\Psi}\mathbb{G}_M^*\Psi = \begin{cases} -\bar{\Psi}\mathbb{G}_4\Psi \\ \quad\bar{\Psi}\mathbb{G}_5\Psi \\ -\bar{\Psi}\mathbb{G}_6\Psi \end{cases} \\ \bar{\Psi}_{c\,M}\Psi_c &= -\bar{\Psi}\mathbb{S}_M^*\Psi = \begin{cases} -\bar{\Psi}\mathbb{S}_4\Psi \\ \quad\bar{\Psi}\mathbb{S}_5\Psi \\ -\bar{\Psi}\mathbb{S}_6\Psi \end{cases} \end{aligned} \tag{39}$$

Finally, we have

$$\begin{aligned} \Psi_{PCT}(-x) &= \mathbb{TCP}\Psi(x) \\ &= i\mathbb{G}\mathbb{S}_6\Psi(x) = i\mathbb{G}_6\Psi(x) \end{aligned} \tag{40}$$

where

$$\mathbb{TCP} = i\mathbb{G}_6 = \begin{Bmatrix} i\gamma_5 & 0 \\ 0 & -i\gamma_5 \end{Bmatrix} \tag{41}$$

3. THE FIRST-ORDER EQUATION

We will try to determine the most symmetric form of a first-order wave equation on the 8-spinors

$$\Psi(x) = \frac{1}{\sqrt{2}}\begin{Bmatrix} \varphi(x) \\ \chi(x) \end{Bmatrix} \tag{42}$$

which is equivalent to the second-order wave equation (5). Here "equivalence" means that there should be a one-to-one correspondence between the solutions $\Psi(x)$ and $\psi(x)$; in the sense that either $\varphi(x)$ or $\chi(x)$ in $\Psi(x)$ is an arbitrary solution of the second-order equation while the other one is a particular solution of the second-order equation that is determined by means of the first solution.

We propose a Dirac-type equation:

$$(\mathbb{G}_\mu D^\mu - \mathbb{M})\,\Psi(x) = 0 \tag{43}$$

First of all, we should choose $\mathbb{M}$ so that $\bar{\Psi}\mathbb{M}\Psi$ is scalar under Lorentz transformations and internal rotations. In fact, it must show the behavior of the first term, which is scalar for internal and external transformations. Moreover, it will be required that

$$\bar{\mathbb{M}} = \mathbb{M} \tag{44}$$

so that usual physical quantities, like Lagrangians, Hamiltonians, and so on, be real, as we will see later. However, this particular requirement is also connected with the derivation of the continuity equation. In fact, if and only if (44) holds, both the equations

$$\begin{aligned} &\mathbb{G}_\mu D^\mu \Psi - \mathbb{M}\Psi = 0 \\ &\overline{D^\mu \Psi}\mathbb{G}_\mu - \bar{\Psi}\mathbb{M} = 0 \end{aligned} \tag{45}$$

hold, so that, by multiplying, respectively, by $\bar{\Psi}$ and Ψ and by subtracting, we get

$$\bar{\Psi}\mathbb{G}_\mu D^\mu \Psi - \overline{D^\mu \Psi}\mathbb{G}_\mu \Psi = 0 \tag{46}$$

or equivalently

$$\partial^\mu(\bar{\Psi}\mathbb{G}_\mu \Psi) = 0 \tag{47}$$

In order to be sure that $\varphi(x)$ and $\chi(x)$ in $\Psi(x)$ obey separately the given second-order equation, we must verify that there exists a matrix $\mathbb{M}'$ such that the equation

$$(\mathbb{G}_\mu D^\mu + \mathbb{M}')(\mathbb{G}_\mu D^\nu - \mathbb{M})\,\Psi(x) = 0 \tag{48}$$

is coincident with the system

$$\begin{aligned} &(\not{D}^2 - I)\,\varphi(x) = 0 \\ &(\not{D}^2 - I)\,\chi(x) = 0 \end{aligned} \tag{49}$$

First of all, we see that from

$$(\mathbb{G}_\mu D^\mu)^2 = \begin{Bmatrix} \not{D} & 0 \\ 0 & -\not{D} \end{Bmatrix} \begin{Bmatrix} \not{D} & 0 \\ 0 & -\not{D} \end{Bmatrix} = \begin{Bmatrix} \not{D}^2 & 0 \\ 0 & \not{D}^2 \end{Bmatrix} \tag{50}$$

it follows that our iterated equation

$$[(\mathbb{G}_\mu D^\mu)^2 - \mathbb{M}'\mathbb{M} + (\mathbb{M}'\mathbb{G}_\mu - \mathbb{G}_\mu \mathbb{M})\, D^\mu]\ \Psi(x) = 0 \tag{51}$$

is equivalent to the second-order system when

$$\mathbb{M}'\mathbb{M} = \mathbb{I}, \qquad \mathbb{M}'\mathbb{G}_\mu - \mathbb{G}_\mu \mathbb{M} = 0 \tag{52}$$

The second relation, taken for $\mu = 0$, implies

$$\mathbb{M}' = \mathbb{G}_0 \mathbb{M} \mathbb{G}_0 = \mathbb{G}_0 \bar{\mathbb{M}} \mathbb{G}_0 = \mathbb{M}^+ \tag{53}$$

Hence we get a set of conditions on $\mathbb{M}$ alone, namely

$$\mathbb{M}^+\mathbb{M} = \mathbb{I}, \qquad \mathbb{M}^+\mathbb{G}_\mu - \mathbb{G}_\mu \mathbb{M} = 0 \tag{54}$$

As a unitary matrix $\mathbb{M}$ can be cast in the form

$$\mathbb{M} = e^{i\mathbb{A}}, \qquad \mathbb{A}^+ = \mathbb{A} \tag{55}$$

so that, by means of the power expansion of the exponential function, our second condition implies

$$\begin{aligned} \mathbb{M}^+\mathbb{G}_\mu - \mathbb{G}_\mu \mathbb{M} &= e^{-i\mathbb{A}}\mathbb{G}_\mu - \mathbb{G}_\mu e^{i\mathbb{A}} \\ &= \sum_{\kappa:0}^{\infty} \frac{1}{\kappa!} [(-i\mathbb{A})^\kappa\, \mathbb{G}_\mu - \mathbb{G}_\mu (i\mathbb{A})^\kappa] \\ &= \sum_{n:0}^{\infty} (-1)^n \left(\frac{[\mathbb{A}^{2n}, \mathbb{G}_\mu]}{(2n)!} - i\, \frac{\{\mathbb{A}^{2n+1}, \mathbb{G}_\mu\}}{(2n+1)!} \right) = 0 \end{aligned} \tag{56}$$

Hence it will be enough to have

$$\{\mathbb{A}, \mathbb{G}_\mu\} = 0 \tag{57}$$

We also get

$$\bar{\mathbb{A}} = \mathbb{G}_0 \mathbb{A}^+ \mathbb{G}_0 = -\mathbb{A} \tag{58}$$

In order to satisfy these anticommutation relations, $\mathbb{A}$ cannot contain the matrices $\mathbb{G}_\mu$ and $\mathbb{S}_{\mu\nu}$. Hence we have

$$\mathbb{A} = 2\pi p\, \mathbb{G} + 2\pi q s_M \mathbb{G}_M, \qquad s_M s_M = 1 \tag{59}$$

where q and p are, respectively, a real scalar and pseudoscalar, and qs_M is an internal real 3-vector, whose behavior under internal and external transformation coincides with that of $\bar{\Psi}\mathbb{G}_M\Psi$. This means also that $\bar{\Psi}\mathbb{A}\Psi$ behaves like a scalar under a transformation whatsoever. Of course, we have also

$$(s_M \mathbb{G}_M)^2 = s_M s_M \mathbb{I} = \mathbb{I} \tag{60}$$

Moreover we can take p, $q \in [0, 1]$; outside of this interval the matrix $\mathbb{M}$ shows a periodic behavior. In conclusion, and since $[\mathbb{G}, \mathbb{G}_M] = 0$, we have

$$\begin{aligned}
\mathbb{M} &= e^{2\pi i(qs_M \mathbb{G}_M + p\mathbb{G})} \\
&= (\mathbb{I} \cos 2\pi q + is_M \mathbb{G}_M \sin 2\pi q)(\mathbb{I} \cos 2\pi p + i\mathbb{G} \sin 2\pi p) \\
&= \mathbb{I} \cos 2\pi q \cos 2\pi p - s_M \mathbb{S}_M \sin 2\pi q \sin 2\pi p \\
&\quad + i\mathbb{G} \cos 2\pi q \sin 2\pi p + is_M \mathbb{G}_M \sin 2\pi q \cos 2\pi p
\end{aligned} \tag{61}$$

One can verify now that our first-order equation is invariant under space and time inversions, in the sense that the transformed 8-spinors satisfy the same equation:

$$\begin{aligned}
(\mathbb{G}_\mu D^\mu - \mathbb{M})\, \Psi_P(x) &= 0 \\
(\mathbb{G}_\mu D^\mu - \mathbb{M})\, \Psi_T(x) &= 0
\end{aligned} \tag{62}$$

Of course, we shall take in account the fact that $\mathbb{M}$ depends on (q, p, s_M), quantities that are not invariant under space and time inversions:

$$\begin{aligned}
&q^P = q, \qquad p^P = -p, \qquad s_M^P = \begin{cases} s_4 \\ s_5 \\ -s_6 \end{cases} \\
&q^T = q, \qquad p^T = -p, \qquad s_M^T = \begin{cases} -s_4 \\ s_5 \\ -s_6 \end{cases}
\end{aligned} \tag{63}$$

Hence, for the space reflections, we have

$$[\mathbb{G}_\mu D^\mu - \mathbb{S}_6 \mathbb{M}^+(q, -p, s_M^P)\, \mathbb{S}_6]\, \Psi_P(x) = 0 \tag{64}$$

and, for the time reversals,

$$[\mathbb{G}_\mu D^\mu - \mathbb{M}^*(q, -p, s_M^T)]\, \Psi_T(x) = 0 \tag{65}$$

Hence the equation is invariant since

$$\begin{aligned}
&\mathbb{S}_6 \mathbb{M}^+(q, -p, s_M^P)\, \mathbb{S}_6 \\
&\quad = \mathbb{I} \cos 2\pi q \cos 2\pi p + s_M^P \mathbb{S}_6 \mathbb{S}_M \mathbb{S}_6 \sin 2\pi q \sin 2\pi p \\
&\qquad + i\mathbb{G} \cos 2\pi q \sin 2\pi p - i s_M^P \mathbb{S}_6 \mathbb{G}_M \mathbb{S}_6 \sin 2\pi q \cos 2\pi p \\
&\quad = \mathbb{M}(q, p, s_M)
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
\mathbb{M}^*(q, -p, s_M^T) &= \mathbb{I} \cos 2\pi q \cos 2\pi p + s_M^T \mathbb{S}_M^* \sin 2\pi q \sin 2\pi p \\
&\quad + i\mathbb{G} \cos 2\pi q \sin 2\pi p - i s_M^T \mathbb{G}_M^* \sin 2\pi q \cos 2\pi p \\
&= \mathbb{M}(q, p, s_M)
\end{aligned} \tag{67}$$

Then we will impose that our first-order wave equation, under a charge conjugation operation, becomes

$$(\mathbb{G}_\mu D^{\mu *} + \mathbb{M})\, \Psi_C(x) = 0 \tag{68}$$

Of course, we should remember that

$$q^C = q, \qquad p^C = -p, \qquad s_M^C = \begin{cases} -s_4 \\ s_5 \\ -s_6 \end{cases} \tag{69}$$

We have now

$$[\mathbb{G}_\mu D^{\mu *} + \mathbb{M}^T(q, -p, s_M^C)]\, \Psi_C(x) = 0 \tag{70}$$

Now we have

$$\begin{aligned}
\mathbb{M}^T(q, -p, s_M^C) &= \mathbb{I} \cos 2\pi q \cos 2\pi p + s_M^C \mathbb{S}_M^T \sin 2\pi q \sin 2\pi p \\
&\quad - i\mathbb{G} \cos 2\pi q \sin 2\pi p + i s_M^C \mathbb{G}_M^T \sin 2\pi q \cos 2\pi p \\
&= \mathbb{M}^+(q, p, s_M)
\end{aligned} \tag{71}$$

Hence the invariance of our equation under charge conjugation requires

$$\mathbb{M}^+ = \mathbb{M} \tag{72}$$

It means that we should have either

$$\cos 2\pi q = \cos 2\pi p = 0 \tag{73}$$

or

$$\sin 2\pi q = \sin 2\pi p = 0 \tag{74}$$

In the first case, we will get

$$\mathbb{M} = s_M \mathbb{S}_M \tag{75}$$

and in the other

$$\mathbb{M} = \pm \mathbb{I} \tag{76}$$

We remark here that, if the physical behavior of our quantum system is completely described by means of the second-order wave equation, the invariance of the first-order equation under external or internal transformations in not strictly needed. In fact, we could allow $\mathbb{M}$ to change itself in another matrix $\mathbb{M}_1$ of the same type leading to the same second-order equation. However, we will require the full invariance of the first-order equation for two main reasons: (1) We want to keep invariant the form of the relation between the upper and lower 4-spinors in Ψ; any change whatsoever in $\mathbb{M}$ would reflect itself in a different dependence between these two 4-spinors; (2) We will derive all of our Lagrangian and Hamiltonian formalism from the 8-spinor representation, and we need an invariant formulation.

Moreover we want to avoid the possibility that $\mathbb{M}$ assumes a quasi-diagonal form in our representation. In fact, since $\mathbb{G}_\mu D^\mu$ is a quasi-diagonal matrix, the first-order 8-component equation would break in two noncoupled first-order 4-component equations, so that φ, χ will be restricted to a subset of solutions of the second-order equation.

Finally we should consider the fact that, in the 8-spinor formalism, we need a supplementary quantum number, with respect to the 4-spinor case. In fact, we know that[1] the space of all of the solutions of (5) is the direct sum either of the subspaces of the solutions of (6) and

$$(I + \not{D})\, \psi(x) = 0, \tag{77}$$

or of some analogous subspaces. Hence we need an operator, different from the identity, which commutes with all of Eq. (42).

By adding up all of these remarks and the relations (74) and (75), we conclude that the more suitable form for $\mathbb{M}$ is to be equal to one of the $\mathbb{S}_M$, so that the equation is, for example,

$$(\mathbb{G}_\mu D^\mu - \mathbb{S}_4)\,\Psi(x) = 0 \tag{78}$$

Hence, from (42), Eq. (78) becomes

$$\begin{Bmatrix} \not{D} & -\gamma_5 \\ -\gamma_5 & -\not{D} \end{Bmatrix} \begin{Bmatrix} \varphi \\ \chi \end{Bmatrix} = 0 \tag{79}$$

i.e.,

$$\begin{aligned} \not{D}\varphi - \gamma_5\chi &= 0 \\ \not{D}\chi + \gamma_5\varphi &= 0 \end{aligned} \tag{80}$$

It is very easy now to see that the relation between the two 4-spinors is

$$\chi = \gamma_5\, \not{D}\varphi \tag{81}$$

where φ is an arbitrary solution of (5). Of course, the operator commuting with (77) is $\mathbb{S}_4$.

4. THE CANONICAL FORMALISM

In our 8-component formalism, we can now define Lagrangian and Hamiltonian densities in a way that is simple and leads to the correct form of the wave equation and of the conserved current density. In fact, we propose as Lagrangian density ($\mathbb{M} = \mathbb{S}_4$)

$$\begin{aligned} \mathscr{L} &= mc^2\, \mathrm{Re}[\bar{\Psi}(\mathbb{G}_\mu D^\mu - \mathbb{M})\Psi] \\ &= \frac{i\hbar c}{2}(\bar{\Psi}\mathbb{G}_\mu \partial^\mu \Psi - \partial^\mu \bar{\Psi}\mathbb{G}_\mu \Psi) \\ &\quad - e\bar{\Psi}\mathbb{G}_\mu \Psi A^\mu - mc^2 \bar{\Psi}\mathbb{M}\Psi \end{aligned} \tag{82}$$

where we consider as independent canonical coordinates Ψ_A and Ψ_A^+. It is easy to see that this Lagrangian density gives, as wave equations, our Eqs. (5). Moreover, we can also define the conjugate momenta

$$\Pi_A = \frac{\partial \mathscr{L}}{\partial \dot{\Psi}_A} = \frac{i\hbar}{2}\Psi_A^+, \qquad \Pi_A^+ = \frac{\partial \mathscr{L}}{\partial \dot{\Psi}_A^+} = -\frac{i\hbar}{2}\Psi_A \tag{83}$$

where $\dot{\Psi}_A = c\partial_0 \Psi_A$. This allows us to define a Hamiltonian density as

$$\begin{aligned}
\mathscr{H} &= \sum_{A:1}^{8} (\Pi_A \dot{\Psi}_A + \Pi_A^+ \dot{\Psi}_A^+) - \mathscr{L} \\
&= \sum_{A:1}^{8} \frac{i\hbar}{2} (\Psi_A^+ \dot{\Psi}_A - \Psi_A \dot{\Psi}_A^+) - mc^2 \operatorname{Re}[\bar{\Psi}(\mathbb{G}_\mu D^\mu - \mathbb{M})\Psi] \\
&= \operatorname{Re}[i\hbar c \bar{\Psi} \mathbb{G}_0 \partial_0 \Psi - mc^2 \bar{\Psi}(\mathbb{G}_\mu D^\mu - \mathbb{M})\Psi] \\
&= \operatorname{Re}[\bar{\Psi}(eA_0 \mathbb{G}_0 + mc^2 \mathbb{G}_\kappa D_\kappa + mc^2 \mathbb{M})\Psi] \\
&= mc^2 \Psi^+ (\mathbb{A}_\kappa D_\kappa + \mathbb{B})\Psi + eA_0 \Psi^+ \Psi \\
&= \Psi^+ \mathbb{H} \Psi
\end{aligned} \tag{84}$$

where

$$\mathbb{H} = mc^2(\mathbb{A}_\kappa D_\kappa + \mathbb{B}) + eA_0 \mathbb{I} \tag{85}$$

which is a Hermitian matrix because

$$\begin{aligned}
\mathbb{A}_\kappa &= \mathbb{G}_0 \mathbb{G}_\kappa = \mathbb{A}_\kappa^+ \\
\mathbb{B} &= \mathbb{G}_0 \mathbb{M} = \mathbb{G}_0 \mathbb{S}_4 = \mathbb{B}^+
\end{aligned} \tag{86}$$

It can be seen that this $\mathbb{H}$ is exactly the same as can be obtained from the wave equation in the form

$$i\hbar \dot{\Psi} = [eA_0 \mathbb{I} + mc^2 \mathbb{G}_0(\mathbb{G}_\kappa D_\kappa + \mathbb{M})]\Psi \tag{87}$$

Of course, this Hamiltonian does not have the form that could be deduced from the second-order wave equation (5), which would contain square-root operators. The two forms are connected by means of a Foldy–Wouthouysen transformation, which will not be discussed here.

The conserved current can now be deduced both from the wave equation [see Eq. (47)] and from $\mathscr{L}$; it has the form

$$j_\mu = \bar{\Psi} \mathbb{G}_\mu \Psi \tag{88}$$

giving rise to positive conserved densities, which can be interpreted as probability densities. As for the spin density, we start from the Belinfante tensor

$$\begin{aligned}
S_{\mu\nu\rho} &= \frac{i}{c} \left[\frac{\partial \mathscr{L}}{\partial(\partial^\rho \Psi)} \mathbb{S}_{\mu\nu} \Psi - \bar{\Psi} \mathbb{S}_{\mu\nu} \frac{\partial \mathscr{L}}{\partial(\partial^\rho \bar{\Psi})} \right] \\
&= -\frac{\hbar}{2} (\bar{\Psi} \mathbb{G}_\rho \mathbb{S}_{\mu\nu} \Psi + \bar{\Psi} \mathbb{S}_{\mu\nu} \mathbb{G}_\rho \Psi)
\end{aligned} \tag{89}$$

and we define the spin pseudovector field as

$$\begin{aligned} s_\lambda &= \frac{1}{3!}\,\varepsilon_{\lambda\mu\nu\rho} S^{\mu\nu\rho} \\ &= -\frac{\hbar}{3!}\,\mathrm{Re}(\bar{\Psi}\varepsilon_{\lambda\mu\nu\rho}\mathbb{G}^\rho\mathbb{S}^{\mu\nu}\Psi) \\ &= -\hbar\,\mathrm{Re}(\bar{\Psi}\mathbb{G}\mathbb{G}_\lambda\Psi) \\ &= \hbar\bar{\Psi}\mathbb{G}_\lambda\mathbb{G}\Psi \end{aligned} \tag{90}$$

Of course, both the current density and the spin density can be expressed by means of 4-spinors obeying the second-order wave equation (5). In the fact, taking into account (42) and (81), we have

$$\begin{aligned} j_\mu &= \bar{\Psi}\mathbb{G}_\mu\Psi \\ &= \tfrac{1}{2}(\bar{\varphi}\gamma_\mu\varphi + \bar{\chi}\gamma_\mu\chi) \\ &= \tfrac{1}{2}(\bar{\varphi}\gamma_\mu\varphi + \overline{\gamma_5\,\not{D}\varphi}\,\gamma_\mu\gamma_5\,\not{D}\varphi) \\ &= \tfrac{1}{2}(\bar{\varphi}\gamma_\mu\varphi + \overline{\not{D}\varphi}\,\gamma_\mu\,\not{D}\varphi) \end{aligned} \tag{91}$$

which is exactly the form (7) proposed for the conserved current. Finally, for the spin density we have

$$\begin{aligned} s_\mu &= \hbar\bar{\Psi}\mathbb{G}_\mu\mathbb{G}\Psi \\ &= \frac{\hbar}{2}\,(\bar{\varphi}\gamma_\mu\gamma_5\varphi + \bar{\chi}\gamma_\mu\gamma_5\chi) \\ &= \frac{\hbar}{2}\,(\bar{\varphi}\gamma_\mu\gamma_5\varphi + \overline{\gamma_5\,\not{D}\varphi}\,\gamma_\mu\gamma_5\,\not{D}\varphi) \\ &= \frac{\hbar}{2}\,(\bar{\varphi}\gamma_\mu\gamma_5\varphi + \overline{\not{D}\varphi}\,\gamma_\mu\gamma_5\,\not{D}\varphi) \end{aligned} \tag{92}$$

5. CONCLUSIONS

The analysis contained in the preceding sections can be considered as an essential step in coherently defining the relations between the physical quantities (Lagrangians, Hamiltonians, current, spin, and so on) and the 4-spinor solutions of the second-order wave equation (5). In fact, in our preceding papers,[1] we have shown that the usual definitions cannot be retained, since the ordinary definition of the conserved current density

$$J_\mu = \mathrm{Re}(\bar{\psi}\gamma_\mu\,\not{D}\psi) \tag{93}$$

cannot be considered as a suitable starting point to build a scalar product and a statistical interpretation of (5). Hence we are obliged to define the different current (7) with a positive zero component. But, of course, the problem arises now that even the definitions of the other physical quantities should be reformulated. Then, in order to establish the theory on a more firm ground, we started from the remark that, for a second-order wave equation, the state of the system (which should determine the time evolution) is completely specified only if we know even the time derivative of ψ, and we arrived at the conclusion that our state will be determined by means of a sort of double spinor, or 8-spinor Ψ. This paper was devoted to the most general formulation of such a theory in terms of 8-spinors obeying first-order wave equations and equivalent to the theory in terms of 4-spinors and second-order equations. The principal interest of this new scheme is, of course, that now there is a natural, canonical way of defining all the physical quantities. The road is now open to define Clebsch parametrization[9] of the 4-spinor field ψ as an essential step to establishing a causal interpretation of the relativistic quantum mechanics of the spin-$\frac{1}{2}$ particles.

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