Notes on Groups and Geometry, 1978-1986
by Steven H. Cullinane

Typewritten notes collected in a 40-page PDF document.

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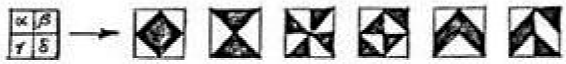
AN INVARIANCE OF SYMMETRY (Written sometime in 1978.)

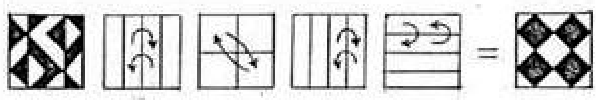
BY STEVEN H. CULLINANE

We present a simple, surprising, and beautiful combinatorial invariance of geometric symmetry, in an algebraic setting.

DEFINITION. A <u>delta transform</u> of a square array over a 4-set is any pattern obtained from the array by a 1-to-1 substitution of the four diagonally-divided two-color unit squares for the 4-set elements.

EXAMPLES.  ....

THEOREM. Every delta transform of the Klein group table has ordinary or color-interchange symmetry, and remains symmetric under the group G of 322,560 transformations generated by combining permutations of rows and columns with permutations of quadrants.

EXAMPLE. 

PROOF (Sketch). The Klein group is the additive group of GF(4); this suggests we regard the group's table T as a matrix over that field. So regarded, T is a linear combination of three (0,1)-matrices that indicate the locations, in T, of the 2-subsets of field elements. The structural symmetry of these matrices accounts for the symmetry of the delta transforms of T, and is invariant under G.

All delta transforms of the 4^5 matrices in the algebra generated by the images of T under G are symmetric; there are many such algebras.

THEOREM. If $1 \leq m \leq n^2+2$, there is an algebra of 4^m $2n \times 2n$ matrices over GF(4) with all delta transforms symmetric.

An induction proof constructs sets of basis matrices that yield the desired symmetry and ensure closure under multiplication.

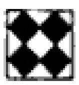
REFERENCE

S. H. Cullinane, <u>Diamond theory</u> (preprint).

Notices of the American Mathematical Society, February 1979, Issue 192, Volume 26, Number 2, pages A-193-A-194:

79T-A37 Steven Hamilton Cullinane

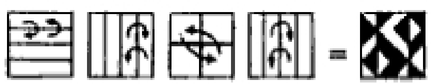
Symmetry invariance in a diamond ring. Preliminary report.



D

We regard the four-diamond figure D as a 4X4 array of 2-color tiles such as . Let G be the group of 322,560 permutations of these 16 tiles generated by arbitrarily mixing random permutations of rows and of columns with random permutations of the four 2X2 quadrants.

THEOREM: Every G-image of D has some ordinary or color-interchange symmetry.

EXAMPLE: D  = Dg (where g ∈ G is a product of two disjoint 7-cycles).

Note that Dg has rotational color-interchange symmetry like that of the famous yin-yang symbol.

REMARKS: G is isomorphic to the affine group on V_4 (GF(2)). The 35 structures of the 840=35·24 G-images of D are isomorphic to the 35 lines in the 3-dimensional projective space over GF(2); orthogonality of structures corresponds to skewness of lines. We can define sums and products so that the G-images of D generate an ideal (1,024 patterns characterized by all horizontal or vertical "cuts" being uninterrupted) of a ring of 4,096 symmetric patterns. There is an infinite family of such "diamond" rings, isomorphic to rings of matrices over GF(4). (Received October 31, 1978.)

STEVEN H. CULLINANE

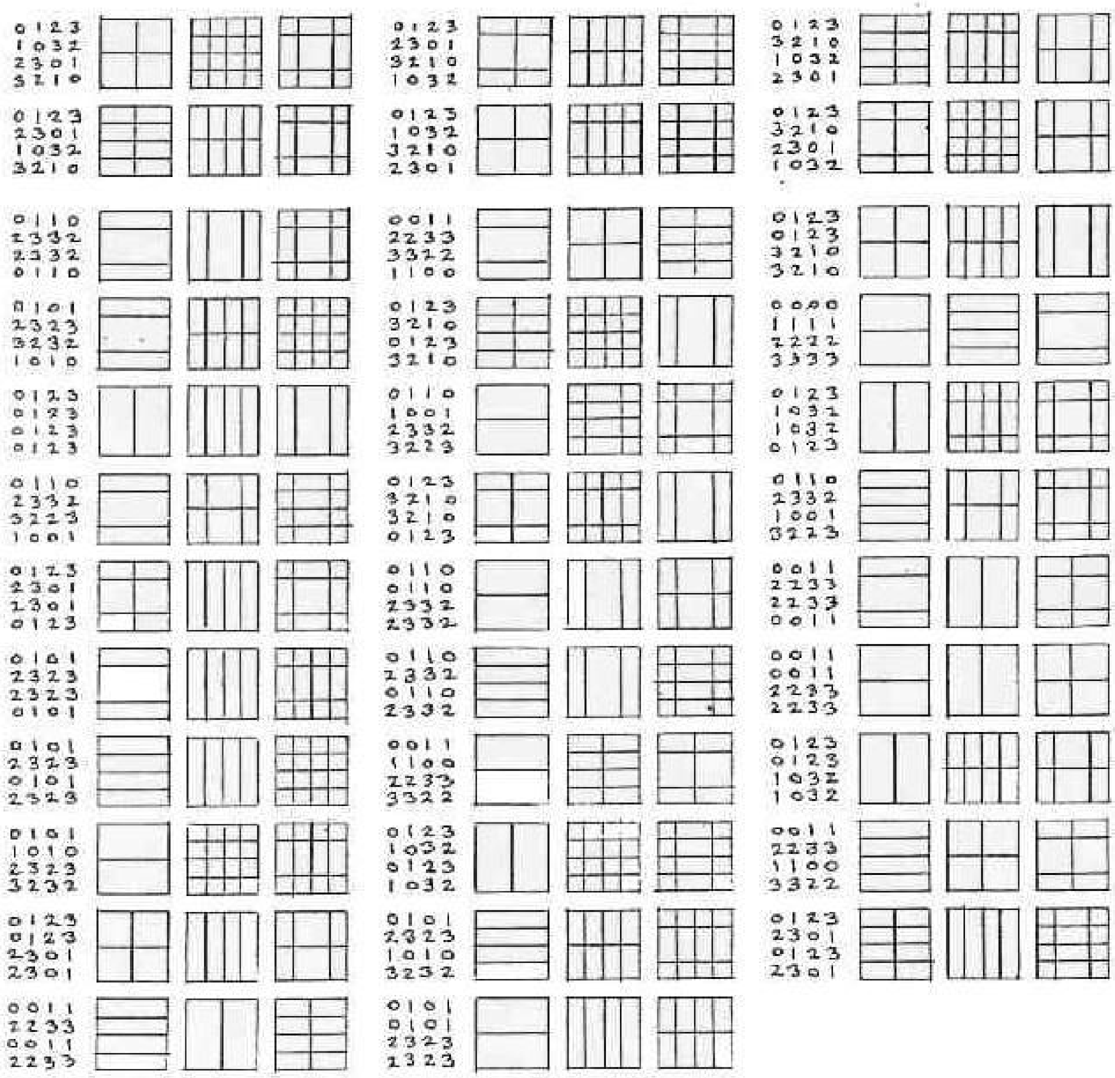
Orthogonality of Latin squares viewed as skewness of lines.

Shown below is a way to embed the six order-4 Latin squares that have orthogonal Latin mates in a set of 35 arrays so that orthogonality in the set of arrays corresponds to skewness in the set of 35 lines of PG(3,2).

Each array yields a 3-set of diagrams that show the lines separating complementary 2-subsets of {0,1,2,3} ; each diagram is the symmetric difference of the other two. The 3-sets of diagrams correspond to the lines of PG(3,2). Two arrays are orthogonal iff their 3-sets of diagrams are disjoint, i.e. iff the corresponding lines of PG(3,2) are skew.

This is a new way of viewing orthogonality of Latin squares, quite different from their relationship to projective planes.

PROBLEM: To what extent can this result be generalized? (Dec. 1978)



Steven H. Cullinane
Patterns invariant, modulo rigid motions, under groups of discontinuous transformations: two examples. Research note. November 5, 1981.

In fig. 1, rigid motions of each cell in a pattern induce rigid motions of the entire pattern. In fig. 3, permutations of cells produce various sectional views of the same (modulo rigid motions) infinite plane pattern. These permutations are derived, as in fig. 2, from motions of a cube.

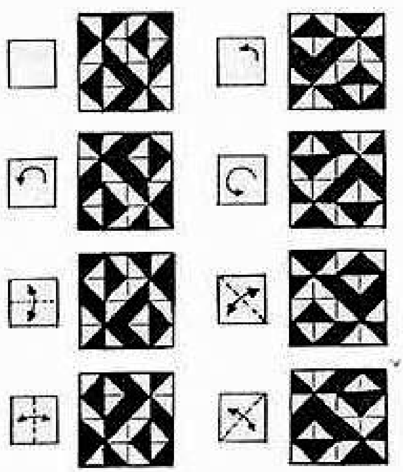


Fig. 1

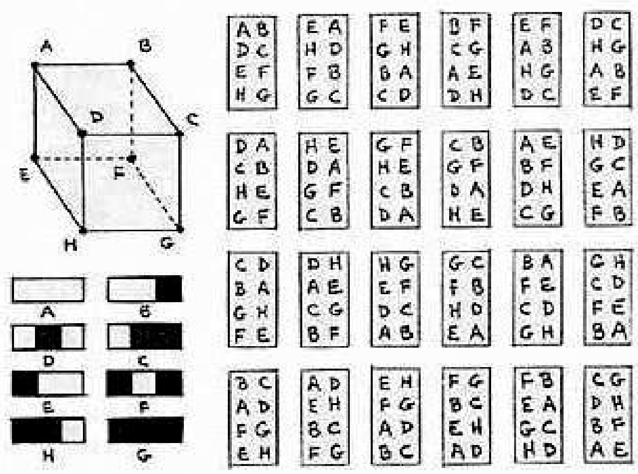


Fig. 2

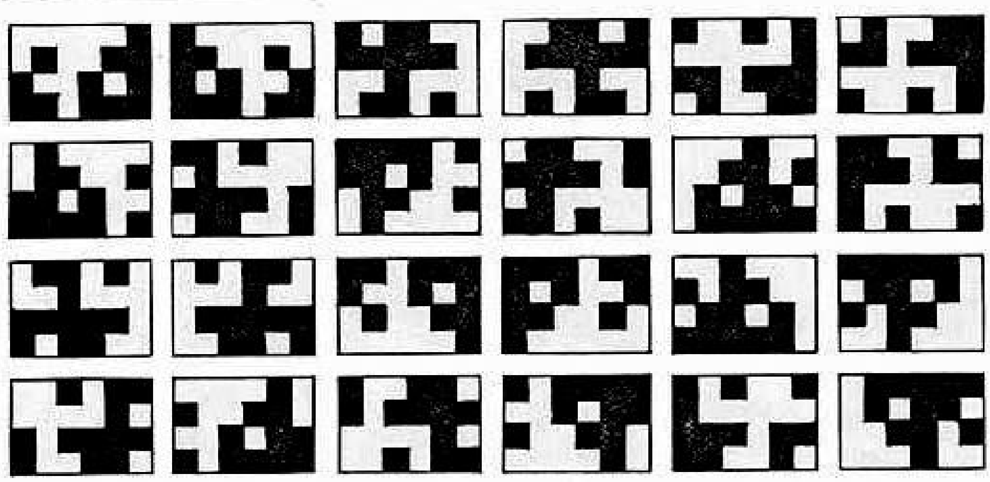
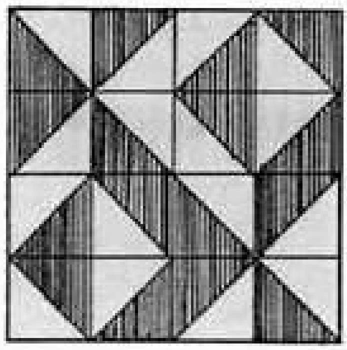


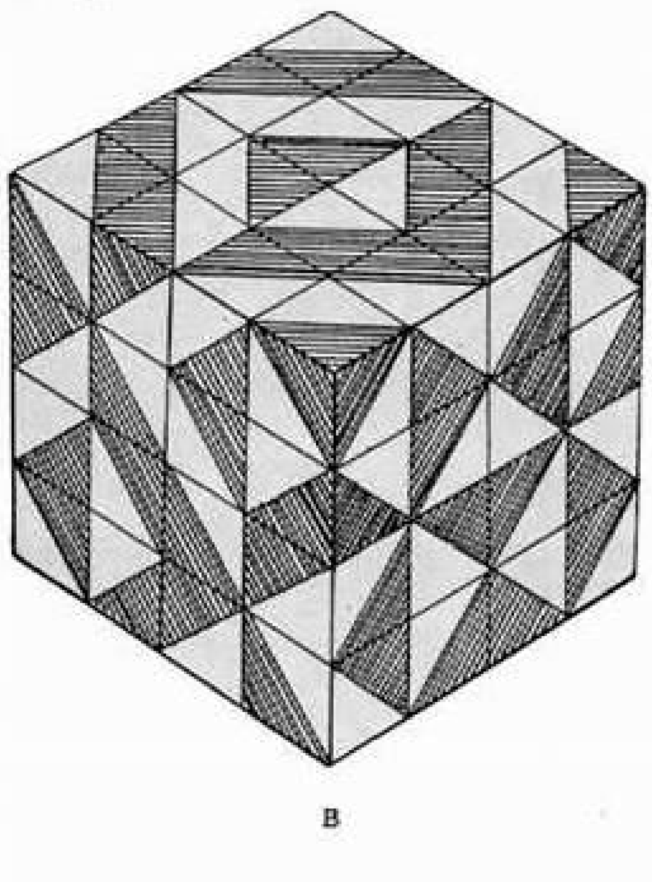
Fig. 3

Steven H. Cullinane
Solid symmetry. Expository note. December 24, 1981.

In pattern A, motions of each cell induce motions of the entire pattern; likewise in B.



A



B

Steven H. Cullinane
Map systems. Query. May 12, 1982.

Definition: Suppose every map ϕ into a given ring M can be written as $\phi = \sum_{i=1}^{N} e_i(\alpha_i \circ \phi)$, where N is a fixed positive integer, the e_i are fixed elements of M, and the α_i are fixed functions from M to P, a proper subset of M.

Let $A = (\alpha_1, \ldots, \alpha_N)$, $O = (e_1, \ldots, e_N)$.

The quintuple (M, A, P, O, N) is a map system.

Example: Using hexadecimal labels for the elements of GF(16), let (M, A, P, O, N) = (GF(16), A, {0,7,E,9}, (6, 8, F), 3), where the functions in A are specified by giving inverse images:

$$\begin{pmatrix} \{0,1,3,2\}\to\{0\} & \{0,4,C,8\}\to\{0\} & \{0,5,F,A\}\to\{0\} \\ \{4,5,7,6\}\to\{7\} & \{1,5,D,9\}\to\{9\} & \{1,4,E,B\}\to\{E\} \\ \{C,D,F,E\}\to\{E\} & \{3,7,F,B\}\to\{7\} & \{3,6,C,9\}\to\{9\} \\ \{8,9,B,A\}\to\{9\}, & \{2,6,E,A\}\to\{E\}, & \{2,7,D,8\}\to\{7\} \end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3).$$

(Multiplication in GF(16) is here defined via the irreducible polynomial $x^4 + x + 1$.)

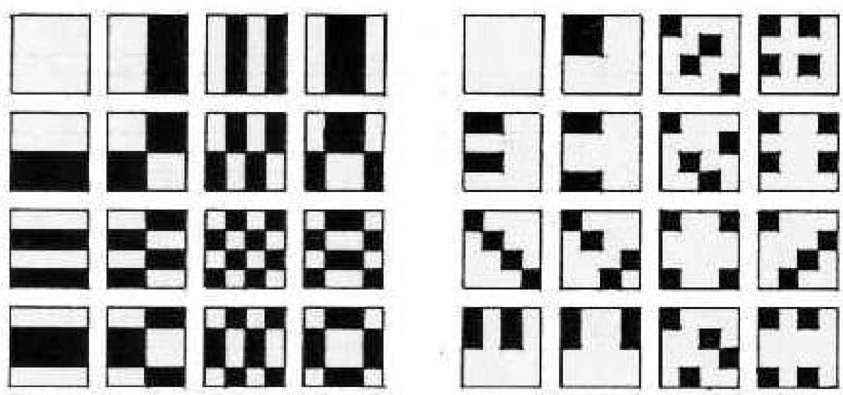
Remarks: Harmonic analysis allows a complicated map to be broken down into, or built up from, simpler maps. Map systems are a different means to the same end.

Query: What is known about such systems?

Steven H. Cullinane
<u>Inscapes</u>. Query. June 12, 1982.

Definition: Let R be an n-ary symmetric relation on a set of t subsets of a t-set, where $n < t = uv$, for positive integers n,t,u,v. Represent each of the t subsets by the 1's in a u×v array a_i over GF(2), where $1 \leq i \leq t$. An <u>inscape</u> of R is a u×v array A of the a_i such that R is true for n of the a_i (that is, for the subsets represented by these a_i) if and only if the arrangement of the a_i within A is the same as the arrangement of the 1's in some $a_i \neq \phi$.

Examples: (Light and dark represent 0's and 1's.)



Remarks: Inscapes are useful for visualizing relations in certain finite geometries. The above examples, for instance, illustrate relations among the 15 hyperplanes of PG(3,2) and among the 15 lines fixed under a particular symplectic polarity of PG(3,2).

Query: What is known about combinatorial systems of this sort?

Note: For some other properties of the a_i in the second example, see E.F. Assmus, Jr. and J.E. Novillo Sardi, "Generalized Steiner systems of type 3-(v, { 4,6 } ,1)", <u>Finite Geometries and Designs</u>, London Math. Soc. Lecture Note Series 49 (Cambridge Univ. Press, Cambridge 1981), pp. 16-21.

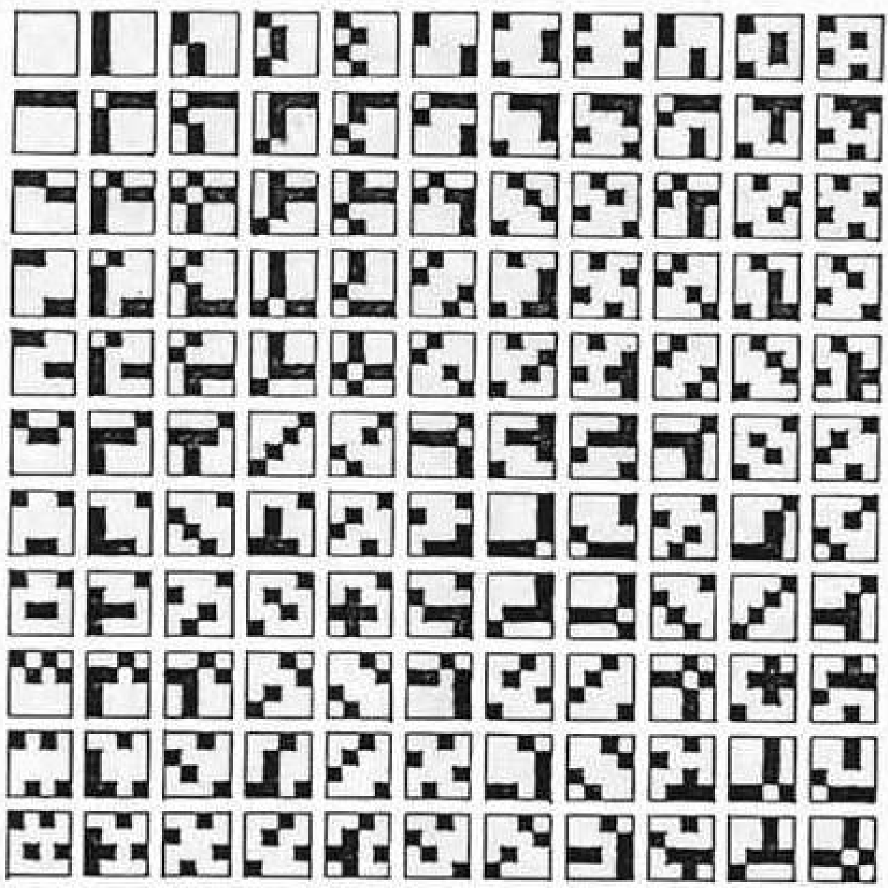
Steven H. Cullinane
A symplectic array. Research note. September 12, 1982.

The 11x11 array below is formed by adding (light = 0, dark = 1, 1 + 1 = 0) the 10 nonempty squares in the first column to the 10 nonempty squares in the first row. These squares represent the 10 pairs of lines interchanged under a particular symplectic polarity of PG(3,2). The array is of interest for several reasons:

1) It serves to illustrate an elementary, but useful, way of constructing a complicated combinatorial object from simpler objects: make an addition table. (Closure is not essential.)

2) Properties of arrays thus formed as addition tables may be of some use in the study of 10x10 Latin squares.

3) Since each of the 121 4x4 squares below represents a set of points in a finite projective space, the array may serve to illustrate or to suggest properties of such spaces.



Steven H. Cullinane
Inscapes II. Query. September 22, 1982.

Given a set X of points, certain families of subsets of X may have, as families, some property s. (Example: the families of spheres that are concentric.) It may be that we can associate to each point of X a subset of X, via an injection $f: X \to 2^X$, in such a way that the f-image, in turn, of this subset of X (i.e., the family of f-images of its points) is in fact one of the families of subsets of X that have property s.
If the map f gives rise in this way to the set S of all such s-families, we can write, in a cryptic but concise way, $S = f(f(X))$, and say that f is an inscape of S.

Query: What known results can be stated, after the appropriate definition of S, in the form "There exists an inscape of S"?

Addendum of Oct. 10, 1982. A more precise definition:

Let X be a non-empty set. Let P(X) denote the set of all subsets of X. Let $S \subset P(P(X))$. Suppose there exists an injection $f: X \to P(X)$ such that, for any $\sigma \in P(P(X))$, $\sigma \in S$ if and only if $\exists x \in X$ such that $\sigma = f(f(x)) = \{f(y) \mid y \in f(x)\}$.
Then f is an inscape of S.

This notion arises naturally in studying the action of a symplectic polarity in a projective space. One of course wonders whether it has arisen previously in any other context.

Steven H. Cullinane
Group scores. Problem. December 27, 1982.

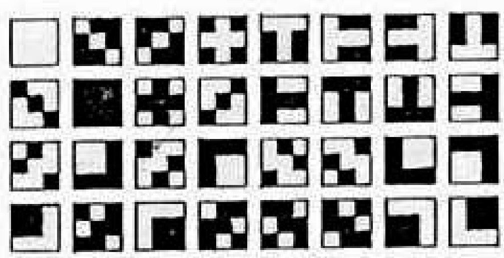
Definition: Let G_1 be a finite permutation group. Represent G_1 as a group of permutation matrices over GF(2), the two-element Galois field, and let $V_1 = V_1(G_1)$ denote the enveloping algebra* of G_1. Suppose there exist subalgebras V_2 and V_3 of V_1 such that

a) V_3 is a transversal of the additive cosets of V_2 in V_1, and

b) V_3 is the enveloping algebra of a subgroup G_2 of G_1.

Then $(G_1, V_1, V_2, V_3, G_2)$ is a group score.

Example (Light and dark represent 0's and 1's):



This array A of matrices is a group score in which

$G_1 = S_3$,

$V_1 = A$,

$V_2 =$ the first row of A,

$V_3 =$ the first column of A, and

$G_2 =$ the permutation matrices in the first column.

Note that in the example neither V_2 nor V_3 is an ideal of V_1.

Problem: What group scores exist?

* See Flath, AMS abstract 797-17-88, the related 797-20-130, and H. Weyl, The classical groups, 2nd ed. (1946), Princeton, p. 79

Steven H. Cullinane
Decompositions of group enveloping algebras. Query. May 31, 1983.

Notation:

Let G be an abstract group, H a subgroup of G. Let $\rho: G \to M$ be a representation of G as a group M of invertible endomorphisms of an R-module V, where R is a commutative ring with unity, and let $N = \rho(H)$. Denote the enveloping algebra of M (i.e., the R-linear closure of M) by $E(M)$, or, more explicitly, by $(E(M), +, \cdot)$. For a, b in $E(M)$ let $[a,b] = a \cdot b - b \cdot a$, and denote the resulting Lie algebra by $(E(M), +, [\,])$.

Query:

1. How can we relate decompositions of $(E(M), +, \cdot)$ to the structure of G?

   (In particular, when can we write $E(M)$ as a direct sum

   $$E(M) = E(N) + A,$$

   where A is a subalgebra of $E(M)$?)

2. How can we relate decompositions of $(E(X), +, [\,])$, where $X = M$ or N, to the structure of G, when M is nonabelian?

   (In particular, how are the Levi direct sum decompositions*

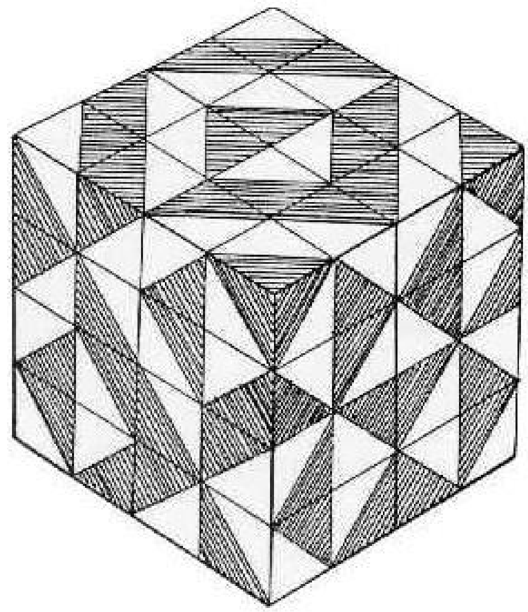
   $$E(M) = R(M) + L(M) \quad \text{and}$$
   $$E(N) = R(N) + L(N),$$

   where $R(X)$ is the radical of $(E(X), +, [\,])$ and $L(X) \cong E(X)/R(X)$ is a semisimple Lie subalgebra of $(E(X), +, [\,])$, related to the structure of G?)

3. How should we restrict the natures of G, H, ρ, M, R, and V in order to answer (1) or (2) above?

* See A. I. Mal'cev, On semisimple subgroups of Lie groups (1944), AMS Translations, series 1, volume 9 (1962).

Steven H. Cullinane
An invariance of symmetry. Research note. June 21, 1983.



B

Theorem: There exists a triply transitive group G of 1,290,157,424,640 permutations of the 64 subcubes of B such that every G-image of B has a rigid-motion symmetry.

(The marking on each subcube of B is identical; each is symmetric under reflection in its center.)

Proof:(sketch): We label the 64 cells of B with the points of the affine 6-space A over GF(2) in such a way that each hyperplane of A is left invariant or is carried to its complement under a group C of 8 rigid motions generated by reflections in midplanes of B. We then define the group G as the group of affine transformations of A. Under G, as under C, the set of hyperplanes and hyperplane-complements is left invariant. This symmetry of hyperplanes is then fairly easily shown to underlie the remarkable invariance of symmetry of B.

(For a geometrically natural way to generate G see AMS abstract 79T-A37.)

Steven H. Cullinane
Group identity algebras. Problem. August 4, 1983.

Definition: Let (S, *) and (S, ∘) be groups with the same set S of element-symbols but with different group tables.
If there is at least one algebraic identity I expressing a nontrivial relationship between * and ∘ then (S, *, ∘) is a sort of algebra, which for lack of any other name we call a group identity algebra.

Example: Let $S = \{e,a,b,c\}$ and let * and ∘ be the operations + and • (or juxtaposition) in the tables below.

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

•	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

The following identity I holds. $\forall\, x,y,z,w \in \{e,a,b,c\}$,

$$((xy)+(zw)) + ((x+y)(z+w)) =$$
$$((xz)+(yw)) + ((x+z)(y+w)) =$$
$$((xw)+(yz)) + ((x+w)(y+z)).$$

The dual identity I' obtained by interchanging + and • also holds.
(Note that in this case I and I' state that certain algebraic forms are invariant under the action of the symmetric group on their indeterminates.)

Problem: Are there infinitely many finite group identity algebras?
(Note the word "nontrivial" in the definition.)

Steven H. Cullinane
Transformations over a bridge. Problem. August 16, 1983.

Let $(G,+,\cdot)$ be the algebra (in the sense of universal algebra) with underlying set $G=\{e,a,b,c\}$ and operations as follows.

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

•	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

(We follow the notational conventions of writing • as juxtaposition and letting • precede + to avoid a proliferation of parentheses.)

(Such an algebra $(G,\ast,\star)$, where $(G,\ast)$ and $(G,\star)$ are groups, we call a bridge.)

Problem.

Part 1. What is the nature of the group T generated by permutations of GxG of the form $t(p,q,r,s)$: $(x,y) \to (px+qy,\ rx+sy)+(x,y)$ where $p,q,r,s \in G$?

Note 1.1. It appears that T is isomorphic to a subgroup of the group of regular affine transformations of the affine 4-space over GF(2).

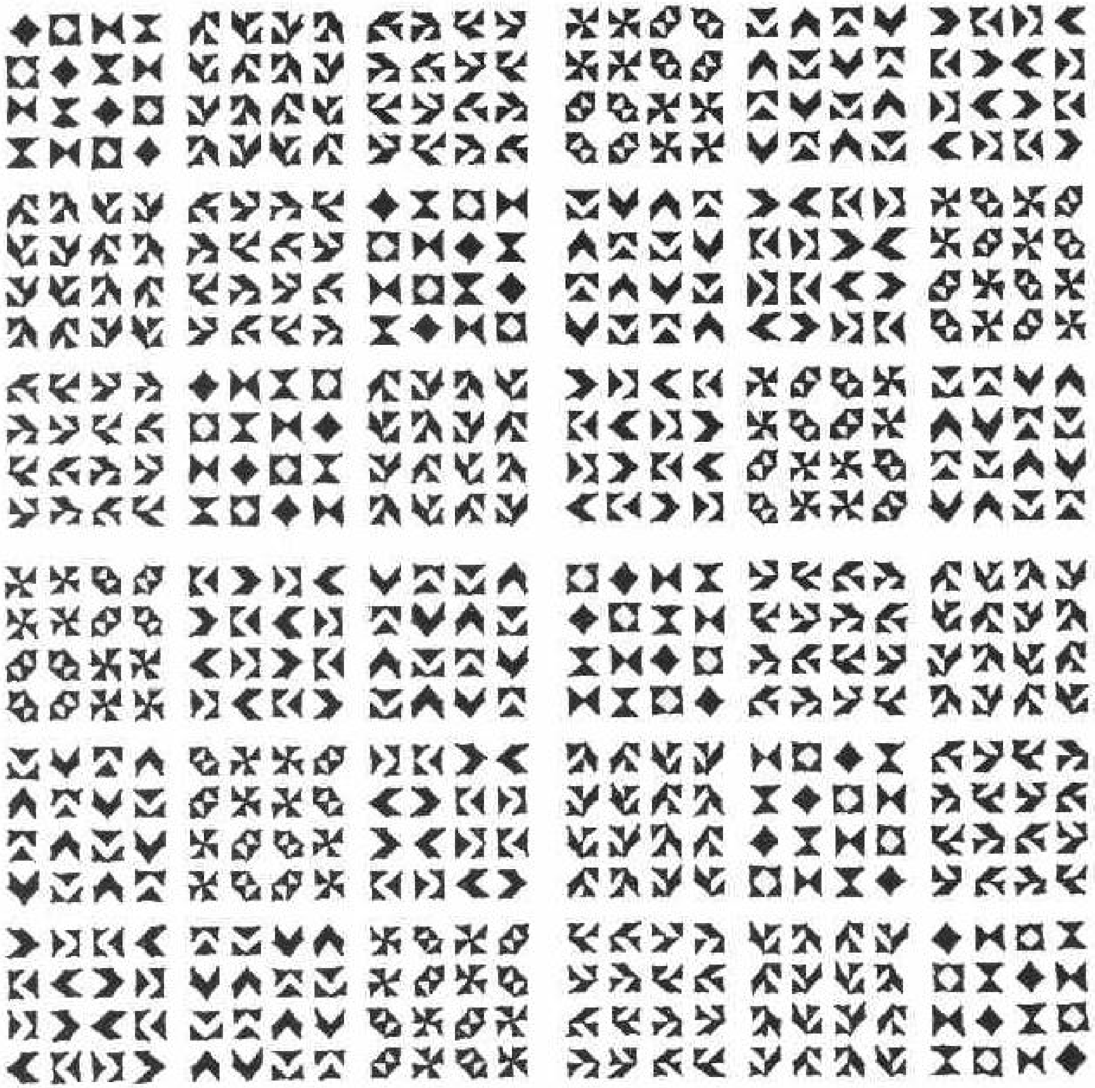
Note 1.2. We can have $t(p,q,r,s) = t(t,u,v,w)$ where $(p,q,r,s) \neq (t,u,v,w)$ since, for instance, $\forall (x,y) \in GxG,\ ax+ay = cx+cy$.

Part 2. Is there some reasonably simple algebraic expression over $(G,+,\cdot)$ for $t(p,q,r,s) \circ t(t,u,v,w)$?

Note 2.1. Not every member of T can be written in the form $t(p,q,r,s)$. Example: $(t(a,c,c,e))^2$.

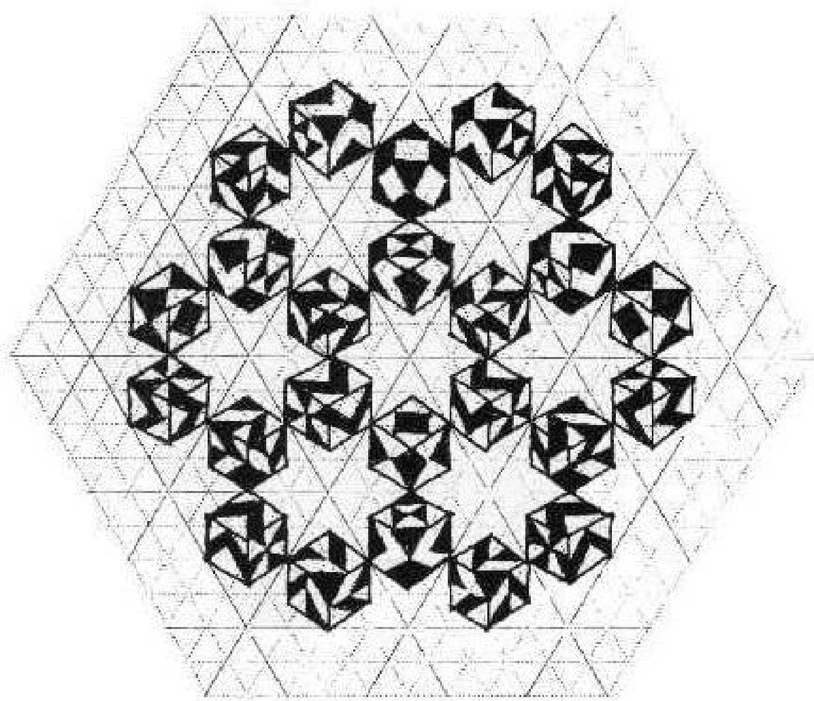
Steven H. Cullinane

Portrait of O. Action of the octahedral group on a diamond. Oct. 1 '83.



Steven H. Cullinane
Study of O. October 16, 1983.

The 24 two-colored 2x2x2 cubes below represent the elements of the octahedral group O, which is viewed as acting in the same way on each of the 8 subcubes of any given 2x2x2 cube. The arrangement of the 24 colored cubes may be of some interest for its combinatorial properties.



Steven H. Cullinane

Compound groups. Problem. November 8, 1983.

Definition: Suppose a finite group G of order n can be represented as a group of permutations $p_1, p_2, \ldots, p_n$ on m objects, where $m < n$. Suppose further that we can take these m objects to be distinct elements $g_1, g_2, \ldots, g_m$ of G in such a way that the n products

$$\begin{array}{c}
(p_1(g_1))(p_1(g_2)) \ldots (p_1(g_m)) \\
(p_2(g_1))(p_2(g_2)) \ldots (p_2(g_m)) \\
\vdots \\
(p_n(g_1))(p_n(g_2)) \ldots (p_n(g_m))
\end{array}$$

are all distinct, i.e. constitute all of G.

If such a group G, compounded in this way from m of its elements, exists, we call G a compound group.

Problem: Which (if any) finite groups are compound?

Steven H. Cullinane

Group compounds. Problem. November 10, 1983.

Definition: Let P be a group of n permutations on a finite m-element set X, let G be a multiplicative group, and let f map X to G.

Let $P = \{p_i : 1 \leq i \leq n\}$, let $X = \{x_j : 1 \leq j \leq m\}$.

Define $\phi : P \rightarrow G$ by $\phi(p_i) = \prod_{1 \leq j \leq m} f(p_i(x_j))$.

The structure (P, G, f, ϕ) is a group compound.

Problem:

(1) Let G be given. For which P and f is $\phi(P)$ a coset of some subgroup of G? When is ϕ a surjection?

(2) Let P be given. For which G and f do inverse images under ϕ form a coset decomposition of P? When is ϕ an injection?

Steven H. Cullinane
Table groups. Problem. November 27, 1983.

We regard the (unbordered) tables of groups of order n as nxn arrays over the symbols 1,2,...,n in which the first row (read from left to right) is the same as the first column (read from top to bottom). (The entry at top left represents the identity but need not be the symbol 1.) Thus we regard each of the arrays

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 3 & 1 & 2 & 4 \\ 1 & 3 & 4 & 2 \\ 2 & 4 & 3 & 1 \\ 4 & 2 & 1 & 3 \end{matrix} \quad \text{as a table of the four-group.}$$

Such a table is determined (since it is a Latin square) by the entries lying below the first row and to the right of the first column. Call this $(n-1)x(n-1)$ portion of a group table an n-box.

(Note that the number B of n-boxes is in general greater than $n!N$, where N is the number of nonisomorphic groups of order n. For instance, for $n=4$ we have $N=2$, but $B=(4!)(4)$ rather than $(4!)(2)$.)

For a given n, we may be able to see something of how the various order-n groups are interrelated by studying group actions on n-boxes.

Definition: Let G(n) denote the direct product of $(n-1)^2$ copies of S_n and regard the components of an element g of G(n) as arranged in an $(n-1)x(n-1)$ array. Such an element g acts componentwise in the obvious way on an n-box to yield an array that may or may not be an n-box. Suppose there exists some subgroup T of G(n) such that T is transitive on some set B(T) of n-boxes that includes N n-boxes representing the N distinct (i.e., pairwise nonisomorphic) groups of order n. (We do not require that B(T) be closed under the action of T, nor even that each T-image of a member of B(T) be an n-box.) We call such a T a table group for n.

Clearly for each n there is at least one table group T, namely G(n). That smaller T's may exist is shown by the following.

Example:* A table group for 4 is generated by the following elements of G(4).

$$\begin{matrix} I & (12) & (12) \\ (12) & (12) & (12) \\ (12) & (12) & (12) \end{matrix} \qquad \begin{matrix} (13) & (13) & (13) \\ (13) & I & (13) \\ (13) & (13) & (13) \end{matrix} \qquad \begin{matrix} (14) & (14) & (14) \\ (14) & (14) & (14) \\ (14) & (14) & I \end{matrix}$$

Problem: What is the order of a smallest table group for n? Is there some way to construct such groups that does not require knowledge of N? (The case for n a prime power seems of particular interest.)

*The example is of course not a smallest table group for 4, but is shown for its structural interest.

Steven H. Cullinane

Linear operators in geometric function spaces. Problem. Jan. 5, 1984.

Let X be a 2n-dimensional linear space over a field K. Let a map c take each subspace S of X to a function f_S: $X \rightarrow K$ that is nonzero on S and zero elsewhere. (Here f_S is a sort of characteristic function representing the subspace S.) Denote by $F = F(2n, K, c)$ the linear space over K spanned by the functions f_S. We call F a geometric function space.

Theorem: There is at least one F(2n, K, c) for which there exists a linear operator T, acting on F, such that T^r takes the 1-dimensional subspaces of X (i.e., functions f_S representing such subspaces) to distinct $r+1$ -dimensional subspaces of X, for $1 \leq r \leq 2n-2$.

Proof: The matrix T at right represents such an operator when X is the linear 4-space over GF(2), the two-element Galois field.

X:

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0

T:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1
1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0
1	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0
1	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0
1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
1	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0
1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0
1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0

Problem: For what other spaces F(2n, K, c) does such a T exist?

Steven H. Cullinane
Diamonds and whirls. Expository note. Sept. 15, 1984.

Modulo color-interchange and rotations, there are exactly 2 ways (see fig. 2) to color the 6 faces of a cube so that
(a) each face is split diagonally into a black half and a white half, and
(b) there are exactly 4 distinct images of the colored cube under the group O of 24 rotational symmetries of the cube.

The rotational symmetries of each such coloring form an order-6 subgroup of O leaving invariant an inscribed hexagon as in fig. 1. This subgroup of O consists of the identity, rotations of 120 and 240 degrees about a diagonal of the cube, and 180-degree rotations about each of 3 axes joining midpoints of opposite edges of the cube.

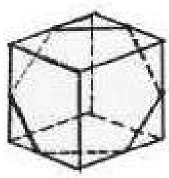


Fig. 1

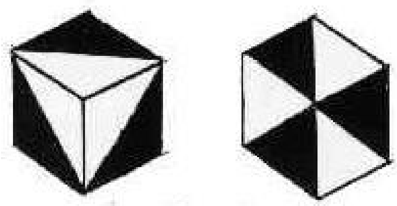


Fig. 2
"Diamond" and "whirl" cubes

Identical copies of these cubes, variously oriented, can be assembled into larger cubical patterns with remarkable symmetry properties.

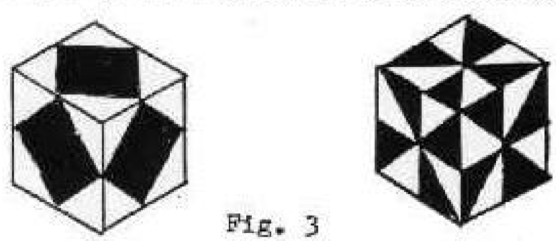
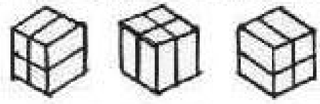


Fig. 3
A: Eight diamond cubes B: Eight whirl cubes

Patterns A and B in fig. 3 yield a number of other symmetric patterns when their subcubes are permuted (without rotation) as follows.

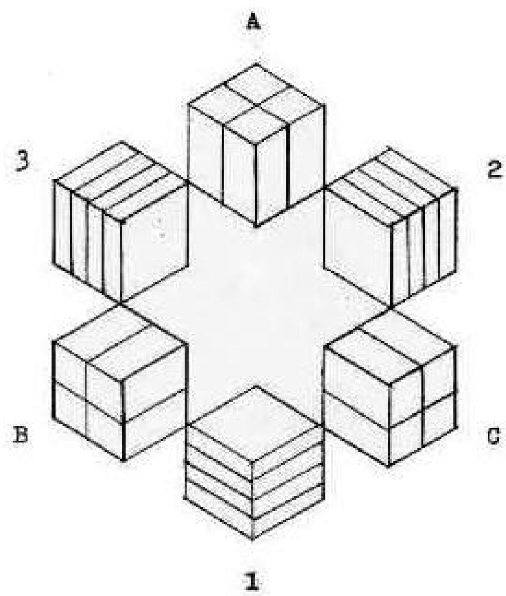
Fig. 4: 

Let S_4 act on the 4 1x1x2 "bricks" in each of the 3 partitions above; the group Δ so generated can be shown to be triply transitive, of order 1344, and isomorphic to the affine group on the linear 3-space over the two-element finite field.

THEOREM: Patterns A and B each have 168 images under Δ. Each of these images has some nontrivial symmetry (ordinary symmetry for A-images, ordinary or color-interchange symmetry for B-images) under at least one of a group of 8 rigid motions of the cube.

Steven H. Cullinane

Affine groups on small binary spaces. Expository note. Sept. 25, 1984.



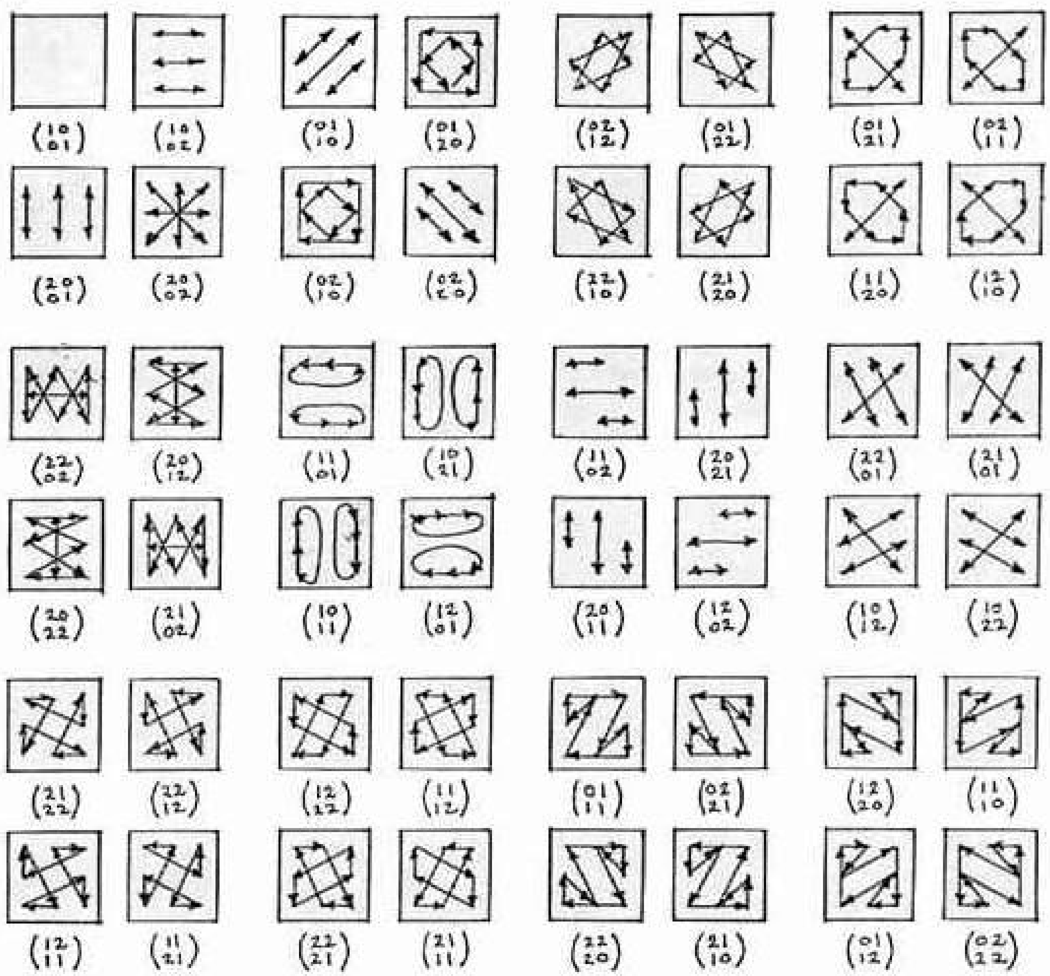
Theorem:

The affine group	of order	is generated by S_4 acting on partitions
AGL(2,2)	24	A
AGL(3,2)	1,344	A, B, C
AGL(4,2)	322,560	A, 2, 3
AGL(5,2)	319,979,520	A, B, C, 2, 3
AGL(6,2)	1,290,157,424,640	A, B, C, 1, 2, 3.

Steven H. Cullinane

Visualizing GL(2,p). Expository note. March 26, 1985.

> "The typical example of a finite group is GL(n,q), the general linear group of n dimensions over the field with q elements." -- J. L. Alperin



The 48 actions of GL(2,3) on a 3x3 coordinate-array A are illustrated above. The matrices shown right-multiply the elements of A, where

$$A = \begin{matrix} (1,1) & (1,0) & (1,2) \\ (0,1) & (0,0) & (0,2) \\ (2,1) & (2,0) & (2,2) \end{matrix} .$$

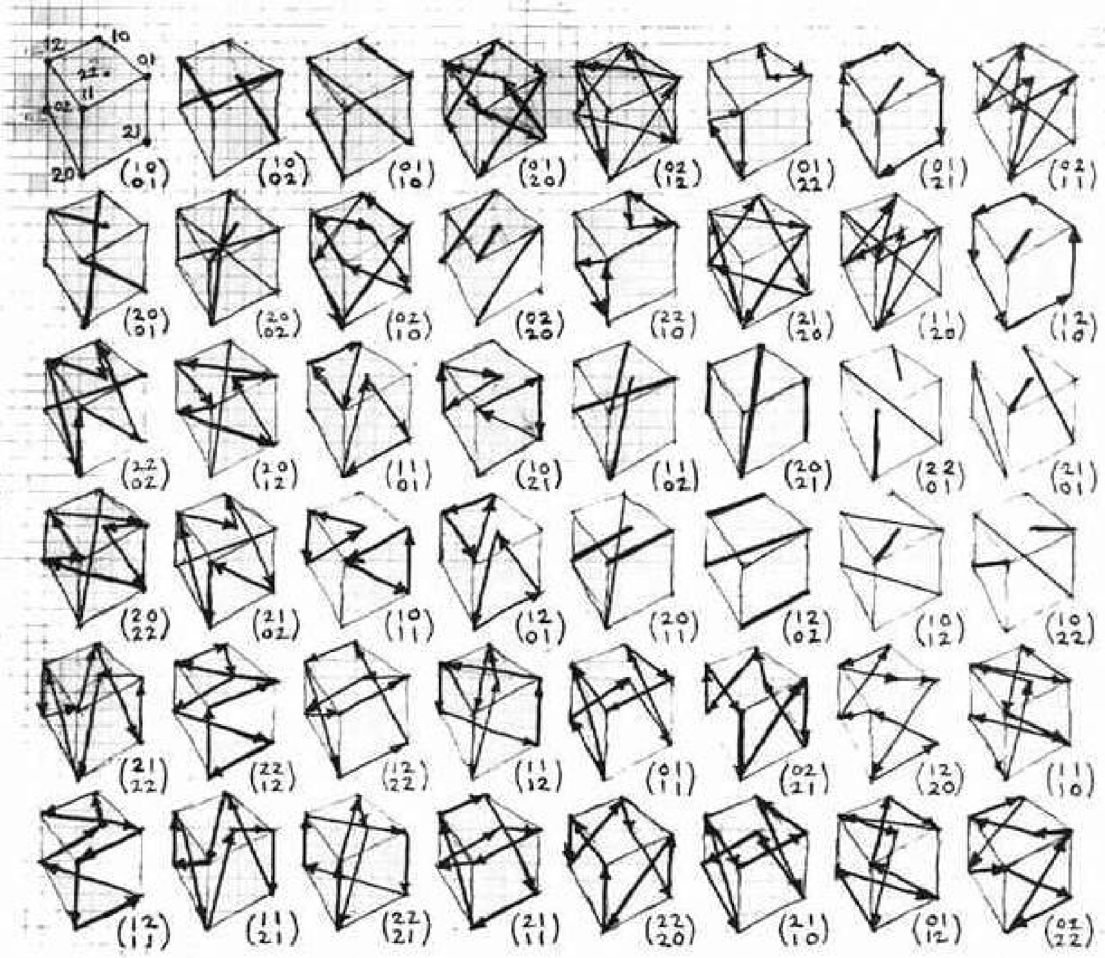
Actions of GL(2,p) on a pxp coordinate-array have the same sorts of symmetries, where p is any odd prime.

Steven H. Cullinane

GL(2,3) actions on a cube. Expository note. April 5, 1985.

The 48 diagrams below illustrate some symmetries of GL(2,3) actions on the 8 nonzero vectors of the linear 2-space over the 3-element field. The vectors are viewed as labeling vertices of a cube (pictured here with a slight distortion, to avoid overlapping lines).

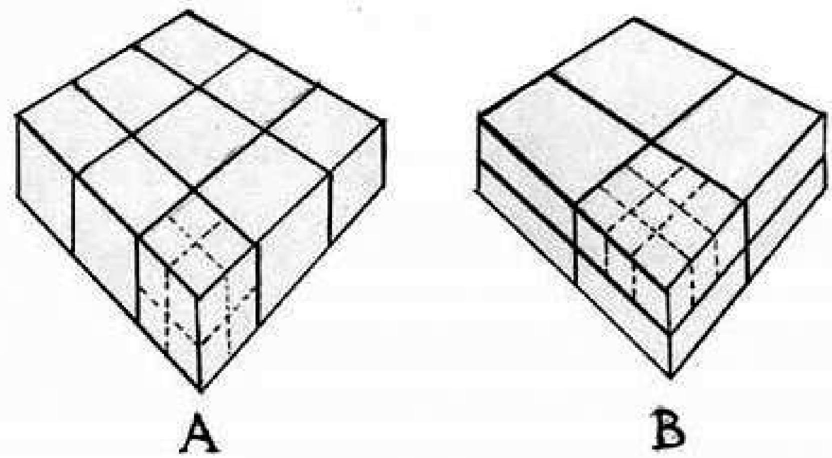
The diagrams may have some heuristic value for the study of groups generated by mixing GL(2,3) actions with those of other groups.



Steven H. Cullinane

Group actions on partitions. Problem and query. April 5, 1985.

Two ways of partitioning a 72-set:



Definition: Let G be the group of degree 72 generated by mixing

(1) actions of the affine group AGL(2,3) on the set of nine 2x2x2 cubes in partition A,

(2) like actions of AGL(2,3) on each of the eight 3x3 sections in B,

(3) actions of AGL(3,2) on the set of eight 3x3 sections in B, and

(4) like actions of AGL(3,2) on each of the nine 2x2x2 cubes in A.

Problem: What is the order of G?

Query: Clearly many similar problems could be posed.

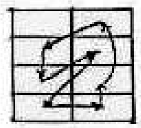
What results or methods are known?

(Note: many equivalent coordinate systems for the affine actions above are available via natural mappings of the respective linear spaces onto 3x3 or 2x2x2 arrays.)

Steven H. Cullinane
Generating the octad generator. Expository note. April 28, 1985.

0	1
x	x+1
x^2	x^2+1
x^2+x	x^2+x+1

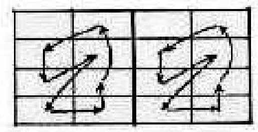
GF(8)
(mod x^3-x-1)



A Singer 7-cycle
S_1 on GF(8)

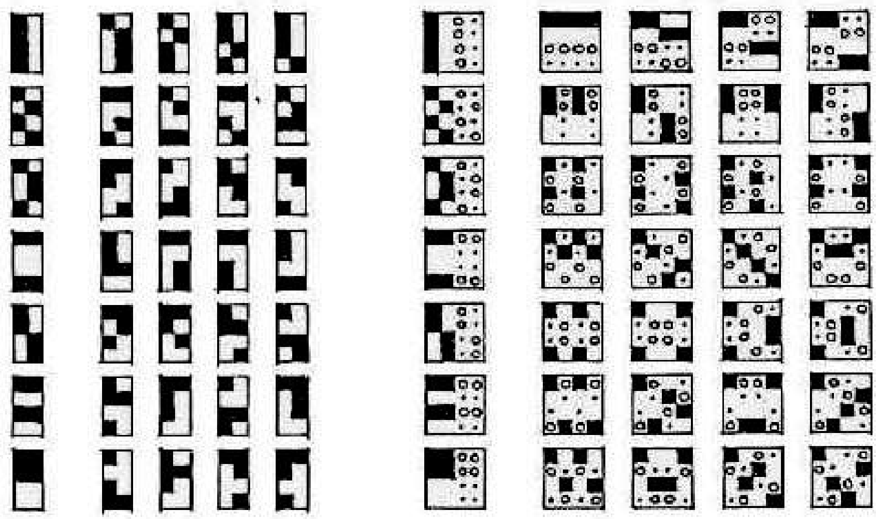
0000	0001	1000	1001
0010	0011	1010	1011
0100	0101	1100	1101
0110	0111	1110	1111

The linear 4-space
L over GF(2)



A linear map S_2 on L
(= 2 copies of S_1)

S_1 and S_2 acting on row 1 below yield the Miracle Octad Generator [3] :



Apart from its use in studying the 759 octads of a Steiner system S(5,8,24) -- and hence the Mathieu group M_{24} -- the Curtis MOG nicely illustrates a natural correspondence C (Conwell [2], p. 72) between

(a) the 35 partitions of an 8-set H
(such as GF(8) above, or Conwell's 8 "heptads") into two 4-sets, and

(b) the 35 partitions of L into four parallel affine planes.

Two of the H-partitions have a common refinement into 2-sets iff the same is true of the corresponding L-partitions. (Cameron [1], p. 60)

Note that C is particularly natural in row 1, and that partitions 2-5 in each row have similar structures.

1. Cameron, P.J., Parallelisms of Complete Designs, Camb. U. Pr. 1976.

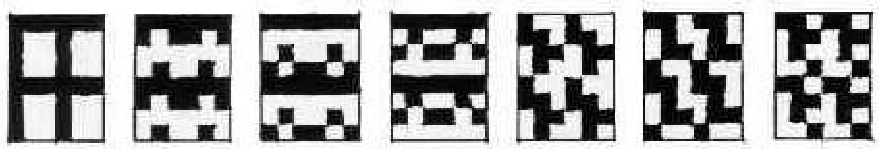
2. Conwell, G.M., The 3-space PG(3,2) and its group, Ann. of Math. 11 (1910) 60-76.

3. Curtis, R.T., A new combinatorial approach to M_{24}, Math. Proc. Camb. Phil. Soc. 79 (1976) 25-42.

Steven H. Cullinane
Symmetry invariance under M_{12}. Expository note. Aug. 22, 1985.

The quintuply transitive Mathieu group M_{12} might be expected to thoroughly scramble any neat pattern it acts on. However, recent work by R. T. Curtis and J. H. Conway [1] has the following remarkable consequence.

Theorem: The set of 7 infinite plane patterns below is invariant (modulo rigid motions of the plane, and color-interchange) under Curtis-Conway M_{12} actions on the 4x3 motifs shown as quadrants.



Note that each pattern has nontrivial symmetry, modulo color-interchange. (The motifs are 7 of the 132 hexads in an S(5,6,12) ingeniously constructed in [1].)

REFERENCE

1. Curtis, R. T., The Steiner system S(5,6,12), the Mathieu group M_{12} and the "kitten," Computational Group Theory, ed. Michael D. Atkinson, Academic Press, 1984, 353-358.

S. H. CULLINANE 11/17/85.

Finite groups of the same order are sometimes related by a nontrivial identity.

Example:

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

·	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

.

We have, $\forall\ w, x, y, z \in \{e, a, b, c\}$,

(D) $x \cdot (y + z) = (x \cdot y) + (x \cdot z) + x$, and hence

(I) $((w \cdot x) + (y \cdot z)) + ((w + x) \cdot (y + z)) =$

$((w \cdot y) + (x \cdot z)) + ((w + y) \cdot (x + z)) =$

$((w \cdot z) + (x \cdot y)) + ((w + z) \cdot (x + y))$.

The dual identity I′ obtained by interchanging $+$ and $\cdot$ in (I) also holds.

Such a structure -- two groups joined together by a nontrivial identity -- might be called a "bridge." Are there infinitely many sorts of bridges? I am grateful to S. Comer for the following reformulation of this rather vague question.

<u>Definitions</u>: Let $B = \{(G, *, \cdot) : (G, *)$ and $(G, \cdot)$ are groups$\}$. For a subvariety $V \subseteq B$ let Δ_* denote the set of identities holding in $(G, *)$ for all $(G, *, \cdot) \in V$. Similarly, define $\Delta_\cdot$. For any set of identities Δ in the language for B let $V(\Delta)$ denote the variety of all members of B that satisfy Δ. Call a variety V <u>reducible</u> if $V = V(\Delta_*) \cap V(\Delta_\cdot)$.

<u>Problem</u>: Are there infinitely many irreducible subvarieties of B?

S. H. Cullinane

Dynamic and algebraic compatibility of groups. Dec. 11, 1985.

(A) Observation -- Nonisomorphic order-n groups, each transitively permuting the same n points, may generate a group smaller than A_n.
Example -- The four group and C_4, acting on the vertices of a square, generate D_4.

(B) Observation -- Nonisomorphic order-n groups are sometimes related by a nontrivial identity.
Example --

+	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

∘	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

,

with $x\circ(y+z) = (x\circ y) + (x\circ z) + x \quad \forall x,y,z \in \{e,a,b,c\}$.

Problems:

(A) For which (n,k) are there k nonisomorphic order-n groups G_i (each with the same elements and the same identity element) and regular permutation representations f_i such that $|\langle f_1(G_1),\ldots,f_k(G_k)\rangle| < |A_n|$?

(B) For which (n,k) are there k nonisomorphic order-n groups G_i (each with the same elements and the same identity element) all interrelated by a nontrivial algebraic identity?

(C) For which (n,k) are there solutions to both (A) and (B)?

S. H. Cullinane

Geometry of partitions II. Problems. January 11, 1986.

Definitions:

Given $0 < a \in \mathbb{R}$, and a finite (or countably infinite) sequence $\xi = (a_1, a_2, \ldots)$ of positive real numbers such that $\Sigma a_i = a$ (or such that the partial sums of ξ converge to a), call ξ a partition of a. Let $L(\xi)$ be the following surface:

$$L(\xi) = \{(x,y,z) \in \mathbb{R}^3 \mid x > 0 \text{ and } (xa)^y = \Sigma (xa_i)^z\}.$$

Thus L is a mapping that lets us represent partitions by surfaces. (If the partial sums of ξ diverge but the corresponding surfaces converge, one might define $L(\xi)$ to be the limit surface.)

Theorem (Nicomachus-Bachet):

The surfaces $L((1, 2, \ldots, n))$ all intersect at $(1,2,3)$.

Problems:

1. Do any other "natural" families of partitions yield intersection theorems of a nontrivial nature?
2. How do families of infinite-series partitions behave under L? (For example, $\xi_s = (1^{-s}, 2^{-s}, \ldots, n^{-s}, \ldots)$, for $s > 1$.)
3. Is the generalization of L by taking $(x,y,z) \in \mathbb{C}^3$ impossibly difficult?

S. H. Cullinane
Inscapes III: PG(2,4) from PG(3,2). Expository note. Feb. 4, 1986.

This note suggests a way to visualize the finite geometries recently described by A. Beutelspacher in an excellent expository article [1].

Notation -- hexadecimal characters for the 15 points of PG(3,2):

1 = 0001	4 = 0100	7 = 0111	A = 1010	D = 1101
2 = 0010	5 = 0101	8 = 1000	B = 1011	E = 1110
3 = 0011	6 = 0110	9 = 1001	C = 1100	F = 1111 .

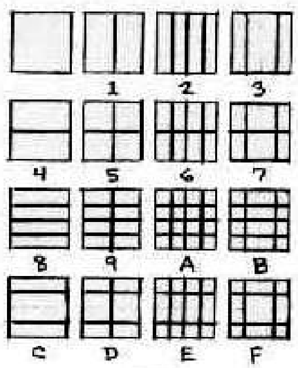
Facts about PG(3,2), the projective 3-space over GF(2):

(A) Each of the 15 points may be expressed as a sum of a unique pair of points from the set $S = \{1,2,3,4,8,C\}$.

(B) Fifteen of the 35 lines of PG(3,2) are distinguished by the fact that their points arise from partitions of S of the form 2 + 2 + 2; e.g., $S = \{1,2\} \cup \{3,4\} \cup \{8,C\}$ yields the line $\{3,4,7\} = \{1+2, 8+C, 3+4\}$. (The remaining 20 lines arise from partitions of S of the form 3 + 3, by summing pairs in the 3-sets.)

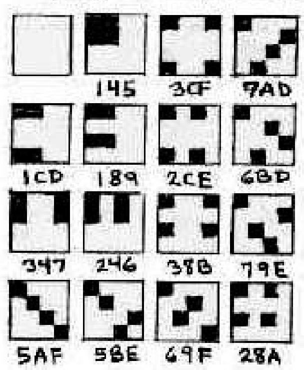
(C) Six spreads, each consisting of 5 mutually skew (i.e., disjoint) lines, can be formed from the 15 distinguished lines in (B).

These facts can be expressed graphically as follows.



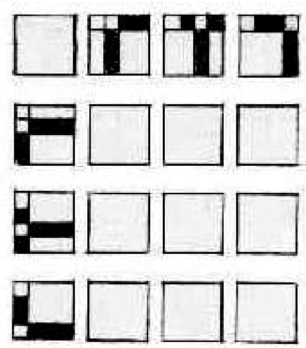
(A)

Points of PG(3,2). (Note symmetric-difference sums.)



(B)

15 distinguished lines. (Called an "inscape" because of part-whole relationship.)



(C)

The 6 spreads in (B). (Note correspondence with S in (A).)

Beutelspacher describes a construction of PG(2,4) with
21 points = the 6 points of S and the 15 distinguished lines (B), and
21 lines = the 6 spreads (C) and the 15 point-pairs (A).

REFERENCE

1. Beutelspacher, A., 21 − 6 = 15: A connection between two distinguished geometries, Am. Math. Monthly 93 (Jan. '86), 29-41.

S. H. Cullinane

The relativity problem in finite geometry. Feb. 20, 1986.

> This is the relativity problem: to fix objectively a class of equivalent coordinatizations and to ascertain the group of transformations S mediating between them.
>
> -- H. Weyl, The Classical Groups, Princeton Univ. Pr., 1946, p. 16

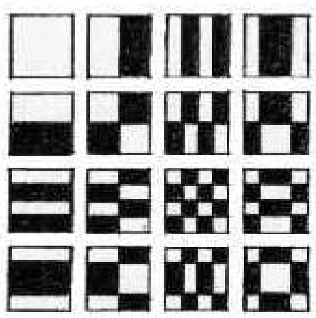
In finite geometry "points" are often defined as ordered n-tuples of elements of a finite (i.e., Galois) field GF(q). What geometric structures ("frames of reference," in Weyl's terms) are coordinatized by such n-tuples? Weyl's use of "objectively" seems to mean that such structures should have certain objective -- i.e., purely geometric -- properties invariant under each S.

This note suggests such a frame of reference for the affine 4-space over GF(2), and a class of 322,560 equivalent coordinatizations of the frame.

The frame: A 4x4 array.

The invariant structure:

The following set of 15 partitions of the frame into two 8-sets.



A representative coordinatization:

0000	0001	0010	0011
0100	0101	0110	0111
1000	1001	1010	1011
1100	1101	1110	1111

The group: The group AGL(4,2) of 322,560 regular affine transformations of the ordered 4-tuples over GF(2).

S. H. Cullinane
Group topologies. Problems. March 31, 1986.

If a group G acts on a set X, there is a natural closure operation on subsets of X: define topological closure as closure under G-actions. Then the closed sets (in both senses) are the empty set, the G-orbits, and arbitrary unions of G-orbits. ($A \subseteq X$ is open iff A is closed.) The result is a group topology T(G,X).

Unfortunately, T(G,X) is trivial if the group action is transitive. But G acts on the power set P(X) as well as on X, and we have

X is nonempty $\Rightarrow$ T(G,P(X)) is not trivial, and
the G-action is nontrivial $\Rightarrow$ T(G,P(X)) is not discrete and not T_1 (i.e., not all singletons are closed).

(That a topology is not T_1 is unfortunate if the underlying set is infinite, but very fortunate if the underlying set is finite.)

Let $P_0 = X$, $P_n = P(P_{n-1}(X))$, and let $T_n = T(G,P_n(X))$.

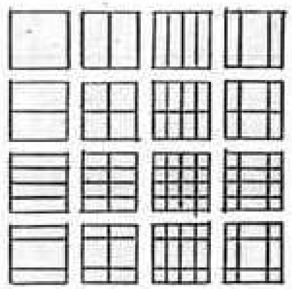
Problems:

(1) Is there a purely set-theoretic characterization of the finite T_n (i.e., among all other topologies on P_n based on partitions that refine the cardinality partition)?

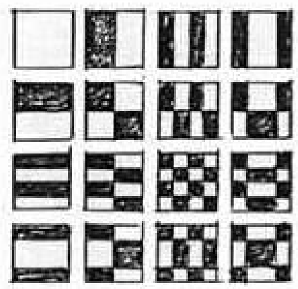
(2) Consider the topologies T_n for a faithful action F of G on X.
  (a) Is F always determined by T_0, T_1, ..., T_n for some $n = n(F)$?
  (b) If $H \triangleleft G$, how are the T_i for H related to the T_i for G?
  (c) If X is countably infinite, can we regard the minimal closed sets of T_1 as "natural" G-orbits on some continuum?

S. H. Cullinane

Picturing the smallest projective 3-space. April 26, 1986.

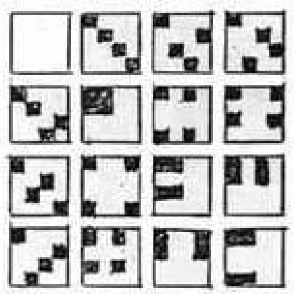


The 15 points



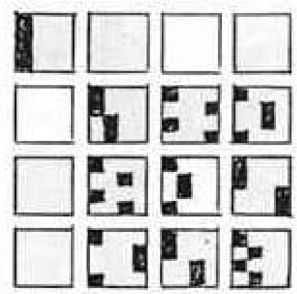
The 15 hyperplanes

The 2 figures at left show a symplectic polarity a; each point lies in its corresponding hyperplane. The 15 lines fixed under a are shown in fig. A below.



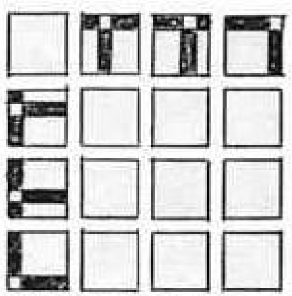
A

B

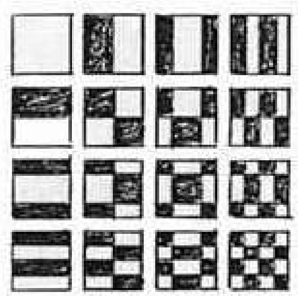


C

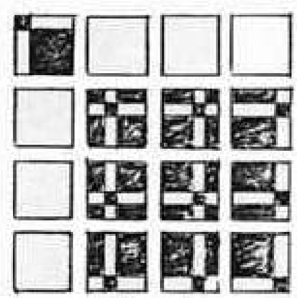
The 35 lines



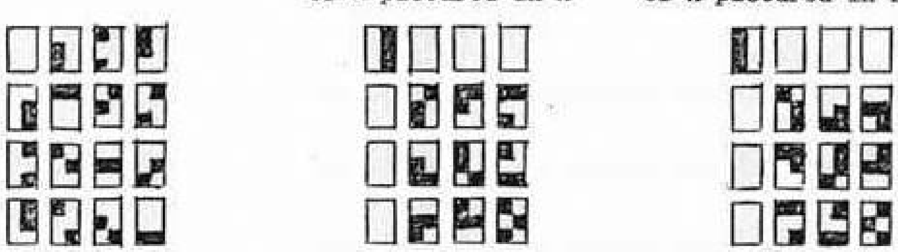
The 6 spreads in A



Sums of the 4-subsets of A pictured in A



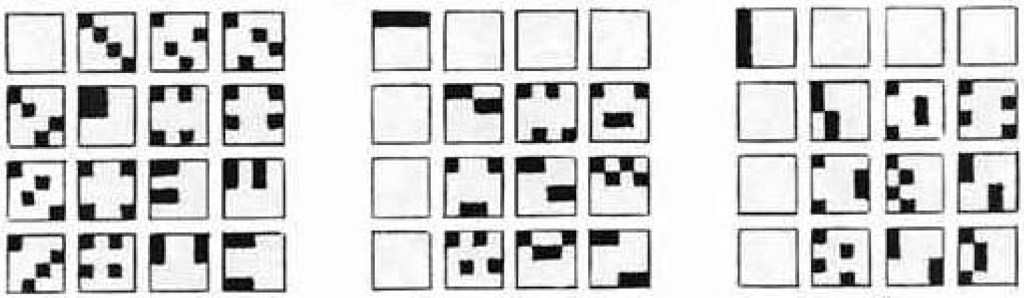
Sums of the 4-subsets of A pictured in B or C



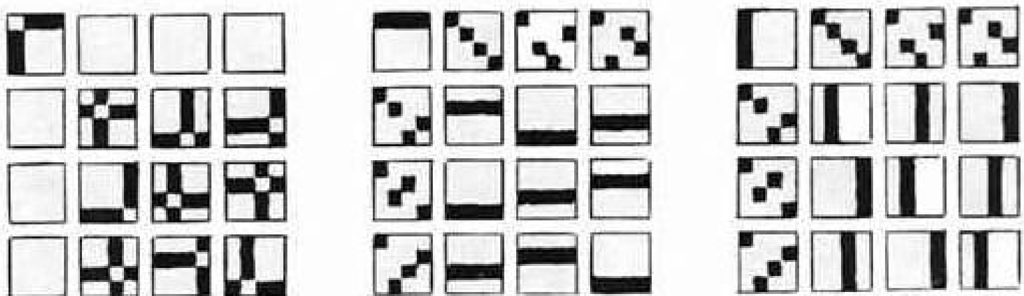
The R. T. Curtis correspondence between the 35 lines and the 2-subsets and 3-subsets of a 6-set. This underlies M_{24}.

S. H. Cullinane
A linear complex related to M_{24}. May 8, 1986.

Figures A,B,C show the 35 lines of PG(3,2); fig. A is a linear complex.

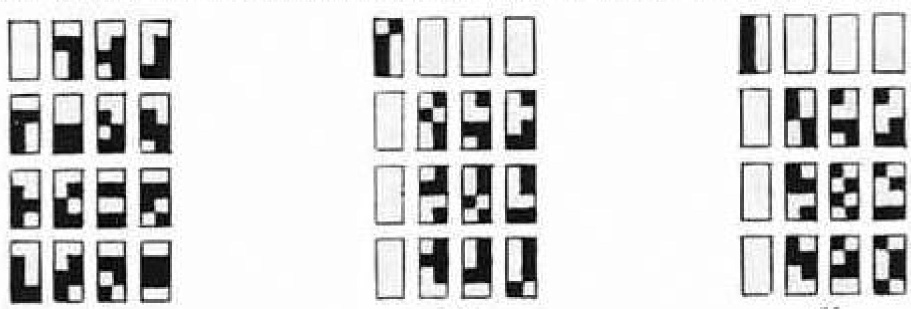


A B C

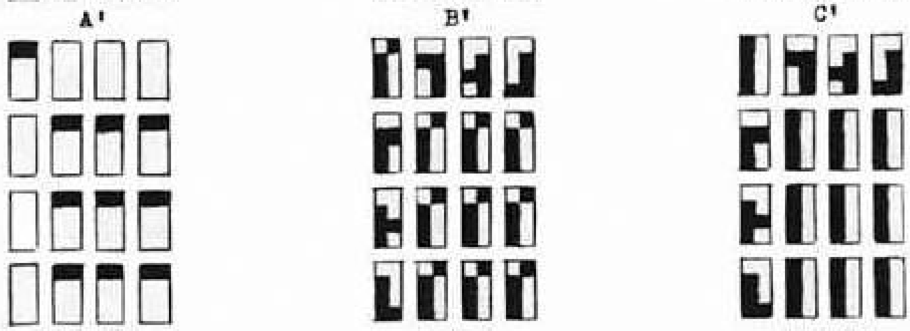


B + C A + B A + C

Figures A',B',C' show the R.T. Curtis correspondence between the 35 lines and the 35 partitions of an 8-set into two 4-sets. This underlies M_{24}.



A' B' C'



B' + C' A' + B' A' + C'

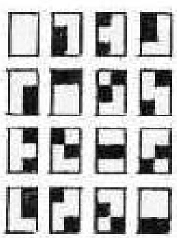
S. H. Cullinane

The 2-subsets of a 6-set are the points of a PG(3,2). May 26, 1986.

This note was suggested by
(1) A. Beutelspacher's model [1] of the 15 points of PG(3,2) as the 15 partitions of a 6-set into three 2-sets, and by
(2) R. T. Curtis's model [3] of the Conwell correspondence [2] between the 35 lines of PG(3,2) and the 35 partitions of an 8-set into two 4-sets.

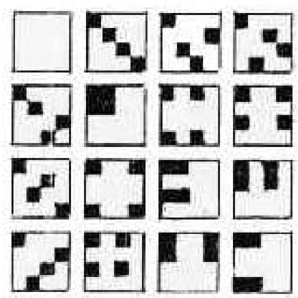
If X is a finite set, we may regard the power set P(X) as an elementary abelian 2-group in which addition is the set-theoretic symmetric-difference operation. Let K(X) be the subgroup of P(X) consisting of $\varnothing$ and X, and let Q(X) = P(X)/K(X).

When X is a 6-set, the 2-subsets form a subgroup A of Q(X) whose nonzero elements we may take as the points of a PG(3,2), with collinearity defined in the obvious way.



A

A subgroup of Q(X) illustrating
(1) the 15 2-subsets of a 6-set
(2) the 15 points of PG(3,2)



B

Subsets of A illustrating
(1) the Curtis correspondence between A-{0} and the 15 partitions of a 6-set into three 2-sets
(2) a linear complex in PG(3,2)

REFERENCES

1. Beutelspacher, A., 21 - 6 = 15: A connection between two distinguished geometries, Amer. Math. Monthly 93 (1986) 29-41.

2. Conwell, G. M., The 3-space PG(3,2) and its group, Ann. of Math. 11 (1910) 60-76 (esp. p. 72).

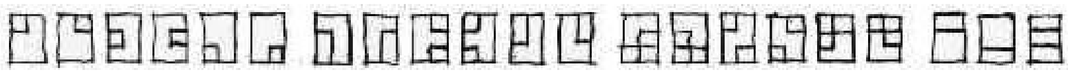
3. Curtis, R. T., A new combinatorial approach to M_{24}, Math. Proc. Camb. Phil. Soc. 79 (1976) 25-42.

S. H. Cullinane

21 projective partitions. Research note. June 6, 1986.

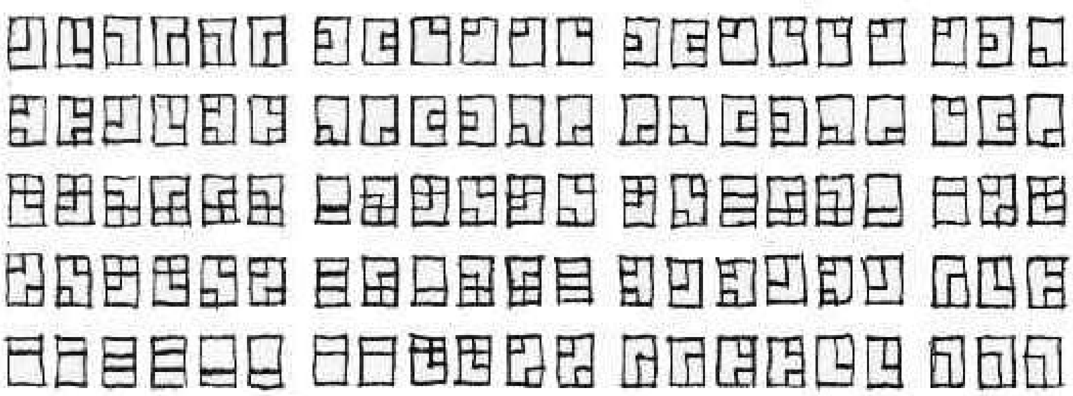
Shown below are the 21-point projective plane PG(2,4) and its dual. The points (or lines) are the 21 partitions of a 6-set into disjoint sets A, B, where |A| = 2 or 1.

Lines (or points):



Points (or lines):

Points on the above lines (or lines on the above points):



The 6-set permutation interchanging points and lines is from the Miracle Octad Generator of R. T. Curtis [1, p. 28].

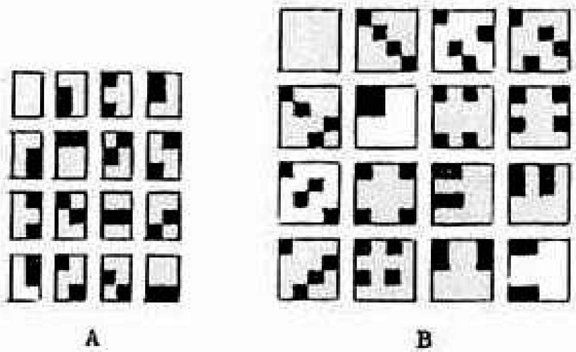
REFERENCE

1. Curtis, R. T., A new combinatorial approach to M_{24}, Math. Proc. Camb. Phil. Soc. 79 (1976), 25-42.

S. H. Cullinane

An outer automorphism of S_6 related to M_{24}. June 11, 1986.

Figure A below shows the 2-subsets of a 6-set S; figure B shows the locations in A of the triples of 2-subsets that partition S.

The map C: 

Together, A and B specify a correspondence C between the 15 subsets and the 15 partitions. This correspondence leads in a natural way to

(1) a model of the projective plane PG(2,4) in which the 21 points (and also the 21 lines) are the 21 partitions of S into subsets X, $\overline{X}$, where $|X| = 2$ or 1;

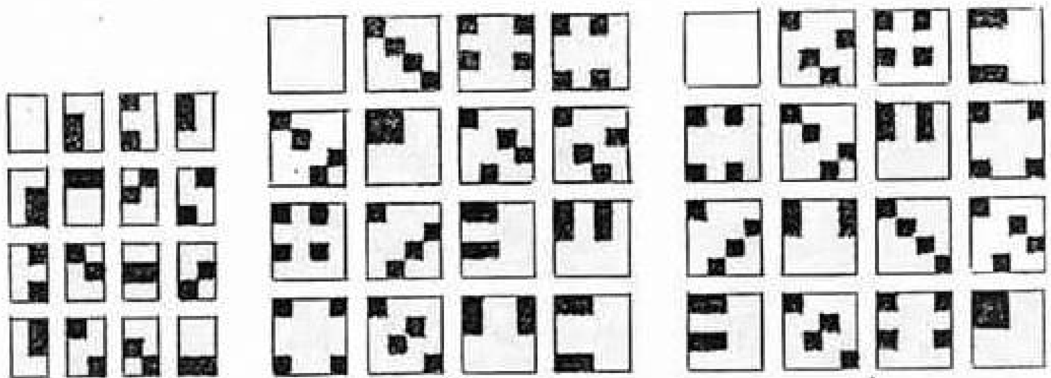
(2) the Conwell mapping of the 35 (4 + 4)-partitions of an 8-set onto the 35 lines of PG(3,2), which preserves certain intersection properties;

(3) the R. T. Curtis "MOG" model of the Steiner system S(5,8,24) and of M_{24} as the model's automorphism group.

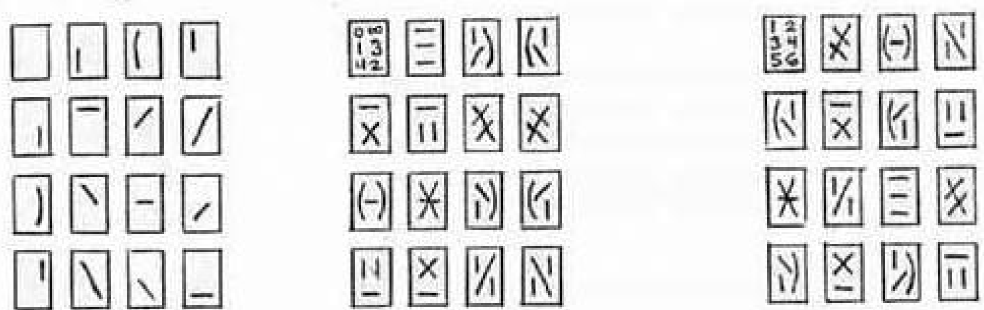
Let $f: s \to s'$ exchange rows 2 and 3 in each 3x2 picture s in A, and let C' map a 2-subset s to C(s'). If we regard the 2-sets and partitions as transpositions and products of transpositions, C' induces an outer automorphism p of S_6. (In the above PG(2,4), S_6 and $p(S_6)$ act in concert as a group of collineations.)

S. H. Cullinane

Picturing outer automorphisms of S_6. Expository note. July 3, 1986.



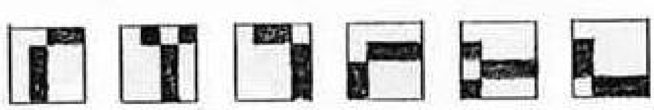
X θ φ



X θ φ

Shown above are two ways to picture some outer automorphisms of S_6 that have been discussed in the literature (θ in (1), φ in (2)). In the top row, figure X shows the 15 2-sets in a 6-set S, and θ , φ show the locations in X of triples of 2-sets that partition S. The second row shows the corresponding permutations.

Each row's θ , φ contain 6 special 5-subsets:



In the top row these 5-subsets are spreads of lines in a PG(3,2); in the second row they are parallelisms of S. Such 5-subsets (each of which can be selected in 6 ways, then arranged in 5! ways) determine the 6! outer automorphisms of S_6.

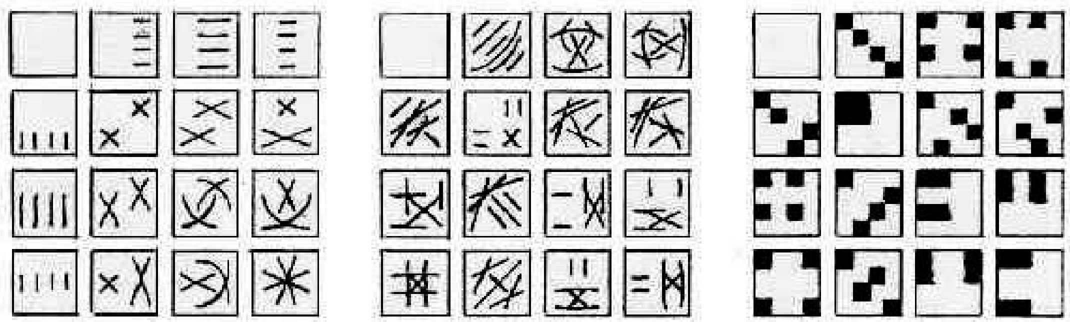
REFERENCES

(1) Conway, J. H., Three lectures on exceptional groups (section 2.3), in Finite Simple Groups, ed. Powell and Higman, Academic Pr., 1971.

(2) Janusz, G., and Rotman, J., Outer automorphisms of S_6, Amer. Math. Monthly 89 (June-July 1982), 407-410.

S. H. Cullinane
Inscapes IV: Inner and outer group actions. July 11, 1986.

This note was suggested by J. H. Conway's construction (1) of an order-2 outer automorphism of S_6.



A B X

Figures A and B above each show 16 permutations of a 16-set that generate groups G(A) and G(B), respectively. Figure X shows 16 subsets of a 16-set. The groups G(A) and G(B) can act on figure X in two ways: by an <u>inner</u> action on each of the 16 4x4 parts individually, or by an <u>outer</u> action permuting the 16 parts.

Theorem: Let a denote any permutation in A, and let b denote the permutation in the corresponding location in B. Then the inner (outer) action of a on X induces (is induced by) the outer (inner) action of b on X. The group G(A), and hence G(B), is isomorphic to S_6, and the map taking each a to its corresponding b extends to an involutive outer automorphism of S_6.

REFERENCE

(1) Conway, J. H., Three lectures on exceptional groups (section 2.3), in Finite Simple Groups, ed. M. B. Powell and G. Higman, Academic Press, 1971.