A Weyl-Type Theorem for Geometrized Newtonian Gravity[†]

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ABSTRACT

I state and prove, in the context of a space having only the metrical structure imposed by the geometrized version of Newtonian gravitational theory, a theorem analagous to that of Weyl's for a Lorentz manifold. The theorem says that a projective structure and a suitably defined compatible conformal structure jointly suffice for fixing the metrical structure of a Newtonian spacetime model up to constant factors. It allows one to give a natural, physically compelling interpretation of the spatiotemporal geometry of a geometrized Newtonian gravity spacetime manifold, in close analogy with the way Weyl's Theorem allows one to do in general relativity.

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1 Weyl's Theorem

Soon after Einstein first promulgated the theory of general relativity, Weyl (1918) formulated and proved a theorem that has since served as the foundation for one of the most influential and compelling ways to give a physical interpretation to the theory's mathematical machinery, Lorentzian geometry:[1]

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[1]Researchers as varied in their backgrounds, aims and temperaments as Eddington (1923), Møller (1952), Trautman (1965), Hawking and Ellis (1973) and Malament (2012) have used it so, to mention only a small sample. This form of interpretation can be enlighteningly contrasted with one based, *e.g.*, on chronometry, as in Einstein (2001) or Synge (1960).

Theorem 1.1 (Weyl 1918) *Given a conformal structure on a differential manifold, and a projective structure agreeing with it on (images) of its null geodesics, there is a Lorentz metric, fixed up to a constant factor, having each as its associated structure of that kind.*

In essence, to know the conformal structure is to know which curves are timelike (the possible paths of massive bodies), which are null geodesics (the possible paths of light-rays *in vacuo*), and which are spacelike (the possible paths of no physical system); to know the projective structure allows one further to say which timelike curves are (images of) geodesics, *i.e.*, the possible paths of freely falling massive bodies. To know both, then, the theorem tells us, is to know the spacetime metric up to a fixed constant, *i.e.*, up to the choosing of a system of units of measurement, thus giving us the physical significance of the metric: it is determined by the behavior of light rays and freely falling bodies.[2] In this paper, I construct the necessary machinery, a Newtonian conformal structure, for a formulation and proof of a natural analogue of this theorem in the context of geometrized Newtonian gravity. It provides in the same way the basis for a compelling physical interpretation of the spatiotemporal structures of that theory.

In §2, I characterize the analogue of a conformal structure in Newtonian spaces. I conclude in §3 with a statement and proof of the theorem, and an explanation of the way it grounds a physical interpretation of the theory. In appendix A, I sketch the machinery of geometrized Newtonian gravity required for the paper's definitions and proofs, following the treatment of Malament (2012, ch. 4, §2) (with minor emendations and simplifications).

2 Newtonian Conformal Structure

The following definition encapsulates almost the entirety of the formal structure of geometrized Newtonian gravity as a physical theory, the remainder being elaboration of and derivations from its elements.

Definition 2.1 *A* Newtonian spacetime model *is an ordered quintuplet* $(\mathcal{M}, \rho, h^{ab}, t_{ab}, \nabla_a)$ *such that:*

1. $\mathcal{M}$ *is a four-dimensional, connected, paracompact, smooth, differential manifold*
2. ρ *is a smooth, non-negative, scalar function on* $\mathcal{M}$
3. h^{ab} *is a smooth, symmetric tensor field on* $\mathcal{M}$ *of signature* $(0, 1, 1, 1)$
4. t_{ab} *is a smooth, symmetric tensor field on* $\mathcal{M}$ *of signature* $(1, 0, 0, 0)$
5. t_{ab} *and* h^{ab} *are compatible,* i.e., $t_{an}h^{nb} = 0$
6. ∇_a *is a smooth derivative operator on* $\mathcal{M}$*, compatible with* t_{ab} *and* h^{ab} *in the sense that* $\nabla_a t_{bc} = 0$ *and* $\nabla_a h^{bc} = 0$

[2]See, *e.g.*, Ehlers, Pirani, and Schild (1972) for a thorough exposition of the mathematics behind the theorem and its intended physical significance, and Malament (2012, ch. 2, §1, pp. 120–121) for a lapidary account of such a physical interpretation, including a discussion of its virtues and possible problems.

$\mathcal{M}$ represents spacetime, the "totality of all point-events", and ∇_a the physically relevant affine structure, *i.e.*, the one whose geodesics represent unaccelerated paths in the spacetime. ρ represents the mass-density distribution of matter. h^{ab} and t_{ab} represent, respectively, the closest we come to having spatial and temporal metric structures on $\mathcal{M}$. (See appendix A for an explanation of the sense in which these tensors represent such structure.) We refer to the ordered pair (h^{ab}, t_{ab}) as a *Newtonian metrical structure*.

From hereon, we assume all Newtonian spacetime models to be spatially flat (*i.e.*, R^{abcd}, the spatialized Riemann tensor, vanishes; see appendix A for an explanation of the significance of this condition). It follows from the results and discussion of Malament (1986) that this represents no real loss of generality, as it is those models that best capture the idea of the possible spaces of classical Newtonian gravitational theory. We also assume in what follows that $\mathcal{M}$ is simply connected and the spacetime model is temporally orientable (*i.e.*, that there exists a globally defined temporal function t such that $t_{ab} = \nabla_a t \nabla_b t$). Again, this is no real loss of generality, for all arguments and conclusions would still go through without the assumption, at the cost of constant hedging about which results are local and which global, and hence much technical work of a nit-picky sort without any counter-balancing gain in physical insight.

From hereon, we will need to keep track of the difference between a curve considered, on the one hand, as a smooth, injective mapping from a real interval to $\mathcal{M}$, and, on the other, as the point-set image in $\mathcal{M}$ of such a mapping. I will use 'curve' when I mean the former, and 'image of a curve' when I mean the latter.

Two affinities ∇_a and $\tilde{\nabla}_a$ are projectively equivalent if they agree on images of geodesics, which is to say, if they agree on geodesics up to arbitrary (smooth, monotonic) reparametrization: $\xi^n \nabla_n \xi^a = 0$ if and only if $\xi^n \tilde{\nabla}_n \xi^a = \lambda \xi^a$ for λ a smooth function on the image ξ. A projective structure, then, is a maximal collection of affine structures on a Newtonian spacetime all of which agree on images of all geodesics ("maximal" in the sense that we throw in every affine structure that meets the criterion). One can equally well define the projective structure as the complete family of images of geodesics on which the affine structures agree. A member of such a family of images of geodesics is a projective geodesic.

Now, to characterize the analogue of conformal structure in geometrized Newtonian gravity. In general relativity, one can define a conformal structure to be an assignment of a smoothly varying field of quadratic cones at every point of the spacetime manifold, the null-cones. Two Lorentz metrics g_{ab} and $\tilde{g}_{ab}$ are conformally equivalent if they have the same null-cone structure. This holds if and only if the two metrics agree on their null geodesics, which holds if and only if $g_{ab} = \Omega^2 \tilde{g}_{ab}$ for some smooth, non-zero scalar field Ω. Because we have no such metric structure in geometrized Newtonian gravity, and correlatively no cone-structure (except the degenerate one, which will not do) or non-trivially null type of vector, one cannot apply such a characterization of conformal structure here.

The following observation provides the required clue for moving forward: two Lorentz metrics are conformally equivalent if and only if they agree on orthogonality relations for all pairs of vectors. Equivalently, g_{ab} and $\tilde{g}_{ab}$ agree in assignments of ratios of lengths to any pair of (non-null) tangent vectors. The conformal structure so characterized then allows one to distinguish among timelike, null and spacelike vectors, to distinguish null geodesics, and to reconstruct the null-cone structure. Null-vectors are those non-zero vectors orthogonal to themselves, which picks out the null-cones;

and a null geodesic is one contained in a null-cone (in an appropriate sense—see Ehlers, Pirani, and Schild 1972). Timelike vectors are those pointing into the interior of the null cones, and spacelike vectors are all the rest.

This suggests that we attempt to characterize conformal structure in the context of geometrized Newtonian gravity by making use of orthogonality among (appropriate) pairs of vectors.

Definition 2.2 *A* Newtonian Conformal Structure $\mathfrak{C}$ *on a four-dimensional manifold* $\mathcal{M}$ *consists of:*

1. *at each point* $p \in \mathcal{M}$*, a three-dimensional vector subspace* $\mathfrak{C}_p$ *of* $T^*_p M$ *(the tensor space of 1-forms over* p*), smoothly varying from point to point*
2. *orthogonality relations fixed for all pairs of elements of* $\mathfrak{C}_p$

This structure suffices for defining a family of symmetric tensor fields characterized by a given h^{ab} with signature (0, 1, 1, 1), fixed up to multiplication by a positive scalar field Ω^2: $\alpha_a, \beta_a \in \mathfrak{C}_p$ are orthogonal if and only if $h^{mn}\alpha_m\beta_n = 0$. We can now distinguish timelike from spacelike vectors: spacelike vectors are those that result from raising an index of a 1-form in a $\mathfrak{C}_p$; timelike are those that cannot be so derived. If $\xi^a = h^{an}\alpha_n$, we say α_a is a spacelike representative 1-form of x^a. We can now also determine a second family of symmetric tensor fields, characterized by a given t_{ab} with signature (1, 0, 0, 0), fixed up to multiplication by a positive scalar field χ^2. A vector ξ^a is timelike if and only if $t_{mn}\xi^m\xi^n > 0$. Clearly, any two such representative tensor fields h^{ab} and t_{ab} are compatible, in the sense that $t_{an}h^{nb} = 0$. We say two symmetric tensor fields h^{ab} and $\tilde{h}^{ab}$ of the appropriate signature are *conformally equivalent* if they live in the same family of tensor fields of a given Newtonian conformal structure, and similarly for t_{ab} and $\tilde{t}_{ab}$.

Because, given two spacelike vectors, there is always a third that, added to the first, makes it orthogonal to the second, the trigonometric functions allow one to define angles among them. One can similarly define hyperbolic angles among pairs of timelike vectors by taking the difference between them, fixing a spacelike vector, and using that as a "unit" to treat the difference vector as, in essence, a velocity difference. One can then use the hyperbolic trigonometric to define the angles. (This is essentially the same procedure one uses in general relativity to define hyperbolic angles between timelike vectors; and, in the same way, this structure does not allow one to define orthogonality relations among timelike vectors.) From this, one immediately derives ratios of lengths between all pairs of spacelike vectors, and between all pairs of timelike vectors, which is exactly the information encoded in the two families of conformally related tensor fields.

A conformal derivative operator, then, is the family of derivative operators each of which is compatible with a pair of compatible spatial and temporal metrics in the conformal families, in the sense that each triplet of representatives satisfy $\nabla_a h^{bc} = 0$ and $\nabla_a t_{bc} = 0$. A *conformal representative* of the conformal structure is a triplet (h^{ab}, t_{ab}, ∇_a) all compatible with each other. By construction, we are always guaranteed that at least one conformal representative has a spatially flat derivative operator.[3]

[3]In general, the derivative operator of a given conformal representative will not be spatially flat, and so cannot represent the affine structure of a Newtonian spacetime model according to the constraints of this paper, but that is not a problem. The conformal structure I sketch here is strictly that of the spatial geometry of a candidate Newtonian spacetime model. A conformal structure in this sense need not embody all the geometry of a complete

It will give some insight into the character of a Newtonian conformal structure, as well as being useful in the proof of theorem 3.1, to show that being spatially twist-free for a vector field ($\nabla^{[a}\xi^{b]} = 0$) is a conformally invariant notion, *i.e.*, a conformal derivative operator allows one to determine whether a given vector field is twist-free or not. This makes intuitive sense, as being twist-free essentially means that "nearby" vectors in the vector field have no "angular velocity" with respect to each other, but angular velocity is a conformal notion. More precisely, fix two conformal representatives (h^{ab}, t_{ab}, ∇_a) and ($\tilde{h}^{ab}$, $\tilde{t}_{ab}$, $\tilde{\nabla}_a$), with $\tilde{h}^{ab} = \Omega^2 h^{ab}$, and the difference vector $C^a{}_{bc}$ between the two derivative operators. First,

$$\begin{aligned} 0 &= \tilde{\nabla}_a \tilde{h}^{bc} \\ &= \tilde{\nabla}_a (\Omega^2 h^{bc}) \end{aligned}$$

so

$$\Omega^2 \tilde{\nabla}_a h^{bc} + h^{bc} \tilde{\nabla}_a \Omega^2 = 0$$

Using $C^a{}_{bc}$ to re-express this, and noting that ∇_a and $\tilde{\nabla}_a$ agree in their action on Ω^2, we get

$$\begin{aligned} 0 &= \Omega^2 \nabla_a h^{bc} + \Omega^2 C^b{}_{an} h^{nc} + \Omega^2 C^c{}_{an} h^{bn} + h^{bc} \nabla_a \Omega^2 \\ &= 2\Omega^2 h^{cn} C^b{}_{an} + h^{bc} \nabla_a \Omega^2 \end{aligned} \tag{2.1}$$

Thus $2\Omega^2 h^{an} C^b{}_{nc} = -h^{ab} \nabla_c \Omega^2$ and so $h^{n[a} C^{b]}{}_{nc} = 0$. Thus

$$\tilde{\nabla}^{[a}\xi^{b]} = 0 \quad \text{if and only if} \quad \nabla^{[a}\xi^{b]} = 0$$

The analagous calculation now shows that $\tilde{\nabla}_a$ is compatible with $\tilde{t}_{ab}$ if and only if $\tilde{t}_{ab} = \Omega^2 t_{ab}$, and $2\Omega^2 t_{an} C_b{}^{nc} = t_{ab} \nabla^c \Omega^2$.

Now, for a Newtonian conformal structure to be a physically meaningful analogue of conformal structure in general relativity, it should allow one to distinguish a preferred family of images of curves, the conformal spacelike geodesics, *i.e.*, images of curves that can be reparametrized so as to be geodesics for each representative derivative operator of the conformal derivative operator associated with the conformal structure.[4] And it does. They are the spacelike curves that, in an appropriate sense, preserve orthogonality relations. Fix two spacelike vector-fields ξ^a and η^a everywhere orthogonal to each other. One can always do this. Pick a flat representative of the conformal derivative operator, ∇_a; fix ξ^a and η^a orthogonal at a point p; then construct a curve (unique up to parametrization) that has ξ^a as its tangent vector at p, and that parallel-transports η^a along it with respect to ∇_a. Then do the same with the roles of ξ^a and η^a reversed. Do the same along every point of the constructed curves. Pick a vector θ^a orthogonal to both ξ^a and η^a at p, and parallel-transport the constructed curves along θ^a. And so on. Because we are working with

Newtonian spacetime model. In particular, although spatial flatness of a complete Newtonian spacetime model may fix the spatial metric up to a constant, that does not imply that spatial conformal factors for a Newtonian conformal structure can be only constants. *Any* $\mathbb{S}^2$-preserving transformation of R^3 will preserve orthogonality, and there are far more of those than just multiplication by a constant—the Möbius transformations, *e.g.*

[4] Although a conformal structure in general relativity fixes the null geodesics as curves, not just images of curves, we cannot expect that here: that happens in the null case only because all vectors tangent to all curves formed by parametrizing the images have the same length (*viz.*, 0), and so the images remain true geodesics under arbitrary parametrizations. That will not be the case for the spacelike curves here, which will be true geodesics (not just re-parametrizable so as to be geodesics) only under a preferred family of parametrizations.

a spatially flat derivative operator, we are guaranteed that parallel-transport of all these vectors is path-independent, so the construction is consistent.

The integral curves of ξ^a and η^a are conformal spacelike geodesics. Now, pick again a representative ∇_a of the conformal derivative operator (not necessarily spatially flat), with its associated h^{ab}. We will use h^{ab} to effectively lower indices in the following calculations, by using it to arbitrarily fix spacelike representative 1-forms of ξ^a and η^a respectively. (It is a simple calculation to show that the following argument is independent of the initial choice of spacelike representative 1-forms.) Now, it is clear that the two vector-fields Lie-derive each other. Thus $\xi^n \nabla_n \eta^a = \eta^n \nabla_n \xi^a$, and, by orthogonality, $\nabla_a \xi^n \eta_n = 0$. Now the result follows by playing these off each other in the standard way.

$$\begin{aligned} \eta_m \xi^n \nabla_n \xi^m &= \xi^m \xi^n \nabla_n \eta_m \\ &= \xi^n \nabla_n \xi^m \eta_m - \eta_m \xi^n \nabla_n \xi^m \end{aligned}$$

The first term in the last line is zero by orthogonality, and so we are left with $\eta_m \xi^n \nabla_n \xi^m = -\eta_m \xi^n \nabla_n \xi^m$, *i.e.*, $\eta_m \xi^n \nabla_n \xi^m = 0$. Since this is true for all vectors orthogonal to ξ^a, we conclude that $\xi^n \nabla_n \xi^a = \lambda \xi^a$, for some non-negative λ. Thus, the integral curves of ξ^a are spacelike geodesics up to reparametrization.

3 The Theorem

A projective structure and a Newtonian conformal structure are compatible with each other if the conformal spacelike geodesics determined by the Newtonian conformal structure are also projective geodesics. We can now state the main result of the paper.

Theorem 3.1 *Two conformal representatives of the same conformal structure have projectively equivalent derivative operators if and only if the two temporal and spatial metrics differ, respectively, only by a constant factor.*

PROOF:
The "if" part of the theorem is immediate. Assume, then, that we have two pairs of conformal representatives , $(h^{ab}, t_{ab}, \nabla_a)$ and $(\tilde{h}^{ab}, \tilde{t}_{ab}, \tilde{\nabla}_a)$, where $\tilde{h}^{ab} = \Omega^2 h^{ab}$ and $\tilde{t}_{ab} = \Omega^2 t_{ab}$. Because their respective compatible derivative operators, ∇_a and $\tilde{\nabla}_a$, agree on images of geodesics, their difference tensor $C^a{}_{bc}$ is of the form $\delta^a{}_{(b}\phi_{c)}$ for some 1-form ϕ_a. In particular, because they are conformally related, from equation (2.1) we know that $C^a{}_b{}^c = -\frac{1}{2} h^{ac} \nabla_b \ln \Omega^2$ and $C_a{}^b{}_c = \frac{1}{2} t_{ac} \nabla^b \ln \Omega^2$. The proof now proceeds as in the Lorentzian case (Malament 2012, ch. 1, §9, p. 83), playing these expressions off each other, until one derives $\nabla_a \Omega^2 = 0$.

First, contract the three equations, yielding

$$\begin{aligned} C^n{}_{na} &= C_a = \delta^n{}_{(n}\phi_{a)} = (n+1)\phi_a \\ C^n{}_n{}^a &= C^a = -\frac{1}{2}\nabla^a \ln \Omega^2 \\ C_n{}^n{}_a &= C_a = \frac{1}{2}\nabla_a \ln \Omega^2 \end{aligned}$$

so

$$(n+1)\phi^a = -\frac{1}{2}\nabla^a \ln \Omega^2$$
$$(n+1)\phi_a = \frac{1}{2}\nabla_a \ln \Omega^2$$

It follows that $\phi^a = \phi_a = 0$, and so $\nabla_a \Omega^2 = 0$.

■

One can now use the theorem to render a physical interpretation to the Newtonian metric structure of a Newtonian spacetime model, in the following way.[5] We will use the following interpretive principles.

C1 timelike curves represent the possible paths of massive bodies

C2 segments of images of spacelike geodesics represent the spatial position and extent of (rigid) yard-sticks

P1 images of timelike geodesics represent the paths of freely falling massive bodies

Now, if we can determine when two coincident rigid rods are orthogonal at an instant of time, then, by **C2**, we can determine the Newtonian conformal structure. This can be operationalized in any of a number of ways, such as by the use of a compass. The projective structure is then fixed by determining the images of paths of freely falling bodies (**P1**). Thus, by the theorem, to know when rigid rods are orthogonal to each other at a given moment of time and to know the images of the paths of freely falling bodies is to fix the entirety of the metrical and affine structure of a Newtonian spacetime model. (A Newtonian metrical structure does not by itself fix a unique affine structure, in contradistinction to the situation in general relativity.)

A Appendix: Geometrized Newtonian Gravity

A Newtonian spacetime model is as defined in definition 2.1.

Definition A.1 *The* temporal length *of a vector ξ^a is $(\xi^m \xi^n t_{mn})^{\frac{1}{2}}$. ξ^a is* timelike *if its temporal length is non-zero,* spacelike *if the vector is non-zero and has a temporal length of zero.*

It is natural to think of the zero-vector as both timelike and spacelike, in so far as we know what it means for two events to occur at the same time as well as at the same place. The addition of any spacelike vector to a timelike one is always timelike. It follows that the family of timelike vectors forms a four-dimensional affine space modeled on the tangent space. (It cannot form a vector-space, as the sum of two timelike vectors may be spacelike.) It also follows that the family of spacelike vectors at a point forms a three-dimensional vector subspace of the tangent space at that point.

The signature of t_{ab} implies that at every point there exists a neighborhood and smooth 1-form t_a defined on that neighborhood such that $t_{ab} = t_a t_b$ on that neighborhood. If there is a globally defined such 1-form, then the spacetime model is *temporally orientable* and t_a is a

[5] I do not claim this is the unique or even just a canonical way of doing so, or the most elegant or concise or what have you, only that it is a perspicuous and natural way, with clear physical significance.

temporal orientation on it; a timelike vector ξ^a is future-directed if $t_n\xi^n > 0$. Item 6 in definition 2.1 implies that t_a is closed, and so, at least locally, there exists a smooth function t such that $\nabla_a t = t_a$, the (local) *time-function*. If $\mathcal{M}$ is simply connected and the spacetime model temporally orientable, t is defined globally. In this case, its constant surfaces represent the Newtonian idea of "all of space at a given moment of time", *i.e.*, a maximal collection of events all (absolutely) simultaneous with each other, a *simultaneity slice*, and so $\mathcal{M}$ has the topology $\mathbb{R} \times \Sigma$, where Σ is a three-dimensional manifold diffeomorphic to a simultaneity slice. This t is unique up to the addition of a constant, which may be thought of as a change of temporal origin. Its scale, however, is fixed by the temporal length of vectors: to multiply t_{ab} by a constant factor changes nothing physically, but rather represents only a change in units of temporal measurement.

t_{ab}, then, determines an absolute temporal structure—the temporal separation of any two events in the spacetime is fixed once and for all, independent of any other state of affairs in the world; the equivalence classes of spacetime points under the relation "having a temporal separation of zero with" are exactly the simultaneity slices. A *timelike curve* is one whose tangent vectors are everywhere timelike. Such curves represent the possible worldlines of ponderable bodies. The signature of t_{ab} has another consequence of note: it does not allow one to define orthogonality between two timelike vectors. This makes physical sense: two orthogonal, timelike vector-fields would define time-functions different in the sense that they would not share surfaces of constancy, *i.e.*, they would define incommensurable temporal structures.[6]

h^{ab} defines the spatial metric structure in a more indirect way.

Proposition A.2

1. *A vector ξ^a is spacelike at a point p if and only if there is a covector α_a at p such that $\xi^a = h^{an}\alpha_n$.*

2. *For all covectors α_a and β_a at a point, if $h^{an}\alpha_n = h^{an}\beta_n$, then $h^{mn}\alpha_m\alpha_n = h^{mn}\beta_m\beta_n$.*

In virtue of this proposition, the following is well formulated.

Definition A.3 *The* spatial length *of a spacelike vector ξ^a is $(h^{nm}\alpha_n\alpha_m)^{\frac{1}{2}}$, where α_a is any covector satisfying $h^{an}\alpha_n = \xi^a$.*

If a spacelike η^a equals $h^{an}\alpha_n$, then we say α_a is a *representative spacelike 1-form* of η^a. In so far as h^{ab} determines lengths only for spacelike vectors, but not at all for timelike vectors (whereas t_{ab} determines lengths for all spacelike vectors: 0), it defines a spatial metric in only a Pickwickian sense, as it ought to according to Galileian relativity. We know what it means in Newtonian theory to assign a definite distance between two simultaneous events, by employing yard-sticks and the like, independent of any other state of affairs in the world. We do not know how to do so for non-simultaneous events.[7]

[6] One can see the physical sense this makes in another, more indirect way: in so far as one can consider all the spacelike vectors at a point in Newtonian spacetime to be the result, in the limit, of "flattening" the null-cones in a relativistic spacetime—"letting the upper bound of possible velocities go to infinity"—the timelike Newtonian vectors at that point encode essentially the same information as the timelike vectors in the interior of the original null-cones, and no two timelike vectors can be orthogonal to each other with respect to a fixed Lorentz metric. See Malament (1986) for a precise characterization and analysis of the process of flattening the null-cones in a relativistic spacetime.

[7] See Stein (1967) for a thorough discussion.

The following construction captures the content of this observation. Specify at a point p a constant timelike vector ξ^a of unit temporal length, and define $\hat{h}_{ab}$ to be the unique symmetric tensor at p satisfying

$$\hat{h}_{an}h^{nb} = \delta_a{}^b - t_{an}\xi^b\xi^n$$
$$\hat{h}_{an}\xi^n = 0$$

$\hat{h}_{ab}$ represents the covariant spatial metric determined by projection relative to ξ^a. Then the spatial length of any vector assigned indirectly by h^{ab} will agree with that assigned directly by $\hat{h}_{ab}$. The spatial separation of two non-simultaneous events, however, as determined by such a $\hat{h}_{ab}$, depends on the choice of ξ^a. Again, this is as it should be. If we knew how to assign a spatial length to timelike vectors, and so a fixed spatial separation between non-simultaneous events, then we could define a notion of absolute rest: "a particle is at absolute rest if the timelike curve representing its worldline has everywhere tangent vectors of zero spatial length". This, however, we cannot do.

In contradistinction to t_{ab}, h^{ab} allows one to define a relation of orthogonality among spacelike vectors: two spacelike vectors η^a and θ^a are orthogonal if $h^{mn}\alpha_m\beta_n = 0$, where α_m and β_n are any two representative spacelike 1-forms of η^a and θ^a respectively. This makes physical sense as well: we know in Newtonian spacetime how to determine whether two yardsticks at a single moment of time are at a right angle to each other. h^{ab} also allows us to define a relation of orthogonality between a spacelike and a timelike vector: given a a timelike vector ξ^a, a spacelike vector η^a, and any representative 1-form α_a for η^a, then $\xi^n\alpha_n = 0$ if and only if $\xi^n\beta_n = 0$ for any other representative spacelike 1-form β_a. Though this may sound a little odd at first, it makes physical sense, too. A simple calculation shows that any timelike curve parametrized by arc-length ("proper time") will satisfy $\xi^n\nabla_n\xi^a = \eta^a$, where ξ^a is the tangent to the curve and η^a is spacelike and orthogonal to ξ^a. This says that for any ponderable body, the worldine of which by definition instantiates a timelike curve, its acceleration is everywhere spacelike, and thus a viable candidate for entering Newton's Second Law on the righthand side. If this were not the case, then one could have ponderable bodies accelerating in a timelike direction, with attendant timelike forces, and I see no possible way to make physical sense of such a conceit, except perhaps in a 1950's B-movie "Great Scot! He's beginning to mutate!"-sort of way. Thus, we can assign determinate magnitudes, fixed once and for all, to the acceleration of bodies, as Galileian relativity allows us to do, and Newtonian mechanics demands we do (Stein 1967). In sum, h^{ab} encodes the right amount of structure of the right kind, nothing more, nothing less. (One may think of this as physics practiced in accord with the Goldilocks Principle.)

In what follows, we assume that $\mathcal{M}$ is simply connected and the spacetime model is temporally orientable; this is no real loss of generality, for all arguments and conclusions would still go through without the assumption, at the cost of constant hedging and much technical work of a nit-picky sort without any counter-balancing gain in physical insight. In particular, we do not need to worry about which results hold only locally and which globally.

Before moving on to the definition of the appropriate analogue of a conformal structure for a Newtonian metric structure, we rehearse a few of its properties and state the most important two theorems in geometrized Newtonian gravity, which will be of use later. When there is no chance for ambiguity, we will use raised indices to represent the action of h^{ab} on a tensorial object, *e.g.*, $\nabla^a := h^{an}\nabla_n$.

Theorem A.4 (The Geometrization Lemma) *Fix a Newtonian spacetime model* $(\mathcal{M},\, \rho,\, h^{ab},\, t_{ab}, \nabla_a)$*, such that* ∇_a *is flat,* i.e., *its associated Riemann tensor* $R^a{}_{bcd} = 0$*, and a smooth scalar field* ϕ *such that* $\nabla^n \nabla_n \phi = 4\pi\rho$ *(the Poisson equation). Define another derivative operator* $\tilde{\nabla}_a = (\nabla_a,\, C^a{}_{bc})$*, where* $C^a{}_{bc} = -t_{cb} \nabla^a \phi$*. Then:*

1. $(\mathcal{M},\, \rho,\, h^{ab},\, t_{ab}, \tilde{\nabla}_a)$ *is a Newtonian spacetime model*
2. $\tilde{\nabla}_a$ *is the unique derivative operator such that*

$$\xi^n \tilde{\nabla}_n \xi^a = 0 \quad \textit{if and only if} \quad \xi^n \nabla_n \xi^a = -\nabla^a \phi$$

3. *the Riemann tensor* $\tilde{R}^a{}_{bcd}$ *associated with* $\tilde{\nabla}_a$ *satisfies*
   a. $\tilde{R}_{ab} = 4\pi\rho t_{ab}$
   b. $\tilde{R}^a{}_b{}^c{}_d = \tilde{R}^c{}_d{}^a{}_b$
   c. $\tilde{R}^{ab}{}_{cd} = 0$

A few remarks are in order to explain the physical significance of the theorem. In essence it says that, given a Newtonian spacetime model with a flat affine structure, we can always construct one in which the geodesics (possible paths of "freely falling bodies") of a curved affine structure are the same paths as those representing bodies moving under the force of the ambient gravitational field associated with ρ in the original model. The constructed model "geometrizes" gravity: it incorporates the effects of gravity into the metric structure of the new model, in analogy with general relativity.

The effective converse of the Geometrization Lemma holds as well.

Theorem A.5 (Trautman Recovery Theorem) *Fix a Newtonian spacetime model* $(\mathcal{M},\, \rho,\, h^{ab},\, t_{ab}, \nabla_a)$ *satisfying*

1. $R_{ab} = 4\pi\rho t_{ab}$
2. $R^a{}_b{}^c{}_d = R^c{}_d{}^a{}_b$
3. $R^{ab}{}_{cd} = 0$

Then there is a derivative operator $\tilde{\nabla}^a$ *and a scalar field* $\tilde{\phi}$ *such that*

1. $\tilde{\nabla}^n \tilde{\nabla}_n \tilde{\phi} = 4\pi\rho$ *(the Poisson equation)*
2. $\tilde{\nabla}^a$ *is compatible with* t_{ab} *and* h^{ab}
3. $\tilde{\nabla}^a$ *if flat*
4. *for all timelike curves with tangent vector* ξ^a

$$\xi^n \nabla_n \xi^a = 0 \quad \textit{if and only if} \quad \xi^n \tilde{\nabla}_n \xi^a = -\tilde{\nabla}^a \tilde{\phi}$$

Moreover, $(\tilde{\nabla}_a,\, \tilde{\phi})$ *is not unique. Any other such pair* $(\hat{\nabla}_a,\, \hat{\phi})$ *will satisfy the stated conditions if and only if*

a. $\nabla^a \nabla^b (\tilde{\phi} - \hat{\phi}) = 0$

b. $\hat{\nabla}_a = (\tilde{\nabla}_a, C^a{}_{bc})$, *where* $C^a{}_{bc} = t_{bc}\nabla^a(\tilde{\phi} - \hat{\phi})$

Given a Newtonian spacetime model in which the gravitational effects of ρ are geometrized (incorporated into the curved affine structure), the theorem tells us that we can recover one in which the gravitational effects of ρ are rather represented by the acceleration of the paths of freely-falling bodies, as defined by a flat affine structure.

In disanalogy with general relativity, and crucially for our purposes, in this context $R^{ab}{}_{cd} = 0$ is *not* equivalent to $R^a{}_{bcd} = 0$. The interpretation of the condition $R^{ab}{}_{cd} = 0$ and its relative strength as compared to $R^a{}_{bcd} = 0$ follow from a proposition whose proof is straightforward, albeit tedious (Malament 2012, §§4.2–4.3). To state it, we need a few more definitions. A Newtonian spacetime model is *spatially flat* if $R^{abcd} = 0$, *i.e.*, if the affine structure on the simultaneity slices derived by restricting to them the action of the global affine structure is flat. A unit, future-directed timelike vector-field ξ^a is *rigid* if $\pounds_\xi h^{ab} = 0$, *i.e.*, if $\nabla^{(a}\xi^{b)} = 0$. (One may think of a rigid vector field as the analogue of a Killing field in general relativity.) ξ^a is *twist-free* if $\nabla^{[a}\xi^{b]} = 0$, and is *acceleration-free* if $\nabla_a\xi^b$.

Proposition A.6 *Fix a spatially flat Newtonian spacetime model. Then:*

1. $R^{ab}{}_{cd} = 0$ *if and only if there exists a rigid and twist-free vector field*
2. $R^{abcd} = 0$ *if and only if there exists a rigid, twist-free and acceleration-free vector field*

Another illuminating characterization of $R^{ab}{}_{cd} = 0$ follows from this proposition: the condition holds if and only if parallel-transport of spacelike vectors is path-independent.

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