A Methodology for Teaching Logic-Based Skills to Mathematics Students

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Abstract: Mathematics textbooks teach logical reasoning by example, a practice started by Euclid; while logic textbooks treat logic as a subject in its own right without practical application to mathematics. Stuck in the middle are students seeking mathematical proficiency and educators seeking to provide it. To assist them, the article explains in practical detail how to teach logic-based skills such as: making mathematical reasoning fully explicit; moving from step to step in a mathematical proof in logically correct ways; and checking to make sure inferences are logically correct. The methodology can easily be extended beyond the four examples analyzed.

Keywords: Inference chain, justifying and checking inferences, matching logical form, mathematical proof, syntactic and semantic validity.

Introduction

Logical reasoning is an absolute requirement of mathematical proficiency and has been since ancient times. The most famous textbook in the history of mathematics, the *Elements* of Euclid,[1] showed by example after example that mathematical propositions[2,3] are to be justified by a non-empirical method: logical argumentation.[4] Thus, measuring the interior angles of a triangle is not how mathematics justifies the proposition that those angles add up to 180

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[1] According to scholars, Pythagoras was probably the source for most of Books I and II of the *Elements*; Hippocrates of Chios for Book III; and Eudoxus of Cnidus for Book V, while books IV, VI, XI, and XII probably came from other ancient Greek mathematicians. See Ball 1960 [1908], 44. See also Kneale and Kneale (1962) and Gabbay and Woods (2004).

[2] The author is aware of philosophical concerns over the ontology of logic, such as whether arguments are composed of sentences (type or token) as opposed to propositions, which are abstract entities expressed by sentences in a language. The sentence ontology is easier for students to understand and will be used here.

[3] In modern terms, mathematical propositions are analytic, a priori, and necessary; while the propositions of science are synthetic, a posteriori and contingent. Though it is beyond the purview of this article to address philosophical controversies stemming from this distinction, they are mentioned here because mathematics students should be encouraged to ask questions that are not, strictly speaking, mathematical.

[4] Logical argumentation in mathematics predated Euclid by several centuries. Thales of Miletus is usually credited with its application in geometry.

degrees.[5] While Euclid makes extensive use of drawings, it is not being suggested that geometry is about figures on paper (or papyrus, in Euclid's case). The drawings are a heuristic device to facilitate comprehension.[6]

Euclid developed a wealth of mathematics using what came to be known as the axiomatic method: First, state some propositions assumed without argument, along with definitions of key terms[7]; and then derive everything else by logical argumentation. Once established, results can be used to derive more results the same way. Euclid made logical argumentation the standard method for deriving mathematical results.

What this method itself is, the *Elements* does not explain. Perhaps it should have. Euclid was trained by students of Plato and as such was probably aware of the Platonist distinction between *F* itself and instances of it; and between a list of instances of *F* itself and a definition of it. Moreover, the *Meno* argues in effect that conceptual analysis is a pre-condition of pedagogy. Given this background, Euclid should have provided an explanation of: (1) the method his examples of logical argumentation instantiated; (2) why he believed it was applied correctly in all of them; and (3) what 'correctly' meant.[8] Generations of students were left to fend for themselves.

(1), (2) and (3) were not really new problems.[9] In his *Prior Analytics*, Aristotle, who slightly preceded Euclid, proposed what he probably thought was a sufficiently general analysis of logical argumentation.[10] Stated in modern terms, his answer was insightful in principle: Logical argumentation means the

[5] Such measurements could be used to convince students that mathematical applications to 'the real world' always involve approximation; and that there is a fundamental difference between mathematics and science in how results are justified. See Cusmariu 2012.

[6] Descartes would later dispense with drawings by introducing algebraic methods into geometry.

[7] Book I of the *Elements* assumes ten propositions without argument: Five "Postulates" and five "Common Notions." Teachers should explain the difference between them as well as the role of definitions.

[8] It is unknown how Euclid proceeded in classroom settings. Berlinski (2013, 17) claims that Euclid "had no interest" in what made his arguments valid and that he "was not a mathematician disposed to step back to catch himself in the act of stepping back." What an odd characterization of Aristotle, and of Frege!

[9] Ball states: "It would appear that he [Euclid] was well acquainted with the Platonic geometry, but he does not seem to have read Aristotle's works." (Ball 1960, 43). Even so, Euclid's fellow mathematicians would have been aware of what Aristotle accomplished.

[10] An *Encyclopedia of Mathematics* article states: "At the time [of Euclid] the problem of the description of the logical tools employed to derive the consequences of an axiom had not yet been posed," evidently unaware that Aristotle had in fact 'posed' this problem and suggested a solution; and that the need for analysis of important concepts is a key theme in Plato. See Novikov citation in References.

application of rules of inference, of which Aristotle supplied 15 in syllogistic form.[11]

Unfortunately, Aristotle seems not to have tested his analysis against logical argumentation as practiced in mathematics; why not is unknown.[12] Had he done so, he would have realized that mathematical propositions were not in general reducible to the four types in his syllogisms;[13] and that his 15 rules were insufficient to capture all logically correct mathematical reasoning.[14] Revision would have followed.

Greek mathematics after Euclid showed no interest in the problems that Aristotle tried to solve. Archimedes, Apollonious, Diophantus, Pappus, Eratosthenes and their contemporaries continued to use logical argumentation to derive results (what else?), also without an analysis of logical argumentation itself.[15] This was true many centuries later also of Descartes, Newton, Euler, Gauss, Cauchy and their contemporaries. We can only wonder what might have been if Euclid had taken a Fregean turn – or even Descartes, who was a philosopher as well as a mathematician. As we shall see, logic today is still a 'silent partner' in mathematics and its instruction, despite Frege's insights.[16]

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[11] Aristotle's analysis yielded 256 syllogisms in standard form, of which 15 are logically correct and effectively can function as rules of inference. Briefly, a syllogism is composed of two premises and a conclusion in subject-predicate form, designated by the letters **A**, **E**, **I** and **O**. **A** is of the form "All S are P," **E** is of the form "No S are P," **I** is of the form "Some S are P," and **O** is of the form "Some S are not P." The syllogism also contains at most three terms, which can occur as subject, predicate, and middle terms. Kant was reporting a historical fact when he stated in the preface to the second edition of his *Critique of Pure Reason* that "logic … is thus to all appearances a closed and completed body of doctrine," (Kant 1929 [1787], 17, Bviii) meaning Aristotelian logic. Gottlob Frege's analysis in the *Begriffsschrift* (1879) showed otherwise. Some elementary logic textbooks, e.g., the highly popular Copi, Cohen and McMahon (2010), still cover Aristotelian logic.

[12] As noted above, many of the results in Euclid's *Elements* predate Euclid and would have been known to students of Plato's Academy, which Aristotle attended, including the use of logical argumentation to derive mathematical results.

[13] Sentences stacking quantifiers are counterexamples. "For any x there is a y such that $f(x, y)$" cannot be analyzed using Aristotelian propositions as, for example, a conjunction of the **A** proposition "for any x, $f(x)$" and the **I** proposition "there is a y, $f(y)$." See Cusmariu 1979A for an explanation why Aristotle's solution to the problem of universals is wrong.

[14] Logical analysis of the Pythagorean proof that $\sqrt{2}$ is not a rational number requires the machinery of first-order logic with the equality symbol. Other examples of mathematical proofs requiring sophisticated logical machinery for a full analysis are in Muller 1981.

[15] Of the five major ancient Greek mathematicians just cited, Kneale and Kneale (1962, 62) only mentions Diophantus and it is a passing reference to algebraic notation he introduced.

[16] Berlinski (2013, 2) claims that in the view of most mathematicians, mathematical logic is not part of mathematics. This attitude toward logic is common among mathematics teachers as well, as the author can testify from his own experience. While in college, the author expressed interest in studying the concept of mathematical proof as a subject in its own right, to which he got the response "that's not mathematics."

From the fact that mathematics has been able to progress without an analysis of logical argumentation it should not be inferred that the problems Aristotle tried to solve are pedagogically unimportant; far from it. Yet, mathematics textbooks continue to assume that students will 'get' logical reasoning on their own just by working through examples of mathematical reasoning.[17] Standard logic textbooks, on the other hand, fail to explain in practical terms why studying the methods of logic can help students learn mathematics and do a better job of solving problems.[18]

Educational systems cannot realistically expect the student population to figure out the principles of logical reasoning and their application in mathematics without the special training necessary to acquire a skill that is absolutely essential to learning and doing mathematics. It is imperative that schools add logic to mathematics curricula. The best way to do that is by means of a sound, practical and clearly laid out methodology. The following must be explained to mathematics students: how to apply rules of logic in ways that are explicitly linked to mathematical contexts; how they can make sure that applications are carried out correctly; and how they can correct errors if they are not. That is the purpose of this article: To provide step-by-step guidance for teaching logic-based skills to mathematics students, suitable even at elementary levels.

The methodology presented here, illustrated in detail with examples from several mathematical disciplines, will enable students to:

- Distinguish assumed from inferred statements in arguments.
- Build a logical sequence of steps from assumptions to conclusion.
- Identify logical links justifying inferences from one step to another.
- Check to make sure inferences are logically correct.
- State proofs in a way that makes the reasoning logically explicit.[19]

Basic Concepts

For purposes of this article, an argument is a finite sequence of sentences such that some, the premises (assumptions), are claimed to logically imply another sentence, the conclusion. The argument is valid when this claim is correct.[20]

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[17] A classic text by Edmund Landau is typical. He writes: "I will ask of you [students] only the ability to read English and to think logically." (Landau 1951 [1929]), v). Landau does not explain what it means to think logically. Like many mathematics teachers, he leaves it to students to 'pick it up' on their own. Example 1 below presents a detailed analysis of one of Landau's proofs using the methodology of this article.

[18] Mathematical logic textbooks, e.g., Kleene 1952, Church 1956, Mendelson 1964 and Schoenfield 1967, are too difficult for most high school mathematics students.

[19] Other benefits of the methodology will be explained along the way in the main text and in footnotes as appropriate.

[20] It is disconcerting to find a professional mathematician writing: "The conclusion of a valid argument is entrained by its premises." (Berlinski 2013, 16). Teachers should avoid using

However, we note right away that validity is an ambiguous concept in logic, something not generally recognized. Validity has a *semantic* and a *syntactic* meaning.[21]

In a semantically valid argument, if premises are true, the conclusion must be true. An argument can be semantically valid even if (a) premises are false and the conclusion is true; (b) premises and conclusion are false; but not if (c) premises are true and the conclusion is false. On the other hand, (d) an argument can be semantically invalid even though premises and conclusion are all true.

Point (a) might seem unintuitive but is nevertheless correct: "All roses have thorns" follows logically from "all roses are purple flowers" and "all purple flowers have thorns" even though both premises are false and the conclusion is true.[22] A semantically valid argument is easy to construct using only falsehoods, as students can verify. To verify point (d), students should be asked to construct a logically incorrect argument using true premises and a true conclusion – also easy to do.

The mathematically relevant and pedagogically useful concept of validity is syntactic, just as Aristotle thought. Syntactic validity means correct application of rules of logic, which involves matching logical form as explained in detail below.[23]

Logical Symbolism

The following symbols, called 'logical connectives,' will be used below to state mathematical arguments.

~	Negation, meaning 'it is not the case that.'
&	Conjunction, meaning 'and' and its cognates.
v	Disjunction, meaning 'or' and its cognates.
→	Material implication, meaning 'if __, then __.'
≡	Material equivalence, meaning 'if and only if.'

informal language to explain the logical concept of validity. For example, Berlinski would not dare use informal language to teach basic concepts of calculus such as 'continuity' and 'integration.'

[21] The two concepts are related. Students should be encouraged to find out how.

[22] Students should internalize as early as possible the difference between validity, which is a property of arguments, and truth, which is a property of argument components. Logic, mathematics and science often use common words in a technical sense, which must be applied as defined in those fields.

[23] An elementary treatment of the subject is Cusmariu 2016. Texts suitable for high-school mathematics courses are Velleman 1994 and Wohlgemuth 1990. Advanced texts are Takeuti 1987 and Kunen 2012.

$(x)\Phi x$[24], meaning "the predicate Φx holds for all x objects."[25]
Example: *All even numbers are sums of primes*.

$(\exists x)\Phi x$[26] , meaning "the predicate Φx holds for some x objects."
Example: *Some numbers are expressible as ratios.*

Mathematics textbooks use ordinary words and phrases to express logical connectives, though it is not always easy for students to match words and phrases with logical connectives and their symbols. For mathematical purposes and depending on context, the following can be taken to mean 'and' and are symbolized as **&**: 'also,' 'however,' 'though,' 'too,' 'but,' 'besides,' 'what's more,' 'in addition,' 'nonetheless,' 'moreover,' 'yet.' The following can be taken to mean 'or' and are symbolized as **v**: 'unless,' 'otherwise,' 'except,' 'else.' It would be useful for students to put together, and share with each other, a vocabulary listing the various ways that logical connectives can be expressed in words. Spotting words and phrases for logical connective and interpreting them correctly is an important skill.

The standard way of defining logical connectives is by means of truth tables.[27]

p	***q***	***~p***	***p & q***	***p* v *q***	***p → q***	***p ≡ q***
T	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**	**T**	**T**

Rules of Logic: Preliminaries

Because syntactic validity means correct application of rules of logic to yield new mathematical knowledge,[28] it will be useful to have a list of such rules up front.

[24] The notation (x) denotes the universal quantifier.
[25] The terms 'predicate' and 'holds for' are used neutrally here without taking a stand on issues associated with the philosophical problem of universals. A concise statement of this problem can be found in Cusmariu 1979A and Cusmariu 2016A. Universal and existential quantifiers along with predicates and variables belong to the predicate calculus of logic, first developed by Frege.
[26] The notation ($\exists x$) denotes the existential quantifier.
[27] It does not matter that connectives as defined in the truth tables are not in complete agreement with common usage. For example, the first row of the truth table for p v q shows that 'or' is defined in the inclusive sense as 'one or the other or both.' The truth table also shows a weaker sense of 'if __, then __' than is used in non-mathematical contexts.

Below are five rules of the propositional calculus and two rules of the predicate calculus that will be applied to the mathematical arguments studied in this article.[29] However, these rules are not sufficient to capture all logically correct mathematical arguments. The object here is to get students used to operating with the concept of syntactic validity, starting with relatively simple examples. More rules can be added after students have become proficient at operating with the ones presented here.

Note that in Rules 1-3 premise components need not occur immediately above or below one other. Thus, *Modus Ponens* has been applied correctly even if $p \to q$ occurs on line 3 of a proof while p occurs on line 10, and vice versa.

Seven Rules of Logic[30]

Rule 1: *Modus Ponens* (MP)[31]

$p \to q$

p

$\therefore q$

Students encountering MP for the first time may find the rule unhelpful if they see the conclusion, q, as 'part of the premise,' $p \to q$. They may take this to mean that the rule is circular or redundant because it seems to assume what is to be proved. To clear up this misunderstanding, it should be pointed out that $p \to q$ is in conditional form and as such does not assert q; only that IF p is the case, THEN q is the case. The expression 'part of' has a specific, defined meaning in $p \to q$, as shown in the truth table. Logic can help mathematics students learn to operate with concepts as defined. Exercises should be devised to show students correct as well as incorrect ways of matching the form of MP or any other rule. Form-matching exercises will also get students used to thinking in abstract terms, which is another critical skill in mathematics.

Asserting the Consequent is a popular but fallacious argument form that closely resembles MP:

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[28] The author is aware of philosophical problems associated with the idea that 'new knowledge' can be generated from 'old knowledge' by means of 'pure reason.' See Cusmariu 2012 and Cusmariu 2016A.

[29] A standard logic text that can be consulted for more rules of logic is Copi, Cohen and McMahon 2010.

[30] There is significant evidence in the developmental psychology literature that students are able to master some rules of inference remarkably early. See Stylianides and Stylianides 2008.

[31] Students should be informed that sentence letters in propositional calculus rules can be replaced by sentences of any logical complexity whatever. Thus, an inference from $(p \vee r) \to (q \,\&\, s)$ and $(p \vee r)$ to $(q \,\&\, s)$ is also an MP inference. This fact is part of the formal nature of rules and should be accepted as early as possible. The formal nature of rules of logic will help students get used to abstraction in mathematics.

$p \rightarrow q$

q

$\therefore p$

p does NOT follow logically from $p \rightarrow q$ and q. Counterexamples are easy to devise and may usefully be posed as homework. Logic can help mathematics students learn to pay careful attention to formal details.

Rule 2: *Modus Tollens* (MT)[32]

$p \rightarrow q$

$\sim q$

$\therefore \sim p$

Students encountering MT for the first time may object that the pairs of statement forms (p, $\sim p$,) and (q, $\sim q$) cannot be part of the rule because they are contradictory but rules of logic cannot contain contradictions. To clear up this misunderstanding, teachers can note that MT neither asserts nor implies p & $\sim p$ and q & $\sim q$, which are contradictions. The rule says, in words, "given $p \rightarrow q$ as well as $\sim q$, it is logically correct to infer $\sim p$." MT is another opportunity for students to learn careful attention to formal details.

A popular misapplication of MT is the fallacy of *Denying the Antecedent*:

$p \rightarrow q$

$\sim p$

$\therefore \sim q$

$\sim q$ does NOT follow logically from $p \rightarrow q$ and $\sim p$.

Proofs by *reductio ad absurdum* rely on MT and MP.[33] We show A by deriving a contradiction (inconsistent sentence) C from the negation of A, $\sim A$, from which A follows because contradictions are false. In outline, the argument looks like this:

1. $\sim A \rightarrow C$

2. $\sim C$

$\therefore \sim\sim A$, by MT

3. $\sim\sim A \rightarrow A$

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[32] MT refutes the popular misconception that "you can't prove a negative." Mathematics proves negatives routinely. Thus, Pythagoras proved that √2 is <u>not</u> a rational number and Bertrand Russell proved that <u>there is no set of just those sets not members of themselves</u>.

[33] The mathematician G.H. Hardy regarded the *reductio* proof as "one of a mathematician's finest weapons." See Hardy, G.H. (1940, 94). On the other hand, mathematician Jordan Ellenberg (2014, 133) describes the *reductio* proof as "a weird trick, but it works." There is nothing 'weird' about *reductio* proofs. One hopes Ellenberg does not say such things in class.

$\therefore A$, by MP

Many proofs in Euclid are in *reductio ad absurdum* form.[34]

Rule 3: Hypothetical Syllogism (HS)

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

Replacing the arrow in HS with the equality symbol yields Euclid's Common Notion I.1 (see below), which, however, is <u>not</u> equivalent to HS. The arrow is a truth-functional symbol; the equality symbol in mathematics usually designates identity.[35]

Rule 4: De Morgan's Theorem (De M.)

$\sim(p \,\&\, q) \equiv \sim p \vee \sim q$

that is,

$(\sim(p \,\&\, q) \rightarrow (\sim p \vee \sim q)) \,\&\, ((\sim p \vee \sim q) \rightarrow \sim(p \,\&\, q))$.

Rule 5: Material Implication (Imp.)

$p \rightarrow q \equiv \sim p \vee q$

that is,

$((p \rightarrow q) \rightarrow (\sim p \vee q)) \,\&\, ((\sim p \vee q) \rightarrow (p \rightarrow q))$

De M. and Imp. are rules of replacement rather than rules of inference, meaning than expressions flanking the equivalence symbol can be replaced for one other without affecting the validity of an argument.

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[34] *Reductio* proofs can also be found in modern physics, e.g., quantum theory. See Jauch 1968, 115, where Proposition 1 – *Every dispersion-free state is pure* – is proved by *reductio* reasoning. An extensive analysis of Jauch's argument and its implications for quantum mechanics is Cusmariu 2016B.

[35] 'Usually' because the equality symbol sometimes is used in mathematics as shorthand for the definition symbol '=df.' The definition symbol does <u>not</u> mean identity and is used rather to introduce terms, specifying under what conditions they hold and how they are to be used. A defined term may be used to define other terms. Thus, Euclid's Definition 11 in Book I of *Elements*, "an *obtuse* angle is an angle greater than a right angle," assumes the definition of 'right angle' provided in Definition 10.

Rule 6: Universal Generalization (UG)[36]

Φy	read, "the predicate Φ holds for arbitrarily selected individual y."
$\therefore\ (x)(\Phi x)$	read, "Φ holds for any x."

Mathematics often proves that Φ holds for all objects of kind K (geometric figures, numbers, etc.) by selecting an arbitrary instance of K and proving that Φ holds for it. Then it is inferred that Φ must hold for all objects of kind K. According to UG, this inference is correct. The proof from Euclid studied in detail below applies UG to derive its main conclusion, the last line of the argument.

Rule 7: Universal Instantiation (UI)[37]

$(x)(\Phi x)$	read, "Φ holds for any x."
$\therefore \Phi v$	read, "the predicate Φ holds for individual v."

UI is used in mathematics more often than realized. Unfortunately, it is common to see inferences 'from the general to the particular' without any hint that such inferences are based on, and therefore justified by, a rule of logic. Applying a definition to a specific case means applying UI, as does assigning values to variables in a formula. Thus, when students encounter the expression "let *x* be such and such," UI has been applied.

Comments on Proofs

- Knowing that syntactic validity means matching the form of rules of logic can simplify the process of argument building and offer useful hints how to proceed. Students familiar with these rules will know what assumptions must be marshaled to match the relevant forms. Thus, applying MP and MP requires conditional premises; while HS requires all sentences be in conditional form. Mathematical arguments frequently omit conditional premises, even though they are necessary for arguments to go through as we shall see below.
- Definitions in mathematics are often key steps in arguments. The symbol =df is often used to write a definition, A =df B, where A is the concept being defined and B the concept(s) used to define A. Because definitions record equivalence, it is helpful to express A =df B as $A \equiv B$. Because $A \equiv B$ is expanded as $(A \rightarrow B)$ & $(B \rightarrow A)$, an argument can use part of the definition, $A \rightarrow B$, as one of its steps. The A component of a conditional is called the 'antecedent' and B the 'consequent.'

[36] UG and UI are rules of the predicate calculus. The careful student will ask, for example, whether propositional calculus rules also apply to the predicate calculus. They do indeed and work the same way.

[37] Students should be informed that UG and UI are correct for Φ of any logical complexity whatever. Moreover, despite appearances to the contrary, UI is not restricted to sentences with a single quantifier.

- Mathematics often requires proving conditionals. There are several strategies to accomplish this. Strategy 1 is to prove that the negation of $p \rightarrow q$, a sentence of the form p & $\sim q$, implies something that is false, thus $\sim(p$ & $\sim q)$ follows by MT and $p \rightarrow q$ follows by De M. and Imp. Strategy 2 is to infer $p \rightarrow q$ from $p \rightarrow r$ and $r \rightarrow q$ using HS. Strategy 3 is to prove q and then infer $(p \rightarrow q)$ from q and the tautology $q \rightarrow (p \rightarrow q)$ by MP. Euclid uses Strategy 1 to prove his Proposition III.6 – see below. Strategy 3 is a modern development.
- Proofs in mathematics frequently use results that cut across disciplines. This clarifies further the sense in which rules of logic are formal. Thus, MP has the same meaning in all of mathematics, so that p and q can be replaced with formulas of different disciplines and still yield a syntactically valid argument.
- Proofs in mathematical textbooks follow Euclid in presenting what might be called 'proof sketches.' As we shall see, they do not list all the steps necessary and sufficient to derive the final conclusion, or indicate which rules of logic have been applied to justify moving the argument from one step to the next.
- Students should be encouraged to ask probing questions about mathematics and its methods. For example, as they work through proofs to identify assumptions driving a result, students will come to realize, as Euclid did, that mathematics must make some assumptions without argument. It is an interesting and important question how such assumptions are to be justified and in what sense of 'justified.'

Seven Logic Lessons

The 17th century French mathematician Pierre de Fermat famously stated that "*la qualité essentielle d'une démonstration est de forcer á croire*" ("the essential attribute of a proof is that it compels belief" (Fermat 1891-1912, Vol. II, 483). However, a line of reasoning can "compel belief" only if proof elements and their logical links are readily apparent. This is not always the case in mathematics, as Fermat's own "Last Theorem" showed. Students encountering proof narratives may well have difficulty 'tracking' the reasoning from beginning to end because mathematical arguments often omit assumed as well as inferred steps deemed 'obvious'; and there is near universal absence of the rules of logic used to derive steps. It is assumed that the student will 'see' the logic without instruction. Keeping logic a 'silent partner' in mathematics instruction is pedagogically unwise to say the least. The dreaded 'fear of math' can be traced in part to the fact that the logic of mathematical reasoning is not transparent, leaving students confused and discouraged if they fail to 'get it.' Learning can be stifled by negative emotional reactions to the subject matter.

It is preferable to teach students how to build logically explicit arguments in stages. We believe the seven lessons explained and applied to mathematical examples below represent a practical methodology.

Lesson I: Distinguishing assumed from inferred steps.

Lesson II: Identifying assumed steps needed in the argument.
Lesson III: Justifying assumed steps.
Lesson IV: Displaying the entire inference chain of the argument.
Lesson V: Identifying rules of logic applied.
Lesson VI: Checking that rules of logic have been applied correctly.
Lesson VII: Building a fully explicit argument.

Lesson I will teach students that every step in an argument is either assumed or inferred from one or more assumptions. It is critical that students develop an ability to work with both types of argument components. Information that is of neither type should be discarded when constructing proofs, as it is logically irrelevant.

Lesson II will teach students to make sure they have compiled a complete list of assumptions before going forward with an argument. Missing assumptions can easily wreck an argument, sow confusion, and slow down the inference process.

Lesson III will teach students that assumptions may also need justification and they should be prepared to provide it. It is impossible to justify all assumptions, of course. In mathematics we can take for granted axioms, definitions, and previously established results. Justifying a step on grounds that it is a bad idea, certainly in the beginning stages.

Lesson IV will teach students how to lay out argument components sequentially so that the chain of inferences can be checked easily. Inferential chains contain many steps on the way to the final one; how many such steps will be needed is not predictable. Mathematical arguments seldom prove a result in single inference.

Lesson V will teach students how to be explicit about the rules of logic applied to derive inferred steps and how rules work to move from step to step in an argument.

Lesson VI will teach students how to check that rules of logic have been applied correctly to every inferred step.

Lesson VII will teach students how to build mathematical arguments that are fully explicit in all relevant respects: assumptions, inferred steps, and rules of logic. This is a stronger concept of rigor than is customary in mathematics.[38] The stronger concept has many pedagogical advantages, as the following four examples show.

[38] Proofs in Hilbert's axiomatization of geometry (Hilbert 1902) do not cite the rules of logic but they are cited in proofs in his mathematical logic book (Hilbert and Ackermann 1950 [1938]). This may create the (false) impression – not intended by Hilbert – that 'proof' does not mean the same thing in both subjects; that rules of logic are not what determines the validity of mathematical proofs; or that mathematical logic applies different rules of logic than mathematics itself.

EXAMPLE 1

Consider the following theorem and its proof in Edmund Landau's classic textbook on analysis (1951[1929], 9):

Theorem 12: If x < y, then y > x.

Proof: Each of these [$x < y$ and $y > x$] means that $y = x + v$ for some suitable v.

Lesson I: Distinguishing Assumed From Inferred Steps

The Theorem-Proof format, which students encounter routinely in mathematics, illustrates the sense in which logic is a 'silent partner' in mathematics. This format is actually shorthand for an argument, claiming that a sentence, labeled Theorem, follows logically from sentences listed in the Proof. The student is challenged to discover the inferential chain from sentences in the Proof to the conclusion, the Theorem.

A good deal of confusion and misunderstanding can be avoided by telling students right away that the relationship between Theorem and Proof is purely logical, which may not be obvious to all of them. Students should also be told that the proof component may well contain additional arguments. That is, the proof component may well contain other theorems, even though these are not always labeled as such and justification for them is not always included. Mathematical tradition might have evolved differently had Euclid stated explicitly that the relationship between his Propositions and the sentences in the narrative below was purely logical; that justification in mathematics does not mean taking measurements of any kind.

In the example at hand, Landau asserts that his Theorem 12 follows logically from the sentence listed in the proof. In other words, he is claiming that

(a) Each of $x < y$ and $y > x$ means that $y = x + v$ for some suitable v,

logically implies

(b) If $x < y$, then $y > x$.

Students are likely to find this claim mystifying, for several reasons.

To begin with, it is not apparent that sentence (a) is a conjunction of sentences, which it is. Let us label all components, Theorem and Proof, and arrange them vertically:

(a1) $x < y$ means that $y = x + v$.

(a2) $y > x$ means that $y = x + v$.

(b) If $x < y$, then $y > x$.

Landau is claiming that (b), Theorem, follows logically from the conjunction of (a1) and (a2), in Proof. The familiar three dot symbol $\therefore$ is a useful way to distinguish inferred from assumed steps and we shall do so henceforth.

The sequence of steps and their logical relationship in Landau's argument as stated is this:

(a1) $x < y$ means that $y = x + v$.

(a2) $y > x$ means that $y = x + v$.

∴ (b) If $x < y$, then $y > x$.

Lesson II: Identifying Assumed Steps

The second reason students may find Landau' argument hard to fathom is this. Intuitively, if A means B and C means B, it follows that A and B mean the same thing, from which it follows that $x < y$ and $y > x$ mean the same thing. This is not literally true. However, Landau is not suggesting that $x < y$ and $y > x$ mean the same thing just because he says in the Proof component that "each means that $y = x + v$." He is not deliberately sowing confusion.

Rather, (a1) is intended to suggest

(1) If $x < y$ then $y = x + v$,

while (a2) is intended to suggest

(2) If $y = x + v$ then $y > x$.

It is (1) and (2) that are the 'real' premises of Landau's argument. This fact is by no means obvious. Landau evidently expects students to know already how think through information and piece it together into a logically correct argument. He assumes, as do mathematics texts in general, that students already understand logical argumentation, in theory as well as in practice. This assumption is by no means obvious.

Students can realize that (1) and (2) must replace (a1) and (a2) by focusing on the logical form of the conclusion sentence, which is always a good place to start, even though it means starting at the end of an argument. Having noticed that Theorem 12 is in conditional form, students should next consider which rules of inference are relevant to its derivation. This means determining which rules have a conditional sentence in the conclusion; namely, the component in the rule prefixed by the ∴ symbol. In this case, it is HS, which requires conditional premises. This is a kind of 'working backwards' from rules of inference to the argument structure but it is helpful and it works.

Lesson III: Justifying Assumed Steps

Landau provides justification for (1) and (2) in the form of definitions stated a few lines above Theorem 12.

Definition 2: *If* $x = y + u$ *then* $x > y$.

Definition 3: *If* $y = x + v$ *then* $x < y$.

Here students may well wonder what concepts Landau is trying to define. It is clear that he is not trying to define = nor +, which leaves > and <. In general, it is preferable to be explicit about the concepts (terms) being defined.

A second, related criticism is that Landau fails to make clear that definitions in mathematics are not conditionals but rather biconditionals, i.e., conjunctions of conditionals. Students should be warned that definitions in mathematics are not always stated in proper form as biconditionals. Landau is the norm rather than the exception. Even if a mathematics text shows a definition only in conditional form, students should assume that a biconditionals is intended. Stating the above definitions in biconditional form in a way that makes it clear they define the symbols > and < yields:

Definition 2a: $x > y \equiv x = y + u$.

Definition 3a: $x < y \equiv y = x + v$.

Because (3a) is a conjunction of conditionals, we are entitled to use half of it:[39]

(3b) If $x < y$ then $y = x + v$.

(3b) is premise (1) above, which completes the justification for premise (1).

Because (2a) is a conjunction of conditionals as well, we are also entitled to use half of it:

(2b) If $x = y + u$ then $x > y$.

However, there are some discrepancies between (2b) and premise (2)

(2) If $y = x + v$ then $y > x$

that make it less easy to see a definition as the justification for this premise.

First, the variables *x* and *y* are reversed. Second, (2b) has *u* where (2) has *v*. These discrepancies can safely be ignored here. Landau could have made life simpler by writing his definitions consistently. By way of general warning, students should be prepared for less than complete logical rigor in mathematics textbooks, even those of celebrated teachers like Landau. In this case no harm is done because the discrepancies are easily resolved. However, the author urges teachers to avoid 'looseness' and informality and strive for complete clarity as much as possible.

Lesson IV: Displaying the Entire Inference Chain

Here is the inference chain from assumptions to the conclusion, Theorem 12:

(1) If $x < y$ then $y = x + v$.

---

[39] Some students will ask if a rule of logic is being applied when we use only one component of a conjunction p & q. The answer is in Example 4, Lesson III below.

(2) If $y = x + v$ then $y > x$.

∴ (b) If $x < y$ then $y > x$.

Lesson V: Identifying Rules of Logic Applied

Once the elements of an argument have been identified, it should be evident which rules apply – provided, of course, students know rules of inference and how they work. Only one rule of logic was applied to infer (b) from (1) and (2), HS.

A comment is in order about the justification of (1) and (2), which were based on the two definitions provided. Students should realize that when a mathematics text justifies a sentence using the words ‘from definition so-and-so’ a rule of logic is in fact being applied, though this rule is seldom if ever identified. The rule is UI. Here is why.

Definitions in mathematics stipulate the meaning of a term and usually contain no limitation on the range of values of their variables.[40] Thus, the definition of a circle holds for any circle whatever, meaning that definitions are actually universally quantified biconditionals, something of the form ‘$(x)(Fx \equiv Gx)$,’ where ‘Fx’ abbreviates the term being defined and ‘Gx’ its necessary and sufficient defining conditions. This is also not always apparent in mathematics, as the two examples from Landau illustrate. When a mathematics text claims that a sentence about a specific circle ‘follows from’ the definition of a circle, the explanation for this inference – yes, it is an inference – is UI. Students should be alerted to this fact.

Lesson VI: Checking Logical Justification

First, we display an argument structure side by side with the rule of inference applied to derive the conclusion of the argument, Theorem 12, inserting the arrow symbol → for the ‘if __then__’ material conditional:

HS Form	Inference Chain to Step (b)
$p \rightarrow q$	$x < y \rightarrow y = x + v$
$q \rightarrow r$	$y = x + v \rightarrow y > x$
$\therefore\ p \rightarrow r$	$\therefore\ x < y \rightarrow y > x$

Inspection shows that sentence letters in HS have been correctly and consistently replaced by the sentences of the argument. Thus, p has been replaced by $x < y$; q by $y = x + v$; and r by $y > x$. The argument matches the form of HS and, accordingly, is syntactically valid.

[40] Definitions in mathematics are often stipulative, setting down the meaning of a term rather than making a common term (or concept) precise through analysis. A useful text on the subject is Robinson 1954.

Lesson VII: Building a Fully Explicit Argument

Here we display the argument structure side by side with the justification for all steps, assumed as well as inferred. Logic is no longer a 'silent partner':

(1) $x < y \rightarrow y = x + v$……………………From definition component (3a), UI.

(2) $y = x + v \rightarrow y > x$……………………From definition component (2a), UI.

$\therefore$ (b) $x < y \rightarrow y > x$……………………From (1), (2), HS.

EXAMPLE 2

Addition and multiplication are commutative operations, meaning that the order in which they are carried out does not affect the result. Both of these equations are correct:

$a + b = b + a$ $\qquad\qquad$ $a \cdot b = b \cdot a$

What about division? Is it true that $a \div b = b \div a$? In different notation, is it true that $a/b = b/a$?

Here is how a standard mathematics textbook (Edwards, Gold and Mamary 2001, 35) argues that division is not commutative, using the numbers 12 and 3 as examples and displaying argument components on the page this way:

12/3 = 4	3/12 = 1/4
4 ≠ 1/4	≠ means unequal.
Therefore, 12/3 ≠ 3/12.	Division is <u>not</u> commutative.

This is another proof sketch of the sort typically found in mathematics textbooks. The term 'therefore' signals an inference from information presented, so we know which formula is the conclusion. It is left to the student to figure out what information belongs in the premises and how the conclusion is derived from them.

Lesson I: Distinguishing Assumed From Inferred Steps

We begin once again by labeling all sentences provided and arranging them in vertical order.

(a) 12/3 = 4

(b) 3/12 = 1/4

(c) 4 ≠ 1/4

(d) 12/3 ≠ 3/12

(e) Division is not commutative.

It is a relatively simple matter to see that (d) and (e) are inferred steps, while (a), (b) and (c) are assumed steps.

(a) 12/3 = 4

(b) 3/12 = 1/4

(c) 4 ≠ 1/4

∴ (d) 12/3 ≠ 3/12

∴ (e) Division is not commutative.

Figuring out which steps should be prefixed with the three-dot symbol is not always this easy. Mathematics textbooks present proofs in narrative form. Words and phrases commonly used to distinguish assumed from inferred steps are not always present. Inferred-step indicators include: 'therefore,' 'consequently,' 'it follows that,' 'as a result,' 'so,' 'thus,' 'hence,' 'we conclude,' 'we infer,' and 'accordingly.' Words and phrases used to indicate assumptions are: 'assuming,' 'because,' 'since,' 'as,' 'supposing,' 'for the reason that,' 'given that,' and 'seeing that.'[41]

Lesson II: Identifying Assumed Steps

Creativity is required once again to compile the list of assumptions necessary and sufficient to derive a result. In a simple example such as this, the effort is minimal but this is very much the exception. A useful exercise is to provide students with one rule of logic and have them build an inference chain that will apply only that one rule.

A moment's thought will show that premises (a), (b) and (c) are insufficient to derive the two inferred steps (d) and (e). In this case, it is a fairly simple matter to supply missing premises by bearing in mind that rules of logic made available in this article feature conditionals. With a bit of work we get the following:

(1) Division is a commutative operation → $a/b = b/a$.

(2) $a/b = b/a$ → 12/3 = 3/12.

(3) 12/3 = 3/12 → 4 = 1/4.

(c) 4 ≠ 1/4.

Lesson III: Justifying Assumed Steps

Once a complete list of assumed steps has been compiled, we need to explain the basis for each one. Assumed steps are usually axioms, definitions, or theorems.

We get Step (1),

(1) Division is a commutative operation → $a/b = b/a$.

[41] These lists are by no means complete. Students should assemble a vocabulary of premise and conclusion words and phrases and share them with each other.

from

(DC*) R is a commutative operation → $R(a,b) = R(b,a)$,

which is half the definition of commutation

(DC) R is a commutative operation ≡ $R(a,b) = R(b,a)$,

by applying UI and letting R be division in the antecedent of (DC*) and replacing R with the division symbol / in the consequent on (DC*).

We get Step (2),

(2) $a/b = b/a \rightarrow 12/3 = 3/12$

also by applying UI and assigning arbitrary values to a and b, letting $a = 12$ and $b = 3$. As an exercise, students should work out the application of UI to derive steps (1) and (2).

We get Step (3),

(3) $12/3 = 3/12 \rightarrow 4 = 1/4$

by carrying out division on the numbers in the antecedent of (3).

Step (c),

(c) $4 \neq 1/4$

is an arithmetical fact that can be assumed here without argument.

Lesson IV: Displaying the Entire Inference Chain

Listing argument components vertically once again makes is easy to see how the inference chain proceeds from link to link.

(1) Division is a commutative operation → $a/b = b/a$.

(2) $a/b = b/a \rightarrow 12/3 = 3/12$

(3) $12/3 = 3/12 \rightarrow 4 = 1/4$

(c) $4 \neq 1/4$

∴ (d) $12/3 \neq 3/12$

∴ (2*) $a/b \neq b/a$

∴ (e) Division is not a commutative operation.

Lesson V: Identifying Rules of Logic Applied

Inferred steps (d), (2*) and (e) were derived using the same rule, MT. UI was used to justify (1) and (2).

Lesson VI: Checking Logical Justification

Inference chains to inferred steps (d), (2*) and (e) match the form of MT. The application of UI to justify derive steps (2) and (3) was explained above.

MT Form	Inference Chain to Step (d)
$p \rightarrow q$	$12/3 = 3/12 \rightarrow 4 = 1/4$
$\sim q$	$4 \neq 1/4$
$\therefore \sim p$	$\therefore 12/3 \neq 3/12$

MT Form	Inference Chain to Step (2*)
$p \rightarrow q$	$a/b = b/a \rightarrow 12/3 = 3/12$
$\sim q$	$12/3 \neq 3/12$
$\therefore \sim p$	$\therefore a/b \neq b/a$

MT Form	Inference Chain to Step (e)
$p \rightarrow q$	Division is a commutative operation $\rightarrow a/b = b/a$
$\sim q$	$a/b \neq b/a$
$\therefore \sim p$	$\therefore$ Division is not a commutative operation.

Students should be assigned exercises that will help them see the form of rules of logic being matched by different content. Perception of logical form will lead to more advanced abstraction skills essential in mathematics and will make it easier to understand complex mathematical structures at a glance. Students need to get to the point where they can operate with symbols in mathematics without having to ask for specific examples.

The fact that the same sentence letter in a rule of logic can be replaced by different mathematical formulas shows the sense in which rules of logic are formal. MT is a correct rule of logic and has been matched correctly in all three cases, so the argument to show that division is not commutative is syntactically valid.

Lesson VII: Building a Fully Explicit Argument

(1) Division is a commutative operation $\rightarrow a/b = b/a$.....Definition of commutation, **UI**

(2) $a/b = b/a \rightarrow 12/3 = 3/12$……………………………Value assignments to a and b, **UI**

(3) $12/3 = 3/12 \rightarrow 4 = 1/4$………………………………Application of division

(c) $4 \neq 1/4$……………………………………………………Arithmetic fact

$\therefore$ (d) $12/3 \neq 3/12$……………………………………………(3), (c), MT

$\therefore$ (2*) $a/b \neq b/a$………………………………………………(2), (d), MT

$\therefore$ (e) Division is not a commutative operation……(1), (2*), MT

EXAMPLE 3

Algebra is an opportunity to show students that logical reasoning occurs also in mathematical contexts not associated with proving a result.

Suppose a student is asked to 'solve for *x*' in a simple equation such as

(1) x - 7 = 6.

Textbooks usually state that the procedure to follow is to add 7 to both sides of the equation, resulting in 13 as the answer (Edwards, Gold & Mamary 2001, 130).

(1) $x - 7 = 6$

(2) $x - 7 + 7 = 6 + 7$

(3) $x = 13$

The word 'therefore' does not occur anywhere here to help students realize that argumentation is taking place. Students are merely instructed to carry out an operation according to a routine to be memorized. Before showing that computing a solution does in fact rely on logical argumentation, we consider issues in algebra that logic can clarify and will help students get used to the algebra environment.

Let us explain what '*x*' means in the argument. Some algebra textbooks state that '*x*' designates an 'unknown' and equations are described as containing several 'unknowns.' This creates the impression that algebra is about subjective, mysterious things called 'knowns' and 'unknowns.' Algebra – mathematics in general – is not about subjective or mysterious things of any kind. The helpful answer is that *x* is variable – as compared to a constant, such as *a*, also called in logic an 'individual symbol.' As such, *x* is ambiguous between 'for <u>some</u> *x*' and 'for <u>all</u> *x*.'[42]

To resolve this ambiguity for the equation "$x - 7 = 6$," notice that "for <u>all</u> positive integers *x*, $x - 7 = 6$" is false, so that *x* in $x - 7 = 6$ must mean "for <u>some</u> positive integer *x* …" Teachers should make clear that *x* in an algebraic equation means either 'for any *x*' or 'for some *x*.' There is no third alternative, e.g., 'for many *x*' or 'for most *x*.' When equations contain more than one variable, we must be clear which quantifier is intended. Equations can combine quantifiers, e.g., 'for some *x*, for any *y*, $\Phi(x, y)$.'

Lesson I: Distinguishing Assumed From Inferred Steps

In light of the above comments, we need to start by inserting quantifiers in addition to distinguishing assumed from inferred steps. (2) and (3) are inferred steps, while step (1) is assumed:

(1) $(\exists x)(x - 7 = 6)$

---

[42] Algebra is also an opportunity to draw a basic distinction between bound and free variables. In the example under study where we wish to find the value of a variable, the variable is bound by a quantifier, in this case an existential quantifier. Some mathematics texts refer to bound variables as 'apparent variables' and to free variables as 'real variables.' It is not always obvious which variables in mathematical formulas are free and which are bound.

∴ (2) $(\exists x)(x - 7 + 7 = 6 + 7)$

∴ (3) $(\exists x)(x = 13)$

It is possible to simplify matters by eliminating quantifiers, provided we do it carefully. Thus, the existential quantifier $(\exists x)$ can be replaced with an individual symbol such as *a* but only if we have made sure that *a* has not occurred earlier in the argument. This requirement is necessary because using *a* again to get *Ga* from $(\exists x)(Gx)$ after we have proved *Fa* would have the strange consequence that we 'proved' *Fa* & *Ga*! At an elementary level, it may be better to eliminate quantifiers, provided we do not make such mistakes, we can replace $(\exists x)$ with an individual symbol and add $(\exists x)$ after we have proved *Ga* to get $(\exists x)(Gx)$.[43] In the example under discussion, we may safely remove the existential quantifier and replace it with an individual symbol. Students will find it easier to work with the quantifier-free version in the beginning stages.

(1*) $a - 7 = 6$

∴ (2*) $a - 7 + 7 = 6 + 7$

∴ (3*) $a = 13$

Lesson II: Identifying Assumed Steps

The inference to step (2*) is not apparent. To make the inference explicit, we need once again a step that links steps (1*) and (2*) by means of material implication:

(2**) $a - 7 = 6 \rightarrow a - 7 + 7 = 6 + 7$

We also need a step in conditional form that carries out the computation in the consequent of (2**).

(3**) $a - 7 + 7 = 6 + 7 \rightarrow a = 13$

Lesson III: Justifying Assumed Steps

Step (1*)

(1*) $a - 7 = 6$

is stipulated as part of the exercise.

The new step (2**)

(2**) $a - 7 = 6 \rightarrow a - 7 + 7 = 6 + 7$,

follows by UI from an addition property of equality:

---

[43] The inference from *Ga* to $(\exists x)(Gx)$ applies the rule of *Existential Generalization* (EG), not included in the list above. The fourth predicate logic rule is *Existential Instantiation* (EI). We leave it to teachers to research the matter. It is beyond the scope of this article to explain how the rules of predicate logic apply to sentences containing multiple quantifiers.

(AP) $a = b \rightarrow a + c = b + c$

For the time being, we can take it for granted that (AP) is true. No doubt (AP) will seem 'obvious' at a glance. This does not mean that a proof of (AP) is easy or that it is not necessary.

The new step (3**)

(3**) $a - 7 + 7 = 6 + 7 \rightarrow a = 13$

is a conditional that carries out the computation in the consequent of (2**).

Lesson IV: Displaying the Entire Inference Chain

Here is the sequence of steps making explicit the inference to the final step (3*):

(1*) $a - 7 = 6$

(2**) $a - 7 = 6 \rightarrow a - 7 + 7 = 6 + 7$

$\therefore$ (2*) $a - 7 + 7 = 6 + 7$

(3**) $a - 7 + 7 = 6 + 7 \rightarrow a = 13$

(2*) $a - 7 + 7 = 6 + 7$

$\therefore$ (3*) $a = 13$

Lesson V: Identifying Rules of Logic Applied

MP justifies steps (2*) and (3*) and UI justifies step (2**).

Lesson VI: Checking Logical Justification

Inspection shows that inference chains match MP.

MP Form	Inference Chain to Step (2*)
$p \rightarrow q$	$a - 7 = 6 \rightarrow a - 7 + 7 = 6 + 7$
p	$a - 7 = 6$
$\therefore q$	$\therefore a - 7 + 7 = 6 + 7$

MP Form	Inference Chain to Step (3*)
$p \rightarrow q$	$a - 7 + 7 = 6 + 7 \rightarrow a = 13$
p	$a - 7 + 7 = 6 + 7$
$\therefore q$	$\therefore a = 13$

Sentence letters in MP (or any other rule) can be matched by formulas even though formulas are not sentences in the grammatical sense of containing a subject and a predicate like "snow is white." It was one of Frege's discoveries that subject-predicate form, which is assumed in Aristotelian logic, is not sufficiently general for mathematical purposes and had to be replaced by a more powerful analysis.

Lesson VII: Building a Fully Explicit Argument

(1*) a - 7 = 6……………………………………………Assumption

(2**) a - 7 = 6 → a - 7 + 7 = 6 + 7…………………Addition property of equality by UI

∴ (2*) a - 7 + 7 = 6 + 7………………………………(2**), (1*), MP

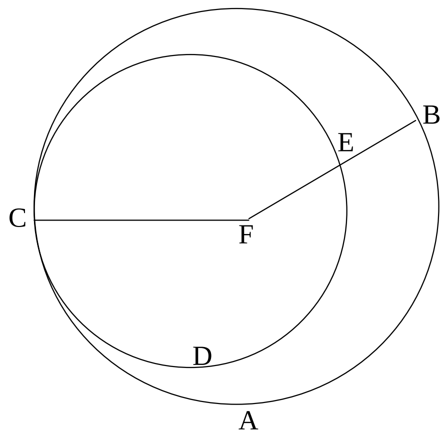
(3**) a - 7 + 7 = 6 + 7 → a = 13………………………Arithmetic addition on (2*)

∴ (3*) a = 13…………………………………………….(3**), (2*), MP

EXAMPLE 4

This example from Euclid is much more challenging to analyze, despite the appearance of simplicity. Students will find that mathematicians often strive for simplicity or elegance of presentation at the expense of logical rigor, leaving out information they consider 'obvious' to avoid cluttering the text – including, as we have already seen, rules of logic used to derive results at various stages of an argument.

Proposition III.6 of the *Elements* states that if two circles touch one another, they will not have the same center. Below is the language of Euclid's proof (2013, 55) along with the drawing he used in presenting the proof.



"For let the two circles *ABC*, *CDE* touch one another at the point *C*; I say they will not have the same center. For, if possible, let it be *F*; let *FC* be joined, and let *FEB* be drawn through at random. Then, since the point *F* is the center of the circle *ABC*, *FC* is equal to *FB*. Again, since the point is the center of the circle *CDE*, *FC* is equal to *FE*. But *FC* was proved equal to *FB*; therefore, *FE* is also equal to *FB*, the less to the greater: which is impossible. Therefore *F* is not the center of the circles *ABC*, *CDE*. Therefore etc., Q.E.D."

Lesson I: Distinguishing Assumed From Inferred Steps

Extracting argument components from Euclid's language and translating them into symbolic notation using truth-functional connectives requires interpretation.

Euclid: "Since the point *F* is the center of the circle *ABC*, *FC* is equal to *FB*."

(1) If *F* is the center of circle *ABC*, then $FC = FB$.

Euclid: "Since the point [F] is the center of the circle *CDE*, *FC* is equal to *FE*."

(2) If *F* is the center of circle *CDE*, then $FC = FE$.

Usually, the term 'since' is a premise indicator but in (1) and (2) it is correct to interpret Euclid as asserting conditionals.

Euclid: "*FC* was proved equal to *FB*."

(3) $FC = FB$.

Euclid: "Therefore, *FB* is also equal to *FE*."

∴ (4) $FB = FE$.

Euclid: "[Therefore], the less [is equal] to the greater."

∴ (5) $FE = FE + EB$.

Euclid: "which is impossible."

(6) $FE \neq FB$.

Euclid: "Therefore, *F* is not the center of the circles *ABC*, *CDE*."

∴ (7) *F* is not the center of the circle *ABC* and the circle *CDE*.

Euclid: "Therefore, etc. Q.E.D."

∴ (III.6) If two circles are tangent, then they do not have the same center.

Lesson II: Identifying Assumed Steps

Euclid argues for III.6 by *reductio at absurdum*. That is, he derives a contradiction from the negation of III.6

(III.6*) Two circles are tangent and have the same center,

from which he concludes that (III.6*) is false, therefore its negation, (III.6) is true.

Euclid does not explain why *reductio* arguments work. Specifically, he does not explain why deriving a contradiction from a sentence *p* proves that *p* false and thus its negation, $\sim p$, is true. If only he had! Here is the explanation.

The preferred term in logic for 'contradiction' is 'inconsistent sentence.' An inconsistent sentence is a sentence of the form:

(IS1) $p \,\&\, \sim p$.

This is the form Euclid applies in his argument. (IS1), however, is not the only inconsistent sentence form; so is this form:

(IS2) $p \equiv \sim p$.[44]

Truth-tables introduced earlier to define logical connectives explain what's wrong with (IS1) and (IS2). We will leave it to the student to apply them to (IS1) and (IS2), which will show that only the truth value False occurs in columns under & in (IS1) and under ≡ in (IS2). What's wrong with contradictions, then, is that they are false no matter what truth values we assign to *p* in (IS1) and (IS2).

There is also the general question why contradictions cannot be allowed in mathematics. The short answer is that any sentence whatever can be proved to follow logically from an inconsistent sentence, meaning that every (well-formed) sentence is a theorem so that the entire system is inconsistent. We leave it to students to carry out the proof. Contradiction is 'check-mate' in mathematics.

Continuing with the analysis of Euclid's argument, let us start by translating its components into symbolic notation to identify logical connectives involved. Let us use Euclid's notation for referring to triangles, points, and segments – a notation that in fact 'hides' an inference to be explained later.
Thus, Euclid's (III.6*)

(III.6*) Two circles are tangent and have the same center

is symbolized as,

(a) Circles *ABC* & *CDE* are tangent & *F* is the center of both circles.

Repeating this procedure the rest of Euclid's argument yields:

(b) ((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → $FC=FB$ & $FC=FE$.

(c) $FC=FB$ & $FC=FE \rightarrow FB=FE$.

(e) $FB=FE + EB$ & $EB>0 \rightarrow FB > FE$.

(f) $FB=FE + EB$ & $EB>0$.

(h) $FB > FE \rightarrow FB \neq FE$.

(k) ~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → (~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles)).

(l) (~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles)) → ((*ABC* & *CDE* are tangent → ~(*F* is the center of both circles.))

This example should give students a better idea than the previous three examples how much work is required in mathematics to compile even a

[44] In the paradox that bears his name, Bertrand Russell uses (IS2). As an exercise, students should show that (IS2) implies (IS1).

reasonably complete list of assumed steps. Mathematical proof sketches contain gaps that must be filled to produce a syntactically valid argument. Practice is the only way to acquire this skill.

Lesson III: Justifying Assumed Steps

The justification for assumed steps needed to make the argument explicit is this:

- Step (a) is an arbitrary instance of III.6*.
- Step (b) combines Euclid's Steps (1) and (2).
- Step (c) applies Euclid's Common Notion I.1 (2013, 2): *Things which are equal to the same thing are also equal to one another.*
- Step (e) applies Euclid's Common Notion I.5 (2013, 2): *The whole is greater than the part*.
- Step (f) is apparent from Euclid's diagram.
- Step (h) follows from the definition of 'greater than,' assumed without statement in Common Notion I.5.
- Step (k) applies half of De M., bringing the negation sign inside brackets.
- Step (l) applies half of Imp., replacing ~__v__ with __→__.

Regarding steps (k) and (l), students should recall that $p \equiv q$ is equivalent with $(p \rightarrow q)$ & $(q \rightarrow p)$ and that, accordingly, 'applying half' of De M. and Imp. to these steps is in fact an inference from a sentence of the form p & q to p, which is justified by a rule of logic called 'Simplification.' Steps (k) and (l) are deliberately not stated as inferences to help teachers determine which students are paying attention to details. The author believes this is good pedagogy in general.

Lesson IV: Displaying the Entire Inference Chain

(a) Let circles *ABC* & *CDE* be tangent & *F* be the center of both circles.

(b) ((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → $FC=FB$ & $FC=FE$.

(c) $FC=FB$ & $FC=FE \rightarrow FB=FE$.

∴ (d) ((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → $FB=FE$.

(e) $FB=FE+EB$ & $EB>0 \rightarrow FB > FE$.

(f) $FB=FE+EB$ & $EB>0$.

∴ (g) $FB > FE$.

(h) $FB > FE \rightarrow FB \neq FE$.

∴ (i) $FB \neq FE$.

∴ (j) ~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)).

(k) ~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → (~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles)).

(l) (~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles)) → ((*ABC* & *CDE* are tangent → ~(*F* is the center of both circles.))

∴ (m) ~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → ((*ABC* & *CDE* are tangent → ~(*F* is the center of both circles)).

∴ (n) *ABC* & *CDE* are tangent → ~(*F* is the center of both circles.)

∴ (III.6) Two circles are tangent → they do not have the same center.

Lesson V: Identifying Rules of Logic Applied

The logical justification for inferred steps is as follows:[45]

- Step (d) follows from steps (b) and (c) by HS.
- Step (g) follows from steps (e) and (f) by MP.
- Step (i) follows from steps (g) and (h) by MP.
- Step (j) follows from steps (d) and (i) by MT.
- Step (k) follows step (j) by De M.
- Step (l) follows from step (k) by Imp.
- Step (m) follows from steps (k) and (l) by HS.
- Step (n) follows from steps (j) and (m) by MP.
- Step (III.6) follows from step (n) by UG.

Lesson VI: Checking Logical Justification

All inference chains match the form of the corresponding rule.

HS Form	Inference Chain to Step (d)
$p \rightarrow q$	(*ABC* & *CDE* are tangent & *F* is the center of both circles) → (*FC*=*FB* & *FC*=*FE*)
$q \rightarrow r$	(*FC*=*FB* & *FC*=*FE*) → *FB*=*FE*
$\therefore\ p \rightarrow r$	∴ (*ABC* & *CDE* are tangent & *F* is the center of both circles) → *FB*=*FE*

MP Form	Inference Chain to Step (g)
$p \rightarrow q$	$FB = FE + EB\ \&\ EB > 0 \rightarrow FB > FE$
p	$FB = FE + EB\ \&\ EB > 0$
$\therefore\ q$	$\therefore\ FB > FE$

MP Form	Inference Chain to Step (i)
$p \rightarrow q$	$FB > FE \rightarrow FB \neq FE$.
p	$FB > FE$.
$\therefore\ q$	$\therefore\ FB \neq FE$.

[45] Steps (c) and (e) are also inferred steps because they apply UI. We leave this as an exercise to the student.

MT Form	Inference Chain to Step (j)
$p \rightarrow q$	((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → *FB*=*FE*.
$\sim q$	*FB*≠*FE*.
$\therefore \sim p$	∴ ~(*ABC* & *CDE* are tangent & *F* is the center of both circles).

De M. Form	Inference Chain to Step (k)
$\sim(p \,\&\, q) \rightarrow (\sim p \vee \sim q)$	~(*ABC* & *CDE* are tangent & *F* is the center of both circles) → (~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles))

Imp Form	Inference Chain to Step (l)
$(\sim p \vee \sim q) \rightarrow (p \rightarrow \sim q)$	(~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles)) → ((*ABC* & *CDE* are tangent → ~(*F* is the center of both circles)).

HS Form	Inference Chain to Step (m)
$p \rightarrow q$	~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → (~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles))
$q \rightarrow r$	(~(*ABC* & *CDE* are tangent) v ~(*F* is the center of both circles)) → ((*ABC* & *CDE* are tangent → ~(*F* is the center of both circles)).
$\therefore p \rightarrow r$	∴ ~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → ((*ABC* & *CDE* are tangent → ~(*F* is the center of both circles)).

MP Form	Inference Chain to Step (n)
$p \rightarrow q$	~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)) → (*ABC* & *CDE* are tangent → ~(*F* is the center of both circles)).
p	~((*ABC* & *CDE* are tangent) & (*F* is the center of both circles)).
$\therefore q$	∴*ABC* & *CDE* are tangent → ~(*F* is the center of both circles).

UG Form	Inference Chain to Step (III.6)
$\Phi(a,b) \rightarrow \sim(\Psi(c,a) \,\&\, \Psi(c,b))$	*ABC* and *CDE* are tangent → *F* is not the center of both *ABC* and *CDE*.
$\therefore (x)(y)(z)(\Phi(x,y) \rightarrow \sim(\Psi(z,x) \,\&\, \Psi(z,y))$	∴ If two circles are tangent, they do not have the same center.

where Φ is 'tangent with' and Ψ is 'center of.'

Euclid selected circles *a* and *b* and center *c* arbitrarily, therefore, UG is satisfied and the inference from (n) to what he set out to prove, (III.6), goes through.

(b) ((Circles *ABC* and *CDE* are tangent) &

(*F* is the center of both circles)) →

FC=*FB* & *FC*=*FE*………………………………………………Assumption for *reductio*

(c) *FC*=*FB* & *FC*=*FE* → *FB*=*FE*. ……………………………Common Notion I.1, UI

∴ (d) (Circles *ABC* and *CDE* are tangent) &

(*F* is the center of both circles) → *FB*=*FE*. ………………………(b), (c), HS

(e) *FB*=*FE* + *EB* & *EB*>0 → *FB* > *FE*. ………………………………Common Notion I.5, UI

(f) *FB*=*FE* + *EB* & *EB*>0. ……………………………………………Assumption

∴ (g) *FB* > *FE*. ……………………………………………………………(e), (f), MP

(h) *FB* > *FE* → *FB*≠*FE*. ………………………………………………Assumption

∴ (i) *FB*≠*FE*. ………………………………………………………………(h), (g), MP

∴ (j) ~((*ABC* and *CDE* are tangent) &

(*F* is the center of both circles.)) ……………………………………(d), (i), MT

∴ (k) ~((*ABC* and *CDE* are tangent) &

(*F* is the center of both circles)) →

(~(*ABC* and *CDE* are tangent) v

~(*F* is the center of both circles))…………………………………(j), De M.

∴ (l) (~(*ABC* & *CDE* are tangent)

v ~(*F* is the center of both circles)) → ((*ABC* & *CDE*

are tangent → ~(*F* is the center of both circles))……………(k), Imp.

∴ (m) ~((*ABC* & *CDE* are tangent) &

(*F* is the center of both circles)) → ((*ABC* & *CDE*

are tangent → ~(*F* is the center of both circles))……………(k), (l), HS

∴ (n) *ABC* and *CDE* are tangent →

~(*F* is the center of both circles). …………………………………(m), (j), MP

∴ (III.6) Two circles are tangent →

they do not have the same center. ……………………………………(n), UG

Logic courses focusing on making Euclid's proofs logically explicit as explained above would be more useful to mathematics students than standard logic courses.[46]

Concluding Remarks

We consider briefly comments by professors of mathematics Robin Hartshorne, Kenneth Kunen, Reuben Hersh, and David Berlinski.

Hartshorne writes:

> Euclid's proof [of Proposition I.1] depends only on the definitions, postulates, and common notions set out at the beginning of Book I. (Hartshorne 2002, 20)

Left unstated (here and elsewhere in the book) is the fact that transitions from one step to the next in a mathematical proof are justified by ('depend on') rules of logic. We have here a 21st century mathematics textbook that still treats logic as a 'silent partner' and presents proofs as they were in Euclid. Hartshorne, however, is hardly alone.

There is also this comment:

> Among experienced mathematicians, there would be little disagreement about what constituted a valid proof, once it was found. (Hartshorne 2002, 11)

It is not made clear how students are to resolve such disagreements; or what 'valid proof' means. In any case, logical argumentation as explained here is sufficient for the purpose of determining whether a mathematical proof is (syntactically) valid.

Kunen writes:

> The justification for the axioms (why they are interesting, or true in some sense, or worth studying) is part of the motivation, or physics, or philosophy; not part of the mathematics. The mathematics itself consists of logical deductions from the axioms. (Kunen 2012, 3-4)

Indeed, the reason theorems are justified (derivation from axioms) does not apply to axioms themselves. This, however, does not mean that the epistemic status of axioms must be extra-mathematical; that students are to look for it elsewhere – not that there is anything wrong with looking in philosophy! For example, an axioms system is justified to the extent that it leads to the derivation of useful results. Sometimes axioms must be rejected because they lead to contradiction, as did Frege's Law V; or because one or more are redundant; these are legitimate, and mathematical, reasons for deciding the epistemic status of axioms. In any case, students should not believe that axioms are 'arbitrary' or 'mere conventions' and thus not objectively true; or true only in some special sense of 'true.' Mathematics must assume that axioms are true (under an

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[46] More advanced proof sketches such as those presented in Aigner and Ziegler 2000 would require a great deal more work to make logically explicit. See also Mueller 1981.

interpretation) and that it is this sense of 'true' that logical argumentation moves from axioms to theorems. The truth of a mathematical result does not appear by magic once it is proved!

Hersh writes:

> I say the 3-cube or the 4-cube – any mathematical object you like – exists at the social-cultural-historic level, in the shared consciousness of people (including retrievable stored consciousness in writing). In an oversimplified formulation, "mathematical objects are a kind of shared thought or idea." (Hersh 1997, 19)

Students told that mathematical objects exist in a 'shared consciousness' may well conclude that the objectivity of mathematics is in doubt. They might be led to infer, as Hersh seems to suggest, that sentences about mathematical objects are true only in a 'social-cultural-historic' sense. Teachers will sow confusion and inhibit learning if they tell students that the interpretation under which the axioms of mathematics are true also depends on 'social-cultural-historic' factors; or that rules of logic and the concept of syntactic validity depend on such factors as well. Mathematics teachers should avoid suggesting that sociology has anything to do with mathematics.

Berlinski writes:

> If the theorems of an axiomatic system follow from its axioms, it is reasonable to ask what *following from* might mean. What *does* it mean? The image is physical, as when a bruise follows a blow, but the connection is metaphorical. The connection between the axioms and the theorems of an axiomatic system is, when metaphors are discarded, remarkably recondite, invisible for this reason to all of the ancient civilizations but the Greek. (Berlinski 2013, 14, original italics)

The logical sense of 'follows from' is not the same as the causal sense Berlinski describes as a 'physical image,' a common (and elementary) misunderstanding. As to the relation between axioms and theorems, Aristotle made it clear in principle long ago, to which modern logic added technically correct details: Derivations of inferred steps are syntactically valid if and only if they match the form of rules of logic.[47] There is nothing 'recondite,' 'metaphorical' or 'invisible' going on here. The seven lessons presented in this article should enable mathematics teachers to make logic transparent and useful to students, who are much more likely to become mathematically proficient as a result.

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[47] Berlinski writes (2013, 16): "Good arguments are good by virtue of their form" and credits Aristotle with this insight; yet, curiously, he seems not to grasp that he has in effect described the syntactic nature of mathematical proof as something not the least bit 'recondite.' Incidentally, Berlinski's claim that only the ancient Greeks understood the formal nature of logical reasoning is debatable. See Gabbay and Woods 2004, Vol. 1, *Greek, Indian and Arabic Logic*.

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