Deduction, Induction and Probabilistic Support

James Cussens

Abstract

Elementary results concerning the connections between deductive relations and probabilistic support are given. These are used to show that Popper-Miller's result is a special case of a more general result, and that their result is not "very unexpected" as claimed. According to Popper-Miller, a purely inductively supports b only if they are "deductively independent"—but this means that $\neg a \vdash b$. Hence, it is argued that viewing induction as occurring only in the absence of deductive relations, as Popper-Miller sometimes do, is untenable. Finally, it is shown that Popper-Miller's claim that deductive relations determine probabilistic support is untrue. In general, probabilistic support can vary greatly with fixed deductive relations as determined by the relevant Lindenbaum algebra.

1 Introduction

The publication of the paper, *A Proof of the Impossibility of Inductive Probability* by Popper and Miller (Popper and Miller, 1983), marked the beginning of a vigorous debate on the relationship between deductive relations and probabilistic support. Popper and Miller (hereafter PM) made the claim that any positive probabilistic support of evidence e for a hypothesis h, as measured by $s(h, e) = p(h, e) - p(h)$ is due solely to deductive relations (properly understood) between e and h. An immediate corollary is that inductive (i.e. non-deductive) probabilistic support does not exist. All probabilistic support is deductive.

It is not my intention to give an account of the ensuing controversy, the interested reader is referred to (Mura, 1990), which contains a list of the relevant papers. There is also a detailed survey in (Cussens, 1991). Instead, I will focus on two points. First, I identify and take issue with the concept of 'induction' employed by PM. Second, I refute the claim that deductive relations, however widely conceived, determine probabilistic support. In order to do this, I will first give a brief account of the Popper-Miller argument against inductive probabilistic support, followed by some elementary results concerning some of the relevant concepts.

2 Aspects of the Popper-Miller Argument

2.1 The "Main Thesis"

Consider any two propositions h and e. We have the following from elementary logic.

$$\vdash h \leftrightarrow ((h \leftarrow e) \wedge (h \vee e)) \tag{1}$$

$$e \vdash h \vee e \tag{2}$$

PM describe $h \leftarrow e$, as "that part of the hypothesis [h] that is not deductively entailed by the evidence [e]". It is then simple to show that $s(h \leftarrow e, e) < 0$ so long as $p(h, e) < 1$ and $p(e) < 1$. This is the core of PM's argument that probabilistic support derives from deductive relations. Jeffrey (Jeffrey, 1984) criticises this argument, using the example of $h = e \wedge f$. f is part of h that does not follow deductively from e and yet is not contained in $h \leftarrow e$ ($h \leftarrow e \nvdash f$ with $h \vdash f$ and $e \nvdash f$). Jeffrey appears to refute the Popper-Miller position by showing that $h \leftarrow e$ *does not* "contain all of h that does not follow deductively from e".

PM can argue as follows, however. Although $e \nvdash f$, e may deductively entail *part* of f. If we 'extract' from f all that is deductively entailed by e, it is *only then* that we are left with "all of h that does not follow deductively from e". We find that if we perform this extraction in a natural way we are indeed left with $h \leftarrow e$. As PM put it "we find that what is left of h once we discard from it everything that is logically implied by e, is a proposition that in general is counterdependent on e" (Popper and Miller, 1987).

This observation constitutes PM's "main thesis". Their argument involves a consideration of the consequence classes of propositions. For PM, the deductive relations between two propositions are contained in the totality of logical implications between elements of their consequence classes. In particular, for PM, *deductive dependence* is measured by the amount of overlap between consequence classes. Deductive dependence and, in particular, deductive dependence of magnitude zero (deductive independence), plays a crucial rôle in the PM account of probabilistic support, so we will now state three elementary results concerning deductive independence.

2.2 Deductive Independence

Definition 2.1 *For any proposition b, $Cn(b)$ is the class of all consequences of b which are not logical truths.* $Cn(b) = \{x : b \vdash x \text{ and } \nvdash x\}$.

Definition 2.2 *Two propositions, a and b, are* deductively independent *iff* $Cn(a) \cap Cn(b) = \emptyset$. *Otherwise they are* deductively dependent.

Lemma 2.3 *For any two propositions a and b,* $Cn(a) \cap Cn(b) = Cn(a \vee b)$.

Proof of 2.3

$$\begin{aligned}
x \in Cn(a) \cap Cn(b) &\Leftrightarrow a \vdash x,\, b \vdash x,\, \nvdash x \\
&\Leftrightarrow \vdash \neg a \vee x,\, \vdash \neg b \vee x,\, \nvdash x \\
&\Leftrightarrow \vdash (\neg a \vee x) \wedge (\neg b \vee x),\, \nvdash x \\
&\Leftrightarrow \vdash (\neg a \wedge \neg b) \vee x,\, \nvdash x \\
&\Leftrightarrow \vdash \neg(a \vee b) \vee x,\, \nvdash x \\
&\Leftrightarrow a \vee b \vdash x,\, \nvdash x \\
&\Leftrightarrow x \in Cn(a \vee b)
\end{aligned}$$

□

Corollary 2.4 *For any two propositions a and b, a and b are deductively independent iff* $\vdash a \vee b$.

Proof of 2.4 Using Lemma 2.3, $Cn(a) \cap Cn(b) = \emptyset \Leftrightarrow Cn(a \vee b) = \emptyset \Leftrightarrow \vdash a \vee b$. □

Corollary 2.5 *b is deductively independent of a iff* $\neg a \vdash b$.

Proof of 2.5 $\vdash a \vee b \Leftrightarrow \neg a \vdash b$. The result then follows immediately from Corollary 2.4 and the definition of deductive independence. □

2.3 Deductive Independence and Probabilistic Countersupport

We now give the essential connections between deductive independence and probabilistic support. We begin with a useful lemma connecting $s(a, b)$ and $s(a, \neg b)$.

Lemma 2.6 *Assuming* $p(b), p(\neg b) > 0$*,*

$$s(a, b)p(b) = -s(a, \neg b)p(\neg b)$$

Proof of 2.6 $s(a, b)p(b) = [p(a \wedge b)/p(b) - p(a)]p(b) = p(a \wedge b) - p(a)p(b)$. Similarly, $s(a, \neg b)p(\neg b) = p(a \wedge \neg b) - p(a)p(\neg b)$. So $s(a, b)p(b) + s(a, \neg b)p(b) = p(a \wedge b) + p(a \wedge \neg b) - p(a)p(\neg b) - p(a)p(b) = p(a)[1 - (p(b) + p(\neg b))] = 0$. The result follows. □

Theorem 2.7 *If* a *and* $\neg b$ *are deductively independent and* $p(b) > 0$ *then* $p(a) < 1 \Rightarrow s(a, b) > 0$ *and* $p(a) = 1 \Rightarrow s(a, b) = 0$

Proof of 2.7 If a and $\neg b$ are deductively independent then it follows from Corollary 2.4 that $\vdash a \vee \neg b$. Hence $p(b) > 0 \Rightarrow p(a, b) = 1$. So

$$s(a, b) = 1 - p(a) \begin{cases} > 0 & \text{if } p(a) < 1 \\ = 0 & \text{if } p(a) = 1. \end{cases}$$

□

Corollary 2.8 *If* $s(a, b) < 0$ *then* a *is deductively dependent on* $\neg b$*.*

Proof of 2.8 If $s(a, b) < 0$ then $s(a, b)$ is defined, hence $p(b) > 0$. The result then follows from Theorem 2.7. □

Theorem 2.9 *If* a *and* b *are deductively independent and* $p(b) > 0$ *then*

$$s(a, b) = p(a, b) - p(a) = \begin{cases} = 0 & \textit{if } p(a) = 1 \textit{ or } p(b) = 1 \\ < 0 & \textit{otherwise} \end{cases}$$

(This result is a direct consequence of Theorem 1 in (Popper and Miller, 1987).)

Proof of 2.9 If $p(b) = 1$ then it follows trivially that $s(a, b) = 0$, so from now on assume that $p(b) < 1$, or equivalently that $p(\neg b) > 0$. If $p(a) = 1$, then, since $p(\neg b) > 0$, $Cn(a) \cap Cn(b) = \emptyset \Rightarrow s(a, \neg b) = 0$ by Theorem 2.7, and therefore $s(a, b) = 0$ by Lemma 2.6. If $p(a) < 1$, then $Cn(a) \cap Cn(b) = \emptyset \Rightarrow s(a, \neg b) > 0$ by Theorem 2.7, and therefore $s(a, b) < 0$ by Lemma 2.6. □

Corollary 2.10 *If $s(a, b) > 0$ then a is deductively dependent on b.*

Proof of 2.10 If $s(a, b) < 0$ then $s(a, b)$ is defined, hence $p(b) > 0$. The result then follows from Theorem 2.9. □

We can now apply the general results above to the specifics of the Popper-Miller argument. PM note that $h \leftarrow e$ is "what is left of h once we discard from it everything that is logically implied by e" (Popper and Miller, 1987). In other words $Cn(h \leftarrow e)$ is the class of consequences of h that are not deductively dependent on e, in particular $h \leftarrow e$ is deductively independent of e. Now it follows from Theorem 2.9 that $s(h \leftarrow e, e) < 0$ unless $p(h) = 1$ or $p(e) = 1$, in which case $s(h \leftarrow e, e) = 0$. PM state that this is a "very unexpected result" (Popper and Miller, 1987), but surely the result is not surprising, since $\neg(h \leftarrow e) \vdash e$. This is a special instance of the relationship between deductive independence and deductive inference given in Corollary 2.5.

3 Induction in the Popper-Miller Argument

In the Popper-Miller argument, PM are concerned with *pure* inductive support, i.e. inductive support that occurs with deductive independence, rather than with inductive support in general.[1]

> ... unless h happens to be deductively independent from e, the values of $d(h, e)$ and $s(h, e)$ are deductively contaminated. If there is such a thing as pure *inductive* dependence at all, there seems nothing for it but to measure it by something like $s(h \leftarrow e, e)$ or $d(h \leftarrow e, e)$. (Popper and Miller, 1987)

[1] $d(a, b) = p(a \wedge b) - p(a)p(b)$, a function closely related to covariance. We will not examine this function in the present paper.

There are two problems with this focus on pure inductive support. Firstly, for the dependence between a and b to be purely inductive, it apparently has to be the case that a and b are deductively independent, but this is equivalent to having $\neg a \vdash b$. So, if the dependence between a and b is purely inductive, the deductive dependence between $\neg a$ and b has to be maximal. This connection between pure inductive dependence and deductive consequence shows that it is not possible to define a notion of purely inductive dependence, which is free from 'deductive contamination'.

The second related problem is that PM seem to view "deductively contaminated" inductive support as not really inductive. This is made evident by the claim that inductive support can only occur in the presence of deductive independence. However, there seems no reason to suppose that inductive (i.e. ampliative) inference should not be deductively contaminated. There can be a *relation* between deduction and induction, without the two types of inference being equivalent, or one reducible to the other. In fact, I take the investigation of this relation by PM to be the most useful contribution made by the Popper-Miller argument.

Although their concern is with pure inductive support, PM usually omit the word 'pure' in their arguments. However, it is clear that there is a big difference between pure inductive support, as defined by PM, and inductive support as it is usually conceived (by inductivists, for example). The latter conception is given by Pierce, who is quoted by PM (Popper and Miller, 1987), where he characterises "*amplifative, synthetic*, or (loosely speaking) *inductive*" reasoning occuring when "the facts summed up in the conclusion are not amongst those stated in the premisses".

It is clear that these two conceptions of inductive inference are not co-extensive. Recall Jeffrey's example, as reported in Section 2.1. On the PM view, "the 'ampliative' part of h relative to e was identified with this conditional $h \leftarrow e$" (Popper and Miller, 1987) and f is not an ampliative part of h relative to e, because e and f are, in general, deductively dependent. On the other hand, according to Pierce's definition, f would count as an ampliative part of h relative to e, simply because $e \not\vdash f$.

This difference between the usual view of induction, as ampliative inference as defined by Peirce, and the notion of induction as the absence of deductive relations, employed by PM, explains the incredulity with which many of PM's critics view the Popper-Miller argument. The usual complaint is that for a proposition to 'go beyond' evidence e, it is not necessary for it to be deductively independent of e, in the sense defined above. It also accounts for the superficially puzzling remark of Miller (quoted in (Mura, 1990)) that the prediction $h =$ 'The sun will rise tomorrow.' is

> not itself 'about' the future; indeed, it consists of a conjunction of a sentence 'about' the past and a sentence countersupported by the evidence.

Now on a view of inductive inference as merely ampliative as defined by Peirce, h will count as an inductive inference, since its truth is not established by evidence. It makes no difference that, if $e =$ 'The sun has risen every day so far.', then $h \vee e$ is a sentence 'about' the past and $h \leftarrow e$ is a sentence countersupported by e. Alternatively, on a view of induction that sees it as a sort of complement to deduction, then it is natural to see only $h \leftarrow e$ as an inductive inference, a proposition that is indeed countersupported by the evidence e.

4 Can Deductive Relations Explain Probabilistic Support?

PM claim that

> Although evidence may raise the probability of a hypothesis above the value it achieves on background knowledge alone, every such increase in probability has to be attributed entirely to the *deductive connections* that exist between the hypothesis and the evidence. (Popper and Miller, 1987)

Now it is clear that PM have given *a necessary condition for the existence of (positive) probabilistic support*: $s(a, b) > 0$ implies a and b are deductively dependent, or, equivalently, $\neg a \vdash b \Rightarrow s(a, b) \leq 0$. However, if the existence of probabilistic support could really be "attributed entirely to *deductive connections*" then surely we would need a condition on a and b, defined entirely in terms of deductive relations, which would hold if *and only if* $s(a, b) > 0$. PM have not supplied such a deductive condition, and we will see below that one is not possible.

PM sometimes seem to go further than merely asserting that the *existence* of probabilistic support is attributable to deductive connections; in some places they appear to be claiming that the *extent* of probabilistic support is decided by deductive relations. For instance, when discussing the properties of the two support functions $\sigma(h,e) = p(h,e)/p(h)$ and $s(h,e) = p(h,e) - p(h)$, they state:

> The measure σ, no less than the measure s, sustains the view that differences in probabilistic support are to be attributed entirely to differences in degrees of deductive dependence. (Popper and Miller, 1987)

However, any claim that the extent of probabilistic support is determined by deductive relations is demonstrably false as we shall now show.

Eells (Eells, 1988) has already cast serious doubt on the thesis that all probabilistic support is deductive in nature. He notes that if we measure the support that e gives to h with support functions based on *different probability measures*, then we will get different levels of support, notwithstanding the fact that the arguments of the functions (h and e) are identical. We can choose s and s' such that $s(h,e) \neq s'(h,e)$.

However, Eells has what I take to be an unduly conservative view of what can count as deductive support.

> Popper and Miller associate a *degree* with the component of an evidence's support of a hypothesis that they call purely deductive support... But, properly understood, it seems that support that is purely deductive in nature is an 'all or nothing' affair; either the evidence fully guarantees the truth of the hypothesis (deductively implies it) or it does not fully guarantee the truth of the hypothesis (does not deductively imply it). Purely deductive support does not come in degrees. (Eells, 1988)

One of the central claims of PM is that probabilistic support is determined by deductive dependence, as measured by the overlap of consequence classes. To be able to examine this claim it is necessary to allow that deductive support may occur in degrees. We will show that deductive relations, no matter how widely coneived, do not determine probabilistic support. We do this by first noting that, for a given first-order language L, deductive relations between formulae in L are completely determined by the algebraic structure of the Lindenbaum algebra of L.

The Lindenbaum algebra of L, denoted $B(L)$ is a Boolean algebra the members of which are equivalence classes of logically equivalent formulae in L. So if $|\alpha| = \{\beta \in L : \vdash \alpha \leftrightarrow \beta\}$, then $B(L)$ is the set $\{|\alpha| : \alpha \in L\}$, equipped with the relation $\leq$, where $|\alpha| \leq |\beta| \Leftrightarrow \vdash \alpha \rightarrow \beta$. The logical operations of conjuction, disjunction and negation in L, then correspond to the algebraic notions of meet, join and complement in $B(L)$ as follows:

$$|\alpha| \wedge |\beta| = |\alpha \wedge \beta|, \quad |\alpha| \vee |\beta| = |\alpha \vee \beta|, \quad \neg|\alpha| = |\neg\alpha| \tag{3}$$

It is then simple to define probability functions on this algebra, so that the value of $s(a, b)$ varies. Since the various probability functions are defined on one and the same algebra, it follows immediately that deductive relations of any sort remain constant, while $s(a, b)$ varies. This is sufficient to refute the claim that deductive relations determine probabilistic support. This is essentially the same argument used by Howson (Howson, 1973), when criticising Popper's claim that the 'logical' probability of universal laws is always zero.

It might be argued that a given probability measure *implicitly* assumes certain deductive relations between the elements of the algebra on which it is defined. These deductive relations can, perhaps, not be expressed adequately in the language used. A sufficiently rich language is required to express all the necessary deductive relations, and it is only when using such a language that deductive relations fix probabilistic support. On this view, a language rich enough to express all the relevant deductive relations will engender a rich and complex Lindenbaum algebra, where deductive relations *do* fix $s(a, b)$. However, against this, the following (rather elementary) lemma and theorem show that, in general, deductive relations do not restrict possible values for $s(a, b)$ at all, whatever the identity of the relevant Lindenbaum algebra.

Lemma 4.1 $p(b) > 0 \Rightarrow s(a, b) = -s(\neg a, b)$

Proof of 4.1 If $p(b) > 0$ then $s(a, b) + s(\neg a, b) = [p(a, b) - p(a)] + [p(\neg a, b) - p(\neg a)] = [p(a, b) + p(\neg a, b)] - [p(a) + p(\neg a)] = 1 - 1 = 0$. The result follows. $\square$

Theorem 4.2 *Let a and b be members of some arbitrary Lindenbaum algebra $B(L)$. If $a \neq \bot$, $b \neq \bot$, $a \neq \top$, $b \neq \top$, then exactly one of the following five possibilities holds.*

1. *$a \wedge b = \bot$. $s(a,b)$ can take any value in the interval $(-1, 0]$.*

2. *$a \wedge \neg b = \bot$. $s(a,b)$ can take any value in the interval $[0, 1)$.*

3. *$\neg a \wedge b = \bot$. $s(a,b)$ can take any value in the interval $[0, 1)$.*

4. *$\neg a \wedge \neg b = \bot$. $s(a,b)$ can take any value in the interval $(-1, 0]$.*

5. *None of the above four conditions obtains. $s(a,b)$ can take any value in the interval $(-1, 1)$.*

Proof of 4.2 The condition that $a \neq \bot$, $b \neq \bot$, $a \neq \top$, $b \neq \top$ is enough to ensure that exactly one of the above conditions obtains. We have that

$$s(a,b) = p(a \wedge b)/(p(a \wedge b) + p(\neg a \wedge b)) - p(a \wedge b) - p(a \wedge \neg b)$$

We will consider how admissible choices of values for $p(a \wedge b)$, $p(\neg a \wedge b)$ and $p(a \wedge \neg b)$ constrain the value of $s(a,b)$. We will take each case in turn.

1. If $a \wedge b = \bot$ then $p(a \wedge b) = 0$ and, on the condition that $p(\neg a \wedge b) > 0$ (so that $p(b) > 0$) it follows that $s(a,b) = -p(a \wedge \neg b)$. Let $x \in (-1, 0]$. By choosing $p(a \wedge \neg b) = -x$ and $p(\neg a \wedge b) > 0$, it follows that $s(a,b) = x$.

2. If $a \wedge \neg b = \bot$ then $p(a \wedge \neg b) = 0$ and $s(a,b) = p(a \wedge b)/[p(a \wedge b) + p(\neg a \wedge b)] - p(a \wedge b)$. Let $x \in [0, 1)$. By choosing $p(\neg a \wedge b) = 0$, we have that $p(a \wedge b) > 0 \Rightarrow s(a,b) = 1 - p(a \wedge b)$. So by choosing $p(a \wedge b) = 1 - x$, it follows that $s(a,b) = x$.

3. From the proof of Case 1, it follows that $s(\neg a, b)$ can take any value in $(-1, 0]$. The result then follows from Lemma 4.1.

4. From the proof of Case 2, it follows that $s(\neg a, b)$ can take any value in $[0, 1)$. The result then follows from Lemma 4.1.

5. If none of $a \wedge b$, $a \wedge \neg b$, $\neg a \wedge b$ and $\neg a \wedge \neg b$ equal $\bot$, then each can take any value in the interval $[0, 1]$, subject to the constraint that $p(a \wedge b) + p(a \wedge \neg b) + p(\neg a \wedge b) + p(\neg a \wedge \neg b) = 1$. From above, by choosing $p(a \wedge b) = 0$ we can have $s(a,b)$ take any value in $(-1, 0]$ and by having $p(\neg a \wedge b) = 0$, we can have $s(a,b)$ take any value in $[0, 1)$. So in this case, $s(a,b)$ can take any value in $(-1, 1)$.

□

The following shows that even in the special case of a hypothesis h and evidence e, where $h \vdash e$, $s(h, e)$ can vary considerably with fixed deductive relations between h and e.

Corollary 4.3 *If $h \neq \bot$, $e \neq \bot$, $h \neq \top$, $e \neq \top$ and $h \vdash e$, where h and e are members of some arbitrary algebra, then $s(h, e)$ can take any value in $[0, 1)$.*

Proof of 4.3 $h \vdash e \Leftrightarrow h \wedge \neg e = \bot$. The result follows from above. □

5 Conclusion

Case 5 in Theorem 4.2 is sufficient to prove the non-existence of a deductive condition equivalent to the existence of (positive) probabilistic support, since we can have either $s(a, b) > 0$ or $s(a, b) < 0$ by varying the probability distribution on $B(L)$, whilst keeping a, b and $B(L)$ (i.e. the deductive relations) constant. We have also found in Corollary 4.3, that in the special case where $h \vdash e$, $s(h, e)$ can vary considerably with fixed deductive relations.

Our main finding, then, is that the Popper-Miller argument is invalid in two major ways. Any notion of 'induction' as a sort of complement to deduction seems untenable. According to PM pure inductive dependence between a and b can only occur only when there is no deductive dependence, as they define it, between a and b. But this occurs if and only if $\neg a \vdash b$, which is a highly deductive relation. It is therefore not surprising that deductively independent propositions give negative probabilistic support to each other. For example, $s(h \leftarrow e, e) \leq 0$ simply because $\neg(h \leftarrow e) \vdash e$ which is not a "very unexpected result" by any means.

Also, the claim that the probabilistic support of evidence for a hypothesis "has to be attributed entirely to *deductive connections*" is demonstrably false. In fact, except in a few special cases, there will be uncountably infinitely many possible values for $s(h, e)$, with deductive relations between h and e fixed, no matter how widely the latter are conceived. This is simply due to vast number of probability distributions which are definable on most algebras. Picking on a particular probability function seems to be an irreducibly inductive step. All this leads inescapably to the conclusion that *probabilistic support cannot be explained in terms of deductive relations*: it must therefore depend, at least partly, on non-deductive factors.

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