The Square of Opposition and Generalized Quantifiers

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Abstract. In this paper I propose a set-theoretical interpretation of the logical square of opposition, in the perspective opened by generalized quantifier theory. Generalized quantifiers allow us to account for the semantics of quantificational Noun Phrases, and of other natural language expressions, in a coherent and uniform way. I suggest that in the analysis of the meaning of Noun Phrases and Determiners the square of opposition may help representing some semantic features responsible to different logical properties of these expressions. I will conclude with some consideration on scope interactions between quantifiers.

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1. Noun Phrases as Generalized Quantifiers

Let M be a non-empty set, the universe, of a model $\mathcal{M}$ and Q_M any set of subsets of M. Then, in the Lindström terms [3], Q_M is a type $\langle 1 \rangle$ Generalized Quantifier (GQ, here and henceforth) whose meaning is given by:

$$Q_M(A) \text{ is true iff } A \in Q_M \tag{1}$$

where the Q_M on the left side of the iff-statement is meant to be a quantificational operator of the functional type $(et)t$ (a function mapping predicates onto truth-values, A being a first-order predicate, of type et), and the Q_M on the right side is to be interpreted as a subset of $\wp(M)$, the power set of M (Q_M is a second order unary predicate).

The type $\langle 1 \rangle$ generalized quantifier is a straightforward illustration of the "set-theoretical" conception of quantifiers in the GQ theoretical framework: a quantifier is conceived as a second order n-ary relation between sets (i.e. between first-order predicates). For instance, a type $\langle k_1 ... k_n \rangle$ quantifier, where $k_1 = k_2 = ... = k_n = m$, is a second order n-ary relation between m-ary predicates.

Generalized quantifiers can be written as a variable-binding operators. As a simple case, a type $\langle 1 \rangle$ quantifiers Q_M can be notated in the usual way as $Q(x)\phi(x)$. Let $\mathcal{M} \vDash \phi$ be the *satisfaction* relation between a model and a formula, and $[\![\phi(x)]\!]_{\mathcal{M}}$ the "extension" of $\phi(x)$ in $\mathcal{M}$ (the set of the elements of M satisfying ϕ). The following statement provides the semantics of a formula in which x is bound by the generalized quantifier Q_M, as variable-binding operator:

$$\mathcal{M} \vDash Qx\phi(x) \text{ iff } [\![\phi(x)]\!]_{\mathcal{M}} \in Q_M \tag{2}$$

This notation suggests a compositional account for the first-order logic formulas with quantifiers. It is well-known that standard universal and existential quantifiers do not have a denotation in the predicate calculus, and that, as a consequence, it is not possible to compositionally interpret formulas in which they occur. But, as type $\langle 1 \rangle$ quantifiers, standard universal and existential quantifiers denote sets of sets:

$$\forall_M = \{M\}$$
$$\exists_M = \{A \subseteq M : A \neq \varnothing\}$$

Now, we can easily express, in a "compositional fashion", the meaning of a formulas such as $\forall x\phi(x)$:

$$\mathcal{M} \vDash \forall x\phi(x) \text{ iff } [\![\phi(x)]\!]_{\mathcal{M}} \in [\![\forall_M]\!] \text{ iff } [\![\phi(x)]\!]_{\mathcal{M}} = M \tag{3}$$

The denotation of $\forall x\phi(x)$ is constructed out by applying the denotation of the quantifier to the denotation of the open formula. So conceived, standard quantifiers end up to show the same functional property of Noun Phrases in natural languages.

1.1. Noun Phrases as type $\langle 1 \rangle$ quantifiers

Noun Phrases (NP), typed $(et)t$, are uniformly interpreted (following Montague) as type $\langle 1 \rangle$ quantifiers, in their role of mapping the predicates expressed by the Verb Phrases onto $\{True, False\}$. The following 'parse tree' illustrates the point.

S
t

NP $(et)t$ — VP et

This tree is the representation of the syntactic and semantic structure of the nuclear sentence[1] (in English, but also in other natural languages). As well as a sentence is constructed out by combining a NP with a VP, its meaning, a truth-value, is obtained by a step of functional application of the meaning of the NP to the meaning of the VP. In the following, is showed this process of meaning computation, for a simple sentence like "every sailor drink":

$$\begin{aligned} &[\![[every\ sailor]_{NP}[drink]_{VP}]\!]_t \text{ iff} \\ &[\![every\ sailor]\!]_{(et)t}([\![drink]\!]_{et}) \text{ iff} \\ &drink \in \{A \subseteq M : sailor \subseteq A\} \Leftrightarrow drink \subseteq sailor \end{aligned} \tag{4}$$

[1] S is the symbol for the syntactic category of 'sentence', VP for 'verb phrase'.

provided that the meaning of "every sailor" is conceived as a generalized quantifier:

$$[\![every\ sailor]\!]_{(et)t} = (every\ sailor)_M = \{A \subseteq M : sailor \subseteq A\} \tag{5}$$

The uniform treatment of NPs as type $\langle 1 \rangle$ quantifiers (Mostowski GQ, [5]) is assured by the "type-raising" of the meaning of proper names, as showed by the meaning computation for a sentence like "John drinks":

$$\begin{array}{l} [\![[john]_{NP}[drinks]_{VP}]\!]_t \text{ iff} \\ [\![john]\!]_{(et)t}([\![drink]\!]_{et}) \text{ iff} \\ drink \in \{A \subseteq M : john \in A\} \Leftrightarrow john \in drink \end{array} \tag{6}$$

provided that the meaning of "John" is conceived as a generalized quantifier:

$$[\![john]\!]_{(et)t} = (john)_M = \{A \subseteq M : john \in A\} \tag{7}$$

1.2. Determiners as type $\langle 1, 1 \rangle$ quantifiers

As a natural consequence of Lindström generalization of type $\langle 1 \rangle$ quantifiers to type $\langle k_1 ... k_n \rangle$ quantifiers, it is possible to semantically construe NPs as the result of the combination of Determiners (DET) with common nouns (that are first-order predicates). DETs are interpreted as type $\langle 1, 1 \rangle$ quantifiers, i.e. functions mapping predicates into type $\langle 1 \rangle$ quantifiers. As for sentences, the meaning of NPs is the result of a compositional process, in which the meaning of a DET is applied to that of a common noun:

NP
$(et)t$

DET $(et)((et)t)$ CN et

Type $\langle 1, 1 \rangle$ quantifiers can be defined along the same line of the previous definition for type $\langle 1 \rangle$ quantifiers. Let M be the universe of a model $\mathcal{M}$ and Q_M any set of pairs of subsets of M. Then Q_M is a type $\langle 1, 1 \rangle$ GQ whose meaning is given by:

$$Q_M(A, B) \text{ is true iff } \langle A, B \rangle \in Q_M \tag{8}$$

where $Q_M(A, B)$ on the left side of the iff-statement is a quantificational operator of type $(et)((et)t)$ and $Q_M(A, B)$ on the right side is a subset of $\wp(M) \times \wp(M)$.

Again, an example could help illustrating this definition at work. Let consider the meaning of a definite NP like "the boy", resulting from the application of the meaning of "the" to the meaning of "boy":

$$[\![the\ boy]\!]_{(et)t} = [\![the]\!]_{(et)((et)t)}([\![boy]\!]_{et})$$

Since the meaning of the definite determiner is:

$$the(A, B) \text{ is true iff } A \subseteq B \wedge |A| = 1 \tag{9}$$

we have the following semantic computation for deriving the meaning of the definite NP:

$$\begin{aligned}&[\![the]\!]([\![boy]\!]) \text{ iff}\\ &the(boy, B) \text{ iff}\\ &boy \subseteq B \wedge |boy| = 1\end{aligned} \tag{10}$$

In the last line of this derivation there is a type $\langle 1 \rangle$ quantifier that can equivalently be written as: $the(boy)_M = \{A \subseteq M : boy \subseteq A \wedge |boy| = 1\}$. Moreover, note that (9) explicates the meaning of the singular *the*; for the plural case we have to modify the expression as follows:

$$the_{pl}(A, B) \text{ iff } A \subseteq B \wedge |A| > 1 \tag{11}$$

In order to show the expressive power of GQ theory, and its wide applicability to the analysis of the semantics of natural language determiners, a list of DETs as type $\langle 1, 1 \rangle$ quantifiers is presented. The four quantifiers of the Aristotelian square of opposition are:

$$\begin{aligned}&all(A, B) \Leftrightarrow A \subseteq B\\ &no(A, B) \Leftrightarrow A \cap B = \varnothing\\ &some(A, B) \Leftrightarrow A \cap B \neq \varnothing\\ &not\ all(A, B) \Leftrightarrow A - B \neq \varnothing\end{aligned} \tag{12}$$

Below some other well-known quantifiers are listed:

$(Q^R)_M \Leftrightarrow |A| > \frac{|M|}{2}$(the Rescher quantifier for finite universe)
$most(A, B) \Leftrightarrow |A \cap B| > |A - B|$
$ten(A, B) \Leftrightarrow |A \cap B| \geq 10$
$the\ ten(A, B) \Leftrightarrow |A| = 10 \wedge A \subseteq B$
$John's(A, B) \Leftrightarrow A \cap \{b : owner(j, b)\} \subseteq B \ \wedge \ |A \cap \{b : owner(j, b)\}| > 1$

(in the last line is represented the case of possessive construction in plural NPs, like "John's bikes", while $|A \cap \{b : owner(j, b)\}| = 1$ is the GQ for the case of singular NP, like "John's bike".)

2. Squaring Generalized Quantifiers

As set-theoretical entities, generalized quantifiers can be the object of application of the boolean operations. Among the other boolean operations, in what follows I will focus on negative operations and on the structural and logical properties showed by quantifiers when they are undergoing those operations. The two negative operations on GQs are the outer negation and the inner negation.

Let Q be a GQ. We adopt the following notational convention:

i. $\neg Q$ is the outer negation of Q
ii. $Q\neg$ is the inner negation of Q
iii. $\neg Q \neg = Q^d$ is the dual of Q (outer negation of the inner negation).

A quantifier, together with its outer negation, inner negation and dual forms a natural unit, the *square*:

$$square(Q) = \{Q, \neg Q, Q\neg, Q^d\} \tag{13}$$

Since a type $\langle 1 \rangle$ quantifier Q_M, on a domain M, is a subset of $\wp(M)$, the square of Q may straightforwardly be characterized in terms of set-theoretical formulas as follows:

i. $\neg Q_M(A) = \{\wp(M) - Q(A)\}$
ii. $Q_M\neg(A) = \{Q(M - A)\}$
iii. $Q^d(A) = \{\wp(M) - Q(M - A)\}$

This square is naturally extensible for a type $\langle 1, 1 \rangle$ quantifier $Q(A, B)$:

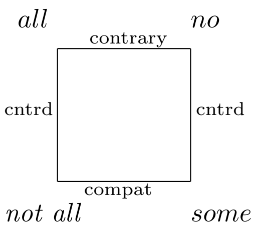
i. $\neg Q_M(A, B) = \{\wp(M)^2 - Q(A, B)\}$
ii. $Q_M\neg(A, B) = \{Q(A, M - B)\}$
iii. $Q^d(A, B) = \{\wp(M)^2 - Q(A, M - B)\}$

The following are some obvious general facts about $square(Q)$:

- if Q is non-trivial ($Q_M \neq \wp(M)$ and $Q_M \neq \varnothing$), so are the other quantifiers in its square
- if $Q' \in square(Q)$ then $square(Q') = square(Q)$
- $square(Q)$ has either two or four members
- if $Q' = Q^d$ then Q and Q' are inter-definable.

2.1. Classical vs modern square

The debate on the difference between classical and modern square of opposition is nowadays still on (see [7] and [6] for a wide discussion on the topic). The classical square represents logical relations among sentences in which a single quantificational determiners occurs in subject position, the so-called "categorical judgments". The modern version is intended to represent the "span" of a quantifier when undergoing the negative operations.



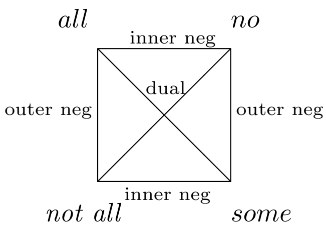


FIGURE 1. On the left is showed the classical square of opposition, on the right the modern interpretation. The logical relations in the classical square involve the existential import.

Clearly, the modern version is a generalization of the classical square of opposition. By interpreting the modern square in set-theoretical terms, the logical relations holding among quantifier of the classical square are easily reproducible, under some conditions on the set-theoretical objects involved. The main case is that of the existential import of ALL in the classical square. In the modern sense, ALL is conceived without existential import, but, given that $ALL_M(A,B) \Leftrightarrow A \subseteq B$ and $ALL_M\neg(A,B) \Leftrightarrow A \subseteq M - B$, to ensure that they can not both be true (for reproducing the contrary relation) is simply required that A is not empty.

Other consequence of the more generality of the modern square is that it is applicable to every quantifier and thus it displays more utility for investigating the semantics of NPs. Moreover, it is worth noting that, in many natural languages, there are pairs of quantificational expressions (DETs), but also of expressions belonging to other categories, like adverbials (*still/already*) or conjunctions (*because/although*), whose denotation exhibits the pattern of duality. This may be a sign that the square of logical duality codifies some deep and pervasive pattern of the semantics of natural languages expressions embodying some quantificational feature.

3. Squaring NPs properties

The semantic homogeneity of NPs is due to their functional homogeneity as type $\langle 1 \rangle$ generalized quantifiers. But type $\langle 1 \rangle$ quantifiers have different logical properties, splitting them into three fundamental categories: ultrafilter, filter and "intersective" quantifiers, as outlined in the following table:

ultrafilter	Proper Names, definite NP$_{sg}$s, Pronouns *John, the boy, I*, etc.
filter	definite quantified NPs *every boy, no student*, etc.
intersective	indefinite quantified NPs *some boys, a boy, two boys*, etc.

Briefly, what distinguishes a filter Q from an intersective quantifier Q' is that Q and $Q\neg$ cannot be both true (they are contrary), while Q' and $Q'\neg$ do (they are compatibles). What distinguishes an ultrafilter Q from a filter Q' is that Q and $Q\neg$ cannot be neither both true nor both false (they are contradictory), while Q' and $Q'\neg$ can be both false[2]. This way of distinguishing filters from ultrafilters

[2]The distinction here introduced is slightly different from the original characterization of definite NPs as filters due to Barwise and Cooper [2] and revisited by Loebner [4]. For example, the classification here proposed has the concept of filter in common with the characterization of NPs in Loebner [4], but the method of "squaring" logical properties of quantifiers involves some not negligible differences. The effect of the inner negation in determining contradiction or contrariness is here the only parameter used in distinguishing quantifiers. As a consequence, three classes of quantifiers are proposed, while Loebner essentially distinguishes only between

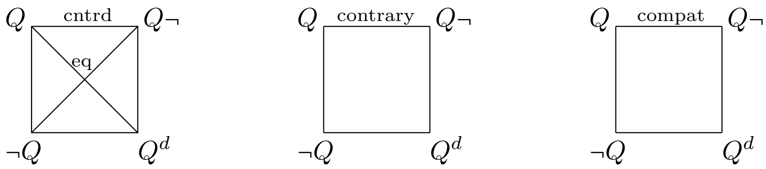


FIGURE 2. Using the modern version of the square, we can easily visualize different patterns of logical relations in the $square(Q)$ of a quantifier Q. From left to right, the first square sketches the logical pattern for ultrafilters (Q and $Q\neg$ are contradictory), the second for filters (Q and $Q\neg$ are contrary), the third for intersective quantifiers (Q and $Q\neg$ are compatible).

and intersective quantifiers suggests the use of the square in order to figure out logical properties of these classes of NPs. Ultrafilters are individuals, inner and outer negation having the same effect of determining contradiction. Filters and intersective quantifiers are "quantirelations": they have the so-called "isomorphism" (ISOM) property, whereas ultrafilters are non-ISOM. ISOM means that for holding the relation expressed by the quantifiers only the cardinality of sets involved is relevant.

The difference between filters and intersective quantifiers is a matter of different logical effect of inner negation, as a set-theoretical operation. For instance, *all* is a filter because $all_M(A,B) = |A-B| = 0$ and $all_M\neg(A,B) = |A-(M-B)| = |A\cap B| = 0$ cannot both be true, if A is not empty, while *some* is an intersective quantifier because $some_M(A,B) = |A\cap B| \neq 0$ and $some_M\neg(A,B) = |A\cap(M-B)| = |A-B| \neq 0$ can obviously be both true but not both false (if A is not empty). Moreover, we can notice that if Q is a filter, Q^d is an intersective quantifier. This is the case of a GQ like $most_M(A,B) = |A\cap B| > |A-B|$, as showed by its square:

i. $most_M\neg(A,B) = |A\cap B| < |A-B|$
ii. $most^d_M(A,B) = |A\cap B| \geq |A-B|$
iii. $\neg most_M(A,B) = |A\cap B| \leq |A-B|$

By reading this square it is immediately verified that $most^d$ and its inner negation $most^d\neg = \neg most$ can be both true.

filters and ultrafilters, as his main concern, in the cited article, is the semantics of definite NPs, and not NPs overall considered as generalized quantifiers.

3.1. Nested quantifiers

In natural language sentences is usual to have NPs both in subject and object position. And, given the formal under-determination of natural language grammar, each NP, as type $\langle 1 \rangle$ quantifier, can take scope over the other. If this scope commutativity between the two NPs produces a semantic ambiguity or not, is a matter of "relative logical properties" of the quantifiers involved.

Let briefly examine the "scope interaction" between quantifiers. If Q_1 and Q_2 are type $\langle 1 \rangle$ generalized quantifiers and R is a binary relation, we can abbreviate the formula $Q_1(x)Q_2(y)R(x,y)$ as Q_1Q_2R and its inversion $Q_2(y)Q_1(x)R(y,x)$ as $Q_2Q_1R^{-1}$. We can now distinguish three fundamental cases (for each case an example is reported immediately below):

i. *self-commuting* quantifiers:
$$Q(Q(R)) \Leftrightarrow Q(Q(R^{-1}))$$
$$all(all(R)) \Leftrightarrow all(all(R^{-1}))$$
ii. *indipendent* quantifiers:
$$Q_1(Q_2(R)) \Leftrightarrow Q_2(Q_1(R^{-1}))$$
$$all(the(R)) \Leftrightarrow the(all(R^{-1}))$$
iii. a quantifier is (scopally) *dominant* over the other:
$$Q_1(Q_2(R)) \Rightarrow Q_2(Q_1(R^{-1}))$$
$$some(all(R)) \Rightarrow all(some(R^{-1}))$$

Semantic ambiguity rises only in the third case, i.e. when Q_1 is dominant over Q_2. In order to illustrate this fact I will proceed with an example. Let consider the following two sentences:

$$\text{All boys read the books} \tag{14}$$
$$\text{All boys read some book} \tag{15}$$

For sentence (14), the *object narrow scope* (ons) reading and the *object wide scope* (ows) reading are equivalent, as a matter of logic of set-theoretical formulas:

$$\begin{array}{ll} \text{(ons)} & all(boy)[the_{pl}(book)[read]] = \\ & boy \subseteq \{x : book \subseteq \{y : x\ read\ y\} \wedge |book| > 1\} \\ \text{(ows)} & the_{pl}(book)[all(boy)[read^{-1}]] = \\ & book \subseteq \{y : boy \subseteq \{x : x\ read\ y\}\} \wedge |book| > 1 \end{array} \tag{16}$$

For sentence (15), (ons) and (ows) readings are not equivalent, given that (ows) $\Rightarrow$ (ons), as a matter of logic:

$$\begin{array}{ll} \text{(ons)} & all(boy)[some(book)[read]] = \\ & boy \subseteq \{x : book \cap \{y : x\ read\ y\} \neq \varnothing\} \\ \text{(ows)} & some(book)[all(boy)[read^{-1}]] = \\ & book \cap \{y : boy \subseteq \{x : x\ read\ y\} \neq \varnothing\} \end{array} \tag{17}$$

In literature [1], scope dominance is defined as follows: in finite domain Q_1 is dominant over Q_2 iff Q_1 is an "exist" quantifier or Q_2 is a "universal" quantifier. On the square, this condition could be expressed as follows: Q_1 is *dominant* over

Q_2 iff Q_1 is an intersective quantifier or Q_2 is a filter. For what concerns independent quantifiers, retracing the definition in Westersthål [8], we can say that Q_1 and Q_2 are *independent* iff at least one of the following holds: (i) both Q_1 and Q_2 are intersective; (ii) both Q_1 and Q_2 are filters; (iii) one of the two is an ultrafilter. Actually, this definition should be refined, in order to account for cardinal quantifiers. In fact, cardinal quantifiers are intersective but it is easy to see that the bidirectional entailment is not valid:

$$n(m(R)) \not\Leftrightarrow m(n(R^{-1})).$$

However, the undirectional entailment continues to hold:

$$n(m(R)) \Rightarrow m(n(R^{-1})).$$

In other words, in the scope interaction of two cardinal quantifiers Q_1 and Q_2 it is the quantifier that takes scope over the other that is scopally dominant. Thus, each of two cardinal quantifier can be dominant over the other, according to the scope it takes relatively to the other. To accommodate this situation, the definition of independent quantifiers could be refined as follows: Q_1 and Q_2 are independent iff

$$Q_1(Q_2(R)) \Leftrightarrow Q_2(Q_1(R^{-1}))$$

or

$$Q_1(Q_2(R)) \Rightarrow Q_2(Q_1(R^{-1})) \wedge Q_2(Q_1(R)) \Rightarrow Q_1(Q_2(R^{-1})).$$

The latter disjunct is required to account for the "reciprocity" of scope dominance characterizing cardinal quantifiers.

For filters (type $\langle 1,1 \rangle$), we need a more subtle distinction between *co-intersective* (let $Q(A,B)$: only $|B - A|$ is relevant) and *proportional* (both $|A \cap B|$ and $|B - A|$ are relevant). Thus, although $all(A,B)$ and $most(A,B)$ are both filter, *most* is dominant over *all*, because of the relevance, in its meaning, of $|A \cap B|$. The (ons) interpretation of a sentence like "all boys read most books" implies the (ows) reading, but the vice-versa does not hold:

$$most(boy)[all(book)[read]] \Rightarrow all(book)[most(boy)[read^{-1}]]$$

but

$$most(boy)[all(book)[read]] \not\Rightarrow all(book)[most(boy)[read^{-1}]]$$

To sum up, in the paper I tried to apply the square of opposition, conceived as the explication of the logical dualities among quantifiers, to characterize some semantic features of natural language NPs. A strong hypothesis, to be validated, could be that all quantificational NPs fall into one of the three categories (ultrafilters, filters, intersective quantifiers) depending on the logical pattern they give rise when subjected to negative operations, i.e. depending on the "collocation" they find in the square.

Generally, the methodology of "squaring" logical relations of quantificational operators can reveal some utility, for various purposes, and this utility is undoubted in the analysis of the semantic and logical properties of natural language expressions such as NPs, in the Generalized Quantifier framework.

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