The Square of Opposition and Generalized Quantifiers

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Outline

type ⟨1⟩ GQ: definition and notation (I)

Let M be a non-empty set, the universe, of a model $\mathcal{M}$ and Q_M any set of subsets of M.
Then Q_M is a type ⟨1⟩ GQ whose meaning is given by:

$$Q_M(A) \text{ iff } A \in Q_M$$

quantificational operator
of the functional type $(et)t$
(A is of type et)

subset of $\wp(M)$
(second order predicate)

type $\langle 1 \rangle$ GQ: definition and notation (II)

The same written as variable-binding operator (where $\mathcal{M} \vDash \phi$ is the usual *satisfaction* relation between a model and a formula, and $[\![\phi(x)]\!]_{\mathcal{M}}$ is the "extension" of $\phi(x)$ in $\mathcal{M}$):

$$\mathcal{M} \vDash Qx\phi(x) \text{ iff } [\![\phi(x)]\!]_{\mathcal{M}} \in Q_M$$

Standard Quantifiers as type $\langle 1 \rangle$ quantifiers

This notation suggests a compositional account for the FOL formulas with quantifiers.
It is well-known that standard universal and existential quantifiers do not have a denotation in FOL formulas, and that, as a consequence, it is not possible to compositionally interpret formulas in which they occur.
But, as type $\langle 1 \rangle$ quantifiers they denote sets of sets:

$$\forall_M = \{M\}$$
$$\exists_M = \{A \subseteq M : A \neq \emptyset\}$$

compositionality in FOL

Now, we can compositionally express the meaning of a formulas such as $\forall x \phi(x)$:

$$\mathcal{M} \models \forall x \phi(x) \iff [\![\phi(x)]\!]_{\mathcal{M}} \in [\![\forall_M]\!] \iff [\![\phi(x)]\!]_{\mathcal{M}} = M$$

The denotation of $\forall x \phi(x)$ is constructed out by applying the denotation of the quantifier to the denotation of the open formula.

So conceived, standard quantifiers end up to show the same functional property of Noun Phrases (NP) in natural languages.

Noun Phrases as type $\langle 1 \rangle$ quantifiers (I)

NPs, typed $(et)t$, are uniformly interpreted (following Montague) as type $\langle 1 \rangle$ quantifiers, in their role of mapping the predicates expressed by Verb Phrases onto $\{True, False\}$.

S
t

NP $(et)t$ $\qquad$ VP et

$$[\![[every\ sailor]_{NP} [drink]_{VP}]\!]_t \Longleftrightarrow$$
$$[\![every\ sailor]\!]_{(et)t}([\![drink]\!]_{et}) \Longleftrightarrow$$
$$drink \in \{A \subseteq M : sailor \subseteq A\} = drink \subseteq sailor$$

In fact, the set denoted by $(every\ sailors)_M$ is $\{A \subseteq M : sailor \subseteq A\}$

Proper Names as type ⟨1⟩ quantifiers

The uniform treatment of NPs as type ⟨1⟩ quantifiers (Mostowski GQ) is obtained by the “type-raising” of the meaning of Proper Names.

$$[\![[john]_{NP} [drinks]_{VP}]\!]_t \Longleftrightarrow$$
$$[\![john]\!]_{(et)t} ([\![drink]\!]_{et}) \Longleftrightarrow$$
$$drink \in \{A \subseteq M : john \in A\} = john \in drink$$

Determiners as type $\langle 1, 1 \rangle$ quantifiers

As a natural consequence of Lindström generalization of type $\langle 1 \rangle$ quantifiers to type $\langle n_1 ... n_k \rangle$ quantifiers, it is possible to semantically analyze NPs as the result of the combination of Determiners (DET) with common nouns (FO predicates).
DETs are interpreted as type $\langle 1, 1 \rangle$ quantifiers, i.e. functions mapping predicates into type $\langle 1 \rangle$ quantifiers.

$$\begin{array}{ccc} & NP & \\ & (et)t & \\ DET & & CN \\ (et)((et)t) & & et \end{array}$$

type $\langle 1,1 \rangle$ quantifiers: definition

Let M be the universe of a model $\mathcal{M}$ and Q_M any binary relation on subsets of M.
Then Q_M is a type $\langle 1,1 \rangle$ GQ whose meaning is given by:

$$Q_M(A,B) \text{ iff } \langle A,B \rangle \in Q_M$$

quantificational operator of type $(et)((et)t)$

subset of $\wp(M) \times \wp(M)$

An example

$$[\![the\ boy]\!]_{(et)t} = [\![the]\!]_{(et)((et)t)}([\![boy]\!]_{et})$$

Since
$the(A, B) \iff A \subseteq B \wedge |A| = 1$

we have that:
$[\![the]\!]([\![boy]\!]) =$
$the(boy, B) \iff boy \subseteq B \wedge |boy| = 1$
(this is a type $\langle 1 \rangle$ quantifier that we can equivalently write:
$the(boy)_M = \{A \subseteq M : boy \subseteq A \wedge |boy| = 1\}$)

Note that this is the singular *the*; for the plural case we have
$the_{pl}(A, B) \iff A \subseteq B \wedge |A| > 1$

A small list of DETs as type $\langle 1, 1 \rangle$ quantifiers

The four quantifiers of the Aristotelian square of opposition:
$all(A, B) \iff A \subseteq B$
$no(A, B) \iff A \cap B = \varnothing$
$some(A, B) \iff A \cap B \neq \varnothing$
$not\ all(A, B) \iff A - B \neq \varnothing$

$(Q^R)_M \iff |A| > \frac{|M|}{2}$ (the Rescher quantifier for finite universe)
$most(A, B) \iff |A \cap B| > |A - B|$
$the(A, B) \iff A \subseteq B \wedge |A| = 1$
$ten(A, B) \iff |A \cap B| = 10$
$the\ ten(A, B) \iff |A| = 10 \wedge A \subseteq B$
$John's(A, B) \iff A \cap \{b : owner(j, b)\} \subseteq B \ \wedge \ |A \cap \{b : owner(j, b)\}| > 1$
(in the case of plural NP, *John's bikes*, while $|A \cap \{b : owner(j, b)\}| = 1$ in the case of singular NP, *John's bike*)

Outline

Negative operations

The two negative operations (among the other boolean operations) on GQs are the outer negation and the inner negation.
Let Q be a GQ:

- $\neg Q$ is the outer negation
- $Q\neg$ is the inner negation
- $\neg Q \neg = Q^d$ is the dual (outer negation of the inner negation).

A quantifier, together with its outer negation, inner negation and dual forms a natural unit, the *square*:

$$square(Q) = \{Q, \neg Q, Q\neg, Q^d\}$$

Squaring type $\langle 1 \rangle$ and $\langle 1, 1 \rangle$ quantifiers

For a $\langle 1 \rangle$ quantifier $Q(A)$:

i. $\neg Q_M(A) = \{\wp(M) - Q(A)\}$

ii. $Q_M\neg(A) = \{Q(M - A)\}$

iii. $Q^d(A) = \{\wp(M) - Q(M - A)\}$

For a $\langle 1, 1 \rangle$ quantifier Q(A,B):

i. $\neg Q_M(A, B) = \{\wp(M)^2 - Q(A, B)\}$

ii. $Q_M\neg(A, B) = \{Q(A, M - A)\}$

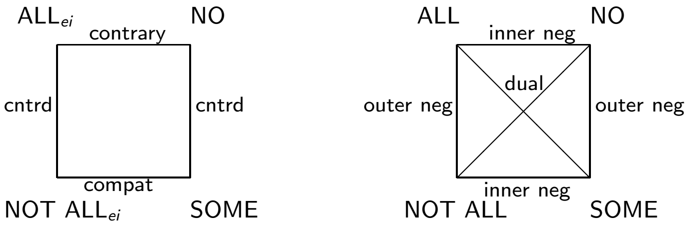
iii. $Q^d(A, B) = \{\wp(M)^2 - Q(A, M - A)\}$

Some properties of squares

These are some obvious general facts about $square(Q)$:

- if Q is non-trivial ($Q_M \neq \wp(M)$ and $Q_M \neq \varnothing$), so are the other quantifiers in its square
- if $Q' \in square(Q)$ then $square(Q') = square(Q)$
- $square(Q)$ has either two or four members
- if $Q' = Q^d$ then Q and Q' are inter-definable.

Classical vs modern square



The classical square represents logical relations among propositions constructed out from quantificational DETs.
The modern version is intended to represent the "span" of a quantifier when undergone to the negative operations.

The existential import

Clearly, the modern version is a generalization of the classical square of opposition.
By interpreting the modern square in set-theoretical terms, the logical relations holding among quantifier of the classical square are easily reproducible, under some conditions on the set-theoretical objects involved. The main case is that of the existential import of ALL in the classical square.

In the modern sense, *ALL* is conceived without existential import, but, given that $ALL_M(A,B) \iff A \subseteq B$ and $ALL_M\neg(A,B) \iff A \subseteq M - B$, to ensure that they can not both be true (for reproducing the contrary relation) is simply required that A is not empty.

Representing a deep semantic pattern

Other consequence of the more generality of the modern square is that it is applicable to every quantifier and thus it displays more utility for investigating the semantics of NP. Moreover, it is well-known that, in many natural languages, there are pairs of quantificational expressions (DETs), but also of expressions of other categories, like adverbials (*still/already*) or conjunctions (*because/although*), whose denotation exhibits the pattern of duality.

This may be a sign that the square of logical duality shows some deep and pervasive pattern of the semantics of natural languages.

Outline

Noun Phrases as Generalized Quantifiers

Squaring Generalized Quantifiers

Squaring NPs properties

Filters

The semantic homogeneity of NPs is due to their functional homogeneity as type $\langle 1 \rangle$ quantifiers. But type $\langle 1 \rangle$ quantifiers have different logical properties, splitting them into three fundamental categories: ultrafilter, filter and “intersective” quantifiers.

ultrafilter	Proper Names, definite NP_{sg}s, Pronouns *John, the boy, I*, etc.
filter	definite quantified NPs *every boy, no student*, etc.
intersective	indefinite quantified NPs *some boys, a boy, two boys*, etc.

Filters

Briefly, what distinguishes a filter Q from an intersective quantifier Q' is that Q and $Q\neg$ cannot be both true (they are contrary), while Q' and $Q'\neg$ do (they are compatibles).

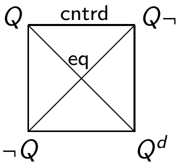
What distinguishes an ultrafilter Q from a filter Q' is that Q and $Q\neg$ cannot be neither both true nor both false (they are contradictory), while Q' and $Q'\neg$ can be both false.

(This way of putting the distinction is slightly different from the original characterization due to Barwise and Cooper.)

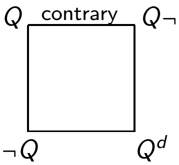
Filters on the square

This way of distinguishing filters from ultrafilters and intersective quantifiers suggests the use of the square to figure out logical properties of these classes of NPs.

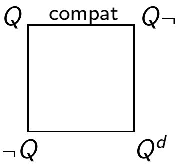
for ultrafilter Q:



for filter Q:



for intersective quantifier Q:



Filters and negative operations (I)

Ultrafilters are individuals, inner and outer negation having the same effect of determining contradiction.

Filters and intersective quantifiers are "quantirelations": they have the so-called "isomorphism" (ISOM) property, whereas ultrafilters are non-ISOM. ISOM means that for holding the relation expressed by the quantifiers only the cardinality of sets involved is relevant.

The difference between filters and intersective quantifiers is due to the different logical effect of inner negation, as set-theoretical operation.

A simple example:

For the filter $Q = ALL$:
$ALL_M(A, B) = |A - B| = 0$ and
$ALL_M\neg(A, B) = |A - (M - B)| = |A \cap B| = 0$
cannot be both true if $A \neq \varnothing$.

For the quantifier $Q = SOME$:
$SOME_M(A, B) = |A \cap B| \neq 0$ and
$SOME_M\neg(A, B) = |A \cap (M - B)| = |A - B| \neq 0$
can obviously be both true but not both false.

Filters and negative operations (II)

Moreover, we can notice that if Q is a filter, Q^d is an intersective quantifier.

An example: the square for $MOST$

- $MOST_M(A, B) = |A \cap B| > |A - B|$
- $MOST_M\neg(A, B) = |A \cap B| < |A - B|$
- $MOST^d_M(A, B) = |A \cap B| \geq |A - B|$
- $\neg MOST_M(A, B) = |A \cap B| \leq |A - B|$

Nested quantifiers

In natural language sentences is normal to have NPs both in subject and object position. And, not surprisingly, each NP, as type $\langle 1 \rangle$ quantifier, can take scope over the other.
If this scope commutability between the two NPs produces an ambiguity or not, is a matter of "relative logical properties" of the quantifier involved.

Scope interaction

There are three fundamental cases:

i. *self-commuting* quantifiers:

$$Q(Q(R)) \Leftrightarrow Q(Q(R^{-1}))$$
$$all(all(R)) \Leftrightarrow all(all(R^{-1}))$$

ii. *indipendent* quantifiers:

$$Q_1(Q_2(R)) \Leftrightarrow Q_2(Q_1(R^{-1}))$$
$$all(the(R)) \Leftrightarrow the(all(R^{-1}))$$

iii. a quantifier is *dominant* over the other:

$$Q_1(Q_2(R)) \Rightarrow Q_2(Q_1(R^{-1}))$$
$$some(all(R)) \Rightarrow all(some(R^{-1}))$$

Ambiguity raises only in the third case.

The logical base of ambiguity (I)

Let consider the two sentences:

(1) All boys read the books
(2) All boys read some book

For sentence (1), the *object narrow scope* reading and the *object wide scope* reading are equivalent, as a matter of logic of set-theoretical formulas:

(ons) $\quad all(boy)[the_{pl}(book)[read]] =$
$boy \subseteq \{x : book \subseteq \{y : x \ read \ y\} \wedge |book| > 1\}$

(ows) $\quad the_{pl}(book)[all(boy)[read^{-1}]] =$
$book \subseteq \{y : boy \subseteq \{x : x \ read \ y\}\} \wedge |book| > 1$

The logical base of ambiguity (II)

For sentence (2) (All boys read some book) (ons) and (ows) readings are not equivalent, given that (ows) $\Rightarrow$ (ons), as a matter of logic:

(ons) $\quad all(boy)[some(book)[read]] =$
$boy \subseteq \{x : book \cap \{y : x \; read \; y\} \neq \varnothing\}$

(ows) $\quad some(book)[all(boy)[read^{-1}]] =$
$book \cap \{y : boy \subseteq \{x : x \; read \; y\} \neq \varnothing\}$

Scope dominance (I)

In literature, scope dominance is defined as follows:
in finite domain Q_1 is dominant over Q_2 iff Q_1 is an "exist" quantifier or Q_2 is a "universal" quantifier.

On the square, this condition could be expressed as follows:
Q_1 is *dominant* over Q_2 iff Q_1 is an intersective quantifier or Q_2 is a filter.

We can add that, if Q_1 and Q_2 are both intersective they are each other scopeless quantifiers:

$$some(some\neg(R)) \Leftrightarrow some\neg(some(R^{-1}))$$

Scope dominance (II)

For filters (type $\langle 1,1\rangle$) , we need a more subtle distinction between *co-intersective* (let $Q(A,B)$: only $|B - A|$ is relevant) and *proportional* (both $A \cap B$ and $|B - A|$ are relevant).
Thus, $all(A,B)$ and $most(A,B)$ are both filter, but *most* is dominant over *all*, because of the relevance, in its meaning, of $|A \cap B|$:

$$most(boy)[all(book)[read]] \Rightarrow all(book)[most(boy)[read^{-1}]]$$

but

$$most(boy)[all(book)[read]] \nRightarrow all(book)[most(boy)[read^{-1}]]$$

Thanks!