Logique & Analyse **205** (2009), 21–37

PHILOSOPHICAL LOGIC IN A FRAMEWORK OF PROPOSITIONAL LOGIC*

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(1) The classic approach to formal logic is twofold. First, describe the syntactical perspectives of a formal language; and second, define a natural semantic interpretation for it (and investigate whether or not there is a complete calculus for this logic, whether or not the logic is decidable, etc.). In that classic approach we have to discuss some more or less complicated and complex meta-logical questions for each new logic. Those questions are obviously important from a mathematical point of view. But they are less important from a philosophical point of view. Thus, for philosophical discussions it would be desirable to have a simple semantic framework that has some simple and straightforward meta-logical properties, and in which we are able to implement any philosophical logic we like. Such a framework quite possibly has no value for mathematical discussions, but it has great value for philosophical discussions, because it keeps those discussions free of complicated and more or less irrelevant meta-logical questions. The present paper describes how to use *propositional logic* as a framework in this sense. This framework, taken from a purely formal point of view, is nothing new. But its exposition as a *sufficient* framework for philosophical purpose *is* new. And this is one general point of the present paper: that it would be a good idea for most philosophical discussions to use this strategy of *semantic restriction* rather than the classic strategy that dogmatically ties each language together with its 'natural ontology'.

(2) There are two different (formal) ways of taking the *expressive power* of a logic: a *meta-logical* and a *metaphysical* way. In the *meta-logical* sense, the expressive power of a logic is given by model theoretic criteria like maximal cardinality of sets that can be characterized in this language up to isomorphism. In the *metaphysical* sense, the expressive power is given by all

*This paper is a condensed version of parts of my PhD-thesis [4]. The work was supported by the Austrian Science Fund (FWF research projects P18066 and P18596). For comments I am grateful to Richard Dawid, Marie Duží, Don Faust, Chris Fermüller, Sy David Friedman, Edwin Glassner, Georg Gottlob, Manfred Kohlbach, Matthias Neuber, Richard Nickl, Gerhard Schurz, and the *Philosophy of Science Colloquium, Institute Vienna Circle* (`http://www.univie.ac.at/ivc/koll/index.html`).

those formal *features* of a language that can be interpreted as expressions of some metaphysical properties. Having higher order quantification, having modal operators, having many truth values, are formal features in this metaphysical sense. In the classic understanding of logic, as driven forward most prominently by W. V. O. Quine, these two notions of expressive power are *convergent*. Having higher order quantification, to mention the most important example for this reading, is strongly connected with the mathematical property of having more expressive power than in the first order case. Taking this for granted, it would not make sense to give another specification of higher order logic than in the so-called standard interpretation, because every other specification (e. g. in many-sorted first order logic) would violate *the natural connection* between philosophical expressivity and mathematical expressive power.[1]

We argue, on the contrary, that there is *no connection a priori* between metaphysical expressivity and meta-logical expressive power, or, to put it in different words: the connection between these two things is *a philosophical* and *not a formal question*. Thus, in principle, if we have a formal language with expressivity A (in the metaphysical sense) we can search for a meta-logical specification for this formal language that equips it with mathematical expressive power B. For example, we can equip a higher order language with the standard interpretation but we also can specify it in a first order and even in a propositional framework. For the philosophical purpose, the procedure must be this: stipulate the mathematical expressive power B that you need for your language, and then develop the language with the desired metaphysical features A, in a meta-logical framework that secures B. The present paper demonstrates this philosophical 'technique' for the case of a philosopher who needs the expressive power of propositional logic.

(3) The best known example for this philosophical 'technique' is the so-called Henkin-trick that allows us to reduce logics with arbitrary philosophical features A to first-order logic.[2]

(4) The stipulation of meta-logical expressive power B, first of all, is a purely formal decision. It is the question of whether our meta-logical framework should be decidable or complete, etc. In this sense we *could* stipulate a meta-logical feature (decidability, completeness, etc.), and stipulate the meta-logical framework, on the basis of these purely formal and technical questions only.

But the meta-logical question is at the same time philosophical in an immediate sense. It seems obvious that we buy some metaphysical features, on

[1] For a discussion of the standard interpretation of second order logic and alternatives to it cf. [16, section 4].

[2] See [9]. Cf. [11] for a specification of numerous logics in such a first-order framework.

the basis of our meta-logical decision, at least in the sense of being *natural* in our framework. Every meta-logical framework seems to have something like a *natural metaphysical reading* or a *natural ontology*. Quine's reading of first order logic, for example, can be understood in this sense.[3] If we accept the thesis that meta-logical and metaphysical expressivity are non-identical, we could adopt Quine's ontology and reject at the same time his general understanding of logic; we could *accept* Quine's ontological claims in [13] as the development of the *natural ontology* of first-order logic and *reject* them at the same time, because we prefer some different features, beyond the meta-logical level, e. g. because we prefer to combine the first order framework with modal metaphysics, etc.

(5) Ultimately, this distinction between meta-logical and metaphysical features is nothing else than a particular interpretation of the old distinction between *syntax* and *semantics*. What this new version of the distinction only rejects are claims concerning a *natural* and even a *necessary* connection between syntax A and semantics B. In this domesticated and liberalized reading of logic we have these two *perspectives* of logic as totally *independent* perspectives. We now can have a logic which is syntactically first order and semantically propositional, syntactically modal and semantically first order, etc. What we have to do is to divide the discussion of our logic into two different parts: first, a discussion of the semantic framework and the merits of its natural ontology; second, a discussion of the metaphysical features that we would like to establish inside of this framework (on the level of syntax). To think little of such a 'syntactization of metaphysics', again, would be just a symptom of the old understanding of logic that dogmatically ties the syntax together with its natural semantic interpretation.

(6) From an historical point of view, this understanding of logic is opposed to Quine, and it follows Carnap. Quine has a rather *monolithic* understanding of logic. He claims that ontology and logic are ultimately the same thing. Carnap, on the other hand, differentiates between *internal* and *external* questions of logic.[4] External questions are questions of the general layout of a 'formal framework', whereas internal questions are questions that can be asked inside of such a framework. Although Carnap obviously does not see a direct connection between internal questions and syntax on the one hand, and between external questions and semantics on the other, he differentiates between *two different levels of ontological questions*. And this is exactly what we do here. On the external level, we are aiming at a principal ontological layout, we establish a formal framework with a particular natural

[3] Cf. [12, 14].

[4] Cf. [2].

ontology. But the internal structure of the language also allows us to introduce ontological features that go beyond the immediate level of the natural ontology of our language. We can *overrule* the natural ontology, so to speak, by introducing powerful metaphysical features on the internal level, on the level of syntax. Therefore, Carnap's famous *principle of tolerance*[5] appears to be nothing other than the principle of a modern understanding of logic that gains enormous flexibility from its deep understanding of the fundamental *distinction* between the internal and the external level, between syntax and semantics, between metaphysical and meta-logical questions. Quine's monolithic understanding, on the other hand, seems to be the most important limiting factor for the development of logic during the last decades, because it prevents us from seeing the fundamental distinction which Carnap had investigated.

(7) The present paper presents a particular example for a *formal framework*, in the semantic sense – propositional logic – and it shows how to integrate some metaphysical features – the features of first order logic and modal logic – into this framework. This propositional framework is of particular interest chiefly for two reasons: for its *natural ontology* and its *simplicity*. Whereas the natural ontology establishes a rather neutral characterization of the propositional framework, and a more technical reading of it, simplicity provides us with a *direct* argument in favor of the propositional framework:

(8) The propositional framework is *simple*, insofar as it represents *the most basic version* of a semantic framework. Candidates for such a framework are propositional logic, first order logic and more complex systems like second order logic with the standard interpretation. These candidates are obviously frameworks of increasing complexity. Whereas it seems to be always possible to implement the metaphysical features of more complex frameworks in simpler ones, more complex frameworks appear to be *extensions* of simpler ones in a pretty straightforward sense. First order logic is propositional logic *plus* first order machinery, higher order logic is first order logic *plus* higher order machinery, propositional modal logic is propositional logic *plus* modal machinery, and so on. Now, if the question of the framework turns out to be a question of pure *convention* (because, for the philosopher, the natural ontological features of the framework are less important than the metaphysical features which we can formalize inside of it), it seems to be extremely plausible to choose the most simple framework.

(9) The *natural ontology* of propositional logic is *rigid* and *finite* in the following sense. Rigidity is a feature that stems from the specific function of *constants* in propositional logic (according to the classic Fregean understanding). Whereas in predicate logic (first or higher order) an individual

[5] See [3, § 17].

constant is *extensionally interpreted* by associating an arbitrary entity with the name, the extensional interpretation of a constant, in the case of propositional logic, associates just a *truth value*. *The answer*, so to speak, that semantics gives to a constant, is simply *yes or no*, whereas in the case of predicate logic, it can be any object, whatsoever. Therefore, we can say that the *natural ontology* of (two valued) propositional logic is simply this: to interpret *every* constant as the proposition of something, to which we can answer with yes or no.

The most important example for an external application of this propositional ontology is (first order) predicate logic. An *individual constant*, in a propositionally interpreted predicate logic, is not interpreted as a name that denotes an external object, but simply as *the proposition that a particular object exists* (the semantics answers yes or no). A *predicate constant* is the proposition of a particular *property*. If we bind this constant to names, then the semantics responds no, if some of the objects denoted by a name does not exist; otherwise, it answers yes, iff the objects have this property.

Quantification, in such a rigid setting is realized straightforwardly via conjunction. $\forall x P(x)$ can be defined as the formula $P(c_1) \wedge P(c_2) \wedge \ldots$ where the $c_1, c_2, \ldots$ are all individual constants of the language, representing all possible objects. Therefore it is only possible to quantify over *infinitely* many objects, if we allow infinitely long formulas. And this, of course, is not natural. Thus, the natural ontology of propositional logic is also *finite*.[6]

This does not mean that every logic we specify in the propositional setting must be a finite logic with only finitely many objects. If we *need* infinitely many objects, we simply have to accept infinitely long formulas. The problem that we obtain in such a case is only a technical one: finite versions of propositional logic are always decidable (regarding both satisfaction and validity), whereas in infinite versions the meta-logical situation is more complicated. Thus, the finite case is the *natural* one, but nevertheless it is *possible* to consider also logics with an infinite universe. Particularly, the ontological question whether a finite ontology is *sufficient* (e. g. for the purpose of physics) seems to have no *direct* connection with the question of finiteness in logic; therefore, this question shall not be discussed here.

[6] A finite logic, in our sense, has a universe of a finite number of n objects. This case has to be distinguished from the case of so-called *finite model theory*, where first order structures are restricted to finite domains, but where the universe is the same as in non-restricted first order logic. Cf. [5]. Note also that we discuss here only *the natural ontology* of predicate logic in a propositional framework. This ontology in fact leads to a layout of *free logic*. Nevertheless, we *can* implement a different – non-free and extensional – account of first order logic inside of the propositional framework. This will be illustrated in section 2, below.

(10) An important aspect of the strategy pointed out in this paper is *semantic restriction*: to have *a simple formal framework* with convenient meta-logical properties, and to develop a language for this framework that implements *a very high range of metaphysical features*. Such ‘formal ontologies’ tend to buy with each metaphysical enrichment of a language a significant modification of the logic on the meta-logical level, i. e. they are *semantically boundless*.[7] Our proposal in this paper shows how to break this rather unfortunate connection. We now can choose the meta-logical and the metaphysical properties (more or less) *independently*. We demonstrate the general function of this strategy, for the case of a meta-logical framework of propositional logic, and we illustrate the metaphysics of two of the most simple and straightforward examples of logical modes of expression inside of this framework: first order logic and modal logic. How to implement a metaphysically extremely rich language in this framework shall be demonstrated in a future paper.

1. *The propositional framework*

Throughout this paper a *logic* is understood as an algebraic structure $\mathcal{L} = (F_{\mathcal{L}}, \mathbb{S}_{\mathcal{L}}, \vDash_{\mathcal{L}})$ that consists of a set $F_{\mathcal{L}}$ of *formulas* (sentences) plus a class $\mathbb{S}_{\mathcal{L}}$ of *structures* and a *relation of satisfaction* $\vDash_{\mathcal{L}}$ between them. We add some metalogical notions: a formula ϕ is the *logical consequence* of a set of formulas Γ, and we write $\Gamma \vDash \phi$, if every structure which is a model of every formula out of Γ is also a model of ϕ. Two formulas of a logic are *logically equivalent*, if they are satisfied in exactly the same structures.

The propositional logic $\mathsf{RIG_a}(A)$ is built over a set A of propositional constants which can be either finite or infinite (countable or uncountable). We have the logical connective $\neg$ and the generalized conjunction $\bigwedge$, the latter shall be defined for sets of formulas whose maximum cardinality is the cardinality of A. Other logical connectives like disjunction $\bigvee$ and implication $\rightarrow$ are defined in a natural way:

$$\bigvee \Gamma := \neg \bigwedge_{\psi \in \Gamma} \neg \psi,$$

$$\Gamma \rightarrow \phi := \neg \bigwedge \left(\Gamma \cup \{\neg \phi\} \right).$$

[7] Examples for languages that are metaphysically extremely rich, in this sense, can be found in the writings of Richard Montague and Edward Zalta. Cf. [17, 19]. Because both Montague and Zalta use the Henkin-trick, in order to keep there framework semantically first order, there strategy is somewhat intermediate between semantic restriction and semantical boundlessness.

We also define $\wedge$, $\vee$ and $\rightarrow$ for pairs of single formulas. *Structures* are introduced as subsets of A (i. e. sets of true propositions). The relation of satisfaction $\vDash_a$ is defined for structures $\mathfrak{S}$, propositions p, finite formulas ϕ and sets of finite formulas Γ:

$\mathfrak{S} \vDash_a p$ iff $p \in \mathfrak{S}$,
$\mathfrak{S} \vDash_a \neg\phi$ iff not $\mathfrak{S} \vDash_a \phi$,
$\mathfrak{S} \vDash_a \bigwedge\Gamma$ iff $\mathfrak{S} \vDash_a \psi$, for every $\psi \in \Gamma$.

We call a logic $\mathcal{L} = (F_{\mathcal{L}}, \mathbb{S}_{\mathcal{L}}, \vDash_{\mathcal{L}})$ *rigid*, if the following holds. There exists a set $F_{\mathsf{at}} \subseteq F_{\mathcal{L}}$ that defines the set $\hat{F}_{\mathcal{L}}$ of all formulas

$$\phi ::= \ \phi_{\mathsf{at}} \mid \neg\phi \mid \ \bigwedge\Gamma,$$

where ϕ_{at} ranges over F_{at} and Γ over sets of finite formulas. Then there exists a function Θ that (1) maps $\mathbb{S}_{\mathcal{L}}$ injective to $\wp(F_{\mathsf{at}})$ and (2) maps $F_{\mathcal{L}}$ onto $\hat{F}_{\mathcal{L}}$ so that for every $\mathfrak{S} \in \mathbb{S}_{\mathcal{L}}$ and every $\phi \in F_{\mathcal{L}}$ it holds:

$$\mathfrak{S} \vDash_{\mathcal{L}} \phi \quad \text{iff} \quad \Theta(\mathfrak{S}) \vDash_a \Theta(\phi).$$

Here $\vDash_{\mathcal{L}}$ and $\vDash_a$ are the respective consequence relations of $\mathcal{L}$ and $\mathsf{RIG_a}(A)$. If the set F_{at} of a rigid logic is finite and the function Θ is recursive, then we call this logic *finite*.

To each rigid/finite logic $\mathcal{L}$ we assign a triple $\tilde{\mathcal{L}} = (F_{\mathsf{at}}, \Theta, \tilde{\mathbb{S}}_{\mathcal{L}})$, where F_{at} and Θ are defined as above and $\tilde{\mathbb{S}}_{\mathcal{L}}$ is the image $\Theta(\mathbb{S}_{\mathcal{L}})$. This $\tilde{\mathcal{L}}$ is a fragment of the logic $\mathsf{RIG_a}(F_{\mathsf{at}})$, because it holds that $\tilde{\mathbb{S}}_{\mathcal{L}} \subseteq \wp(F_{\mathsf{at}})$.

Proposition 1: *Every finite logic is generally decidable, i. e. it is decidable regarding both satisfaction of a formula in a particular structure and logical consequence.*

Proof. Using the reduction to finite propositional logic we can decide satisfaction and validity via truth-tables. Logical consequence must be decidable, because there are only finitely many logically equivalent formulas, in a finite logic, thus we can restrict ourselves to situations $\Gamma \vDash \phi$ where Γ is a finite set of formulas. □

It is a central point in the finite setup that there is *no need in principle* for deductive systems in the traditional sense, because logical consequence is decidable in a straightforward way (via truth-tables). Of course, the task of deciding validity in a finite logic will also be a question of complexity. Therefore, traditional deductive systems (i. e. polynomial algorithms for the deduction of valid formulas) appear to be of interest, even for finite logics.

Moreover, in the general case of a possibly non-finite rigid logic, deductive systems are indispensable. Unfortunately, there is no general rule how to construct a *complete* deductive system for a rigid logic. Thus, the questions of completeness and decidability have to be discussed independently for every single rigid logic. Obviously, there *are* complete (and polynomial) deductive algorithms and even decision procedures for numerous rigid logics, but, because of questions of space, we do not discuss this point any further here. We only prove, for the logics described below, that they are rigid and that there is a particular class of finite instances of them.

2. *First-order logic*

Classic first-order logic is not rigid, because of the possibility of varying the domains of structures freely (the class of all structures of a rigid logic must by definition be a set). The only straightforward way to construct a rigid version of first-order logic seems to be the following. First, we have to restrict the amount of possible entities in the domains of structures to a particular (possibly infinite) set D. Second, we have to introduce the individual constants of the logic in such a way that they always denote the same object out of D. (In other words: individual constants in a rigid logic are similar to Kripke's *rigid designators*.[8]) Then we are able to formulate every possible statement about the universe D in terms of individual constants, in particular quantifiers like $\forall$ can be defined as

$$\forall x\, \phi \text{ iff } \phi\left[\tfrac{c}{x}\right], \text{ for every individual constant } c.$$

The first-order logic $\mathsf{RIG}_\mathsf{p}(D, \mathsf{P}, \alpha)$ is built over a (possibly infinite) set D of *individuals*, a finite or countable set P of *predicates* and a function $\alpha : \mathsf{P} \mapsto \mathbb{N}$ that assigns to each predicate its 'arity'. (We do not introduce functions and identity here.)

The idea is that the domain of a structure of the logic is always a subset of the 'universe' D. Because the non-logical constants serve both as 'signature' and as 'domain' of this logic (i. e. they provide both the non-logical *constants* and the *objects* of the logic, following the rigid layout we described in the introduction, above), we also call the collection of non-logical names of the logic its *domain-signature*. It turns out that the domain-signature (D, P, α)

[8] 'Let's call something a *rigid designator* if in every possible world it designates the same object.' [10, p. 48] We follow this conception here, with a slight modification. For us a designator is rigid if it denotes either a definite object or nothing, i. e. there could be some possible worlds in which the designator exists, however, the object it denotes does not exist.

determines the whole *ontological complexity* of the logic $\mathsf{RIG_p}$, i. e. it determines every possibility of producing non-tautological expressions.

A predicate logic, in this setting, turns out to be a *free logic*[9], because we will have structures where the individual(-constant) c is not an element of the domain. But note that this layout of our logic is a direct consequence of the rigid 'natural ontology' of the propositional framework. Of course, it *would* be possible to define a non-free version of first-order logic also in this framework (using semantic interpretations, in the usual sense). But the specification of such a logic would be *formally* complicated and by no means straightforward. If we would prefer a non-free account (because of philosophical reasons), we can easily obtain this account as a special case of our logic, as specified at the end of this section. (The logic $\mathsf{RIG_p}$ is a *free logic* only on a technical level, on the level of the natural ontology of the propositional framework. *Metaphysically*, it can be interpreted any way we like it: as free or non-free, etc.)

Another significant point of this rigid setting is given by the fact that constants and objects are identified. A classic argument against such an identification urges that it leads to a collapse of the syntax-semantics-distinction. This, however, is not true. We *do have* a semantics in the rigid setting. This semantics is *weaker* than in the traditional first order case because the universe of the logic is restricted to a particular set D, whereas in the classic case the universe is built by the class of all sets. But nevertheless, we have the notion of a semantic structure here. The only technical difference between the classic and the rigid case is this: in the classic case the domain of a structure is *an arbitrary set*, whereas in the rigid case *it is a subset of the universe* D:

A *structure* $\mathfrak{S}$ over the domain-signature (D, P, α) is defined by a pair $(D_\exists, \pi)$, where $D_\exists \subseteq D$ provides the set of 'existing entities' of $\mathfrak{S}$ and π is a function that assigns to every predicate $P \in \mathsf{P}$ with $\alpha(P) = i$ a set $\pi(P) \subseteq D_\exists^i$. We call $\mathbb{S}_p$ the set of all possible structures over (D, P, α). It is easy to see that $\mathbb{S}_p$ is finite, iff D and P are finite. If n is the number of elements of D we obtain, for finite domain-signatures:

$$|\mathbb{S}_p| = \sum_i \left[\binom{n}{i} \prod_{P \in \mathsf{P}} 2^{\left(i^{\alpha(P)}\right)} \right].$$

The elements of D are the *individual constants* of our logic. There is also a countable set of *variables*. Individual constants and variables are also called *terms*. If P is a predicate with $\alpha(P) = i$ and $t_1, \ldots, t_i$ are terms, then

[9] Cf. [1] and [8, section 1].

$P(c_1,\ldots,c_i)$ is an *atomic formula*. Additionally we have atomic formulas $\mathsf{E}(t)$, where t is any term and $\mathsf{E} \notin \mathsf{P}$ is an *existence predicate*. The set of all $\mathsf{RIG_p}$-formulas is defined as

$$\phi ::= \ p \mid \forall x\phi \mid \exists x\phi \mid \neg\phi \mid \phi \wedge \phi.$$

Here p ranges over atomic formulas and x over variables. We define satisfaction $\vDash_p$. $\phi\left[\frac{c}{x}\right]$ we call the formula ψ which results from ϕ by replacing every instance of x (if there is any) with c. For structures $\mathfrak{S} = (D_\exists, \pi)$, atomic formulas $P(c_1,\ldots,c_i), \mathsf{E}(c)$ where the c_j and c are constants, variables x and formulas ϕ, ψ we have:

$$\begin{array}{lll}
\mathfrak{S} \vDash_p P(c_1,\ldots,c_i) & \text{iff} & (c_1,\ldots,c_i) \in \pi(P),\\
\mathfrak{S} \vDash_p \mathsf{E}(c) & \text{iff} & c \in D_\exists,\\
\mathfrak{S} \vDash_p \forall x\phi & \text{iff} & \mathfrak{S} \vDash_p \phi\left[\frac{c'}{x}\right], \text{ for every } c' \in D,\\
\mathfrak{S} \vDash_p \exists x\phi & \text{iff} & \text{there is a } c' \in D \text{ with } \mathfrak{S} \vDash_p \phi\left[\frac{c'}{x}\right],\\
\mathfrak{S} \vDash_p \neg\phi & \text{iff} & \text{not } \mathfrak{S} \vDash_p \phi,\\
\mathfrak{S} \vDash_p \phi \wedge \psi & \text{iff} & \mathfrak{S} \vDash_p \phi \text{ and } \mathfrak{S} \vDash_p \psi.
\end{array}$$

Proposition 2: *Every instance of* $\mathsf{RIG_p}(D, \mathsf{P}, \alpha)$ *is rigid; it is finite, iff the sets* D *and* P *are finite.*

Proof. We restrict ourselves to formulas without free variables and define F_{at} as the set of all atomic $\mathsf{RIG_p}(D, \mathsf{P}, \alpha)$-formulas without variables. Θ is defined for atomic formulas p, variables x and formulas ϕ, ψ:

$$\begin{aligned}
&\Theta(p) := p,\\
&\Theta(\forall x.\phi) := \bigwedge_{c\in D} \Theta\left(\phi\left[\frac{c}{x}\right]\right),\\
&\Theta(\exists x.\phi) := \bigvee_{c\in D} \Theta\left(\phi\left[\frac{c}{x}\right]\right),\\
&\Theta(\neg\phi) := \neg\Theta(\phi),\\
&\Theta(\phi \wedge \psi) := \Theta(\phi) \wedge \Theta(\psi).
\end{aligned}$$

Further, Θ assigns to each structure $\mathfrak{S} = (D_\exists, \pi)$ the subset of F_{at} which contains (1) a formula $\mathsf{E}(c)$ iff $c \in D_\exists$, (2) a formula $P(c_1,\ldots,c_i)$ iff $(c_1,\ldots,c_i) \in \pi(P)$. Thus the logic is rigid and it is finite, if D and P are finite. If D or P are infinite, then the logic is not finite, because the set of all structures $\mathbb{S}_p$ is infinite and Θ must be an injective function from $\mathbb{S}_p$ to the finite set $\wp(F_{\mathsf{at}})$. $\square$

RIG_p is not the only possible first-order logic that is rigid in the *technical* sense of section 1. But it is the only *straightforward* version of a rigid first-order logic that implements *the natural ontology* of the propositional framework, as pointed out in the introduction, paragraph (9).

With $\forall$ and $\exists$ we quantify over the whole range of individuals of the logic, no matter whether they exist in a structure or not. In order to quantify only over those individuals which do exist in the actual 'world', we define the (dual) quantifiers $\forall^{\mathsf{E}}$ and $\exists^{\mathsf{E}}$:

$$\exists^{\mathsf{E}} x\, \phi := \exists x\, (\mathsf{E}(x) \wedge \phi),$$
$$\forall^{\mathsf{E}} x\, \phi := \forall x\, (\mathsf{E}(x) \rightarrow \phi).$$

If we now restrict our logic to formulas that does contain individual constants *only* in the context of quantifiers $\exists^{\mathsf{E}}$ and $\forall^{\mathsf{E}}$, i. e. if we not make any *direct* use of individual constants and quantify only over existing individuals, then we clearly obtain a straightforward version of *extensional first order logic*, in the tradition of Russell and Quine.[10]

3. *Modal logic*

Given any logic $\mathcal{L}$, we are able to construct a first-order logic over it in order to quantify over the structures of $\mathcal{L}$. This section will describe how to implement this well-known technique of modal model theory and *correspondence theory*[11] in a semantic framework of propositional logic. The advantage of this framework is that it allows us to *extend* in a straightforward sense any basic logic with first-order features for quantification over structures. This is *not* possible, in the more general case of a classic first order framework (as it is usually used in correspondence theory), because in the case of non-rigid logics the class of all structures of a logic mostly is not *a set* (but rather the class of all sets or the class of all finite sets) and therefore we cannot introduce this class as a simple domain for (set theoretical) quantification.

A rigid layout of modality must be similar to the first-order logic which we specified in the preceding section. Given a basic logic $\mathcal{L} = (F_{\mathcal{L}}, \mathbb{S}_{\mathcal{L}}, \models_{\mathcal{L}})$,

[10] The trick of avoiding statements which contain individual constants was already used by both Russell and Quine. Cf. [15, 13].

[11] Cf. [18].

over which we wish to quantify, we have to choose a set $\mathfrak{M}$ of objects out of $\mathbb{S}_{\mathcal{L}}$ as the *modal domain* of our logic (i. e. the set of all possible worlds). Again, like in the rigid logic $\mathsf{RIG}_{\mathsf{p}}$, the objects of $\mathfrak{M}$ function as individual constants and as objects, and we have some relations for quantification over $\mathfrak{M}$. However, we do not introduce *modal operators* directly into the syntax of our language. Instead, we have the syntactic element $\Vdash$, defined as a binary relation between formulas and possible worlds. $\mathfrak{S} \Vdash \phi$ shall express that formula ϕ is true in the possible world $\mathfrak{S}$. Given this device we can define modal operators from scratch, in the sense of 'ϕ is true in every possible world related to the actual' (cf. formula (N), below).

Let $\mathcal{L} = (F_{\mathcal{L}}, \mathbb{S}_{\mathcal{L}}, \vDash_{\mathcal{L}})$ be any logic. Then we define $\mathsf{FLP}_{\mathcal{L}}(\mathsf{P}_w, \alpha_m, \mathfrak{M})$ as a logic over $\mathcal{L}$, where P_w is a set of *modal predicates*, α_m is a function that assigns to each modal predicate its arity and $\mathfrak{M} = (\mathfrak{W}, \Pi)$ is a *modal structure*. Here, $\mathfrak{W}$ is a set of structures out of $\mathbb{S}_{\mathcal{L}}$, called the set of all *possible worlds* of the modal structure. Π assigns to every modal predicate $P \in \mathsf{P}_w$ with $\alpha_m(P) = n$ a set $\Pi(P) \subseteq \mathfrak{W}^n$.

The elements of the set $\mathfrak{W}$ of possible worlds are defined as *constants*. Additionally, there is a constant $\mathtt{SELF} \notin \mathfrak{W}$. (Note the difference between constants and possible worlds, according to our definition: every possible world is a constant, but the constant $\mathtt{SELF}$ is not a possible world.) We define, for possible worlds $\mathfrak{S}, \mathfrak{S}'$ and $\mathfrak{W}$-constants $a_1, \ldots, a_n$, a functional expression that assigns to every finite sequence of $\mathfrak{W}$-constants $a_1, \ldots, a_n$ and to every possible world $\mathfrak{S}$ a value $\mathfrak{S}(a_1, \ldots, a_n)$:

$$\begin{aligned}
&\mathfrak{S}(\mathfrak{S}') := \mathfrak{S}',\\
&\mathfrak{S}(\mathtt{SELF}) := \mathfrak{S},\\
&\mathfrak{S}(a_1, \ldots, a_n) := \mathfrak{S}(a_1), \ldots, \mathfrak{S}(a_n).
\end{aligned}$$

We need this functional expression, simply because the semantic has to replace $\mathtt{SELF}$ by the possible world that is 'actual' on the respective place of a formula. (Technically, the $\mathfrak{W}$-constants of the object language are interpreted as functions, on the level of the meta-language.)

We have a countable set of *variables* $w, w', \ldots$, every constant and every variable is defined as a *term*. If $P \in \mathsf{P}_w$ is a predicate-constant with $\alpha_m(P) = n$ and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is an *atomic* $\mathsf{FLP}_{\mathcal{L}}$-formula. The whole range of $\mathsf{FLP}_{\mathcal{L}}$*-formulas* is then defined as:

$$\phi ::= \ \phi_{\mathcal{L}} \mid p \mid t \Vdash \phi \mid \forall w\, \phi \mid \exists w\, \phi \mid \neg\phi \mid \phi \wedge \phi.$$

Here $\phi_{\mathcal{L}}$ ranges over $\mathcal{L}$-formulas, p over atomic $\mathsf{FLP}_{\mathcal{L}}$-formulas, w over $\mathfrak{W}$-variables and t over terms. We define, for $\mathcal{L}$-formulas $\phi_{\mathcal{L}}$, atomic $\mathsf{FLP}_{\mathcal{L}}$-formulas $P(a_1, \ldots, a_n)$ (where the a_i are constants), possible worlds $\mathfrak{S}$, constants a, variables w and $\mathsf{FLP}_{\mathcal{L}}$-formulas ϕ, ψ:

$$
\begin{array}{lll}
\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi_{\mathcal{L}} & \text{iff} & \mathfrak{S} \vDash_{\mathcal{L}} \phi_{\mathcal{L}}, \\
\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} P(a_1, \ldots, a_n) & \text{iff} & \mathfrak{S}(a_1, \ldots, a_n) \in \Pi(P), \\
\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} a \Vdash \phi & \text{iff} & \mathfrak{S}(a) \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi, \\
\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \forall w\, \phi & \text{iff} & \mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi\left[\frac{\mathfrak{S}'}{w}\right], \text{ for every } \mathfrak{S}' \in \mathfrak{W}, \\
\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \exists w\, \phi & \text{iff} & \text{there is an } \mathfrak{S}' \in \mathfrak{W} \text{ with } \mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi\left[\frac{\mathfrak{S}'}{w}\right], \\
\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \neg\phi & \text{iff} & \text{not } \mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi, \\
\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi \wedge \psi & \text{iff} & \mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi \text{ and } \mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \psi.
\end{array}
$$

Proposition 3: $\mathsf{FLP}_{\mathcal{L}}(\mathsf{P}_w, \alpha_m, \mathfrak{M})$ *(1) is rigid, iff the basic logic* $\mathcal{L}$ *is rigid; (2) it is finite, iff the basic logic* $\mathcal{L}$ *is finite and every predicate out of* P_w *is decidable.*

Proof. In the proof we restrict ourselves to the set of all formulas of the form $\mathfrak{S} \Vdash \phi$, which does not contain the constant SELF and which does not contain free variables. Those restrictions are completely insubstantial, because (1) $\mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi$ is true iff $\mathfrak{S} \Vdash \phi$ is true in any possible world; (2) we can assign to every formula ϕ and every possible world $\mathfrak{S}$ a formula $\mathfrak{S} \Vdash \psi$, where ψ results from ϕ by replacing SELF with possible worlds, in a pretty natural way.

Suppose that the basic logic is rigid. Let $\tilde{\mathcal{L}} = (F_{\mathsf{at}}, \Theta, \tilde{\mathbb{S}}_{\mathcal{L}})$ be the propositional variant of $\mathcal{L}$, in the sense specified in section 2. We define F^*_{at} as the set of all formulas $\mathfrak{S} \Vdash \phi_{\mathsf{at}}$, where ϕ_{at} is an element of F_{at} and $\mathfrak{S}$ is a possible world. Now we define the function Θ^* in such a way that it holds, for possible worlds $\mathfrak{S}' \in \mathfrak{W}$ and formulas $\mathfrak{S} \Vdash \phi_{\mathsf{at}} \in F^*_{\mathsf{at}}$ that

$$\mathfrak{S} \Vdash \phi_{\mathsf{at}} \in \Theta^*(\mathfrak{S}') \quad \text{iff} \quad \mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} \phi_{\mathsf{at}},$$

i. e. the set $\Theta^*(\mathfrak{S}')$ is the same, for every $\mathfrak{S}'$. Further we define Θ^* for atomic $\mathsf{FLP}_{\mathcal{L}}$-formulas p (without variables and the constant SELF), $\mathcal{L}$-formulas $\phi_{\mathcal{L}} \notin F_{\mathsf{at}}$, formulas $\phi_{\mathsf{at}} \in F_{\mathsf{at}}$, possible worlds $\mathfrak{S}, \mathfrak{S}'$, variables w and

$\mathsf{FLP}_{\mathcal{L}}$-formulas ϕ, ψ:

$$\Theta^*(\mathfrak{S} \Vdash p) := \begin{cases} \mathfrak{S} \Vdash \top & \text{if } \mathfrak{S} \vDash_{\mathsf{FLP}_{\mathcal{L}}} p, \\ \mathfrak{S} \Vdash \bot & \text{otherwise,} \end{cases}$$

$$\Theta^*(\mathfrak{S} \Vdash \phi_{\mathcal{L}}) := \Theta^*(\mathfrak{S} \Vdash \Theta(\phi_{\mathcal{L}})) \qquad \text{[sic: } \Theta \text{, not } \Theta^*\text{],}$$

$$\Theta^*(\mathfrak{S} \Vdash \phi_{\mathsf{at}}) := \mathfrak{S} \Vdash \phi_{\mathsf{at}},$$

$$\Theta^*(\mathfrak{S} \Vdash (\mathfrak{S}' \Vdash \phi)) := \Theta^*(\mathfrak{S}' \Vdash \phi),$$

$$\Theta(\forall w.\phi) := \bigwedge_{a \in \mathfrak{W}} \Theta\left(\phi\left[\frac{a}{w}\right]\right),$$

$$\Theta(\exists w.\phi) := \bigvee_{a \in \mathfrak{W}} \Theta\left(\phi\left[\frac{a}{w}\right]\right),$$

$$\Theta^*(\mathfrak{S} \Vdash \neg\phi) := \neg\Theta^*(\mathfrak{S} \Vdash \phi),$$

$$\Theta^*(\mathfrak{S} \Vdash \phi \wedge \psi) := \Theta^*(\mathfrak{S} \Vdash \phi) \wedge \Theta^*(\mathfrak{S} \Vdash \psi).$$

Now we can prove by induction that $\mathsf{FLP}_{\mathcal{L}}$ is rigid and it immediately turns out that $\mathsf{FLP}_{\mathcal{L}}$ can not be rigid, if the basic logic $\mathcal{L}$ is not (1). $\mathsf{FLP}_{\mathcal{L}}$ is finite, if the basic logic $\mathcal{L}$ is finite and all the predicates out of P_w are decidable. If $\mathcal{L}$ is not finite, then $\mathsf{FLP}_{\mathcal{L}}$ is also not finite, because the set F^*_{at} then is infinite (2). $\square$

If we take the propositional logic $\mathsf{RIG}_{\mathsf{a}}$ as the basic logic, we obtain the propositional modal logic $\mathsf{FLP}_{\mathsf{RIG}_{\mathsf{a}}(A)}$, what is pretty straightforward. In the first order modal logic $\mathsf{FLP}_{\mathsf{RIG}_{\mathsf{p}}(D,\mathsf{P},\alpha)}(\mathsf{P}_w, \alpha_m, \mathfrak{M})$ we can quantify both over the $\mathsf{RIG}_{\mathsf{p}}(D, \mathsf{P}, \alpha)$-*structures* and over the *domain* D. We literally do nothing else here but to add some more predicates to the basic logic $\mathsf{RIG}_{\mathsf{p}}$ which allow us to quantify over the $\mathsf{RIG}_{\mathsf{p}}$-structures. Let $\Box$ be a modal operator, defined with a relation over possible worlds R as

$$\text{(N)} \quad \Box\phi := \forall w\, R(\mathtt{SELF}, w) \rightarrow w \Vdash \phi.$$

Then, as is easily seen, $\forall^{\mathsf{E}}$ (defined like in section 2) provides a device for *variable domain quantification*, i. e. the Barcan-formulas

$$\forall^{\mathsf{E}} x.\Box\phi \rightarrow \Box\forall^{\mathsf{E}} x.\phi \quad \text{and}$$

$$\Box\forall^{\mathsf{E}} x.\phi \rightarrow \forall^{\mathsf{E}} x.\Box\phi$$

are both not valid. In the case of $\forall$, on the other hand, we have constant domain quantification and the formula

$$\forall x.\Box\phi \leftrightarrow \Box\forall x.\phi$$

is valid. (This is rather trivial, because by using $\forall$ we quantify over the same objects, in every possible world.) In other words: the logic $\mathsf{FLP}_{\mathsf{RIG_p}(D,\mathsf{P},\alpha)}$ can be interpreted in the standard directions, as pointed out in the classic literature.[12]

We can introduce modal operators with many formulas as arguments in every logic $\mathsf{FLP}_{\mathcal{L}}$. As an example we consider a partial order $<$ over the possible worlds (in the sense of a temporal logic) and define the until-operator $U(\phi,\psi)$:

$$U(\phi,\psi) := \exists w\,(\mathtt{SELF} < w \wedge w \Vdash \phi) \wedge \\ (\forall w' : \mathtt{SELF} < w' < w \rightarrow w' \Vdash \psi).$$

On the other hand we can introduce modal operators with predicates having more than two arguments. For example, we introduce a relevance-logical consequence relation $\Rightarrow$ on the basis of the ternary relation R:

$$\phi \Rightarrow \psi := \forall w, w' : R(\mathtt{SELF}, w, w') \rightarrow (w \Vdash \phi \rightarrow w' \Vdash \psi).$$

$\mathsf{FLP}_{\mathcal{L}}$ is not the most general kind of a modal logic in a propositional framework. First, we could use a different logic for quantifying over possible worlds and formulas. Second, we could allow to vary the modal model inside of a logic. In such a logic $\mathsf{M1P}_{\mathcal{L}}$ (whose basic logic $\mathcal{L}$ must be rigid) we have the set of all modal models $\mathbb{M}$, as an additional sort, and satisfaction must be defined, relative to pairs $[\mathfrak{S},\mathfrak{M}]$ of structures and modal models. The predicates that range over $\mathbb{M}$ must be semantically specified here, via a second-order modal model $\mathfrak{M}^2$ that is fixed for this logic. Now, by analogy to this, we also could specify a logic $\mathsf{M2P}_{\mathcal{L}}$, where we are able to quantify also over second-order modal models. Further abstraction leads us to a logic $\mathsf{MnP}_{\mathcal{L}}$, for every finite n, where modal models of order $x \leq n$ are variably defined and where a fixed modal model of order $n+1$ is stipulated. Satisfaction, in such a logic, must be defined relative to a vector $[\mathfrak{S},\mathfrak{M}^1,\ldots,\mathfrak{M}^n]$, consisting of a structure and a modal model of every order $x \leq n$. However, we do not go into the formal details of those logics.

[12] Cf. [6, Chapter 4], for an overview.

A logic like $\mathsf{FLP}_{\mathcal{L}}$ has some crucial advantages over the classic setting of modal logic. Its *modularity* allows us to establish $\mathsf{FLP}_{\mathcal{L}}$ as a *direct extension* of arbitrary basic languages $\mathcal{L}$. The logic $\mathsf{FLP}_{\mathcal{L}}$ *inherits* the meta-logical features from $\mathcal{L}$: if $\mathcal{L}$ is rigid, then also $\mathsf{FLP}_{\mathcal{L}}$; if $\mathcal{L}$ is finite, then also $\mathsf{FLP}_{\mathcal{L}}$. Especially the latter case should be of high interest for the philosopher, because it allows us to concentrate on the really important philosophical questions. *Formally*, the layout of every rigid language is the same, be it finite or not. Thus, if we *assume* the logic to be finite, we do not have to change anything in the principal formal layout. Finiteness, moreover, always implies that a logic is *decidable*, both regarding validity and satisfaction. And that means for us that in assuming finiteness we can assume *every* modal interpretation of our logic to be decidable. Therefore, we can skip the whole range of completeness proofs for particular modal systems, because in our setting *every* modal system is decidable (as long as we can express it in $\mathsf{FLP}_{\mathcal{L}}$, and as long as we assume the system to be finite). This ruling out of such meta-logical questions which seem to be simply irrelevant for the most philosophical discussions allows us to slim down the formal apparatus and to concentrate on the really important philosophical questions (e. g. the question of correspondence theory, of a structural comparison between different modal systems). In this way, a division of labor can be established between a philosophical discussion in a simple framework such as the propositional framework we presented in this paper, and the more technical discussions of the mathematician and the computer scientist who search for particular meta-logical properties, quick decision algorithms and elegant deductive calculi.

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