Derivability and Metainferential Validity

Bruno Da Ré[1,2], Damian Szmuc[1,2], and Paula Teijeiro[1,2]

[1]Department of Philosophy, University of Buenos Aires, Argentina
[2]IIF-SADAF, National Scientific and Technical Research Council (CONICET), Argentina

Abstract

The aim of this article is to study the notion of derivability and its semantic counterpart in the context of non-transitive and non-reflexive substructural logics. For this purpose we focus on the study cases of the logics ST and TS. In this respect, we show that this notion doesn't coincide, in general, with a nowadays broadly used semantic approach towards metainferential validity: the notion of local validity. Following this, and building on some previous work by Humberstone, we prove that in these systems derivability can be characterized in terms of a notion we call absolute global validity. However, arriving at these results doesn't lead us to disregard local validity. First, because we discuss the conditions under which local, and also global validity, can be expected to coincide with derivability. Secondly, because we show how taking into account certain families of valuations can be useful to describe derivability for different calculi used to present ST and TS.

Keywords: derivability, non-reflexive logics, non-transitive logics, local validity, absolute global validity

1 Background and aim

In the past decade, a considerable amount of *substructural* accounts of the paradoxes have been in the spotlight. Allegedly, their main advantage lies in their involvement with the more abstract and general features of logical consequence. Instead of proposing to abandon this or that feature of a logical expression or connective in order to accommodate the paradoxical phenomena, these alternatives provide a more encompassing solution that affects the features of logical consequence itself.

Among the substructural options we can find the so-called *strict-tolerant* approach championed by Cobreros, Égré, Ripley and van Rooij, and the so-called *tolerant-strict* approach defended by French. The literature classifies these as, respectively, *non-transitive* and *non-reflexive* approaches to paradox as the first invalidates the rule of Cut and the second invalidates the rule of Identity—in a sense to be made precise below. These sort of substructural approaches to paradox, embodied in the paradigmatic cases of the logics ST and TS, respectively, will be the focus of our article.[1]

Substructural takes on paradoxical phenomena are crucially said to invalidate some *metainferences* that are valid in Classical Logic. These can be understood by analogy to the ground level inferences: while the latter concern relations between formulas, the former concern relations between inferences themselves. Since these logics are often presented through both proof-theoretic and semantic means (in particular, through sequent calculi and trivaluations, respectively) one ought to consider what standards for metainferential validity are available in these contexts.

On the proof-theoretic side, it is important to highlight that sequent calculi have two different standards in what pertains to metainferential validity: these correspond to the relations of *admissibility* and *derivability.* Roughly speaking, the former corresponds to the question of whether the conclusion of a metainference can be derived in the calculus if all the premises of said metainference are derivable; whereas the latter corresponds to the question of whether the conclusion of a metainference can be derived in the calculus resulting from adding all the premises of the metainference as axioms.

On the semantic side, researchers have so far focused on two different standards for metainferential validity: these correspond to the relations of *global* and *local* validity. To understand them, assume a given concept of counterexample for an inference. Local validity means that any counterexample to the conclusion is a counterexample to one of the premises as well. Global validity means that if there is a counterexample to the conclusion, there must also be a counterexample to one of the premises (where the two counterexamples need not be the same).

The aim of this article is to study the notion of derivability and its semantic counterpart in the context of non-transitive and non-reflexive substructural logics. In this regard, we provide both negative and positive results. The negative result concerns the fact that derivability doesn't coincide, in general, with local validity. The *positive results* are three-fold. First, we prove that in non-transitive and non-reflexive systems derivability can be characterized in terms of a notion we call "absolute global validity". Secondly, we show that arriving at these results doesn't lead us to disregard local validity, because there are certain conditions under which local, and also global validity, can be expected to coincide with derivability. Thirdly, we show that—taking into account certain families of valuations—local validity can indeed be useful to describe derivability

[1]Both the notion of logical consequence associated with ST and TS were presented in [5]. In this regard [4], [5], [6], [7], [25], [26] and [12] are articles where Cobreros, Égré, Ripley and van Rooij, and French—respectively—embrace these approaches. It should be mentioned that the general framework allowing to define logics of the kind of ST is presented in [11], whereas the general framework allowing to define logics of the kind of TS is presented in [21]. Furthermore, as remarked by an anonymous reviewer, the ST setting is already presented in full detail in [15].

for different calculi used to present ST and TS.

With this purpose, the article is structured as follows. In Section 2 we introduce some technical preliminaries that we will assume throughout the article. In Section 3 we present the two most developed standards for metainferential validity, the global and the local notion, and show a negative result, i.e. that none of them is the semantic counterpart of the proof-theoretical notion of derivability. In Section 4 we prove the first positive result: that in non-transitive and non-reflexive logics the semantic counterpart of derivability is the notion of *absolute global validity*. Next, in Section 5 we present our second positive result, regarding some sufficient conditions for collapsing the different notions of validity. In this way, although local validity doesn't coincide in general with derivability, it does in some special cases. Actually, in Section 6 we introduce our final positive result, regarding the coincidence of local validity with derivability, in the context of the logics ST and TS. In Section 7 we conclude with some final remarks and directions for future work.

2 Technicalities

In this section, we present some technical notions that we will use below. We will be working with a propositional language $\mathcal{L}$ which contains a denumerable set Var of propositional variables $p, q, r, \ldots$ and with logical connectives $\neg, \wedge, \vee$—intended to represent negation, conjunction, and disjunction, respectively. Thus, $FOR(\mathcal{L})$ is the set of well-formed formulas, as usually construed. Lower case Greek letters $\varphi, \psi, \chi, \ldots$ will be considered as schematic formulas, whereas upper case Greek letters $\Gamma, \Delta, \Theta, \ldots$ will be considered as schematic sets of formulas.

As we mentioned earlier, our aim is to discuss certain aspects of certain substructural logics. These are to be understood as systems where some of the *metainferences* usually associated with the properties of Reflexivity, Monotonicity, and Transitivity of a consequence relation fail.[2] Just like regular inferences hold between (collections of) formulas, metainferences can be regarded informally as inferences between inferences. More formally, these can be described as in the definitions below.

Definition 1. An *inference token* ρ of a language $\mathcal{L}$ is a pair $\langle \Gamma, \Delta \rangle$, where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$; a *simple inference schema* $\boldsymbol{\rho}$ is the set of all and only the inference tokens that can be obtained from one of its members—its "basic instance"—by uniformly substituting some propositional variables p_i in it by some formulas φ_i. Given a simple inference schema $\boldsymbol{\rho}$ an *inference schema with contexts* is the union of $\boldsymbol{\rho}$ together with some (possibly all) of the sets

[2] In the context of this article, we won't be discussing Contraction or Exchange, as we will be working with sets of formulas.

$\{\langle \Gamma \cup \Sigma, \Delta \cup \Pi \rangle : \langle \Sigma, \Pi \rangle \in \boldsymbol{\rho}\}$, where Γ, Δ are arbitrary sets of formulas.[3]

In what follows, we will divide the premises and the conclusion of the corresponding inferences by a double-lined arrow $\Rightarrow$. For the sake of clarity, let us exemplify the previous definitions with regard to different incarnations of Disjunctive Syllogism. While $\neg p, (p \vee (p \wedge q)) \Rightarrow p \wedge q$ is an inference token, $\neg \varphi, (\varphi \vee \psi) \Rightarrow \psi$ is a simple inference schema, whereas $\Gamma, \neg \varphi, (\varphi \vee \psi) \Rightarrow \psi, \Delta$ is an inference schema with contexts. Also, when it generates no ambiguity, we will indistinctly refer as "schemas" to both simple schemas and to those with contexts.

Definition 2. A *metainference token* P of a language $\mathcal{L}$ is a pair $\langle A, \rho \rangle$, where A is a set of inference tokens and ρ is an inference token of $\mathcal{L}$; a *simple metainference schema* $\mathbf{P}$ of a language $\mathcal{L}$ is the set of all and only the metainference tokens that can be obtained from one of its members—its "basic instance"—by uniformly substituting some propositional variables p_i in it by some formulas φ_i. As before, given a simple metainference schema $\mathbf{P}$ a *metainference schema with contexts* is the union of $\mathbf{P}$ together with some (possibly all) of the sets $\{\langle \{\Gamma_i \cup \Sigma_i \Rightarrow \Delta_i \cup \Pi_i\}, \Xi \cup \Theta \Rightarrow \Upsilon \cup \Omega \rangle : \langle \{\Sigma_i \Rightarrow \Pi_i\}, \Theta \Rightarrow \Omega \rangle \in \mathbf{P}\}$, where $\Gamma_i, \Delta_i, \Xi, \Upsilon$ are arbitrary sets of formulas.[4]

In order to exemplify these notions, let us examine what the literature refers to as meta Explosion—see [1] and [3]. From left to right below, the first is a metainference token, the second is a simple metainference schema, and the third is a metainference schema with contexts.

$$\frac{\Rightarrow p \wedge q \quad \Rightarrow \neg(p \wedge q)}{\Rightarrow r \vee s} \quad \frac{\Rightarrow \varphi \quad \Rightarrow \neg \varphi}{\Rightarrow \psi} \quad \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \quad \Gamma_2 \Rightarrow \neg \varphi, \Delta_2}{\Xi \Rightarrow \psi, \Upsilon}$$

The substructural systems that we will focus mostly on—Cobreros, Égré, Ripley and van Rooij's logics ST and TS (the latter defended by French)—are respectively of a non-transitive and non-reflexive kind, and so it is important to present them clearly. When introducing these logics proof-theoretically, usually their advocates appeal to a Gentzen-style sequent calculus. In this context, calculi are just sets of metainference schemas, where metainference schemas

[3] A few clarifications about these definitions are in order. First of all, it should be highlighted that they are pretty much aligned with the way in which Humberstone approaches inferences and schematic versions thereof in [19]. Regarding inference tokens there seems to be no misunderstanding, whereas the literature has it that what we call simple inference schemas are usually referred to as sequential formula-to-formula rules. In this regard, also, let us notice that these don't have a unique basic instance, but instead have infinitely many such instances as discussed in [19, p. 123]. Furthermore, regarding inference schemas with contexts, although this notion isn't discussed by Humberstone, he states that arriving at such entities through a detour via simple inference schemas (sequential formula-to-formula rules, in his terminology) is "clear enough"—see [19, p. 123].

[4] Once again, a few clarifications are in order. First, what we call metainference tokens are nothing more than members of what are called application-sets of a sequent-to-sequent rule—see [19, pp. 588-589]. In this regard, metainference schemas with or without contexts can be identified with such application-sets.

with empty premises are also called axioms.[5] Before introducing the sequent calculi we will work with, we need first to present some definitions:

Definition 3. A set of inference tokens A is closed under a metainference token $\langle B, \rho \rangle$ if and only if either some inference of B isn't in A, or the conclusion ρ is in A. A set of inference tokens A is closed under a metainference schema if and only if it is closed under all of its instances.

Definition 4. Given a set of metainference schemas $\mathbf{S}$ the set of $\mathbf{S}$-provable inferences $pr(\mathbf{S})$ is the smallest set of inferences closed under $\mathbf{S}$. So, we will say that an inference ρ is provable in $\mathbf{S}$ if and only if $\rho \in pr(\mathbf{S})$.

Both in the case of ST and TS, some modification of the sequent calculus **LK**, appearing in Figure 1, is usually employed.[6]

$$\frac{}{\varphi \Rightarrow \varphi}\ [Id]$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta}\ [KL] \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Pi, \Delta}\ [KR]$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Sigma, \varphi \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ [Cut]$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\ [\neg L] \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ [\neg R]$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ [\vee L] \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ [\vee R]$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ [\wedge L] \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\ [\wedge R]$$

Figure 1: The calculus **LK**

In some of their works, the aforementioned authors describe the logic ST in terms of the set of inferences determined by $\mathbf{LK}^{\setminus Cut}$ (resulting from removing the Cut rule from **LK**) whereas in some other works this role is played by the set $\mathbf{LK}^{\setminus Cut}_{INV}$ (resulting from adding to $\mathbf{LK}^{\setminus Cut}$ all its inverted rules).[7] Wary of these

[5]Notice also that what is usually called *rule* here is a metainference schema with contexts, and what is usually called *sequent* here is an inference.

[6]Following [10], we call this calculus **LK** although it differs from (but it is equivalent to) Gentzen's original calculus in some respects (e.g. Contraction and Exchange are absorbed since we are working with sets).

[7]To be more specific, while Ripley employs the former in [27] and [26], he uses the latter in the more recent [24].

distinctions, in [12] French looks at two ways to describe the logic TS in terms of calculi. For the most part he focuses on $\mathbf{LK}^{\backslash Id}$ (resulting from removing the Identity rule from $\mathbf{LK}$) and later he entertains the set determined by $\mathbf{LK}^{\backslash Id}_{INV}$ (resulting from adding to $\mathbf{LK}^{\backslash Id}$ all its inverted rules).

Interestingly, whether we choose to work with the calculi with or without inverse rules doesn't have any effect on the set of provable inferences. As we will discuss in detail in Section 6, $pr(\mathbf{LK}^{\backslash Cut}_{INV}) = pr(\mathbf{LK}^{\backslash Cut}) = pr(\mathbf{LK})$, whereas $pr(\mathbf{LK}^{\backslash Id}_{INV}) = pr(\mathbf{LK}^{\backslash Id}) = \emptyset$. There are, however, two different ways of lifting the notion of provability from inferences to metainferences (for the second one, as we will see in Section 6, it does make a difference which sequent calculus one chooses):

Definition 5. A metainference token $\langle B, \rho\rangle$ is *A-admissible* in a set of inferences A if and only if A is closed under $\langle B, \rho\rangle$. A metainference schema is A-admissible if and only if all its instances are.

Definition 6 ([18])**.** A metainference token $\langle B, \rho\rangle$ is $\mathbf{S}$-*derivable* from a set of metainference schemas $\mathbf{S}$ if and only if every set of inferences A which is closed under $\mathbf{S}$ is closed under $\langle B, \rho\rangle$. A metainference schema is $\mathbf{S}$-derivable if and only if all its instances are.

Let us pause to state some remarks about the connection between both concepts. First, derivability involves a relation with a set of metainference schemas $\mathbf{S}$, while admissibility—as it stands—involves a relation with a set of inferences. However, we can also say that, given a certain $\mathbf{S}$, a metainference is admissible in $\mathbf{S}$ if it is admissible in the smallest set closed under $\mathbf{S}$ (i.e., it is $pr(\mathbf{S})$-admissible). In this way, derivability is a sort of "super admissibility", as they both are certain types of closure conditions. Second, notice that a metainference token $\langle B, \rho\rangle$ is $pr(\mathbf{S})$-admissible if and only if $\langle\emptyset, \rho\rangle$ is $\mathbf{S}$-derivable or $\langle\emptyset, \rho'\rangle$ isn't $\mathbf{S}$-derivable, for some $\rho' \in B$. Third, a metainference without premises as $\langle\emptyset, \rho\rangle$ is $\mathbf{S}$-derivable if and only if it is $pr(\mathbf{S})$-admissible (if and only if $\rho \in pr(\mathbf{S})$). For this reason, and when no confusion arises we will identify the metainference token $\langle\emptyset, \rho\rangle$ with the inference token ρ. This is especially pertinent for the case of Identity.

Coming back to our target systems ST and TS, they are also defined in a semantic way. For this, three-valued valuations of a special kind are usually employed, and a particular definition of logical consequence is taken into consideration to describe which inferences are valid in these logics. For future reference, let us introduce the following definitions.

Definition 7. A *valuation* is a function from $FOR(\mathcal{L})$ to a set of truth-values. We will work here with two different valuational spaces:

1. the set of bivaluations V_2, which range over $\{1, 0\}$
2. the set of trivaluations V_3, which range over $\{1, i, 0\}$

With regard to these sets of valuations, we can think about four different standards for validity. In the literature, these are called strict-strict, tolerant-tolerant, strict-tolerant, and tolerant-strict accounts—from these, we will focus mainly on the last two, but see [5] for more.

Definition 8. A formula φ is s-satisfied by a valuation v in V if and only if $v(\varphi) = 1$. On the other hand, a formula φ is t-satisfied by a valuation v in V if and only if $0 < v(\varphi) \leq 1$.

Definition 9. An inference token $\Gamma \Rightarrow \Delta$ is xy-satisfied in a valuation v in V if and only if it is not the case that this valuation x-satisfies γ for every $\gamma \in \Gamma$, and at the same time it does not y-satisfy δ for any $\delta \in \Delta$, where $x, y \in \{s, t\}$. An inference token $\Gamma \Rightarrow \Delta$ is xy-valid in the set of valuations V if and only if it is satisfied by all valuations v in V. Similarly, a schema is xy-valid in the set of valuations V if and only if all of its instances are.

Furthermore, these bivaluations and trivaluations can of course be restricted to some subsets thereof respecting certain constraints. Strong Kleene trivaluations V_{sk} are a subset of general trivaluations that respect the compositionality constraints induced by the strong Kleene truth-tables, appearing in Figure 2. Similarly, we will refer to Boolean bivaluations V_b as the bivaluations belonging to V_2 which respect the compositionality constraints induced by the $\{1, 0\}$-reduct of such truth-tables.

	$\neg$
1	0
i	i
0	1

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

Figure 2: The strong Kleene truth-tables

With all these clarifications in mind, let us recall that ST is semantically induced by focusing on the set of inferences that are st-valid in V_{sk}, and TS results from focusing on the set of inferences that are ts-valid in V_{sk}. We will also refer to the logics characterized by the set of inferences that are, respectively, ss-valid and tt-valid in V_{sk}, in Section 6. Finally, Classical Logic CL can be semantically identified with the set of inferences that are ss-, tt-, st-, or ts-valid in V_b. For the sake of readability, when it doesn't make a difference which standards we choose for validity (such as in the case of bivaluations), we will drop reference to standards altogether.

In the next section, we discuss two different semantic standards for metainferential validity, showing that, even in the case of non-transitive and non-reflexive logics, none of them can be taken as the semantic counterpart of derivability.

3 Global and local validity

In this section, we present and analyze the two most developed standards for metainferential validity, namely the global and the local notions, and show our first *negative result*. That is, that neither of them coincides with the proof-theoretical notion of derivability. Although this was proven by Humberstone in [18] for the case of bivaluations, we generalize his results and prove this for the case of trivaluations. Later, we focus on our two case studies ST and TS, showing why some scholars have focused on the local notion.

Let us, then, review *global validity* first. In a nutshell, this account tells us that a metainference is valid whenever it preserves validity from its premises to its conclusion. Below we give a general account, which can be then instantiated for any of the notions of validity discussed:

Definition 10. A metainference token $\langle A, \rho \rangle$ is *globally* xy-valid in a set of valuations V if and only if, if the premises in A are xy-valid in V, the conclusion ρ is xy-valid in V, where $x, y \in \{s, t\}$. A metainference schema is globally xy-valid in V if and only if all its instances are.

It should be pointed out that a certain coincidence between global validity and admissibility can be highlighted. To see this, let $Val(xy, V)$ be the set of inferences which are xy-valid in V, where $x, y \in \{s, t\}$. The connection between global validity and admissibility can be clearly stated by saying that a metainference is globally xy-valid in V if and only if it is $Val(xy, V)$-admissible. Furthermore, when one is given a calculus $\mathbf{S}$ which is sound and complete with respect to xy-validity in V—i.e., such that $pr(\mathbf{S}) = Val(xy, V)$—it can be observed that the admissible metainferences in $pr(\mathbf{S})$ are those that are globally xy-valid in V. In short, granted soundness and completeness, admissibility and global validity coincide.

There is an alternative to this approach, given by the notion of *local validity* for metainferences. Briefly, this means that metainferences should preserve satisfaction by a valuation instead of preserving validity—as usual, a valuation satisfies an inference if and only if it is not a counterexample to it. Below we present this notion with full generality, without making reference to any particular set of valuations:

Definition 11. A metainference token $\langle A, \rho \rangle$ is *locally* xy-valid in V if and only if, if a valuation in V xy-satisfies all the premises in A, then it xy-satisfies the conclusion ρ, where $x, y \in \{s, t\}$. A metainference schema is locally xy-valid in V if and only if all its instances are.

Given that global validity has a natural tie to the property of admissibility, then, one may wonder if local validity has a similar connection with derivability.

Shortly, we will see that this isn't generally the case.[8] However, in order to precisely formulate the sort of correspondence one might expect between these two notions, one should find a way to select a set of valuations that captures at the semantic level the content of the set of metainference schemas whose Derivability relation is in question. A first (though unsuccessful) attempt in this direction is provided by the notion of local range.

Definition 12. The *local xy-range in V* of a set of metainferences $\mathbf{S}$ is the maximum set of valuations $V_i \subseteq V$ such that all members of $\mathbf{S}$ are locally xy-valid in V_i, where $x, y \in \{s, t\}$.

With this notion at hand, one could hypothesize that the following correlation holds: a metainference $\langle A, \rho \rangle$ is $\mathbf{S}$-derivable if and only if $\langle A, \rho \rangle$ is locally xy-valid in the local xy-range in V of $\mathbf{S}$, for some appropriate xy standard, and some appropriate V.

Nevertheless, as an anonymous reviewer points out, the question remains of how to select such standards and such valuations. In this respect, given the characterization results proved in [13, Corollary 1] (which will be discussed in the next section) we can state the following. Since reflexive, monotonic and transitive sets of inferences can be characterised by bivaluations, for the derivability relation of sets of metainferences containing [*Id*], [KL], and [KR] and [*Cut*], the natural choice would be to take the local xy-range in V_2, with respect to any standard xy. On the other hand, since reflexive and monotonic sets of inferences can be characterised by trivaluations with an *st*-standard, for the derivability relation of sets of metainferences containing [*Id*], [KL], and [KR], the natural choice would be to take the local *st*-range in V_3. Finally, since monotone and transitive sets of inferences can be characterised by trivaluations with a ts-standard, for the derivability relation of sets of metainferences containing [KL], [KR] and [*Cut*], the natural choice would be to take the local ts-range in V_3. In what follows, we will show that these three claims fail, and we will provide counterexamples to them.

Regarding the first claim, consider a set containing [*Id*], [KL], [KR], [*Cut*],

---

[8] Let us highlight in passing that local validity is somewhat included in global validity, and sometimes this inclusion is strict. This happens when the premise-inferences of a metainference are invalid, which is easier seen when we consider metainference tokens. For instance, take the set of Boolean bivaluations V_b and consider the following metainference token:

$$\frac{\Rightarrow p}{\Rightarrow q}$$

There are of course valuations satisfying p but not q. But, since p is invalid, the metainference becomes globally valid in V_b. When we move on to schemas, though, examples cannot be taken from Boolean bivaluations, because both approaches to metainferential validity coincide for such a family of valuations. That is to say, global validity in V_b and local validity in V_b coincide with regard to metainference schemas—a proof of this fact can be found in [30]. We will expand on this point in Section 5.

and the following set of simple metainference schemas, which we will call $\mathbf{S_1}$:[9]

$$\frac{\Rightarrow \varphi}{\Rightarrow \varphi \vee \psi} \qquad \frac{\Rightarrow \psi}{\Rightarrow \varphi \vee \psi} \qquad \frac{\varphi \Rightarrow \quad \Rightarrow \varphi \vee \psi}{\Rightarrow \psi}$$

It is straightforward to check that the valuations belonging to the local range of $\mathbf{S_1}$ in V_2 are all those satisfying the following conditions:

- If $v(\varphi) = 1$ or $v(\psi) = 1$, then $v(\varphi \vee \psi) = 1$
- If $v(\varphi) = v(\psi) = 0$, then $v(\varphi \vee \psi) = 0$

Now, consider the following simple metainference schema:

$$\frac{\varphi \Rightarrow \quad \psi \Rightarrow}{\varphi \vee \psi \Rightarrow} \ \mathbf{P_1}$$

Its local range is determined by the following weaker restrictions:

- If $v(\varphi) = v(\psi) = 0$, then $v(\varphi \vee \psi) = 0$.

This means that the local range in V_2 of $\mathbf{S_1}$ is included in the local range in V_2 of $\mathbf{P_1}$, which in turn means that $\mathbf{P_1}$ is locally valid in the local range of $\mathbf{S_1}$ in V_2. Regarding the underivability of $\mathbf{P_1}$ from the set $\mathbf{S_1}$, in Section 5 we will prove it using semantical tools and appealing to the main result of the Section 4.

For the non-transitive case, take the following metainferential schemas without context, and add them to a set containing $[Id]$, $[KL]$ and $[KR]$:

$$\frac{\psi \Rightarrow \quad \Rightarrow \varphi \vee \psi}{\Rightarrow \varphi} \qquad \frac{\varphi \Rightarrow \quad \Rightarrow \varphi \vee \psi}{\Rightarrow \psi} \qquad \frac{\Rightarrow \varphi \quad \varphi \Rightarrow \quad \Rightarrow \psi \quad \psi \Rightarrow \quad \Rightarrow \varphi \vee \psi}{\Rightarrow}$$

Let us call this set $\mathbf{S_2}$. The valuations in the local *st*-range of $\mathbf{S_2}$ in V_3 are all those satisfying the following restrictions:

- If $v(\varphi) = 0$ and $v(\psi) \in \{0, i\}$, then $v(\varphi \vee \psi) = 0$.
- If $v(\psi) = 0$ and $v(\varphi) \in \{0, i\}$, then $v(\varphi \vee \psi) = 0$.
- If $v(\varphi) = v(\psi) = i$, then $v(\varphi \vee \psi) = 0$.

Then consider again the following metainference schema:

$$\frac{\varphi \Rightarrow \quad \psi \Rightarrow}{\varphi \vee \psi \Rightarrow} \ \mathbf{P_1}$$

Its local *st*-range in V_3 is determined by the following weaker restrictions:

[9] This example builds on the one provided by Humberstone in [18]. We present this modified version because his example only works in a SET-FMLA setting (i.e. the conclusion of an inference is a formula), while in this article we're working in a SET-SET framework (i.e. the conclusion of an inference is a set of formulas). We'd like to thank an anonymous reviewer for pointing this out to us.

- If $v(\varphi) \in \{0, i\}$ and $v(\psi) \in \{0, i\}$, then $v(\varphi \vee \psi) \in \{0, i\}$.

This means that the local *st*-range in V_3 of $\mathbf{S_2}$ is included in the local *st*-range in V_3 of $\mathbf{P_1}$, which in turn means that $\mathbf{P_1}$ is locally *st*-valid in V_3 according to $\mathbf{S_2}$. Once again, a formal argument concerning the underivability of $\mathbf{P_1}$ will have to wait until later in the article.

For the non-reflexive case, take instead a set $\mathbf{S_3}$ with $[KL]$, $[KR]$, and $[Cut]$ and the following metainference schemas without context:

$$\frac{\Rightarrow \varphi \vee \psi}{\psi, \varphi \Rightarrow \varphi, \psi} \qquad \frac{\psi \Rightarrow \quad \Rightarrow \varphi \vee \psi}{\varphi \Rightarrow \varphi} \qquad \frac{\varphi \Rightarrow \quad \Rightarrow \varphi \vee \psi}{\psi \Rightarrow \psi} \qquad \frac{\varphi \Rightarrow \quad \psi \Rightarrow}{\varphi \vee \psi \Rightarrow}$$

The valuations in the local *ts*-range in V_3 of $\mathbf{S_3}$ are all those satisfying the following restrictions :

- If $v(\varphi) = v(\psi) = i$, then $v(\varphi \vee \psi) \in \{0, i\}$.
- If $v(\varphi) = i$ and $v(\psi) = 0$, then $v(\varphi \vee \psi) \in \{0, i\}$.
- If $v(\psi) = i$ and $v(\varphi) = 0$, then $v(\varphi \vee \psi) \in \{0, i\}$.
- If $v(\varphi) = v(\psi) = 0$, then $v(\varphi \vee \psi) = 0$.

The metainference schema which is $\mathbf{S_3}$-underivable is in this case the following:

$$\frac{\Rightarrow \varphi \vee \psi}{\Rightarrow \varphi, \psi} \ \mathbf{P_2}$$

whose local *ts*-range in V_3 is determined by:

- If $v(\varphi) \in \{0, i\}$ and $v(\psi) \in \{0, i\}$, then $v(\varphi \vee \psi) \in \{0, i\}$.

Which means that the local *ts*-range of $\mathbf{S}_3$ in V_3 is included in that of $\mathbf{P_2}$. Once more, a technical explanation of the underivability of $\mathbf{P_2}$ will have to wait until later in the article.

Below, we will see how to fill the gap between derivability and an adequate semantic notion of metainferential validity. However, to close the present discussion we will comment on some reasons why, in the literature on ST and TS, people have opted to work with the local notion when assessing the metainferential validity of Cut and Identity, from a semantic point of view. The reader eager to see the appropriate semantic approach to derivability for non-transitive and non-reflexive systems can skip the following paragraphs and go directly to the next section.

In the case of ST, although the non-transitive nature of the strict-tolerant account has been predicated on the theories (of truth and vagueness) that are built on top of the base logic ST, some have observed that this is a symptom of certain non-transitive character of said base logic. This can be thought in analogy to the non-transitive essence of the calculi through which ST is usually presented. In both of them, although Cut is admissible, it fails to be derivable. Thus, given that highlighting this non-transitivity was their aim, it is easy to observe that the global notion was of no good for this purpose. In fact, per the global reading, Cut is a valid metainference. In light of these considerations, it is obvious that if we want to semantically emphasize the substructural nature of ST, then another notion needs to be taken into account. Probably these considerations led some scholars to work with the local notion, for which Cut is an invalid metainference. However, as we discussed earlier, although local validity might serve the purpose of highlighting the invalidity of Cut in ST, it isn't the case that in general local *st*-validity in V_3 coincides with derivability.

In the case of TS, as far as we can tell, no one has intended to defend any particular notion of validity for its metainferences. So, let us briefly analyze how the local and the global notion of metainferential validity behave for this logic. Recall, first, that the set of valid inferences of TS is empty. This is so, because the V_{sk}-valuation which assigns to every propositional letter the intermediate value is a counterexample to every inference. Given this, no matter what the notion for metainferential validity one adopts, all of the inferences will be invalid. However, regarding metainferences with non-empty sets of premises, some distinctions can be made.

If we take the global approach, all of the metainferences with non-empty sets of premises would be valid. This is so because any instance of the metainference schema appearing below has as a *ts*-counterexample the V_{sk}-valuation which assigns to every propositional letter the intermediate value. Thus, in some sense, the global notion makes TS *metainferentially trivial*, and so it seems like a rather useless conception of metainferential validity.

On the other hand, the local notion, as in the case of ST, is more useful since it allows us to make finer distinctions, e.g. invalidating many metainferences. For instance, the metainference token appearing below, whose conclusion is an instance of Identity, is locally *ts*-invalid in V_{sk}—as it can be seen by taking any strong Kleene trivaluation that assigns q a classical value and p the intermediate value. Thus, also for the case of TS, the local notion seems to have some preeminence.

$$\frac{q \Rightarrow q}{p \Rightarrow p}$$

We have given then enough reasons to want to work with semantic notions of metainferential validity that are closer to derivability and, thus, far from global validity. We also showed that in general, the notions of local validity and derivability don't coincide in the way one might expect. In the next sections, we are going to provide a number of positive points in this regard. First, we are going to show that there is a semantic counterpart to the notion of derivability that in all cases coincides with it, which we call "absolute global validity". Secondly, we are going to show under which circumstances this notion coincides with global and local validity. Finally, in light of these considerations, we are going to demonstrate how to properly work with local validity as a faithful surrogate for the notion of derivability in the calculi used to present the logics ST and TS.

4 Absolute global validity

In the previous section, we provided a *negative result* exposing the difference between local validity and derivability, which was originally discussed by Humberstone for logics semantically induced by bivaluations, and which was then generalized by us for logics semantically induced by trivaluations. In this section, we provide a *positive result* consisting of establishing the semantic counterpart for derivability. This was already done by Humberstone for the case of logics characterized by bivaluations (that is, Tarskian logics, i.e., Reflexive, Monotonic and Transitive logics), and thus our contribution in this section is to generalize this to logics characterized by trivaluations (that is, monotonic logics of the non-transitive or non-reflexive kind).

The first step to the positive results is to have in mind the following crucial definitions.

Definition 13. A set of inference tokens A is:

- *Reflexive* if and only if $\varphi \Rightarrow \varphi \in A$;
- *Monotonic* if and only if $\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \in A$, whenever $\Gamma \Rightarrow \Delta \in A$;
- *Completely Transitive* if and only if for all Σ, if for all $\Sigma_1 \cup \Sigma_2 = \Sigma$, $\Sigma_1, \Gamma \Rightarrow \Sigma_2, \Delta \in A$, then $\Gamma \Rightarrow \Delta \in A$.[10]

Definition 14. A metainference token $\langle B, \rho \rangle$ is $\mathbb{MR}$-derivable ($\mathbb{MT}$-derivable) from a set of metainference schemas $\mathbf{S}$ if and only if $\langle B, \rho \rangle$ is $\mathbf{S}$-derivable and any set of inferences closed under $\mathbf{S}$ is Monotonic and Reflexive (Monotonic and Completely Transitive). A metainference schema is $\mathbb{MR}$-derivable ($\mathbb{MT}$-derivable) from a set of metainference schemas $\mathbf{S}$ if and only if all its instances are. A metainference token (a metainference schema) is $\mathbb{MRT}$-derivable from a set of metainference schemas $\mathbf{S}$ if and only if it is $\mathbb{MR}$-derivable and $\mathbb{MT}$-derivable.

Furthermore, in our quest for a semantic counterpart to the concept of derivability we will replace the concept of a local range by the following broader one, which Humberstone calls global range, but that we will call absolute global range for matters of disambiguation.

Definition 15. The *absolute global xy-range in V* of a set of metainferences $\mathbf{S}$ is the maximum set $\mathcal{V}$ of sets of valuations $V_i \subseteq V$ such that all members of $\mathbf{S}$ are globally xy-valid in V_i, for every $V_i \in \mathcal{V}$, where $x, y \in \{s, t\}$.

[10] Complete Transitivity can be seen as an infinitary rule, a kind of Cut for sets (see [13]). In any compact consequence relation, if a set of inferences is closed under the ordinary rule of Cut then it is Completely Transitive (see [28, p. 436]). So, although for the familiar logics with which we deal, the rule of Cut is enough, the results we prove in this section are more general, so we stick to this broader terminology.

With this in mind, Humberstone introduces an alternative conception of metainferential validity—a notion he refers to as global, but that in order to minimize ambiguity we choose to call *absolute global validity.* We provide its definition, in full generality, below.

Definition 16. A metainference token $\langle B, \rho \rangle$ is *absolutely globally xy*-valid in a set of sets of valuations $\mathcal{V}$ if and only if it is globally xy-valid in V, for every $V \in \mathcal{V}$, where $x, y \in \{s, t\}$. A metainference schema is absolutely globally xy-valid in $\mathcal{V}$ if and only if all its instances are.

With these tools, he proved his main result [18] (which we state in our terminology):

Theorem 17. *Let* $\mathbf{S}$ *be a set of metainferences such that any set of inferences closed under* $\mathbf{S}$ *is Monotonic, Reflexive and Completely Transitive; let* $\mathcal{V}$ *be the absolute global range in* V_2 *of* $\mathbf{S}$*. A metainference schema* $\mathbf{P}$ *is* $\mathbb{MRT}$*-derivable from* $\mathbf{S}$ *if and only if it is absolutely globally valid in* $\mathcal{V}$*.*

In what remains of this section, we generalize these results showing that absolute global validity can be understood as the semantic counterpart of derivability, also for non-transitive and non-reflexive logics. To do this we deploy a strategy in two stages. For this, we recall that on the one hand, derivability was defined as a sort of "super admissibility", while now absolute global validity is a sort of "super global validity". That is, derivable metainferences are those which are admissible in many inference sets, and absolutely globally valid ones are those which are globally valid according to many sets of valuation. Thus, the first stage of our strategy consists in proving the correspondence between global validity and admissibility with full generality, whereas the second stage consists in proving the correspondence between absolute global validity and derivability in non-transitive and non-reflexive systems.

For the first stage, then, as we already informally stated in section 3, soundness and completeness for inferences is all that is needed to prove the correspondence between global validity and admissibility. In particular, we will work with one special, general version of soundness and completeness, which appeals to the concept of a Galois connection.

Definition 18. A *Galois connection* between sets A and V and a relation R between them is a pair of functions $f : \wp(A) \longrightarrow \wp(V)$ and $g : \wp(V) \longrightarrow \wp(A)$ such that $f(x) = \{y \mid Rzy, \text{ for all } z \in x\}$ and $g(x) = \{y \mid Ryz, \text{ for all } z \in x\}$.

The paradigmatic example in [18] consists of taking A to be a set of people, V to be a set of cities, and R to be the relation of visiting. Then, the function f gives you, for each group of people, the set of cities that they all visited, while the function g gives you, for each group of cities, the set of people who visited them all. The interesting thing about Galois connections is that, if you compose the functions, you get a closure operation.[11]

[11] As discussed in many places of the specialized literature, an operation $\mathcal{C}$ over a set Z is a *closure operation* whenever, for all $X, Y \subseteq Z$: (1) $X \subseteq \mathcal{C}(X)$, (2) If $X \subseteq Y$, then $\mathcal{C}(X) \subseteq \mathcal{C}(Y)$, (3) $\mathcal{C}(X) = \mathcal{C}(\mathcal{C}(X))$.

Definition 19. If $\langle f, g\rangle$ is a Galois connection between A and V, then a set $A_i \subseteq A$ is *Galois-closed* if and only if $g(f(A_i)) = A_i$, and a set $V_i \subseteq V$ is Galois-closed if and only if $f(g(V_i)) = V_i$.

Fact 20. *If $\langle f, g\rangle$ is a Galois connection between A and V and R, then for all $A_i \in A$ and $V_i \in V$ the sets $f(A_i)$ and $g(V_i)$ are Galois-closed.*

All this can be applied to logic when, instead of cities and people, one takes R to be the satisfaction relation between a set of inferences A and a set of valuations V. Then, the function $f(A_i)$ gives you the set V_i of valuations for which the set A_i of inferences is sound, and if you go back with the function $g(V_i)$, you get the superset of A_i which is also complete with respect to V_i. We say that this is a generalization of the usual soundness and completeness theorems because it doesn't rely on the sets of inferences A_i being generated by any specific proof theory, or the valuations V_i to respect any previously defined restrictions on the sets of interpretations. Therefore, by appealing to this sort of soundness and completeness, we can finish the first stage and prove that global validity and admissibility match under very general conditions:

Lemma 21. *Let A be a set of inferences, V a set of valuations, R a satisfaction relation between them, and $\langle f, g\rangle$ the corresponding Galois connection. Then, (a) if $A_i \subseteq A$ is a Galois-closed set, then a metainference token $\langle B, \rho\rangle$ is globally valid in $f(A_i)$ if and only if it is A_i-admissible ; and (b) if $V_i \subseteq V$ is a Galois-closed set, then a metainference token $\langle B, \rho\rangle$ is $g(V_i)$-admissible if and only if it is globally valid in V_i.*

Proof. We show here only the proof of (a), since proving (b) is symmetrical.

If: Assume $\langle B, \rho\rangle$ is globally valid in $f(A_i)$ and its premises belong to A_i. By definition of f, A_i is a sound set with respect to $f(A_i)$, that is, all its members are valid in $f(A_i)$. Thus, given the global validity of $\langle B, \rho\rangle$, its conclusion has to be valid in $f(A_i)$ too. But by assumption, A_i is a Galois-closed set, and thus, it isn't only sound with respect to $f(A_i)$, but also complete. This implies that the conclusion of $\langle B, \rho\rangle$ has to be in A_i. Hence, $\langle B, \rho\rangle$ is A_i-admissible.

Only if: Assume that $\langle B, \rho\rangle$ is A_i-admissible —that is A_i is closed under $\langle B, \rho\rangle$ —and that its premises are valid in $f(A_i)$. Since A_i is, by assumption, a Galois-closed set, it is complete, i.e., every inference which is valid in $f(A_i)$ is in A_i. Thus, the premises of $\langle B, \rho\rangle$ must be in A_i. And since A_i is closed under $\langle B, \rho\rangle$, the conclusion must belong to A_i too. But by definition of f, A_i is sound with respect to $f(A_i)$, and hence, the conclusion of $\langle B, \rho\rangle$ is also valid in $f(A_i)$. This means $\langle B, \rho\rangle$ preserves validity in $f(A_i)$, that is, it is globally valid in $f(A_i)$. □

This result could seem a bit too abstract, since these sets of inferences are not being obtained for instance, by a compelling proof-theory. Thus, it may raise doubts on whether this Lemma can be applied, in particular, to logics. The answer is that Galois-closed sets have in fact a strong connection to logic: when the set of valuations is that of the bivaluations, and the satisfaction relation is any of the ones discussed above, Galois-closed sets are precisely those which are Tarskian—i.e., Reflexive, Transitive and Monotonic. This theorem proved by [16] shows that Tarskian logics are, in a deep sense, the logics of bivaluations.[12]

In this vein, we can state that Lemma 21 shows that global validity and admissibility coincide *at least* in Tarskian logics. Also, as proved by [13], when the set of valuations is that of the trivaluations, and the satisfaction relation is the strict-tolerant one (respectively, the tolerant-strict relation), Galois-closed sets are precisely those which are Reflexive and Monotonic (respectively, Completely Transitive and Monotonic). This shows that non-reflexive and non-transitive logics are the logics of trivaluations. In what follows, we will state this result in more formal terms. We will set $\langle f_r, g_r \rangle$ as the Galois connection between the set of $\mathcal{L}$-inferences, the valuations in V_3 and the relation of *st*-validity, and $\langle f_t, g_t \rangle$ as the same connection, but with the relation of *ts*-validity instead. Also, we will take fg_r and fg_t to be the respective closure operations. Thus, the theorem proved in [13] can be formulated as follows.

Theorem 22. *A set of inference tokens is fg_r-closed if and only if it is Monotonic and Reflexive, and it is fg_t-closed if and only if it is Monotonic and Completely Transitive.*

This closes the first stage of our strategy, because Theorem 22 shows that we can apply the previous Lemma 21 to the systems we are interested in, showing that also in these substructural logics admissibility and global validity coincide.

The second stage of our strategy, as stated above, builds on this very general correspondence between admissibility and global validity. Keeping in mind that the theories we are interested in are not necessarily transitive and reflexive sets of inferences, will help us remind that they won't be Galois-closed with respect to bivaluations, but to trivaluations. Then, we move on to prove that the concept of absolute global validity works as a valuational counterpart of the concept of derivability in non-transitive and non-reflexive logics as well:

Theorem 23. *Let* $\mathbf{S}$ *be a set of metainferences such that any set of inferences closed under* $\mathbf{S}$ *is Monotonic and Reflexive (respectively, Monotonic and Completely Transitive); let* $\mathcal{V}$ *be the absolute global st-range (respectively, ts-range) in* V_3 *of* $\mathbf{S}$*. A metainference schema* $\mathbf{P}$ *is* $\mathbb{MR}$*-derivable (respectively,* $\mathbb{MT}$*-derivable) from* $\mathbf{S}$ *if and only if it is absolutely globally st-valid in* $\mathcal{V}$ *(respectively, ts-valid).*

Proof. We show here the proof for the set of Reflexive valuations and $\mathbb{MR}$-derivability. The proof for transitive valuations and $\mathbb{MT}$-derivability is analogous.

Only if: Suppose $\mathbf{P}$ is absolutely globally *st*-invalid in $\mathcal{V}$. Thus, there is a set of valuations $V \in \mathcal{V}$ such that every metainference in $\mathbf{S}$ is globally *st*-valid in V but there is a metainference token $\langle B, \rho \rangle$ of $\mathbf{P}$ which isn't. We need to show

[12] Though these bivaluations are not necessarily Boolean, of course, since that depends not only on the structural properties, but also on the interpretations of the connectives.

that **P** is $\mathbb{MR}$-underivable, that is, that there is a set of inferences which is (i) Reflexive and Monotonic, (ii) closed under **S**, but (iii) not closed under **P**. This will be the set $g_r(V)$, that is, all *st*-valid inferences in V. Notice that, because of the Fact 20, we know this set of inferences is Galois-closed. Hence first, Theorem 22 applies to $g_r(V)$, and (i) follows because of Theorem 22. Second, (ii) follows because, given that metainferences in **S** are by assumption globally *st*-valid in V, then because of Lemma 21, they are all $g_r(V)$-admissible. Lastly, (iii) follows because, since $\langle B, \rho \rangle$ is by assumption globally *st*-invalid in V, again Lemma 21 implies it isn't $g_r(V)$-admissible. Hence, **P** isn't $\mathbb{MR}$-derivable.

If: Suppose **P** isn't $\mathbb{MR}$-derivable. Thus, there is a set of inferences A closed under Monotonicity and Reflexivity and **S**, and a metainference token $\langle B, \rho \rangle$ of **P** which isn't A-admissible. We need to show that there is a set of valuations which: (i) is in the global range of **S** and (ii) **P** is globally invalid according to it. This will be the set $f_r(A)$, that is, the valuations satisfying all inferences in A. Notice that, because of the Fact 20, we know this set of valuations is Galois-closed, which guarantees part (b) of Lemma 21 applies to it. First, (i) follows because, given that metainferences in **S** are by assumption A-admissible, then because of Lemma 21, they are all globally *st*-valid in $f_r(A)$. Second, (ii) follows because, since $\langle B, \rho \rangle$ is by assumption not A-admissible, Lemma 21 applies, and thus $\langle B, \rho \rangle$ isn't globally *st*-valid in $f_r(A)$. Hence, **P** is absolutely globally *st*-invalid in $\mathcal{V}$. □

Having shown our first positive result, that the notion of absolute global validity appropiately coincides with the notion of derivability, we now move on to the final part of the article. In what follows, we discuss that what we said above doesn't entail that we must get rid of the notions of local and global validity—even when we want to discuss derivability. The reason for this is, as will become clear shortly, that under some conditions these approaches to metainferential validity do indeed collapse.

5 Collapsing validities

Up until this section we have mainly focused on the differences between global, local, and absolute global validity. However, it is a fairly standard goal to have a logic in which they all match. In this section, we comment on the conditions under which each pair of these notions collapse or coincide. It is worth noting that all the results in what follows will pertain not only to trivaluations, as discussed in the previous section, but also to bivaluations and valuations over any space of truth-values. We consider this to be an advantage and an interesting new piece of information in this regard.

First, let us start by the conditions under which *local validity collapses into global validity.* Both concepts are in fact not that far away as one may think by reading some of the literature. In fact, as long as one considers only metainference schemas and not merely metainference tokens, they coincide in many logics whose languages are powerful enough, as established in [30]. How to characterize, in full generality, the necessary expressive strength to achieve this collapse isn't an easy task. One can say, for instance, that functional completeness is sufficient; but to offer necessary conditions applicable to any logical system—even to non-truth-functional systems—is harder to do. The informal and intuitive idea, however, is that one needs to somehow express the truth-values of the domain of the corresponding valuations through formulas, so that one can get a metainference token of the invalid metainference schema where the formulas involved have the same value in every valuation—i.e., the value they get in the local counterexample. As an example of this, take the following metainference schema, which is locally ts-invalid in V_3:

$$\frac{\Rightarrow \varphi}{\psi \Rightarrow \psi}$$

but it is globally valid if the language lacks truth-value constants. This is so, because every instance of the premise of the schema is invalid (recall there are no valid inferences in TS, since the valuation which assigns the intermediate value to every formula is a counterexample to every inference). However, once a 1-constant $\top$ is introduced into the language, the metainference schema becomes globally invalid as shown by the following token:[13]

$$\frac{\Rightarrow \top}{p \Rightarrow p}$$

As the previous example illustrates, the use of schemas with the addition of truth-value constants allows each valuation to be represented by a token, and once this happens the difference between global and local validity vanishes.

Secondly, in a really similar fashion we can show sufficient conditions under which *global validity collapses into absolute global validity*. However, in order to even compare these notions, some clarifications are in order. A set of valuations might be selected in different ways. From the semantic side, one could select a subset which complies with previously established truth-theoretical meaning constraints on some vocabulary—e.g., excluding valuations assigning a certain truth-value to both a formula and its negation. In the same fashion, instead of thinking about truth-conditions, one can proof-theoretically restrain the available valuations so that they respect the aforementioned meanings. That is, one chooses a set of metainference schemas of some sort, and then determines a set of valuations by taking, for instance, its local or its absolute global range, as we did in previous sections.

If we want to compare our semantic notions by themselves, without appealing to their proof-theoretical counterparts, a small difference between global and local validity on the one hand, and absolute global validity on the other, emerges. While a set of valuations and a validity standard uniquely fixes both the corresponding local and global notions of metainferential validity, many possible notions of absolute global validities can be defined from that very same

[13]Notice that with the addition of $\top$, TS has valid inferences, e.g. $\Rightarrow \top$, $\neg\top \Rightarrow$, $\Rightarrow \top \wedge \top$, and so on.

starting set of valuations. In order to compare these three notions, one may want to have some privileged way of going about and defining a notion of absolute global validity in such cases.

Thus, given a fixed base set of valuations V_B, the whole spectrum of choices from which absolute global validity can be defined is obtained by taking the powerset of the powerset of V_B. We have then a lattice of options, where the maximum is the powerset $\wp(V_B)$, and the minimum is the empty set. Any of the elements of the lattice can be used to provide a definition of absolute global validity. However, notice that for some elements of the lattice, the union of all its members won't be V_B. Thus, defining local and global validity over it won't be equivalent to defining it over V_B. Moreover, the union of the maximum is of course V_B, and therefore if we define absolute global validity over $\wp(V_B)$, we can prove the following result:

Fact 24. *Let $V_B \subseteq \{v \mid v : FOR(\mathcal{L}) \to \{a_1, ..., a_n\}\}$ be a set of valuations and $\mathcal{L}$ a propositional language that contains a truth-value constant $\overline{a_i}$ for each truth-value a_i. Then, a metainference schema $\mathbf{P}$ is globally valid in V_B if and only if it is absolutely globally valid in $\wp(V_B)$.*

Proof. **If:** Assume $\mathbf{P}$ is globally invalid in V_B. Since $V_B \in \wp(V_B)$, $\mathbf{P}$ is absolutely globally invalid in $\wp(V_B)$.

Only if: Assume $\mathbf{P}$ is absolutely globally invalid in $\wp(V_B)$. Therefore, there is a metainference token $\langle A, \rho \rangle$ of $\mathbf{P}$ and a set of valuations $V_i \subseteq V_B$ such that each $v \in V_i$ satisfies all of the inferences of A but dissatisfies the conclusion ρ. Take one of these valuations, say v_i and define a new metainference token $\langle A', \rho' \rangle$, uniformly substituting in $\langle A, \rho \rangle$ each propositional letter p occurring in $\langle A, \rho \rangle$ by $\overline{a_k}$, with $v_i(p) = a_k$. Now, it is easy to see that $\langle A', \rho' \rangle$ is an instance of $\mathbf{P}$, such that in every valuation of V_B, all of the premises are satisfied, but the conclusion isn't. Therefore, $\mathbf{P}$ is also globally invalid in V_B. □

It is worth noticing that in this result, we assume that the language $\mathcal{L}$ is rich enough to contain a truth-value constant for each truth-value, just like it was done in [30] to prove the collapse between global and local validity. As we will see in the collapse result between local validity and absolute global validity (see Fact 25), this isn't required.

Thirdly, and at last, this same general definition of absolute global validity can be used to establish the third point of comparison, which is the conditions under which *local validity collapses into absolute global validity*:

Fact 25. *Given a set of valuations V_B, the set of metainferences which are absolutely globally valid in $\wp(V_B)$ is identical to the set of metainferences which are locally valid in V_B.*

Proof. Assume a metainference is locally invalid in V_B. Thus, there is a valuation $v \in V_B$ satisfying the premises but not the conclusion. Take the set $\{v\}$. This set belongs to the absolute valuational space (given that it is $\wp(V_B)$), and the metainference doesn't preserve validity in $\{v\}$. Thus, it isn't absolutely globally valid in $\wp(V_B)$. Assume a metainference is absolutely globally invalid in $\wp(V_B)$. Thus, there is a $V_i \subseteq V_B$ such that the premises are valid in V_i but the conclusion isn't. Hence, there is a $v \in V_i$ such that it is a counterexample to the conclusion but not the premises. But $v \in V_B$ and thus the metainference isn't locally valid either. □

Notice that the conditions for matching global validity with absolute global validity are stronger than those for matching the latter with local validity. Once we start with the same set V_B, then both local and global validity can be seen as particular cases of absolute global validity. In this vein, notice that local validity can be defined as absolute global validity over the set $\{\{v\} : v \in V_B\}$, i.e. the set which only contains the singletons of valuations. Global validity in turn can be defined as absolute global validity over the set $\{V_B\}$, i.e. the set which only contains the whole set V_B.

All these coincidences could lead someone to reason along the following lines. Given that local validity seems to be conceptually more palatable than absolute global validity, and also more practical to implement, and given that we have a collapsing result, one may dispense with the latter and just work with the former. However, that reasoning rests on a fallacy.

Notice that the previous fact doesn't constitute a recipe to match absolute global validity and local validity: it only works when absolute global validity is defined over the power set of the set used to define local validity. But not all sets of metainferences have as their absolute global range such a set. That is, if $\mathcal{V}$ is the absolute global range of a set of metainferences, and we take the union of all its elements, it won't always be the case that $\mathcal{V}$ is the power set of that base set.

Actually, all of the sets of metainference schemas $\mathbf{S_1}$, $\mathbf{S_2}$ and $\mathbf{S_3}$ introduced in Section 3 provide examples of this statement. Let's start by showing this for $\mathbf{S_1}$. We have shown that the local range of $\mathbf{S_1}$ in V_2 is the set of bivaluations respecting the clauses of the Boolean disjunction. Now consider the following two valuations:

$$v_1(\varphi) = v_1(\psi) = 0 \text{ and } v_1(\varphi \vee \psi) = 1 \qquad v_2(\varphi) = v_2(\psi) = v_2(\varphi \vee \psi) = 0$$

The set $V_{\mathrm{I}} = \{v_1, v_2\}$ belongs to $\mathcal{V}_{S_1}$, the absolute global range of $\mathbf{S_1}$ in V_2, since all of the metainferences of $\mathbf{S_1}$ are globally valid in V_{I}. However, notice that v_1 doesn't belong to the local range of $\mathbf{S_1}$ in V_2 (it isn't a Boolean bivaluation). In other words, we have just shown that there is a valuation v_1 which belongs to some set V_{I} included in the absolute global range of $\mathbf{S_1}$, but not in its local range. Putting things differently, if we take the base set V_B to be the union of all of the members of $\mathcal{V}_{S_1}$, then there are valuations in V_B that are not in the local range of the set $\mathbf{S_1}$.

Also, using these valuations, it's easy to check that $\mathbf{P_1}$ from Section 3 is globally invalid in V_{I}, since the premises of $\mathbf{P_1}$ are valid, but its conclusion is invalid. Given that absolute global validity coincides with derivability, this proves that $\mathbf{P_1}$ isn't derivable from $\mathbf{S_1}$.

The same reasoning shows that $\mathbf{P_1}$ isn't derivable from $\mathbf{S_2}$ and that $\mathbf{P_2}$ isn't derivable from $\mathbf{S_3}$, where the corresponding absolutely global counterexamples are given by the sets $V_{\text{II}} = \{v_2, v_4\}$ and $V_{\text{III}} = \{v_1, v_3\}$.

$$v_3(\varphi) = v_3(\psi) = v_1(\varphi \vee \psi) = 1 \qquad v_4(\varphi) = v_2(\psi) = i \text{ and } v_1(\varphi \vee \psi) = 1$$

All metainferences in $\mathbf{S_2}$ are globally *st*-valid in V_{II}, whereas $\mathbf{P_1}$ isn't, while all the metainferences in $\mathbf{S_3}$ are globally *ts*-valid in V_{III}, whereas $\mathbf{P_2}$ isn't.

This is in fact another way to look at what we claimed in Section 3, i.e. that the local range is sometimes incomplete when it comes to counterexamples to underivable metainferences. Put in the terms of the present section, what happens in those cases is that the base set of V_B properly includes the local range of that set of metainferences. And this incompleteness cannot be amended by adding valuations coming from the union of the elements of its absolute global range, since the local range is the *maximum* set of valuations which satisfy all metainferences. This means that any other valuation outside of it will be locally unsound with respect to the metainferences.[14]

Thus, not all logics will have a local characterization of their derivability relation, because their local range can diverge from the union of the elements of their absolute global range. If the former is included in the latter, we have soundness—but we might fail completeness, as mentioned before. On the other hand, if the latter is included in the former—although, as we said, we might fail soundness—we have completeness as the following fact shows:

Fact 26. *Let* $\mathbf{S}$ *be a set of metainferences containing* $[KL]$ *and* $[KR]$*,* $\mathcal{V}$ *its absolute global range, and* V_B *the union of the elements of* $\mathcal{V}$*. A metainference schema* $\mathbf{P}$ *is locally valid in* V_B *only if it is derivable from* $\mathbf{S}$*.*

Proof. Assume a metainference $\mathbf{P}$ isn't derivable from $\mathbf{S}$. By theorem 23, all metainferences in $\mathbf{S}$ are absolutely globally valid in $\mathcal{V}$, but $\mathbf{P}$ isn't. Thus, there is a $V \in \mathcal{V}$ such that all members of $\mathbf{S}$ are globally V-valid but $\mathbf{P}$ isn't. Hence, there is a token $\langle A, \rho \rangle$ of $\mathbf{P}$ and a valuation $v \in V$ satisfying all premises of $\langle A, \rho \rangle$ but not its conclusion. And given that v is in V_B, $\mathbf{P}$ is locally invalid in V_B. □

To sum up, we think these collapsing conditions help us see these three notions not as rivals of each other but as different perspectives on the same relation. Moreover, each perspective has its own merit thanks to their different applications. Absolute global validity can work as the main bridge with proof-theory, global validity is arguably the one which has the richer conceptual ground, and local validity is the easiest to implement. We want to present an illustration of this last point in the next section, given that it explains the wide use of local validity in the literature.

[14] Having unsound valuations inside some elements of the absolute global range isn't an issue, because in those cases, their singletons are never part of the range.

6 Local validity revisited

We now know that the counterpart of derivability in non-transitive and non-reflexive logics isn't necessarily local validity, but absolute global validity. Hence, we cannot in general adopt the former merely relying on the fact that the rules of our proof system are sound and complete with respect to a base set of valuations. However, provided we keep this warning in mind, there are still some cases where local validity can satisfactorily be implemented, because it does in fact coincide with derivability. In this section, we aim at illustrating how local validity remains useful by presenting the relation between this notion and derivability, regarding the two logics we have been using as study cases: ST and TS, and their respecting sequent calculi introduced in Section 2.

Let us start with ST. As mentioned in Section 2 and following [10], it should be remarked that ST has received at least two different sequent calculus presentations. One of them through the system $\mathbf{LK}^{\setminus Cut}$ and the other through the calculus $\mathbf{LK}^{\setminus Cut}_{INV}$. Both are sound and complete with regard to inferential st-validity in V_{sk}.

In fact, thanks to the famous *Hauptsatz*, both of them have the same set of provable inferences, encoding exactly those inferences valid in Classical Logic, i.e. $pr(\mathbf{LK}^{\setminus Cut}) = pr(\mathbf{LK}^{\setminus Cut}_{INV}) = pr(\mathbf{LK})$. However, they have different derivable metainferences. In fact, just like the authors mention in [10, p. 395], the set of $\mathbf{LK}^{\setminus Cut}$-derivable metainferences is strictly included in the set of $\mathbf{LK}^{\setminus Cut}_{INV}$-derivable metainferences. So, does local st-validity in V_{sk} coincide with either of them? Fortunately, in [10] the authors give a positive answer to this question, that we detail below adapting it to our current terminology.

Fact 27 ([10])**.** *A metainference is $\mathbf{LK}^{\setminus Cut}_{INV}$-derivable if and only if it is locally st-valid in V_{sk}.*

Also, as Barrio et al. showed [2] the set of metainferences which are locally st-valid in V_{sk} can be translated into the valid inferences of LP, and vice versa.[15] LP—presented by Priest in [22] among other places—is the famous logic induced by those trivaluations complying with the strong Kleene truth-tables for which validity is defined using the tt-standard, i.e., the LP-valid inferences are those tt-valid in V_{sk}. It is, thus, a structural or Tarskian logic which is also paraconsistent, meaning that the inference schema $\varphi, \neg\varphi \Rightarrow \psi$ is invalid in it. This collapse result led some authors to say that LP is "the logic of the metainferences" of ST, and ultimately led others—like Dicher and Paoli in [10]—to claim that the latter is just a metainferential presentation of the former, since the only interesting notion of consequence in the context of a substructural logic is that holding between inferences and not that holding between formulas.

Furthermore, as a corollary of Facts 25 and 27 we obtain that $\wp(V_{sk})$ is the absolute global st-range in V_{sk} of the metainference schemas in $\mathbf{LK}^{\setminus Cut}_{INV}$. Given these, and the collapse with LP, we also have the following result.

[15] Interestingly, a version of this result was also proved by Pynko in [23].

Corollary 28. *A metainference is absolutely globally st-valid in $\wp(V_{sk})$ if and only if its translation is valid in* LP.

From these facts it can be concluded that if the right standard for determining the validity of a metainference is its derivability, then ST can be identified with LP only if we grant that $\mathbf{LK}^{\setminus Cut}_{INV}$ is the preferred calculus presentation of ST. However, the identification can be rejected if one endorses the weaker calculus $\mathbf{LK}^{\setminus Cut}$, given that its derivable metainferences cannot be characterized by local *st*-validity in V_{sk}. Can we find another set of valuations for which local validity does coincide with this derivability relation?

A deeply interesting observation in this regard has been proved by Girard in [14, p. 163-166], which went, as far as we know, relatively unnoticed by people working in these issues. What Girard shows is precisely that a particular set of (reflexive) trivaluations can be implemented to characterize the notion of derivability in $\mathbf{LK}^{\setminus Cut}$. They are called *Schütte valuations*, i.e., trivaluations satisfying the non-deterministic truth-tables appearing in Figure 3, with the strict-tolerant standard—depicted here following the clauses described by Girard, adapting things a little bit to our terminology. In this regard, and for the purpose of establishing the next results, let us denote by V_{Sch} the subset of the trivaluations that comply with the Schütte truth-tables.

	$\neg$
1	$\{0,i\}$
i	$\{i\}$
0	$\{1,i\}$

$\wedge$	1	i	0
1	$\{1,i\}$	$\{i\}$	$\{0,i\}$
i	$\{i\}$	$\{i\}$	$\{0,i\}$
0	$\{0,i\}$	$\{0,i\}$	$\{0,i\}$

$\vee$	1	i	0
1	$\{1,i\}$	$\{1,i\}$	$\{1,i\}$
i	$\{1,i\}$	$\{i\}$	$\{i\}$
0	$\{1,i\}$	$\{i\}$	$\{0,i\}$

Figure 3: The Schütte truth-tables

Moving onto the proper results, then, the first one is due to Girard and the second is easily obtained as a corollary of Girard's result and Fact 25.

Fact 29 ([14])**.** *A metainference is $\mathbf{LK}^{\setminus Cut}$-derivable if and only it is locally st-valid in V_{Sch}.*

Corollary 30. *A metainference is $\mathbf{LK}^{\setminus Cut}$-derivable if and only it is absolutely globally st-valid in $\wp(V_{Sch})$.*

Now the question arises whether there is some logic that under translations corresponds to the $\mathbf{LK}^{\setminus Cut}$-derivable metainferences, as it happens between LP and $\mathbf{LK}^{\setminus Cut}_{INV}$. One sensible guess might be that the logic induced by the *tt*-valid inferences in V_{Sch}, which we may call *paraconsistent Schütte logic*, is such a system. This guess is based on the fact that this logic, as LP, is paraconsistent—but also on the fact that the logic invalidates every elimination inference (as happens metainferentially in the case of $\mathbf{LK}^{\setminus Cut}$). So, what one would be inclined to think is that there could be some kind of translation from the $\mathbf{LK}^{\setminus Cut}$-derivable metainferences into the valid inferences of the paraconsistent Schütte logic. Ultimately, if this translation were provided and the conjecture were proved with all the details, one could argue that ST is no more LP than the paraconsistent Schütte logic is ST. Interesting as it is, we leave this exploration for future work.

Now, let us turn our attention to TS. As mentioned before, this logic was much less explored in the recent literature, and as far as we know, no one has tried to philosophically argue that TS should be collapsed into a Tarskian logic (as we saw for the case of ST and LP). However, in what follows we will present some results that in some sense dualize those for ST, perhaps implicitly emphasizing the symmetries of these two systems.[16]

Firstly, as mentioned in Section 2, the logic TS can be presented using at least two different (but natural) sequent calculi: $\mathbf{LK}^{\setminus Id}$ and $\mathbf{LK}^{\setminus Id}_{INV}$. Each of them is the result of modifying in some sense the well-known sequent calculus for Classical Logic. As before, both are sound and complete with regard to inferential ts-validity in V_{sk}. In other words, the set of provable sequents on both systems coincides with the set of valid inference tokens in TS. However this is somewhat trivial, since both of these sets are empty, i.e. $pr(\mathbf{LK}^{\setminus Id}) = pr(\mathbf{LK}^{\setminus Id}_{INV}) = \emptyset$. In this context, once we drop Identity we cannot prove anything. So, from an inferential point of view, this collapse between provable inferences and different semantics isn't very informative, in the sense that it is enough for a logic to be an empty consequence relation to extensionally coincide with the set of provable inferences in both of these calculi.

However, things become more interesting when we consider the notion of derivability of these systems. In this sense, it is straightforward to check that the set of $\mathbf{LK}^{\setminus Id}$-derivable metainferences must be included in the set of $\mathbf{LK}^{\setminus Id}_{INV}$-derivable metainferences. This is so, because in order to derive an inverse metainference, one needs to use Cut (which is available in $\mathbf{LK}^{\setminus Id}$) but also the instances of Identity corresponding to the active formulas in the metainference, and these are not available in this context. So, now that we know that both of the natural proof systems for TS have different derivability relations, one could wonder whether there are sets of valuations such that local validity over them coincides with these systems. The answer is that there are, as we will now see.

Let us start with the case of $\mathbf{LK}^{\setminus Id}_{INV}$ (i.e. the system resulting from dropping $[Id]$ and adding the inverse metainferences from $\mathbf{LK}$). It can be proved that the very set of V_{sk}-valuations is what we were looking for:[17]

Fact 31. *A metainference is $\mathbf{LK}^{\setminus Id}_{INV}$-derivable if and only if it is locally ts-valid in V_{sk}.*

[16]In fact, in [9] and [8] the authors even claim these logics are properly speaking metainferential duals of each other.

[17]As far as we can tell, there is nowhere in the literature an actual written proof of this statement. Although we think such proof can be given (especially regarding the results we mentioned for ST and the duality between these systems ([9]), we are not providing it here, since it would lead us astray from the purpose of the present section.

Not only—as French suggested in [12, p.9], and as it arises of the investigations of Barrio, Pailos and Szmuc [3] and Scambler [29]—can the set of metainferences which are locally ts-valid in V_{sk} be translated into the set of valid inferences of K_3 (and vice versa). Furthermore, K_3 is the logic induced by the trivaluations satisfying the strong Kleene truth-tables, for which validity is defined using the ss-standard, i.e., the K_3-valid inferences are the inferences ss-valid in V_{sk}. It is, thus, a Tarskian logic which is paracomplete—meaning that the inference schema $\Rightarrow \varphi, \neg\varphi$ is invalid in it. So, as a corollary of Facts 25 and 31 we obtain that $\wp(V_{sk})$ is the absolute global ts-range in V3 of the metainferences in $\mathbf{LK}^{\setminus Id}_{INV}$. Thus, given that the metainferences which are locally ts-valid in V_{sk} coincide via translations with the inferences of K_3 we have the following result.

Corollary 32. *A metainference is absolutely globally ts-valid in $\wp(V_{sk})$ if and only if its translation is valid in K_3.*

Now, let us take into account the weaker system $\mathbf{LK}^{\setminus Id}$. As in the case of $\mathbf{LK}^{\setminus Cut}$, to characterize its set of derivable metainferences using the notion of local validity over a given set of valuations, we have to deal with some non-deterministic valuations.[18] In [17], the authors introduce the following non-deterministic semantics, which they call *strong Schütte valuations*:

	$\neg$
1	$\{0\}$
i	$\{0, i, 1\}$
0	$\{1\}$

$\wedge$	1	i	0
1	$\{1\}$	$\{0, i, 1\}$	$\{0\}$
i	$\{0, i, 1\}$	$\{0, i, 1\}$	$\{0\}$
0	$\{0\}$	$\{0\}$	$\{0\}$

$\vee$	1	i	0
1	$\{1\}$	$\{1\}$	$\{1\}$
i	$\{1\}$	$\{0, i, 1\}$	$\{0, i, 1\}$
0	$\{1\}$	$\{0, i, 1\}$	$\{0\}$

Figure 4: The strong Schütte truth-tables

So, let us denote by V_{SSch} the subset of the trivaluations that comply with the strong Schütte truth-tables. The authors in [17] prove the following:

Fact 33 ([17])**.** *A metainference is $\mathbf{LK}^{\setminus Id}$-derivable if and only it is locally ts-valid in V_{SSch}.*

And again, as a corollary of the above and of the Fact 25, we can state the following that relates derivability with absolute global validity:

Fact 34. *A metainference is $\mathbf{LK}^{\setminus Id}$-derivable if and only it is absolutely globally ts-valid in $\wp(V_{SSch})$.*

It is still an open question whether there is some logic that under translations corresponds to the $\mathbf{LK}^{\setminus Id}$-derivable metainferences, as we mentioned to happen between K_3 and $\mathbf{LK}^{\setminus Id}_{INV}$. The natural conjecture is that the logic of the

[18]See [20] for a comprehensive study about non-deterministic semantics for Cut-free and Identity-free sequent calculi.

inferences that are ss-valid in V_{SSch}, which we may call *the paracomplete strong Schütte logic*, is such a system. Firstly, as in the case of K_3 this logic is paracomplete. Also, as every elimination metainference is $\mathbf{LK}^{\setminus Id}$-underivable, in the paracomplete strong Schütte logic it holds that every elimination inference is invalid. However, as in the case of $\mathbf{LK}^{\setminus Cut}$ and the paraconsistent Schütte logic, we don't provide any particular translation here to do the job, as this would take us outside the bounds of this article. So, although we hope to have motivated enough this connection, this guess still needs a proof, which we leave for further research.

7 Conclusions and future work

In this article, we showed that, if we care about substructural logics, and think that a semantic interpretation for them should provide a notion of consequence that matches a syntactical notion of derivability, then one should be careful not to assume that a local definition of validity is always the way to go. Instead, we showed how, for some families of substructural logics of the non-transitive and non-reflexive kind the semantic counterpart of derivability is the concept of absolute global validity. However, since the latter is harder to implement, we showed how in many cases they coincide, as it can prove useful to appeal to local validity in order to relate substructural logics with structural ones.

We also left some open questions, which we think are possible routes to continue this investigation. First of all, as we said, even though absolute global validity is an essential valuational tool, the absence of an intuitive reading for it seems to limit its value as a stand-alone, *bona fide* semantic notion. We consider an interesting task for future endeavors to search for a philosophically compelling interpretation thereof.[19]

Secondly, there is the issue of characterizing the derivability relation of a certain calculus in terms of the notion of local validity over a certain family of valuations. We proved, in Section 5, that this task isn't always achievable. However, we also showed that in certain cases this is plausible. The question is, then, what proof-theoretic properties should it have in order for this characterization to be in fact possible?

Thirdly, and following these lines, whenever we do get a characterization of the notion of derivability of a certain set of metainferences in terms of local validity, this isn't always guaranteed to be describable in terms of a deterministic

[19] An anonymous reviewer suggested the following interpretation of absolute global validity. A given set of metainference schemas **S** determines a set of valuations, which can be seen as a set of "good" or acceptable scenarios, and adding any axioms constitutes a way of restricting those valuations (alternatively, those acceptable scenarios). Whence, the reviewer suggests that assessing the derivability of a given metainference would amount to quantifying over all those valuations determined (alternatively, those scenarios deemed "good") by the joint force of **S** and its premise inferences, and assessing whether in all these cases the conclusion inference of such a metainference is valid (respectively, considering whether these are "good" scenarios too). We think this might be an interesting collection of ideas, and we hope to discuss them in future works.

semantics. Indeed, in Section 6, we showed that some calculi—like $\mathbf{LK}^{\backslash Cut}_{INV}$ and $\mathbf{LK}^{\backslash Id}_{INV}$—have deterministic semantics for this purpose, whereas some other—like $\mathbf{LK}^{\backslash Cut}$ and $\mathbf{LK}^{\backslash Id}$—have non-deterministic semantics for this purpose. The question is, then, what proof-theoretic conditions should be imposed on a calculus in order to secure a deterministic semantics for its notion of derivability.

Finally, an exploration of the inferential correlates of the substructural logics defined by $\mathbf{LK}^{\backslash Cut}$ and $\mathbf{LK}^{\backslash Id}$ is due. We already ventured some hypothesis, and there is work here to be done. In this vein, we would like to emphasize here that we are neither giving reasons for adopting $\mathbf{LK}^{\backslash Cut}$ nor $\mathbf{LK}^{\backslash Cut}_{INV}$ as the preferable calculus for ST, nor for adopting either $\mathbf{LK}^{\backslash Id}$ or $\mathbf{LK}^{\backslash Id}_{INV}$ as the preferable calculus for TS. All of them have their benefits, some based on their metainferential weakness (in the case of $\mathbf{LK}^{\backslash Cut}$, for instance, the fact that it doesn't absorb any instance of Cut), some on their strength (in the case of $\mathbf{LK}^{\backslash Cut}_{INV}$, it absorbs some maybe unproblematic instances of Cut). Be that as it may, we hope to explore these and many other issues in future works soon.

Acknowledgments

We would like to thank two anonymous reviewers for their comments and suggestions which improved the clarity and the content of the article. Also, we would like to thank the audience of the Workshop on Substructural Logics and Metainferences (Buenos Aires, 2020). Our thanks also go to the members of the Buenos Aires Logic Group. This paper could not have been written without the financial aid of the National Scientific and Technical Research Council (CONICET).

References

[1] E. Barrio, F. Pailos, and D. Szmuc. What is a Paraconsistent Logic? In W. Carnielli and J. Malinowski, editors, *Between Consistency and Inconsistency*, Trends in Logic, pages 89–108. Springer, Dordrecht, 2018.

[2] E. Barrio, L. Rosenblatt, and D. Tajer. The logics of strict-tolerant logic. *Journal of Philosophical Logic*, 44(5):551–571, 2015.

[3] E. A. Barrio, F. Pailos, and D. Szmuc. A hierarchy of classical and paraconsistent logics. *Journal of Philosophical Logic*, 49(1):93–120, 2020.

[4] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Tolerance and mixed consequence in the s'valuationist setting. *Studia Logica*, 100(4):855–877, 2012.

[5] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Tolerant, classical, strict. *Journal of Philosophical Logic*, 41(2):347–385, 2012.

[6] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Priest's motorbike and tolerant identity. In R. Ciuni, H. Wansing, and C. Wilkommen, editors, *Recent Trends in Philosophical Logic*, pages 75–85. Springer, Cham, 2014.

[7] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Reaching transparent truth. *Mind*, 122(488):841–866, 2014.

[8] P. Cobreros, E. La Rosa, and L. Tranchini. (I can't get no) Antisatisfaction. *Synthese*, 2020. Forthcoming.

[9] B. Da Ré, F. Pailos, D. Szmuc, and P. Teijeiro. Metainferential Duality. *Journal of Applied Non-Classical Logics*, 30(4):312–334, 2020.

[10] B. Dicher and F. Paoli. ST, LP, and tolerant metainferences. In C. Başkent and T. M. Ferguson, editors, *Graham Priest on Dialetheism and Paraconsistency*, pages 383–407. Springer, Dordrecht, 2019.

[11] S. Frankowski. Formalization of a plausible inference. *Bulletin of the Section of Logic*, 33(1):41–52, 2004.

[12] R. French. Structural reflexivity and the paradoxes of self-reference. *Ergo*, 3(5):113–131, 2016.

[13] R. French and D. Ripley. Valuations: Bi, Tri, and Tetra. *Studia Logica*, 107:1313–1346, 2019.

[14] J. Girard. Proof theory and logical complexity. *Annals of Pure and Applied Logic*, 53(4):197, 1991.

[15] J.-Y. Girard. Three valued logics and cut-elimination: the actual meaning of Takeuti's conjecture. *Dissertationes Mathematicae*, 136:1–49, 1976.

[16] G. M. Hardegree. Completeness and super-valuations. *Journal of Philosophical Logic*, 34(1):81–95, 2005.

[17] B. Hösli and G. Jäger. About some symmetries of negation. *Journal of Symbolic Logic*, 59(2):473–485, 1994.

[18] L. Humberstone. Valuational semantics of rule derivability. *Journal of Philosophical Logic*, 25(5):451–461, 1996.

[19] L. Humberstone. *The Connectives*. MIT Press, 2011.

[20] O. Lahav. Studying sequent systems via non-deterministic multiple-valued matrices. *Journal of Multiple-Valued Logic and Soft Computing*, 21(5/6):575–595, 2013.

[21] G. Malinowski. Q-consequence operation. *Reports on Mathematical Logic*, 24(1):49–59, 1990.

[22] G. Priest. The logic of paradox. *Journal of Philosophical logic*, 8(1):219–241, 1979.

[23] A. Pynko. Gentzen's Cut-Free Calculus Versus the Logic of Paradox. *Bulletin of the Section of Logic*, 39(1/2):35–42, 2010.

[24] D. Ripley. Uncut. *Unpublished Manuscript.*

[25] D. Ripley. Conservatively extending classical logic with transparent truth. *Review of Symbolic Logic*, 5(02):354–378, 2012.

[26] D. Ripley. Paradoxes and failures of cut. *Australasian Journal of Philosophy*, 91(1):139–164, 2013.

[27] D. Ripley. Revising up: strengthening Classical Logic in the face of paradox. *Philosophers' Imprint*, 13(5):1–13, 2013.

[28] D. Ripley. On the 'transitivity'of consequence relations. *Journal of Logic and Computation*, 28(2):433–450, 2017.

[29] C. Scambler. Classical Logic and the Strict Tolerant Hierarchy. *Journal of Philosophical Logic*, 2(49):351–370, 2020.

[30] P. Teijeiro. Strenght and Stability. *Análisis Filosófico*, 2019. Forthcoming.