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Is Hume's Principle Analytic?

Eamon Darnell · Aaron Thomas-Bolduc

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Abstract The question of the analyticity of Hume's Principle (HP) is central to the neo-logicist project. We take on this question with respect to Frege's definition of analyticity, which entails that a sentence cannot be analytic if it can be consistently denied within the sphere of a special science. We show that HP can be denied within non-standard analysis and argue that if HP is taken to depend on Frege's definition of number, it isn't analytic, and if HP is taken to be primitive there is only a very narrow range of circumstances where it might be taken to be analytic.The latter discussion also sheds some light on the connections between the Bad Company and Caesar objections.

Keywords Neo-logicism · Non-standard Analysis · Hume's Principle · Frege · Analyticity

1 Introduction

Is Hume's Principle analytic? Several authors have discussed this question according to the "classical account" of analyticity (Wright, 1999; Boolos, 1997). Yet, few seem to have devoted special attention to addressing whether or not the Principle can be considered analytic according to *Frege's* account of *analyticity*. Crispin Wright describes the classical account of *analyticity* as holding (minimally) that, "the analytical truths...[are] those which follow from

Eamon Darnell
Department of Philosophy, University of Toronto, 27 King's College Circle Toronto, ON, M5S 1A1
E-mail: eamon.darnell@mail.utoronto.ca

Aaron Thomas-Bolduc
Department of Philosophy, University of Calgary, 2500 University Dr. Calgary, AB, T2N 1N4
E-mail: athomasb@ucalgary.ca
(Author's listed alphabetically.)

logic and definitions." (Wright, 1999, p. 8). For Frege, a statement is analytic, roughly, if it is provable from *general logical laws* and *admissible definitions*. The latter charactarisation is *similar* to the former, but the ways in which it differs suggest that the latter cannot be adopted if Hume's Principle is to be deemed analytic. Below, we will explain why we take this to be the case. We begin with a brief elucidation of Hume's Principle (HP) and then sketch Frege's characterisation of analyticity. Next, we argue that, if HP is taken as a definition (as neo-logicists take it to be), its admissibility depends on a proposition which is not, itself, analytic (according to Frege's conception). Specifically, we show that the proposition belongs to the "sphere of a special science" (namely, standard analysis) because it can be denied, without contradiction, within another (non-standard analysis). We conclude on this basis that Hume's Principle fails to satisfy Frege's definition of analyticity.

We then go on to argue that even if we follow the neo-logicist line taking HP as primitive—not dependant on an explicit definition of "cardinal number"—there is a very narrow set of conditions that would have to be met for HP to be considered analytic in Frege's sense. These conditions have to do with three of the most pressing concerns for contemporary neo-logicists: Bad Company, Good Company, and Caesar. We conclude with some ways our results might be expanded or generalised.

We should note at his point that although we are not aware of anyone who currently subscribes to Frege's understanding of analyticity, it is similar to certain modern conceptions. The methodology and results established below, therefore, may be of interest not only to those concerned with Frege's program, but also those interested in investigating the epistemic status of Hume's principle through a modern, neo-logicist lens.

2 Hume's Principle

We will follow Wright (1999, p. 6) in formulating Hume's Principle as follows. For any (appropriate[1]) concepts F and G,

(HP) The Number of Fs is the same as the Number of Gs if and only if there is a one-to-one correspondence between the Fs and the Gs.[2]

We will interpret HP as follows. First, we will take there to be a one-to-one correspondence between the Fs and the Gs just in case there is a *bijection* between F and G (hereafter, we will use the expressions, 'F is *equinumerous* with G' and '$F \approx G$', as synonymous with, "there is a one-to-one correspondence between the Fs and the Gs"). Given these conventions, the formal version of HP with which we are operating is (the universal closure of) the following.

HP: $\#F = \#G \leftrightarrow F \approx G$

[1] See footnote 4 below.

[2] Frege's formulation of HP is developed in (Frege, 1980, §§63–73).

where '#' is a function from Fregean concepts to objects, and '$\approx$' is a second-order formula asserting the bijectability of the Fs and Gs.[3]

Second, we will take the Number of Fs to be the same as the Number of Gs iff there are *exactly as many* Fs as there are Gs. Accordingly, and as is implied by Wright in (Wright, 1999, p. 12), we will take the referent of 'the Number of Fs' to be that which (correctly) answers the question: How many Fs are there? (and likewise for G).[4]

2.1 What does 'less-than' Mean?

There is one other important feature of HP, and of Frege's account of cardinality more generally that may seem obvious and natural now, but is in the background of much of what follows (and the foreground in §4.5): in asserting HP, Frege is, like Cantor, asserting that one-to-one correspondence is the correct criterion for cardinal identity for both finite and infinite collections.[5] Another way of putting this is that if we say that there are fewer Fs than Gs, i.e. the number of Fs is less than ($<$) the number of Gs, an injective function from F into G could not be surjective—there would be Gs "left over". This is now the standard way to think about the less-than relation, at least among those of us familiar with 20th century mathematical logic. There is, however, another common intuition about the meaning of less-than relation having to to with the part-whole relation.[6] This is perhaps best expressed using set/subset discourse but that does not mean that it is only applicable in formal set-theoretic settings. The principle is roughly this: if the Fs are a proper subset of the Gs, then the number of Fs is less than the number of Gs.

Take a bowl of fruit as a toy example. There are some mangoes and some figs in our fruit bowl. Without having to count either all of the pieces of fruit or the just the figs, we know that the number of figs is less than the number of pieces of fruit because the figs are a proper subset of the fruit. Likewise, if we were to eat all of the figs we would know that the number of mangoes is equal to the number of pieces of fruit because the mangoes are *not* a *proper* subset of the fruit. Notice in this case nothing was ever *counted*, but answers

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[3] Some authors prefer 'NxFx = NxGx...' or something similar, treating the cardinality operator as a variable-binding term-forming operator. In the presence of full second-order comprehension, which is part of the background logic, the two formulations are equivalent (Burgess, 2005, §2.6).

[4] Wright introduces this question as part of his response to the objection that HP is not analytic on the grounds that not every concept has a number (e.g. *is self-identical*). In short, his point is that a restriction is needed such that substitutions for 'F' are restricted to those concepts such that the question, "How many Fs are there? makes sense—or at least has a determinate answer..." (Wright, 1999, p. 12), e.g. *count nouns* and expressions for *sortal concepts*.

[5] The ideas in the subsection owe much to the reading of Mancosu (2009, 2015, 2016). We encourage readers interested in what follows to look at those works.

[6] For the history of the use of these different conceptions among mathematicians dating back to the medieval period see especially (Mancosu, 2016, chapter 3).

to “number of” questions were compared with respect to the less-than and equal-to relations.[7]

We will have more to say about the two intuitions about the less-than relation after developing our central argument against the Fregean analyticity of HP, but two things are worth keeping in mind for what follows. First, that the two conceptions of the meaning of ‘less-than’ agree entirely for finite cases. Second, that the one-to-one correspondence (Fregean/Cantorian) understanding entails the subset/ part-whole understanding in the infinite case *but the converse does not hold* (Mancosu, 2015, p. 384).

3 Frege’s account of *Analyticity*

With the above interpretation of HP in place, we will now turn to Frege’s account of *analyticity*. To begin, consider Frege’s contrast between *analytic* and *synthetic* truths. He writes:

> The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one, bearing in mind that we must take account also of all propositions upon which the admissibility of any of the definitions depends. If however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one. (Frege, 1980, §3)

Here, Frege claims that a statement φ, is *analytic* just in case there is a proof of φ and that proof relies only on general logical laws and admissible definitions. A definition is *admissible*, in this context, only if the propositions upon which that definition deductively depends are, themselves, analytic.[8] *General logical laws*, as opposed to truths that belong to the sphere of some special science, apply to any subject matter whatsoever.[9]

Matthias Schirn (2006, pp. 199–200) provides a useful (and we would argue correct) interpretation that brings the positive parts of Frege’s account of analyticity together nicely. It can be summarised as follows. For any statement φ, φ expresses an analytic truth just in case:

(1_F) φ expresses a general logical law or,
(2_F) φ expresses an admissible definition or,

[7] We realize that more work will have to be done to compare the cardinalities of disjoint collections, but as we will see, that is possible. See also the references in footnote 5.

[8] Note, Frege may also include principles governing definitions (perhaps like those in (Frege, 1997a, §33), (Frege, 1997b, §65), (Frege, 1997c, p. 316), and elsewhere) among the propositions upon which a definition depends. Though, he does not specify how to account for the analyticity of such principles.

[9] How (or whether) one is to account for the analyticity of *general logical laws* is left unspecified.

(3_{F}) There is a proof of φ such that that proof begins with primitive truths (of logic) and each of its steps appeals (only) to general logical laws or (admissible) definitions.[10]

But of particular relevance to the arguments in the following two sections is that, for Frege the following claim holds.
If it is not possible to prove a statement φ, without making use of truths that belong to the sphere of some special science, then φ is not analytic.

In order to understand this claim, it is useful to consider Frege's explanation as to why the truths of geometry are *synthetic* and not *analytic*. He states,

> For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-contradictions when we proceed to our deductions, despite the conflict between our assumptions and our intuition. The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic, and consequently are synthetic. (Frege, 1980, §14)

It is worth specifically highlighting two of the points that Frege makes here. First, a (true) statement can fail to be analytic even if its denial, or the consequences of its denial, are not intuitable. Frege does not think that one can intuit any non-Euclidean space.[11] Yet, he does think that, for the purposes of conceptual thought, one can consistently assume that there are such spaces. Second, if a particular statement (say, the Parallel Postulate) can be denied within the sphere of a special science (like a non-Euclidean geometry) without contradiction, then that statement is *synthetic*.[12]

[10] It is important here that "primitive truths" are understood as definite propositions that are among the *general logical laws*.

[11] Frege makes this point explicitly stating, "To study such conceptions is not useless by any means; but it is to leave the ground of intuition entirely behind. If we do make use of intuition even here, as an aid, it is still the same old intuition of Euclidean space, the only one whose structures we can intuit." (Frege, 1980, §14)

[12] It is worth mentioning that one may view what Frege says in (Frege, 1980, §14) to be in tension with the statements he makes in (Frege, 1980, §3). In particular, if within a theory T, there is a proof of a statement φ from the axioms (i.e. primitive truths) of T and each step of the proof appeals (only) to general logical laws and (admissible) definitions, but there is a consistent theory T^* in which $\neg\varphi$ does not entail a contradiction, should φ be considered analytic? If one privileges the narrower criterion in (Frege, 1980, §3), one might be inclined to answer: *yes*. Yet, if one privileges the wider criterion in (Frege, 1980, §14), one might be inclined to answer: *no*. We will understand the relationship between the narrower and the wider criteria as follows. The wider §14 criterion is to be used when assessing statements that do not admit of proof (in a theory): the primitive truths of a theory (to ensure that the primitive truths of the theory are *logical*) and definitions (to ensure that the definiendum clearly has the same *Sinn* as the definiens in all domains—see, footnote 16 below). Now, if there is a proof in a theory T', of a statement φ, that begins with primitive truths of logic (i.e. axioms which satisfy the §14 criterion) and each step of that proof appeals only to general logical laws and definitions (i.e. definitions which satisfy Frege's §14 criterion), then φ satisfies (3_F) and is analytic according to Frege's criteria in §3 *and* §14 (since φ is provable by general logical laws from primitive truths and definitions that hold with respect to every

Accordingly, we think that it is safe to say that Frege's conception of *analyticity* entails the following:

(A) If it is not the case that $\neg\varphi$ entails a contradiction within the sphere of some special science (like Euclidean or non-Euclidean geometry), then it is not the case that φ is analytic.

4 Is HP *Analytic*?

Under the assumption that HP is a definition, does HP satisfy Frege's account of *analyticity*? For Frege, a definition is admissible (i.e. *analytic*[13]) only if the propositions upon which it depends are analytic. HP depends, at least, upon the following: For any (appropriate) F,

(N) The Number of Fs is the extension of the concept *equinumerous with* F.

We will also express (N) as, $N(F) = Eq(F)$ (Where '$N(F)$' means "The Number of Fs" and '$Eq(F)$' means "is the extension of the concept *equinumerous with* F).

We will take the analyticity of HP to depend upon (N) for two reasons. First, Frege understands 'the Number of Fs' in terms of (N) (Frege, 1980, § 68). Second, HP holds if (N) does. If the analyticity of HP depends upon (N), then, according to Frege's account of analyticity, HP is analytic only if (N) is.

It is well known that Frege took HP to depend on (N) in the sense that he finds it necessary to derive HP from Basic Law V (BLV—more about this below) and (N).[14] The former is (meant to be) a *basic logical law* and (N), or its expression in the concept-script, to be an admissible definition. But we are in a different position than pre-1902 Frege, so it's worth investigating whether (N) *is* an admissible definition.[15]

4.1 Is (N) *Analytic*?

It looks as though (N) fails to satisfy Frege's conditions for *analyticity*. If, within the sphere of some special science not-(N) is true, then (N) fails to

subject matter, φ holds with respect to every subject matter). See Schirn (forthcoming, §3) for an in depth discussion of the relevant passages.

[13] Henceforth by "analytic" we mean "analytic in Frege's sense" unless it is expressly noted.

[14] Strictly speaking, Frege took HP to depend on the amended version of (N) given in the *Grundgesetze*; however, the differences between the two versions do not have a significant bearing on our argument.

[15] As anonymous reviewer rightly pointed out, the epistemic status of (N) is irrelevant from a neo-logicist perspective, as neo-logicists take HP (or similar principles) as primitive. However, as we will get into nearer the end of the paper, this is not only an interesting application of Mancosu's (2016) use of the part-whole principle in looking at topics related to neo-logicism, but also may provide insight both into Frege's program and neo-logicist conceptions of analyticity.

satisfy Frege's conditions for analyticity. Not-(N) is true within the sphere of some special science Σ, just in case there is at least one F such that $N(F) \neq Eq(F)$ is true in Σ. There is at least one F such that $N(F) \neq Eq(F)$ is true in Σ if, in Σ, one can, consistently, give a value for $N(F)$ (i.e. correctly answer the question: How many Fs are there?) such that that value $\neq Eq(F)$.[16]

Non-standard analysis (NSA) is a branch of mathematics that was developed by Abraham Robinson in the 1960s.[17] NSA introduces hyperreal numbers (an extension of the real numbers into which the real numbers are embedded) which allow for the existence of infinitesimals. There are two approaches to NSA: model theoretic and axiomatic. The former approach, was first presented in Robinson (1966). The latter approach was first presented by Edward Nelson (1977).

Within the sphere of NSA[18] one can, consistently, make $N(F) = num(F)$, where $num(F)$ = the *numerosity* of F, and show that there is at least one F such that $num(F) \neq Eq(F)$. To demonstrate this, we will compare the set of natural numbers including 0 ($\{0, 1, 2, 3, ...\}$), $\mathbb{N}_0$, and the set of natural numbers excluding 0 ($\{1, 2, 3, 4, ...\}$), $\mathbb{N}_1$,[19] and show that where $N(\mathbb{N}_0) = num(\mathbb{N}_0)$ and $N(\mathbb{N}_1) = num(\mathbb{N}_1)$, either $N(\mathbb{N}_0) \neq Eq(\mathbb{N}_0)$ or $N(\mathbb{N}_1) \neq Eq(\mathbb{N}_1)$.

The *numerosity* of a set F, is (roughly) the hypernatural number that answers the question: How many Fs are there? To define the *numerosity* of F, then, requires two things. First, a construction of the hypernatural numbers. Second, a means of mapping F to a particular hypernatural number according to the size of F (i.e. how many Fs there are). Below, we will present each in turn.

4.2 Hypernatural Numbers[20]

The hypernatural numbers can be constructed by injectively mapping $\mathbb{N}_0$ into its hypernatural extension: ${}^*\mathbb{N}_0$. This can be done by, first, defining a free or

[16] In other words, if, with respect to a special science Σ, (N) satisfies (A), then the sense (*Sinn*) of '$N(F)$' is not the same as the sense (*Sinn*) of '$Eq(F)$' within Σ. Hence, (N) does not apply to any subject matter whatsoever (i.e. the sense of '$N(F)$' is not always the same as the sense of '$Eq(F)$'). We understand Frege to be explicitly rejecting such definitions as admissible when he writes, "[T]he laws of logic presuppose concepts with sharp boundaries. . . Accordingly all conditional definitions, and any procedure of piecemeal definition, must be rejected. Every symbol must be completely defined at a stroke so that, as we say, it acquires a *Bedeutung*." (Frege, 1997b, § 65).

[17] It is perhaps more accurate to describe NSA as a branch of mathematical logic as a certain amount of mathematical logic is integral to its presentation.

[18] We are assuming that the sphere of NSA constitutes the sphere of some special science. This is plausible as it is a coherent theory about a particular domain, whose reliance on sets precludes it from belonging to pure logic.

[19] Strictly, using sets here (as opposed to *concepts*) is a departure from Fregean terminology, however, doing so will make things simpler.

[20] Those familiar with the construction of the hypernatural numbers, or who wish to take or take our word on the matter can feel free to skip this section, and similarly for the following section where we define numerosities.

non-principal ultrafilter $\mathcal{U}$,[21] on $\mathbb{N}_0$. We will follow Wenmackers and Horsten (Wenmackers and Horsten, 2013, p. 44) in defining $\mathcal{U}$ such that:

(U1) $\mathcal{U} \subset \mathcal{P}(\mathbb{N}_0)$
(U2) $\emptyset \notin \mathcal{U}$
(U3) $\forall F, G \in \mathcal{U}(F \cap G \in \mathcal{U})$
(U4) $\forall F \subset \mathbb{N}_0(F \notin \mathcal{U} \to \mathbb{N}_0 \backslash F \in \mathcal{U})$
(U5) $\forall F \subset \mathbb{N}_0(F \text{ is finite} \to \mathbb{N}_0 \backslash F \in \mathcal{U})$

(U1) makes $\mathcal{U}$ a proper subset of the power set of $\mathbb{N}_0$. (U2) states that the empty set is not an element of $\mathcal{U}$. (U3) holds that for any pair of sets in $\mathcal{U}$, the intersection of that pair of sets is also in $\mathcal{U}$. (U4) states that for any proper subset of $\mathbb{N}_0$, F, if F is not in $\mathcal{U}$, then the set of all elements in $\mathbb{N}_0$ that are not contained in F is a member of $\mathcal{U}$. Lastly, according to (U5), for any proper subset of $\mathbb{N}_0$, F, if F is finite, then the set of all elements in $\mathbb{N}_0$ that are not contained in F is a member of $\mathcal{U}$. Together, (U1)–(U5) make $\mathcal{U}$ a set of infinite subsets of $\mathbb{N}_0$.[22]

Using $\mathcal{U}$, $\mathbb{N}_0$ can be injected into ${}^*\mathbb{N}_0$ as follows. For all infinite sequences of natural numbers, $\langle s_n \rangle$ and $\langle r_n \rangle$:

(M1) $\langle s_n \rangle \approx_{\mathcal{U}} \langle r_n \rangle \leftrightarrow \{n \mid s_n = r_n\} \in \mathcal{U}$
(M2) $[\langle s_n \rangle]_{\mathcal{U}} = \{\langle r_n \rangle \mid \langle s_n \rangle \approx_{\mathcal{U}} \langle r_n \rangle\}$
(M3) $\forall n \in \mathbb{N}_0 : n = [\langle n, n, n, n, n, ... \rangle]_{\mathcal{U}}$

(M1) says, roughly, that a pair of infinite sequences of natural numbers are $\mathcal{U}$-equivalent just in case the set of numbers that label the places where the terms in each sequence are equal is in $\mathcal{U}$.[23] (M2) defines the *$\mathcal{U}$-equivalence class* of an infinite sequence of natural numbers $\langle s_n \rangle$ as the set of infinite sequences of natural numbers that are $\mathcal{U}$-equivalent with $\langle s_n \rangle$. (M3) states that for any natural number n, n is equal to the $\mathcal{U}$-equivalence class of infinite sequences of natural numbers that has, as a member, the constant sequence, $\langle n, n, n, n, n, ... \rangle$. The set of hypernatural numbers ${}^*\mathbb{N}_0$, is the set of $\mathcal{U}$-equivalence classes of members of the set of all infinite sequences of natural numbers. (M3) serves to embed $\mathbb{N}_0$ in ${}^*\mathbb{N}_0$ (Wenmackers and Horsten, 2013, pp. 44–45).

4.3 Numerosity

The hypernatural number that constitutes a measure of the size of a set F is the *numerosity* of F. Numerosity has been discussed by a number of authors[24] but Wenmackers and Horsten (2013) provide a particularly clear definition of the notion. For this reason, we will closely follow their procedure below (with some minor variance from their original notation).

[21] If there is no finite set in an ultrafilter, it is *non-principal*.

[22] For a more general discussion of ultrafilters, see Komjáth and Totik (2008).

[23] To illustrate with a toy example, if $\langle s_n \rangle = \langle 0, 3, 4 \rangle$ and $\langle r_n \rangle = \langle 1, 3, 4 \rangle$, then $\langle s_n \rangle \approx_{\mathcal{U}} \langle r_n \rangle$ iff $\{2, 3\} \in \mathcal{U}$. Keep in mind this example is meant merely as an illustration. $\mathcal{U}$ does not contain any finite sets. For an actual example, see §4.3 below.

[24] See, especially, Benci and Di Nasso (2003) and Mancosu (2009).

Wenmackers and Horsten define *numerosity* in three steps. First, they define a function $\mathcal{C}$, that gives the *characteristic bit string* of a set of natural numbers (Wenmackers and Horsten, 2013, p. 47). The characteristic bit string of a subset F of $\mathbb{N}_0$ is constructed from the following function:

$$\begin{aligned}\chi_F \colon \mathbb{N}_0 &\to \{0,1\}\\ n &\mapsto \begin{cases} 0 \text{ if } n \in \mathbb{N}_0 \backslash F \\ 1 \text{ if } n \in F \end{cases}\end{aligned}$$

χ_F takes natural numbers as arguments and gives the value 0 if the given natural number is not in F and gives the value 1 if the number *is* in F. The function $\mathcal{C}$ is now defined as follows:

$$\begin{aligned}\mathcal{C} : \mathcal{P}(\mathbb{N}_0) &\to \{0,1\}^{\mathbb{N}_0}\\ F &\mapsto \langle \chi_F(0), \chi_F(1), \chi_F(2), ..., \chi_F(n), ...\rangle\end{aligned}$$

$\mathcal{C}$ maps F to a sequence of 0s and 1s. In particular, the sequence of 0s and 1s that results from applying χ_F to each number in the linearly ordered sequence of natural numbers ($\langle 0, 1, 2, 3, 4, ...\rangle$). To illustrate, if F is $\{0, 2, 3\}$, $\mathcal{C}(F)$ is $\langle 1, 0, 1, 1, 0, 0, ...\rangle$.

The second step in defining numerosity is to define *partial sums of characteristic bit strings* of F: *sum-*$\mathcal{C}(F)$ (Wenmackers and Horsten, 2013, pp. 47–48). Wenmackers and Horsten define this as follows,

$$\begin{aligned}\mathit{sum}\text{-}\mathcal{C} \colon \mathcal{P}(\mathbb{N}_0) &\to \mathbb{N}_0^{\mathbb{N}_0}\\ F &\mapsto \langle S_n\rangle\end{aligned}$$

where,

$$S_n = \chi_F(0) + ... + \chi_F(n).$$

This function maps the sequence given by $\mathcal{C}(F)$ to a new sequence where the value of the term at the n-th place in the new sequence consists of the sum of all of the terms in places $\leq n$ in the sequence $\mathcal{C}(F)$, for all places n. To illustrate, again suppose that F is $\{0, 2, 3\}$. Accordingly, $\mathcal{C}(F) = \langle 1, 0, 1, 1, 0, 0, ...\rangle$ and so, *sum-*$\mathcal{C}(F) = \langle 1, 1, 2, 3, 3, 3, 3, 3, ...\rangle$.

The final step in defining the *numerosity* of a set F, is to give a means of interpreting *sum-*$\mathcal{C}(F)$ as one hypernatural number. This is done with the following function (Wenmackers and Horsten, 2013, p. 48):

$$\begin{aligned}\mathit{num} \colon \mathcal{P}(\mathbb{N}_0) &\to {}^*\mathbb{N}_0\\ F &\mapsto [\mathit{sum}\text{-}\mathcal{C}(F)]_{\mathcal{U}}.\end{aligned}$$

The value of *sum-*$\mathcal{C}(F)$ is an infinite sequence of natural numbers. The $\mathcal{U}$-equivalence class of an infinite sequence of natural numbers is a hypernatural number. Accordingly, the $\mathcal{U}$-equivalence class of the value of *sum-*$\mathcal{C}(F)$ is a single hypernatural number.

The *numerosity* of a set F is the hypernatural number given by $num(F)$: the $\mathcal{U}$-equivalence class of the partial sums of characteristic bit strings of F (i.e. the value of $[sum\text{-}\mathcal{C}(F)]_{\mathcal{U}}$). When the sizes of finite sets are given in terms of their *numerosities*, (N) holds (as does HP). To illustrate, assume that F is $\{0, 2, 3\}$. Now measure the size of F in terms of its numerosity. As before,

$$sum\text{-}\mathcal{C}(F) = \langle 1, 1, 2, 3, 3, 3, 3, ...\rangle$$

and so,

$$num(F) = [\langle 1, 1, 2, 3, 3, 3, 3, ...\rangle]_{\mathcal{U}}.$$

$[\langle 1, 1, 2, 3, 3, 3, 3, ...\rangle]_{\mathcal{U}}$ denotes the set of all sequences $\mathcal{U}$-equivalent with $\langle 1, 1, 2, 3, 3, 3, 3, ...\rangle$ and since $\{4, 5, 6, 7, ...\} \in \mathcal{U}$,

$$\langle 1, 1, 2, 3, 3, 3, 3, ...\rangle \approx_{\mathcal{U}} \langle 3, 3, 3, 3, 3, 3, ...\rangle.$$

Hence, $[\langle 1, 1, 2, 3, 3, 3, 3, ...\rangle]_{\mathcal{U}}$ has $\langle 3, 3, 3, 3, 3, 3, ...\rangle$ as a member and so (by (M3)),

$$[\langle 1, 1, 2, 3, 3, 3, 3, ...\rangle]_{\mathcal{U}} = 3.$$

The value of $Eq(F)$ is 3,[25] as is the value of $num(F)$ and so, (N) holds with respect to F. This result generalises for all finite sets of natural numbers.

If num is applied to infinite sets, it gives a value $\in {}^*\mathbb{N}_0 \backslash \mathbb{N}_0$. To demonstrate this, make $F = \mathbb{N}_0$. Thus, $\mathcal{C}(F) = \langle 1, 1, 1, 1, ...\rangle$ and so, $num(F) = [\langle 1, 2, 3, 4, ...\rangle]_{\mathcal{U}}$. There is no place at which $\langle 1, 2, 3, 4, ...\rangle$ begins (infinitely) repeating some finite number n. Hence there is no n such that $\langle 1, 2, 3, 4, ...\rangle$ is in the $\mathcal{U}$-equivalence class containing $\langle n, n, n, n, ...\rangle$. Thus, $[\langle 1, 2, 3, 4, ...\rangle]_{\mathcal{U}}$ must be larger than any finite number and so, $[\langle 1, 2, 3, 4, ...\rangle]_{\mathcal{U}} \in {}^*\mathbb{N}_0 \backslash \mathbb{N}_0$. We will call this number α (i.e. $num(\mathbb{N}_0) = \alpha$). [26]

4.4 (N) is false in NSA

When the *sizes* of $\mathbb{N}_0$ and $\mathbb{N}_1$ are given by their (respective) *numerosities*, (N) is false of either $\mathbb{N}_0$ or $\mathbb{N}_1$. By stipulation,

$$num(\mathbb{N}_0) = \alpha.$$

Now consider $\mathbb{N}_1$. $\mathcal{C}(\mathbb{N}_1) = \langle 0, 1, 1, 1, 1, 1, ...\rangle$ and so, $sum\text{-}\mathcal{C}(\mathbb{N}_1) = \langle 0, 1, 2, 3, 4, 5, ...\rangle$. Subtracting one hypernatural number from another is done by taking sequences from the relevant $\mathcal{U}$-equivalence classes and then subtracting (in the

[25] It follows from the manner in which Frege defines cardinal numbers (Frege, 1980, §§ 77–86) that, for any set of natural numbers F, the value of $Eq(F)$ is equal to the standard cardinality of F.

[26] We are following the lead of Benci and Di Nasso (2003, p. 52) and Wenmackers and Horsten (2013, p. 48) in calling this number, α. Although, strictly speaking, Wenmackers and Horsten do not call $num(\mathbb{N}_0) = \alpha$. Rather they stipulate that $num(\mathbb{N}_1) = \alpha$. Following their stipulation, $num(\mathbb{N}_0)$ should be $\alpha + 1$. We've chosen to overlook this detail to keep things simple.

standard way) their corresponding entries one by one.[27] Accordingly,

$$\begin{aligned}\alpha - 1 &= [\langle 1, 2, 3, 4, ...\rangle]_{\mathcal{U}} - [\langle 1, 1, 1, 1, ...\rangle]_{\mathcal{U}}\\ &= [\langle 1, 2, 3, 4, ...\rangle - \langle 1, 1, 1, 1, ...\rangle]_{\mathcal{U}}\\ &= [\langle (1-1), (2-1), (3-1), (4-1), ...\rangle]_{\mathcal{U}}\\ &= [\langle 0, 1, 2, 3, ...\rangle]_{\mathcal{U}}.\end{aligned}$$

Hence, the $\mathcal{U}$-equivalence class of $\langle 0, 1, 2, 3, ...\rangle$ is the hypernatural number $\alpha - 1$. Since $num(\mathbb{N}_1)$ is the $\mathcal{U}$-equivalence class of $\langle 0, 1, 2, 3, ...\rangle$,

$$num(\mathbb{N}_1) = (\alpha - 1).$$

Since $(\alpha - 1) < \alpha$,

$$num(\mathbb{N}_1) < num(\mathbb{N}_0).$$

Therefore, when the sizes of $\mathbb{N}_0$ and $\mathbb{N}_1$ are compared in terms of their respective *numerosities*, $N(\mathbb{N}_1) < N(\mathbb{N}_0)$. With respect to the infinite number $\aleph_0$,[28] Frege states that it applies to the concept F (i.e. $Eq(F) = \aleph_0$) just in case, "there exists a relation which correlates one to one the objects falling under the concept F with the finite Numbers." (Frege, 1980, § 84). Since the objects falling under $\mathbb{N}_0$ *are* the finite numbers,

$$Eq(\mathbb{N}_0) = \aleph_0.$$

Let f, be the function from $\mathbb{N}_1$ to $\mathbb{N}_0$:

$$\begin{aligned}f\colon \mathbb{N}_1 &\to \mathbb{N}_0\\ n &\mapsto (n-1).\end{aligned}$$

Accordingly, f correlates one-to-one the objects falling under $\mathbb{N}_1$ with the finite numbers. Thus,

$$Eq(\mathbb{N}_1) = \aleph_0.$$

Hence, $Eq(\mathbb{N}_1) = Eq(\mathbb{N}_0)$ and so, either $N(\mathbb{N}_0) \neq Eq(\mathbb{N}_0)$ or $N(\mathbb{N}_1) \neq Eq(\mathbb{N}_1)$. In either case, it follows that there is at least one F such that $N(F) \neq Eq(F)$ is true.

4.5 HP is not Analytic in Frege's Sense

It follows from the above that (N) is not analytic. There is a sphere of some special science (NSA) such that not-(N) does not lead to contradiction. It is consistent within NSA to make $N(F) = num(F)$. Furthermore, it seems that $num(F)$ does correctly answer the question "How many Fs are there?" As was shown above, with respect to any finite F, $num(F) = Eq(F)$. Hence, with respect to any finite F, if $Eq(F)$ correctly indicates how many Fs there

[27] See (Wenmackers and Horsten, 2013, p. 50) for a brief explanation of how addition on ${}^*\mathbb{N}_0$ is defined.
[28] Frege uses ∞_1 rather than $\aleph_0$.

are, so must $num(F)$. Other than presupposing the analyticity of (N), we see little reason to suppose that $num(F)$ should fail to also correctly indicate how many Fs there are with respect to any infinite F. Hence, by (A), (N) is not analytic. Under Frege's account of *analyticity*, HP is analytic only if (N) is analytic. Therefore, HP is not analytic according to Frege's account of *analyticity*.[29]

As mentioned above, and argued in much greater detail by Mancosu (2016), the question of whether numerosities or Fregean cardinalities should be used to measure infinite cardinalities comes down to whether we wish to privilege the intuition that subsets are always strictly smaller than their supersets (if $F \subset G$ then $F < G$), or the intuition that cardinality is completely captured by bijectability/ one–one correspondence as with HP. Both understandings coincide for finite numbers, but diverge in the case of infinite cardinals. Thus it appears that we're in an even better position than Frege was in the case of geometry, because although Frege found non-Euclidian spaces to be unintuitable, the intuition that proper parts are strictly smaller than their wholes is a common one. Indeed, Mancosu (2016), traces a venerable history of mathematicians relying on that intuition both before and after Frege and Cantor 'decided' on one–one correspondence.

5 An Easier Route

Beginning again with the assumption that the analyticity of HP relies on (N) being an admissible definition, and is thus itself analytic, we can take a much shorter route to the conclusion that HP isn't analytic in Frege's sense. The principle (N) is straightforwardly *in*admissible. Here's why. It says that numbers are a particular class of *extensions*, which are *logical objects* governed by Frege's Basic Law V (BLV) (1997b), which says that two concepts have the same extension just in case exactly the same objects fall under both concepts, i.e. the two concepts are coextensional.[30] But in his (in)famous letter to Frege in 1902 (see van Heijenoort, 1967, pp. 124–126), Russell shows that BLV is inconsistent. Despite its suggestive moniker then, BLV is *not* a basic logical law. Even if we were to find a way to characterise extensions with consistent, analytic axioms, Frege's derivation of HP from (N) relies heavily on BLV.[31] So, HP, if it relies on (N), is not analytic (in Frege's sense).

[29] The result that (N) can be consistently denied within NSA means that (N) is either synthetic or false. However, the result is not, in itself, sufficient to decide between the syntheticity or falsity of (N). For this reason we take no stand on this issue (likewise, for HP).

[30] BLV: $\forall F \forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \equiv Gx))$

[31] This is not to say that it would be impossible to find a consistent theory of extensions, the objects of which could be used in the formulation of (N), and HP derived therefrom. However, our current best theory of extensions is Zermelo-Fraenkel set theory. If we then take extensions to be governed by such a system, we would have to show that the axioms of ZF are analytic. And if we can do *that* we can declare victory for logicism without having to worry about (N) or HP, other than to perhaps pick out which sets to call the natural numbers.

Our appeal to numerosities is not, however, a superfluous exercise of our mathematical muscles. Recall that neo-logicists in the vein of Hale and Wright (see esp. Hale and Wright, 2001) want to take HP as primitive and then argue that it is analytic, or has some equally important epistemic status that will allow us to ground our epistemology of arithmetic.[32] If we are concerned with the analyticity of HP from a neo-logicist perspective, then we need not concern ourselves with (N). In such a case, HP would have to either qualify as a basic logical law, or as an admissible definition.

In the first case, one would be hard pressed to find anyone willing to endorse the claim that HP is a basic logical law, thus we won't go into any great detail here. It is first worth noting though, that if one *were* to claim that HP is a basic logical law, and assuming that basic logical laws are analytic (which is the point), the neo-logicist reduction of arithmetic to logic falls out immediately.[33] But HP almost certainly isn't a basic logical law. The obvious arguments against HP as a basic law are ontological in character. For one, if we accept HP as a basic logical law, then we are already committed to there being infinitely many objects. This is a much larger ontological commitment then first-order logic (1 thing) or second-order logic (1 thing).[34] Furthermore, HP is picking out objects called numbers which, if neo-Fregeans are to believed, are abstract objects accessible *only* via HP. Should a basic logical law be the sole means of picking out an entire category of objects?

There's more to be said here, but to keep laying into a straw man seems unfair. So we're now left with the possibility that HP is an admissible definition. It is here that our earlier development of numerosities will come in handy (again).

5.1 HP isn't an Admissible Definition

Before proceeding, it will be useful to have a couple more definitions at hand. First, HP is an example of a class of principles now known as *abstraction principles* (APs) which neo-logicists hope will play a central role in grounding the epistemology of mathematics beyond just arithmetic. In general, abstraction principles are of the form:

$$\text{(AP)} \quad \partial F = \partial G \leftrightarrow F \sim G$$

[32] It strikes us that Fregean analyticity as we have represented it here differs enough from the standard Kantian or Quinean accounts of analyticity that it may provide such a status even if we may not consider it to be a species of analyticity proper. Discussion of this possibility would take us too far afield, but the second author hopes to address it in the near future.

[33] That is it falls out immediately from a proof of Frege's theorem, which though non-trivial, is by now well known (Boolos, 1996; Heck, 2011).

[34] Quine (1970) was wrong about the vast ontological commitments of second-order logic. See Boolos (1975) for the canonical refutation of Quine on this count.

where '∂' is a function from concepts to objects—the abstraction operator—and '$\sim$' is an equivalence relation.[35]

Both HP and BLV satisfy this schema, as do uncountably many other sentences, including a numerosity AP (Mancosu, 2016, §9). But more about that below. As should be fairly obvious from the beginning of this subsection, we can't simply argue that APs are all analytic or otherwise epistemically privileged, because, for one thing, BLV is inconsistent (and unsatisfiable), while HP is taken to be the paradigm case of a 'good' AP.[36] This is the problem of Bad Company.

What all of this has to do with NSA and analyticity is the following. Establishing HP as an admissible definition requires a solution to Bad Company unless we have some reason to think that HP is privileged even among APs. The reason we would need a solution to Bad Company is that we would presumably need to give strict ground for thinking that HP successfully cashes out phrases like 'the number of Fs is the same as the number of Gs' while at the same time denying that BLV (for example) successfully cashes out phrases like "the Fs and the Gs are coextensional". This isn't the usual way of framing the Bad Company problem, but it is effectively the same problem.

The case could potentially be simplified by arguing that HP is special among APs, but barring such an argument, which to our knowledge has never been successfully made, we are essentially back where we started.[37]

Since we are dealing with a single sentence, what we would need to do to show that HP isn't an admissible definition and thus not analytic, is to show that there is some special science where HP fails, but is itself coherent. NSA looks like a good candidate. Indeed, Heck thinks that the case of numerosities closes the door on HP *qua* conceptual truth. In discussing Mancosu (2009) he writes the following.

> Mancosu's announced goal in his paper is "to establish the simple point that comparing sizes of infinite sets of natural numbers is a legitimate conceptual possiblility" (Mancosu, 2009, p. 642). I think it is clear that he succeeds. But if it is conceptually possible that infinite cardinals do not obey HP, then it is conceptually possible that HP is false, which means HP is not a conceptual truth, so HP is not implicit in ordinary mathematical thought. (Heck, 2011, pp. 265–6)

[35] Again, the abstraction operator is sometimes presented as a variable-binding, term-forming operator. See footnote 3 above. Additionally, the variables bound by the abstraction operators can be of any order or arity, though in general it's APs involving first level concepts that are of particular interest.

[36] It's straightforward to construct a model of HP (see e.g. Boolos, 1998, Chapter 9). Additionally, HP plus full axiomatic second-order logic, known as Frege arithmetic (FA) is equiconsistent with PA^2.

[37] In fact results reported by Cook (2017) and Walsh and Ebels-Duggan (2015) might give us reason to think that HP *is* special from certain mathematical perspectives. As those results have little to do with definitions of number however, we don't think it likely that HP's admissibility as a definition would follow.

This is in fact, very broadly, the argument we're in the midst of giving and it turns out that there are a few more holes to plug. That brings us squarely to:

6 A Final Worry (or Three)

The 800 pound pink gorilla in the foyer happens to be called Caesar, and is concerned with the question of whether the Numbers of HP and the numerosities of NSA are commensurable in the first place. It might be the case that we're equivocating when we say that we can use both Numbers and numerosities to answer the question "how many are there?" To put it another way, invoking NSA as a 'special science' in which HP fails, is to say that we are talking about the *very same* cardinal numbers in both cases. Although there are good reasons to think that they *are* the same cardinal numbers, it's a metaphysical assumption that can be consistently denied.

6.1 Identifying Cardinals

In §56 of the *Grundlagen* (Frege, 1980) Frege famously laments that ". . . we can – to give a crude example – never decide by means of our definition whether Julius Caesar belongs to a number concept, whether this same well-known conqueror of Gaul is a number or not." A passage that has since given a name to the so-called Julius Caesar objection, or Caesar problem. The core of the issue is that HP gives us no way to determine whether an object not identified by an expression of the form $\#\varphi$ is a number or not. Frege gets around this problem by introducing an explicit definition of 'the number of': (N). As we've already shown that that strategy fails, we have to look elsewhere if we want to figure out whether numerosities and numbers can be identified.

Since abstraction principles are (partial) identity criteria (Fine, 2002, ch. 1), an obvious place to start would be to see whether there is a suitable equivalence relation that will allow us to construct an AP for numerosities. Numbers and numerosities would then be on equal conceptual footing, and we could appeal to the literature on the identification of abstracts.[38] Alas this strategy is unlikely to bear fruit. Mancosu (2016, §9) points out that any AP that satisfies the Part-Whole principle will be (massively) inflationary.[39] Because of Cantor's theorem and related arguments, we know that inflationary APs (like BLV) are unsatisfiable in classical, static settings. So we're left with less direct arguments.

To our minds the most compelling evidence that Numbers and numerosities ought to be identified is that they agree for all finite cases. There are practical

[38] See (Mancosu, 2015, §9) for an overview as well as a discussion of some issues related to NSA and Caesar.

[39] An AP, Γ, is inflationary if it entails there be more Γ-abstracts than there were objects in the original domain. BLV is inflationary; HP is inflationary on finite but not infinite domains.

as well as theoretical reasons to identify the finite cardinals, the finite ordinals, the real whole numbers, etc. This is a thorny issue for structural realists, as well as others who take piecemeal approach to the foundations of mathematics—is the natural number structure embedded in the real number structure, the real number structure in the complex number structure, or $2^{\mathbb{N}} \neq 2^{\mathbb{R}} \neq 2^{\mathbb{C}}$?[40] Denying the identity of numbers presented in different ways would wreak havoc on ordinary mathematics, and there don't look to be reasons to uphold such distinctness claims (beyond perhaps some currently unpopular metaphysical theses). It would be much easier to say that the various properties of numbers converge on the naturals, or slowly diverge as they become more complex.

That leaves the possibility, though, that numerosities and Numbers are much like classical cardinals and ordinals, agreeing in finite cases, but diverging for infinite cases. No-one to our knowledge holds that we can't have both infinite cardinals and infinite ordinals, and still maintains the identity of the finite cardinals and ordinals.

Where this breaks down is with the less-than relation. To hold that numerosities and Numbers are fundamentally different we would have to give up the motivation for considering numerosities in the first place, namely that the less-than relation should (or at least could) be defined according to the part-whole principle *rather than* bijectibility. So once again it looks like we have a dilemma. We can insist on a univocal less-than relation, or give up on the identity of finite "natural numbers" that have been defined in different ways.

If we take hold of the first horn, we should conclude that HP isn't an admissible definition, and thus not analytic because it is inconsistent within NSA. If we grasp the second horn, the possibility that HP is an admissible definition is still open, in which case more work needs to be done.

6.2 Companions

Given our arguments thus far, an obvious strategy presents itself: find another "special science" where HP fails. While we admit that such a strategy may eventually be successful, a problem immediately presents itself. Since we will have already given up on identifying Fregean numbers and numerosities, we would be hard pressed to find a domain that meets the requisite criteria, but won't allow us to make a similar move. We could just keep claiming that the objects of the domain under consideration and Fregean Numbers are incomensurable.

Two other issues also arise if we are trying to establish HP as an admissible definition. The first, already briefly introduced, is Bad Company. The second has been recently dubbed Good Company by Paolo Mancosu (2015). Both of these problems are closely related to the criterion of universal applicability that is behind the search for special sciences in which HP fails. Bad Company

[40] See (Cook and Ebert, 2005), who calls this the 'C-R problem, for more discussion in the context of neo-logicism.

asks us to weed out APs that will lead to inconsistency, while Good Company asks us to choose between principles that will do the same work as one another. Indeed, the motivation for worrying about Good Company is essentially the same as our concern about the comensurability of numerosities and Fregean numbers. If we have multiple ways to construct or define a concept like 'cardinal number', in this case different APs, how ought we decide between them?

Bad Company presents a slightly (but only slightly) different problem. If we have a principle that we're claiming to be universally applicable, analytic, an admissible definition, why should we think that it is fundamentally different than other, inconsistent principles of the same form?

These are serious issues for neo-logicists, and there is a great deal of literature proposing and rejecting possible solutions to Bad Company (see Linnebo, 2011; Cook, 2012; Cook and Linnebo, 2017, for the state of the art), much of which will be applicable to Good Company. For our purposes however, the issues are somewhat narrower in scope.

Since we are only concerned with the analyticity of *HP*, we need not worry about delineating the class (or classes) of acceptable APs. Instead we can look at justifying an assumption underlying much of the literature on Bad Company explicitly challenged by Good Company: HP is special. We would need to show that there is something conceptually, and/or logically different about HP that puts it above other similar principles, and also conceptually above other understandings of cardinality such as that provided by NSA. In particular we would need to show that it is broadly applicable in a way that other options are not. We have already argued in §2.1 that our 'natural' understanding of 'number of' won't be enough. That was the point of looking at NSA.

The other option requires solutions to Good Company and Bad Company, issues we won't take a stand on here other than to note that they are both open questions, and the state of the debate on Bad Company suggests that that problem at least won't admit of a static solution.

All of this is to say that even if we assume the incomensurability of Fregean numbers and numerosities, the best hope for establishing the analyticity of HP is to find a very specific kind of solution to problems that have proven extremely contentious and difficult.

7 Concluding remarks

There are some important take-ways from our analysis here. First and foremost, it is exceedingly unlikely that HP is analytic on Frege's understanding of analyticity. This in itself is interesting for a couple of reasons. First, it puts an important bound on how much of *Frege's* logicist project can be reconstructed without BLV, at least with respect to Frege's chosen method for showing the purported analyticity of arithmetic. In another way though, that HP isn't analytic vindicates Frege's desire to ground HP on more fundamental principles and definitions. It furthermore highlights a more insidious aspect of the Caesar problem which arguably was Frege's impetus for that decision: there are apparent abstracta that are much more difficult to differentiate from HP's numbers than the "well-known Conqueror of Gaul".[41]

More generally we have highlighted just how closely entwined the Caesar problem and the Good and Bad Company problems are. This may turn out to shed light on the importance of resolving all of these issues if the epistemic supremacy of HP is to be established as is required by Scottish neo-logicism.

Finally, we contend that the analysis we have herein provided will be useful in showing that HP isn't analytic in senses other than Frege's, but that's a project for another day.

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[41] Again see (Mancosu, 2015, §7), where he also uses the word 'insidious'. We liked it enough to use it and add this footnote.

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