Takeuti's Well-Ordering Proof: Finitistically Fine?

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Abstract If it could be shown that one of Gentzen's consistency proofs for pure number theory could be shown to be finitistically acceptable, an important part of Hilbert's program would be vindicated. This paper focuses on whether the transfinite induction on ordinal notations needed for Gentzen's second proof can be finitistically justified. In particular, the focus is on Takeuti's purportedly finitistically acceptable proof of the well-ordering of ordinal notations in Cantor normal form.

The paper begins with a historically informed discussion of finitism and its limits, before introducing Gentzen and Takeuti's respective proofs. The rest of the paper is dedicated to investigating the finitistic acceptability of Takeuti's proof, including a small but important fix to that proof. That discussion strongly suggests that there is a philosophically interesting finitist standpoint that Takeuti's proof, and therefore Gentzen's proof, conforms to.

1 Introduction

The project of developing constructive foundations for mathematics at the beginning of the 20th century was largely a reaction to the set-theoretic antinomies that looked to threaten both classical mathematics and the alternative foundational programs of Russell and Whitehead, and Frege, though the movement can be traced back at least to Kroenecker and Poincaré. At that time the contenders for constructive foundations were the intuitionism of Brouwer and Heyting, and Hilbert's finitism, though Weyl and Poincaré's predicativism should be mentioned as well.[1]

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[1] The papers in part I of Benacerraf & Putnam (1983) provide a nice overview of foundational programs in the early 20th century. See also (Feferman & Hellman, 1995; Feferman, 2005) for discussions of predicativism.

The focus of this paper will be on Hilbert's program, the goal of which, roughly speaking, was to ground classical mathematics by giving constructive, finitistic consistency proofs for as much of mathematics as possible.[2] An obvious goal of the project would thus have been to prove the consistency of first-order arithmetic with full induction (or pure number theory, it was often called). A plausible candidate for such a consistency proof is Gentzen's (second) proof of the consistency of pure number theory. The goal of this paper then, is to investigate whether Gentzen's and Bernays's suggestions that said proof is finitisically acceptable can be upheld.

We will take a historically informed approach, beginning in §2 with a discussion of what 'finitisically acceptable' should be taken to mean. On the one hand, we might follow W. W. Tait and conclude that PRA exhausts finitism entirely, on the other we might take some remarks from Hilbert, Bernays and others seriously, and conclude that perhaps induction up to certain transfinite ordinals is finitisically acceptable.

In §3 we will look at the controversial move in Gentzen's proof—induction on ordinal notations less than ε_0—starting with a brief discussion of Cantor normal form notions leading to an outline of Gentzen's proof. The upshot of this will be that, unless induction up to ε_0 is already considered finitistically acceptable, a more robust, finitistically acceptable proof of the well ordering of ordinal notions less than ε_0 will be needed.

In §4 we introduce a proof due to Gaisi Takeuti (1987, ch. 2, §11) which is a good candidate for a finitistically acceptable proof of the relevant well-ordering. Once we have introduced that proof, we discuss (§5) whether it is finitistically acceptable, arguing that, once a small repair is made to Takeuti's proof, the question comes down to the status of nested inductions and recursion. We argue that such operations are acceptable according to what Takeuti (1987, p. 101) calls the "Hilbert-Gentzen finitist standpoint", though the situation varies depending on how 'finitism' is defined.

We conclude by taking a brief look at how the notions employed by Gentzen and Takeuti could lead to further progress in what we might call the extended Hilbert's program.

Finally, note that although we raise various issues related to the possible finitistic acceptability of Takeuti's proof and point to solutions, we don't intend to provide definitive answers in most cases. Many of these issues would benefit from further investigation, some of which we plan to do ourselves, but all of which we encourage others to explore.

2 Finitism

The question of what exactly finitism, in the sense of Hilbert and Bernays, amounts to has not been given a definitive answer in the literature, and we will not attempt to give one here. Rather, we will survey some of the evidence, and argue that the

[2] See (Zach, 2006) for a thorough introduction to Hilbert's program.

lower bound is primitive recursive arithmetic (PRA),[3] but that the relevant notion of finitism allows more than PRA and less than transfinite induction, at least beyond certain constructible ordinals, the upper bounds of which should be determined by finitistic considerations. That at least PRA is finistically acceptable is relatively uncontroversial. This is so in part because one of the primary finitist worries has to do with quantification over completed infinities; PRA is quantifier free, so the issue simply does not arise. Furthermore, primitive recursion is easily seen to be finitistic in that, by definition, the values of primitive recursive functions can be found in a finite number of steps.

Tait (1981, 2002) argues that finitism is completely exhausted by PRA, though he explicitly notes that he is more concerned with the technical meaning of the term, separated as much as possible from the historical and philosophical aspects of the concept. Stenlund (2009) argues that the technical and epistemological aspects of Hilbert's finitism cannot be so easily separated, as he takes Hilbert's epistemological aims to be primary. This is because the finite can be the object of intuition in a strongly Kantian sense,[4]and thus can be taken as a firm *philosophical* basis for mathematics. Given Tait's (2002) reservations about Kantian intuition, the reason for this tension should be relatively clear. But, question's about Hilbert's philosophical motivations aside, there are good reasons to think that he and Bernays were open to the possibility that the relevant notion of 'finitist' might go beyond PRA.

The most obvious evidence of this comes from the second volume of *Grundlagen der Mathematik* (1939, pp. 347–8), where they state:

> [W]e have introduced the expression 'finitistic' not as a sharply delineated term, but only as the name of [a] methodological guideline, which enables us to recognize certain kinds of concept-formations and ways of reasoning as definitely finitistic and others as definitely not finitistic. This guideline, however, does not provide us with a precise demarcation between those which accord with the requirements of the finitistic method and those that do not.[5]

This passage at least confirms our conjecture that PRA should not be taken as exhaustive of finitism, as there are no sharp edges to the concept. This may then allow us to place Takeuti's well-ordering proof that will be the main focus of this paper in the grey area above PRA. Further evidence that the finite standpoint might easily extend beyond PRA is the use of induction up $\omega^{\omega^{\omega}}$ by Wilhelm Ackermann in his doctoral dissertation, with which Bernays was familiar.[6] Despite the deficiency of Ackermann's work made obvious by Gödel's incompleteness theorems (Gödel, 1931), neither Hilbert nor Bernays had pointed out anything in Ackermann's work that was not finitistically acceptable, so far as we are aware.

Another telling passage from the same book is discussed in detail by both Zach (1998) and Tait (2002), and gives the impression that Bernays and Hilbert's conception of finitism had changed, likely in response to the results of Gentzen and Gödel (Gödel, 1931; Gentzen, 1936):

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[3] Primitive recursive arithmetic contains the usual recursive definitions of 0, $+$, $\times$, and successor, as well as all other primitive recursive functions, and the quantifier free induction schema.

[4] See (Sieg, 2009) for an interesting look Kant's (and other preceding figures') influence on Hilbert.

[5] (quoted in Zach, 1998, Fn. 16)

[6] The proof-theoretic ordinal of PRA is ω^{ω}, so Ackermann certainly went beyond PRA.

> Certain methods of finitist mathematics which go beyond recursive number theory (in the original sense) have been discussed already in §7 [of vol. I of the *Grundlagen*], namely the introduction of functions by nested recursion and the more general induction schema. (Hilbert & Bernays, 1939, p. 340)

What we should take away from this passage, especially in conjunction with Bernays's (1935, see below) acceptance that Gentzen's first consistency proof was finitisically acceptable, is first that Hilbert and Bernays were willing to reconsider what exactly they meant by 'finitistic' in light of developments in metamathematics. This should not be seen as their giving up on the original conception of finitism, but rather as refinement based on new information. In discussing this passage, Tait (2002, p. 415) points to Hilbert's discussion of 'sharpening' the original conception, and Bernay's discussion of an extension of the finite standpoint. He takes the first as evidence of the point just made, but wonders whether the second might mean that certain methods go beyond finitism. It seems to us that it does not, but we don't deny that the question is open to interpretation.

The second take away from the last quotation is the more obvious point that general induction and nested recursion are finitistically acceptable. In addition to this suggesting that finitism might not be exhausted by PRA, the inclusion of general induction and nested recursion will be important in the discussion of Takeuti's proof of the well-ordering of the ordinal notations less than ε_0 below.

The final piece of evidence we will consider here is the mention by Bernays of Gentzen's first consistency proof for arithmetic (Gentzen, 1936) which also used transfinite induction up to ε_0, though using a completely different notational system than the ordinal notations to be introduced in the next section. The last paragraph of Bernays's (1935) "Hilberts Investigations into the Foundations of Arithmetic" reads, in its entirety:

> During the printing of this report the proof for the consistency of the full number theoretic formalism has been presented by G. Gentzen, using a method that conforms to the fundamental demands of the finite standpoint. Thereby the mentioned conjecture about the range of the finite methods (p. 17) is disproved.

The conjecture referred to is the following:

> ...that it was in general impossible to provide a proof for the consistency of the number theoretic formalism within the framework of the elementary intuitive considerations that conformed to the "finite standpoint" upon which Hilbert had based proof theory.

As Gentzen is explicit about his use of transfinite induction, and even includes a discussion of the finitistic acceptability of the same, in which he tentatively concludes that transfinite induction on ordinal notations up to ε_0 as employed in his proof is finitistically acceptable (see especially Gentzen, 1943), it seems unlikely that Bernays would not have been aware of the potential issue with the proof unless he had not studied the paper before adding that last paragraph to his article.

What may be problematic here is that there does not seem to be a principled reason to stop at ε_0. Why not induction up to Γ_0, or at least to all accessible ordinals?[7] One reason might be that ε_0 is a seemingly natural stopping point, but what this naturalness might consist in from a finitist perspective is unclear. Another might be that to reach ε_0, only ω-many iterations of any given (constructive) operation are needed, whereas this is not the case for the next limit ordinal.

Given this evidence, the likelihood that Hilbert and Bernays's conception of finitism went beyond PRA, and may have even included limited forms of transfinite induction, should be clear. Based on this hypothesis, most of the remainder of this paper will be devoted to the question of whether Takeuti's proof of the well-ordering of ordinal notations for ordinals less than ε_0 shows that the transfinite induction used in Gentzen's *second* proof might be considered finitistically acceptable.[8]

3 Gentzen's Proof

In this section we will give a brief overview of Gentzen's second consistency proof for arithmetic (Gentzen, 1938), beginning with a brief description of ordinal notations in Cantor normal form. The purpose of this is to facilitate discussion of Gentzen's use of induction, so will not include details about the reduction steps themselves. For detailed discussions of this proof we point the reader to Gentzen's original paper, or §12 of (Takeuti, 1987).

3.1 Ordinal Notations

A theorem due to Cantor (1897) says that every ordinal can be written as a sum of 0 and exponents of ω, i.e.

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \omega^{\beta_3} + \omega^{\beta_4} + \dots$$

α, β ordinals, the βs either in this form, or 0s, and

$$\beta_1 \geq \beta_2 \geq \beta_3 \geq \dots$$

Since we need only consider ordinals less the ε_0, and ε_0 is the first ordinal such that $\omega^\alpha = \alpha$ (i.e. $\omega^{\varepsilon_0} = \varepsilon_0$), it is guaranteed that $\alpha > \omega^{\beta_i}$ for all $i > 1$, and $\alpha = \omega^{\beta_1}$ only in the case where the $\omega^{\beta_{i>1}}$ are empty.

[7] An ordinal is accessible, roughly, if it can be reached from below. See §6. Compare to the concept of an inaccessible cardinal for which there is a strong sense in which such cardinals cannot be reached from below.

[8] It is likely much of what follows will apply equally well to the 1936 proof given a finitistically acceptable translation between the ordinal notation systems.

An ordinal written in this notation is said to be in *Cantor normal form*. For ease of notation we can write ω^0 as 1, $\omega^0 + \omega^0 = 2$, and so on for the natural numbers.[9] One further condition, that any '+0' terms are deleted, guarantees that each notation is unique.

We can then write the natural sum of two ordinals, α and μ, $\alpha \# \mu$, as a (possibly) new ordinal in Cantor normal form by interleaving the monomials (terms of the form ω^{β_i}) so that the β's are decreasing.[10] It is the well-ordering of these notations, that Gentzen used for the transfinite induction steps in his proof.[11]

3.2 Gentzen's Second Proof

The general strategy of Gentzen's proof is to take an arbitrary proof in the sequent calculus with arithmetical initial sequents and the inference rule for (full) arithmetical induction (i.e. first-order Peano Arithmetic formulated in the sequent calculus) of the empty sequent and show that such a proof cannot exist.

To do this the 'end-part' of a proof is defined as the largest segment of the proof, looking up from the end-sequent, that contains only structural rules, including inductions. The end-part is then pushed to the top of the proof, which can be done because the end-sequent contains no connectives, so any complex formulae will have to have been removed by a cut at some point.

All inductions in the end part of the proof are replaced with sequences of cuts, and all inessential cuts (cuts on complex formulae) are reduced to essential cuts (cuts on atomic formulae). Obviously all of this is done in a principled way to a 'regular' proof,[12] but the details need not concern us here, as what is important is the use of ordinal notations, and induction thereon.

The key to the proof, and the part that we are here interested in, is showing that the procedure terminates. Gentzen does this by assigning ordinal notations as defined in §3.1, to the each sequent and inference, and showing that each reduction step decreases the ordinal notation assigned to the end-sequent of the proof. Although the procedures for assigning ordinal notations to inductions and cuts are necessarily quite complex, the operations involved are just the stacking of finitely many ω exponents, the taking of natural sums, and ordinary, arithmetical subtractions, all of which can easily be seen to be finitistically acceptable.[13] The only problem for the

[9] Gentzen includes only '1', but Takeuti makes use of this obvious notational extension so we have included it here for completeness.

[10] Note that it may be that $\beta_i = \beta_{i+1} = \ldots = \beta_{i+n}$ for some $i, n > 0$.

[11] From outside the finite standpoint it can easily be seen that these notations are well-ordered, because they are unique, and ε_0 is well-ordered by definition. Whether this can be determined from within the finite standpoint will be discussed in §4 & §5.

[12] A regular proof in one in which all of the non-eigen variables have been replaced with 0s and the eigenvariables have been replaced with appropriate arithmetic terms.

[13] Likewise for other inferences, though those cases are more simple.

finitist then, is being able to see that the ordinal notations are well-ordered, which is to say that the reduction procedure will terminate in finitely many steps.

Of course, a non-finitist will see immediately that the ordinal notations are well ordered, because each notation denotes a unique ordinal less than ε_0, ε_0 is an ordinal, so it is well-ordered by $\in$ (i.e. $< = \in$), thus the ordinal notations are of order type ε_0, so they're well-ordered. But such reasoning requires that one accepts transfinite ordinals (albeit relatively small ones) to be completed infinite sets, which is exactly what the finitist is trying to avoid.

4 Takeuti's Proof

Takeuti proposes a concrete method for demonstrating that the ordinals $<\varepsilon_0$ are well-ordered. His demonstration consists in applications of a series of (concrete) methods, which he calls "eliminators". Eliminators are methods for taking any (concretely) given strictly decreasing sequence of ordinals and (concretely) constructing a new strictly decreasing sequence of ordinals such that if the latter contains no infinitely descending chains neither does the former. Using eliminators, Takeuti gives a demonstration that the ordinals $<\varepsilon_0$ are well-ordered. Takeuti's original proof is both brief (about five pages) and quite dense, and so can be difficult to follow. In this section, therefore, we offer a relatively detailed reconstruction of only a small, initial part Takeuti's proof.[14] This should be sufficient to grasp the general proof strategy, and so understand the issues raised thereafter.

4.1 Introducing "Eliminators"

Takeuti begins his proof by supposing that the natural numbers are well-ordered. That is, for any (strictly) decreasing sequence of ordinals S which begins with some natural number n, the length of S is, at most, $n+1$ (1987 pp. 92–93).[15] Hence, no strictly decreasing sequence of ordinals which begins with a natural number is infinite. Takeuti's *eliminators* capitalize on the well-ordering of $\mathbb{N}$ and enable him to demonstrate that (strictly) decreasing sequences of ordinals beginning with any ordinal, $\alpha <\varepsilon_0$ must be finite.

[14] We plan to publish a full reconstruction of Takeuti's well-ordering proof in the near future.

[15] Takeuti takes this assumption to be uncontroversial because he sees it as an obvious consequence of his definitions of ordinals and the relations: '$=$' , '$+$' and '$<$' on the ordinals (1987 pp. 90–91).

Terms and the 1-eliminator [16]

The first eliminator Takeuti introduces is the *1-eliminator*.[17] The 1-eliminator is a (concrete) method for constructing a *1-sequence*, S_0' from a (concretely) given decreasing sequence of ordinals, S_0. Such that the first ordinal in S_0' is the *1-major part* of the first ordinal in S_0 and if S_0' is finite, then (it can be concretely shown that) S_0 is finite. Takeuti explains these terms (and concepts) by way of an illustration: Consider a (strictly) decreasing sequence of ordinals,

$$(S_0) \qquad\qquad a_0 > a_1 > \dots$$

where a_0 is not a natural number. Each ordinal, a_i in S_0 is to be written Cantor normal form, such that a_i has the form:

$$\omega^{\mu_1^i} + \omega^{\mu_2^i} + \dots + \omega^{\mu_{n_i}^i} + k_i$$

where each $\mu_m^i > 0$, $\mu_{m-1}^i > \mu_m^i$ and k_i is a natural number (the i's are simply meant to index the given Cantor normal form to a_i). Takeuti calls the part of a_i which does not contain k_i the *1-major part* of a_i:

$$\underbrace{\omega^{\mu_1^i} + \omega^{\mu_2^i} + \dots + \omega^{\mu_{n_i}^i}}_{\text{1-major part of } a_i} + k_i$$

Let the 1-major part of a_i be written as: a_i'. Takeuti calls a sequence which consists of just the 1-major parts of each a_i in S_0 a *1-sequence*.

A 1-eliminator is described as a method which takes S_0 and (concretely) produces a decreasing 1-sequence:

$$(S_0') \qquad\qquad b_0 > b_1 > \dots$$

which satisfies the following condition:

(C_1) $b_0 = a_0'$ (i.e. the 1-major part of a_0) and if S_0' is finite, then so is S_0.

At this stage, it is important to emphasize Takeuti's use of '$>$' in S_0'. This indicates that applying the 1-eliminator to S_0 produces a *strictly* decreasing 1-sequence. Hence, S_0' may not be the *very same sequence* as the sequence which would be produced simply by removing each k_i from every a_i in S_0. S_0' does not contain multiple occurrences of identical ordinals, whereas simply removing each k_i from every a_i in S_0 *may* result a sequence that contains multiple occurrences of identical ordinals. This feature of the decreasing sequences constructed from (applying) eliminators is extremely important for Takeuti's argument.

The 1-eliminator is shown to satisfy (C_1) as follows. By definition, each ordinal in S_0 is identical with its 1-major part plus a given natural number. That is, $a_i = a_i' + k_i$ for all a_i in S_0. Hence, S_0 can be written as:

$$a_0' + k_0 > a_1' + k_1 > \dots$$

[16] All content in this section is from or adapted from: (Takeuti 1987 p. 93)

[17] All subsequent eliminators (and their associated terminology) are analogous to the 1-eliminator.

applying a 1-eliminator to this sequence still produces the 1-sequence, S_0'. Now, take some finite part of S_0', say the sequence:

$$b_0 > b_1 > ... > b_m$$

where, $b_0 = a_0'$ and $b_m = a_i'$ (for some i). So, either $a_i' = a_{i+1}' = ... = a_{i+p}'$ (for some p) and a_{i+p} is the last term in S_0,[18] or $a_{i+p}' > a_{i+p+1}'$. If the former is the case, then stop. If the latter is the case, then make $b_{m+1} = a_{i+p+1}'$ and repeat. If one arrives at a sequence of a_i''s such that $b_m = a_i'$ and $a_i' = a_{i+1}' = ... = a_{i+p}' = ...$, it follows that S_0 must be finite. Since S_0 is a strictly decreasing sequence, $a_i' = a_{i+1}' = ... = a_{i+p}' = ...$ entails that $k_i > k_{i+1} > ... > k_{i+p} > ....$ Given the well-ordering of the natural numbers, the latter sequence must be finite. Hence, S_0 must be finite. Given the definition of S_0' (i.e. a strictly decreasing 1-sequence constructed from S_0), if S_0' is finite, then there is a (last) term, b_m in S_0' such that $b_m = a_i'$ and $a_i' = a_{i+1}' = ... = a_{i+p}' = ...$ Therefore, if S_0' is finite, so is S_0.

Proving a well-order with the 1-eliminator[19]

Takeuti uses the 1-eliminator to show that all decreasing sequences of ordinals which begin with an ordinal $< \omega^2$ must be finite. Consider the sequence:

$$(S_1) \qquad a_0 > a_1 > ...$$

where $a_0 < \omega^2$. Applying a 1-eliminator to S_1 enables the construction of the 1-sequence:

$$(S_1') \qquad b_0 > b_1 > ...$$

such that S_1' satisfies condition (C_1)[20] and where $a_0 \geq b_0$. Since $a_0 < \omega^2$, $b_0 < \omega^2$. It follows that each ordinal b_i in S_1' has the form, $\omega \cdot k_i$ (where k_i is a natural number). This is clear because $\omega \cdot \omega = \omega^2$ and any ordinal $< \omega$ is a natural number. Accordingly, since a_0 is not a natural number, each b_i in S_1' will be of the form, $\omega \cdot k_i$ (where k_i is a natural number). Hence, S_1' can be written as:

$$\omega \cdot k_0 > \omega \cdot k_1 > ...$$

It must then be the case that $k_0 > k_1 > ...$ (since $\omega = \omega$). Given the well-ordering of the natural numbers, $k_0 > k_1 > ...$ must be finite. Therefore, $b_0 > b_1 > ...$ must be finite. Since S_1' satisfies (C_1), S_1 must be finite. S_1 was arbitrary, so this result generalizes which means that any decreasing sequence of ordinals which begins with an ordinal $< \omega^2$ must be finite.

The above procedure nicely illustrates Takeuti's strategy throughout his proof. He assumes that the sequence of ordinals whose limit is ω (i.e. the natural numbers)

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[18] That is the last term with a 1-major part.

[19] The content in this section is from or adapted from (Takeuti 1987 p. 93).

[20] Where (C_1) is amended such that S_0 is changed to S_1 and S_0' is changed to S_1'.

is well-ordered. He then defines an eliminator which, when combined with the already established well-ordering of the natural numbers, enables him to show that the sequence of ordinals whose limit is ω^2 is also well ordered. Takeuti continues with this strategy such that, with each new eliminator, combined with the well-ordering of the sequences of ordinals already established, he is able to show that sequences of ordinals with higher and higher limits must be well-ordered, ending with a demonstration of the well-ordering of the ordinals $<\varepsilon_0$.

5 Is this Finitistic?

Having provided a taste of Takeuti's proof that demonstrates his proof strategy, we are now in a position to evaluate the finitistic acceptability of that proof, and by extension the finitistic acceptability of Gentzen's consistency proof. We will first correct an omission of Takeuti's that, while small, is key to the finitistic acceptability of an early part of his proof. We will then address the thornier question of the finitistic acceptability of the multiple nested inductions and recursions needed nearer the end of the proof.

5.1 An Induction Hypothesis

Takeuti's reading of a version of Hilbert's finitist standpoint which he calls 'the Hilbert-Gentzen finitist standpoint' requires concrete method for showing that descending sequences of ordinal notations terminates. Indeed, it is the provision of such a method that is the main purpose of Takeuti's proof. However, it does not appear that such a method has been provided for the case of ω^ω—in that case we seem to be no better off than we were with the proof of the accessibility of ε_0. Furthermore, *prima facie*, (limited) transfinite induction is already assumed.

The core of the problem is that it isn't obvious from the original presentation of the proof that the natural number superscripts (ω^n) should behave as the natural numbers *qua* natural numbers do (without non-finitistic background assumptions). This is due to the lack of a concrete justification of the induction hypothesis: any descending sequence $d_o > d_1 > \ldots$, with $d_0 < \omega^n$ is finite. This **is** transfinite induction. Luckily this problem is easily remedied. To show that any descending sequence $d_o > d_1 > \ldots$, with $d_0 < \omega^n$ where $n \geq 3$ is finite, proceed as follows:

In a manner analogous to demonstrating that the 1-eliminator satisfies (C_1), show that the 2-eliminator satisfies (C_2)[21] by appealing to the now established well-ordering of the ordinals $< \omega^2$. Then use the 2-eliminator to prove the well-ordering of the ordinals up to ω^3. Next, show that the 3-eliminator satisfies (C_3) by appealing

[21] Where, (C_2) is an appropriate analogue of (C_1).

to the well-ordering of the ordinals up to ω^3. Then use the 3-eliminator to prove the well-ordering of the ordinals up to ω^4.

Continue in this way until reaching the $(n-1)$-eliminator. Use the established well-ordering of the ordinals up to ω^{n-1} to show that the $(n-1)$-eliminator satisfies (C_{n-1}). Then use the $(n-1)$-eliminator to prove the well-ordering of the ordinals $< \omega^n$.

This procedure will establish that the induction hypothesis holds for any value of n in $n-2$ steps. $n-2$ is finite and so there is a concrete procedure for demonstrating the induction hypothesis for any value of n that will terminate in a finite number of steps. *Once this is made explicit, the need to assume transfinite induction has been eliminated.*

5.2 A Final Finitistic Worry

Once we've eliminated this last apparent vestige of transfinite induction, we can look more closely at whether Takeuti succeeds in showing that the use of ordinal notations in Cantor normal form in Gentzen's consistency proof, and hence that consistency proof as a whole, is finitistically acceptable. The most serious objection to the finitistic acceptability of Takeuti's proof is his use of multiple nested inductions and recursions later in the proof. Although the method remains the same as the ω^n case, much more nesting is needed to work down to the case of natural numbers where finitude can finally be established. In other words, the closer we get to ε_0 , the more 'simultaneus' recursion and induction steps are needed to build the relevant eliminators.

The question of whether these multiple inductions and recursions can be justified finitistally will be answered differently depending on the conception of finitism you are working with. We'll look at three cases, the first two admit of straightforward solutions, while the third is more difficult.

The easiest case is for those, like Tait, who insist that finitism is exhausted by PRA. The nested recursion needed for the final steps of Takeuti's proof are not *primitive*, and so the proof as a whole wouldn't be finitisically fine. On the other hand, if our motivation is purely historical—if we wished to vindicate Hilbert and Bernays's assertions that Gentzen's proof conforms to the finitist standpoint—then we have good reason to think that Takeuti's proof *is* fine. The two quotations from the *Grundlagen* in §2 are good evidence; the second appearing to explicitly allow just the sorts of constructions that we're concerned with. Here it is again:

> Certain methods of finitist mathematics which go beyond recursive number theory (in the original sense) have been discussed already in §7 [of vol. I of the *Grundlagen*], namely the introduction of functions by nested recursion and the more general induction schema. (Hilbert & Bernays, 1939, p. 340)

The more difficult case arises when the interest in finitism or constructivism comes from contemporary philosophical considerations. For example, we might be concerned about the justification of our arithmetic beliefs. In such cases more principled justifications would need to be given for accepting or rejecting Takeuti's methods. We won't attempt to survey the conceptual space here,[22] but rather content ourselves with a few thoughts relating to the Hilbert-Gentzen standpoint that we believe may be more widely applicable.

Takeuti charactarises his *Hilbert-Gentzen standpoint* as one which

> avoids abstract notions as much as possible, except those which are eventually reduced to concrete operations or Gedankenexperimente on concretely given sequences. (p. 100-101)

He takes this to be a "natural extension" of Hilbert's standpoint, and goes on to very briefly address the question we're now concerned with, saying:

> Of course we also have to deal with operations on operations, etc. However, such operations, too, can be thought of as Gedankenexperimente on (concrete) operations. (p. 101)

The latter quotation appears to justify nested recursion or induction by appealing to what we might call 'meta-recursion'. Concretely given sequences are concrete by definition, operations on those are either concrete or *Gedankenexperimente*, which are finitisically fine, then operations on operations are just *Gedankenexperimente* on operations and sequences already determined to be finitistically acceptable. More obviously needs to be said here, but something first needs to be said about *Gedankenexperimente*.

Gedankenexperimente include at least those iterated procedures that can be seen to apply and lead to a stated conclusion, but where all of the steps are not explicitly performed. Our fix to Takeuti's proof in the previous subsection is an example. We can see that the procedure for the cases of ω^3 and ω^4 can be applied for each successive value of n for ω^n. The inference from that observation to the existence of a $(n-1)$-eliminator is a *Gedankenexperiment*. More generally, *Gedankenexperimente* allow us to employ small amounts of reasoning that isn't strictly finitistic to make meta-logical inferences with concrete start- and end-points, with the understanding that those inferences are in principle finitistically fine.

Returning to the question of "operations on operations" in the Hilbert-Gentzen standpoint, we need only observe that we are only ever operating on concrete sequences and operations. You might say that concreteness is passed upwards through concrete operations and *Gedankenexperimente*. As long as the latter are finitistically fine, then so are Takeuti's nested inductions and recursions (with respect to the Hilbert-Gentzen standpoint).

6 Beyond the Proofs

Before moving on the the consistency proof for arithmetic, Takeuti (pp. 97–100) makes two relevant points. The first is that his method of proof, as discussed in the previous section, is closely related to the notion of accessibility, discussed by Gentzen in his (1936). The second is that his method can be extended beyond ε_0.

[22] But see (Incurvati, 2005)

The latter should give us pause when wearing our finitist hats, as it suggests that if Takeuti's proof is finitistically acceptable, then transfinite induction (on ordinal notations) extending beyond ε_0 should be as well.

Takeuti (1987, p. 98) characterises accessible ordinals by saying that they are such that every descending sequence of ordinals beginning with that ordinal is finite, and proceeds to sketch a proof showing that ε_0 is accessible. From the definitions it is easy to see that, at the very least all of the notations for the constructible ordinals below the next limit ordinal greater than ε_0 are accessible, which gives us the accessibility of that limit. Then, if we take accessibility as sufficient for finitistic acceptability, we will get a great deal of transfinite induction beyond ε_0-induction. Of course, there may be principled reasons for a finitist to stop at ε_0, but that would have to be separate from its accessibility. It is not clear how far you could take the procedure Takeuti's proof, but it appears that if the proof he gave shows that induction up to ε_0 is acceptable from a philosophically interesting finitist standpoint, then we may very well be able to find finitistically acceptable consistency proofs for systems stronger than PA.

Indeed, Takeuti (1987, §26) uses the notion of accessibility in his development of ordinal diagrams for use in the proof of the consistency of Π^1_1-CA (a subsystem of second-order arithmetic or analysis) motivated by constructive considerations. All of this could lead us to two very different conclusions. The first is that Hilbert's program, suitably modified, has actually achieved a lot of success—the consistency of Π^1_1-CA is a significant result. On the other hand, the fact that this result can be achieved might worry a finitist, as analysis involves quantification over completed infinite sets of natural numbers.

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