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Jairo José da Silva

Mathematics and Its Applications

A Transcendental-Idealist Perspective

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To the memory of my sweet Shoshana
April 1, 2006–August 7, 2016
A little fat star that shone across my sky

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Chapter 1
Introduction

The philosophy of mathematics is not, or should not be a therapeutics; its task is not, or should not be to sanitize mathematics. Nor should it be a sub-product of foundational projects, or part of mathematics itself. Mathematics can take care of itself; it has been doing it marvelously well for the last 4000 years or so. Philosophy of mathematics is a chapter of epistemology; its task, that of making sense of mathematics *as actually practiced* and *traditionally conceived* as a legitimate aspect of human knowledge. Mathematics is extremely useful as an instrument of scientific investigation, not only in the empirical sciences, but in the human and social sciences as well. Mathematics permeates science to the point of sometimes being essential to it. Not only as a "language" for expressing ideas that could be expressed otherwise but itself a source of ideas that cannot be expressed in any other way. It is the philosopher's task to explain what kind of knowledge mathematical knowledge is, what its objects are and how they can be accessed, but also to clarify and justify, in some sense of the term, the applicability of mathematics in science. What kind of connection mathematical objects have with objects of other sciences, empirical reality in particular, that it is sometimes impossible, as in modern physics, to have a theory of the latter independently of the former? A philosophy of mathematics that does not answer this question conveniently, or does not even raise it, fails its purpose. This will be my guiding question here. What is mathematics, I ask, which can be so widely and successfully applied in virtually all fields of science, besides so many aspects of our human lives? How can that which seems to be the epitome of pure reasoning play such an important role in the organization of our experience of reality, and do it so well to the point of predicting the outcome of future experiences? This cannot be a mystery; the question must have a rational and ordinary answer.

Philosophy of mathematics, moreover, just like any intellectual endeavor, must be practiced without prejudices and preconceptions. Philosophical theses cannot precede or condition philosophical inquiry. One cannot embrace a readymade ontology without considering the specificity of mathematical existence, for example. Unfortunately, it is not so in the traditional philosophies of mathematics. Despite

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their differences, which are very real, the three or four "schools" that come immediately to mind when we think of philosophy of mathematics, namely, logicism, formalism, intuitionism, and maybe also predicativism, besides being all born as parts of foundational projects, share important philosophical prejudices that obscure the comprehension of the real nature of mathematical objects, mathematical knowledge and the applicability of mathematics, i.e. the ontology, the epistemology and the pragmatics of mathematics.

These schools of thought, which have been fighting each other for over a century without any clear sign of advantage to any of the litigants, are too well known to require yet another exposition. However, digging into the soil where they are rooted and from where they draw their nutrients can be enlightening. One conclusion stands out, no matter their differences, all traditional philosophies of mathematics share a naturalist, or more specifically, an empiricist conception of existence which is particularly misleading when applied to mathematics. Mathematical existence cannot be thought on the model of existence of the natural object. By so doing, the traditional foundational approaches to the philosophy of mathematics condemn themselves to failure. This book will fulfill its goal to a substantial amount if I convince my reader of the truth of this claim. The fact that the philosophy of mathematics is recognized and practiced as a legitimate and independent topic of philosophical investigation mainly in analytic philosophy has the consequence that no matter the particular orientation they take, philosophies of mathematics inevitably share the philosophical *partis-pris* that define this philosophical tradition, empiricism in particular.

Consider the following quote[1]:

> To account for the indubitability, objectivity and timelessness of mathematical results, we are tempted to regard them as true descriptions of a Platonic world outside of space-time. This leaves us with the problem of explaining how human beings can make contact with this reality. Alternatively, we could abandon the idea of a Platonic realm and view mathematics as simply a game played with formal symbols. This would explain how human beings can do mathematics, since we are game players *par excellence*, but it leaves us with the task of specifying the rules of the game and explaining why the mathematical games is so useful – we don't ask chess players for help in designing bridges.

This quote illustrates perfectly the alternatives that analytic philosophy with its empiricist parti-pris gives to the philosopher of mathematics. From an empiricist perspective, the model of existence is the existence of the empirical object, physical or mental. Physical objects exist and subsist independently in space and time; mental objects, on the other hand, are temporal entities whose existence depends on the mental lives of subjects. In empiricist philosophies of mathematics, there are, consequently, three modes of existence available for mathematical objects, all modeled on the empirical mode of existence. A mathematical object can exist independently, albeit not in space-time, subject-dependently as a mental object, or not at all. Platonism (or realism) chooses the first alternative, constructivism the second, and formalism and nominalism the third. The brand of logicism favored by Frege was

[1] Tymoczko 1998, p. xiii.

Platonist, Russellian logicism was nominalist; intuitionism and predicativism are forms of constructivism and the many varieties of nominalism, ficcionalism and modalism all agree that mathematical objects do not exist, if not, maybe, as names, fictions or mere possibilities.

All these views suffer from well-known serious shortcomings – Tymoczko mentions some – that are fundamentally due, I believe, to the inappropriate empiricist conception of existence on which mathematical existence is modeled. Taking mathematics at face value requires that one grants full rights of existence to all the entities which mathematicians deal with and have theories about. But also that mathematical objects be granted *objective* existence as entities that can be accessed from different subjective perspectives and about which truths can be asserted and theories constructed cooperatively by mathematicians who are often separated by historical time and geographical space. Platonism is the perspective that comes closer to satisfying all these desiderata, but at the price of turning mathematical objects into ghostly counterparts of physical objects existing independently in some non-empirical but empirical-like realm of being. The main shortcoming of Platonism is, of course, the epistemological problem of access. How can we access mathematical objects and truths that exist independently of mathematical activity? A notion of intuition has been proposed (by, for example, the Platonist Gödel and improved by modern interpreters), conceived on the model of sensorial perception (consistently with the empiricist conception of mathematical existence), whose modes of operation, however, are far from clear.

Is there a way of enjoying the benefices of Platonism without having to pay its price? Does objectivity necessarily require independent existence? Is there a conception of intuition available that is a natural generalization of sensorial perception but does not require special sensorial organs (which Gödel, by the way, conjectured to exist)? The answers are, I believe, positive for the first and the third questions and negative for the second. This work is in part devoted to providing answers to these questions.

Constructivism has its share of truth, but of falsity too. Mathematical objects are indeed, in some sense, "constructed", but not in the way constructivists construe this notion.[2] Not, for example, in the sense of intuitionism. Obviously, mathematics is a human activity, but by this I mean more than the trivial truth that mathematicians are human beings, not machines. What I have in mind is the much more serious, but easily misinterpreted, although completely obvious fact that the objects of mathematics are human creations. Mathematical entities are *cultural* artifacts, not mental objects. Human culture is both the context of *production* and *objectification* of mathematics. Traditional forms of constructivism, such as, paradigmatically,

[2]The apparently profound question "is mathematics invented or discovered?" has an obvious answer: both. Better, it is invented, even when it is discovered. By this, I mean that proto-mathematical entities can indeed be discovered, but they only become fully mathematical by some intervention on the part of mathematicians, i.e. by being somehow reinvented. As a rule, mathematicians create the objects with which they occupy their minds. This, of course, only makes the problem of the utility and wide applicability of mathematics more puzzling.

intuitionism, commit the serious error of confining mathematical activity to the mind, no matter how idealized this mind is, completely ignoring human culture, language and communication, the very bases on which the *objectivity* of mathematics rests. But also that of dramatically constraining the creative power of mathematical imagination. Mathematical constitutive activity, as I will argue here, is a communal (non-solipsist, non-interiorized) activity capable of producing all the entities mathematics has business with. Part of this work will be devoted to clarifying and substantiating this claim.

It would be strange if the foundational approaches to the philosophy of mathematics did not contain each a decent amount of truth, for otherwise it would be incomprehensible why they have commanded the attention of so many devoted mathematicians and philosophers. But they are also, although in different ways, inadequate accounts of mathematics, for otherwise it would be incomprehensible why they have failed to command the allegiance of so many intelligent philosophers and mathematicians. I have already suggested that this is indeed the case with respect to Platonism and constructivism. Formalism is no exception.

Formalists believe that the true objects of mathematics are symbolic systems, either systems of calculation or, more commonly, systems of logical derivation, where symbols are "operated upon" according to explicit or implicit (but non-ambiguous) rules of manipulation. Games of a sort. It is a fact that mathematicians often invent symbolic systems, but never as games; to treat them as such may dissolve philosophical puzzles, but are falsifications of reality devoid of philosophical value. It is like counterfeit money, it works like money as long as it is taken for money, but it is not money and it is not honest to treat it as if it were.

In mathematics, symbols always stand for, or denote, something, preexisting things or things posited by the symbols themselves. In and for themselves symbols are not objects of interest. In fact, one way – but not the only way – of creating mathematical objects is by introducing them as *correlates* of formal-symbolic systems.[3] The quaternions, for example. They were posited (by fiat) as objects of a domain of calculation, operated upon by more or less arbitrary rules of symbolic manipulation. The symbols i, j, k of quaternion arithmetic stand for objects that did not exist before being meant and exist only as meant, correlates of the symbols that "denote" them, having only the properties that they are ascribed to have, those that follow from definitional stipulations by logical necessity, or those that can be attributed to them as the theory is axiomatically extended consistently. Mathematical objects so posited stand in relation to the systems in which they are meant as linguistic meanings to the written symbols and sounds by which they are expressed. We do not utter sounds in a linguistic context for their own sake (at least not as a

[3] The usual way of positing mathematical objects is the well-known fiat "let A be a domain of objects where such and such (relations, functions, etc.) are defined satisfying such and such properties". Although the modern approach is to view these axiomatic stipulations as non-interpreted or formal systems open to different interpretations, they can also be viewed, as they, in fact, traditionally were, as object-positing stipulations. Objects so posited are purely *formal*, i.e. determined as to form but undetermined as to matter. These notions will be made more precise later.

rule) to the same extent that we do not devise symbolic systems as playing toys. Treating symbolic systems as games for metamathematical reasons is, of course, an altogether different, fully legitimate, matter.[4]

If one wants to understand mathematics as practiced and used – as I said, understanding the role that mathematics plays in the overall scheme of human knowledge is, or should be, one of the tasks of a philosophy of mathematics – one cannot dissociate philosophical investigations from the history of mathematics. This point was brilliantly made by I. Lakatos in his *Proofs and Refutations*.[5] However, historical considerations as I understand them here are of a somehow peculiar nature. In agreement with E. Husserl's essay "The Origins of Geometry" (Husserl 1954b), I think mathematics cannot be properly understood dissociated from its *transcendental history*, i.e. the chronicle of its *transcendental origin*. The focus of transcendental history is not actual and contingent historical facts but necessary genetic development, not actual mathematicians and their deeds but the mathematical constituting subjectivity and its acts. Transcendental history may emerge in factual history, but not necessarily.

The concepts alluded to above will be discussed in due time, but a preliminary explanation may be useful. I take for granted that mathematical truths are not "out there" somewhere, just waiting to be discovered by particularly gifted individuals who know how to get "there". When inquired about these matters, working mathematicians often answer that when working in mathematics they "have the strong feeling" that they are exploring and discovering. Writers of fictional literature sometimes say similar things, that their fictional characters "act by themselves" and their job as writers is mainly to report what their characters think and do. In fact, mathematicians are explorers to the same extent that fictional writers are reporters. Mathematicians and writers are absolutely free to create anything they want, writers more than mathematicians, for unlike mathematicians they do not work in cooperation with others, participating in a communal task of erecting an edifice that is to a substantial part already standing and whose blueprint is, to considerable extent, already drawn. However, once the plot of the novel and the characters are clearly characterized as to their fundamental psychological, social, and human traits, they sometimes must behave in certain ways in order to be believable. This is what we mean by saying that this or that fictional character "comes out alive" or that he is "a three-dimensional character". At this point writers are indeed reporters. Similar things happen to mathematicians; even if their creations are at the onset completely free and new, they must abide internally and externally to logic. In particular, mathematical theories are inserted in a domain of already existing mathematical objects and theories with which they must "talk" under the constraint of the laws of logics. Chess was invented too, but we can explore the game and discover the most beautiful and interesting *necessary* facts about it that their inventors completely ignored.

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[4] Hilbert's formalist approach was originally devised for the sake of metamathematical investigations, not as a philosophy of mathematics, this came later.

[5] Lakatos 1977.

So much for the “natural” Platonism of working mathematicians and the “feelings” that “justify” it.

Mathematics, I claim, is born out of human necessities, both practical and scientific, including the necessities of mathematics itself. Also, sometimes, for the fun of it, under the inspiration of existing mathematics or even by esthetic motivations. The fact, however, remains that mathematics, or at least the bulk of it, allowing for some exceptions, only survives, no matter how beautiful or elegant, if it is useful in practical life, science, or mathematics itself. Mathematics is often created under the pressure of mathematical problems, and some problems are more important for the mathematics they generated than for themselves. But at the basis of all mathematics, if we trace back the steps of mathematical evolution, lies man and his life-world, the pre-scientific world in which he lives his ordinary life. The needs of trade, agriculture and rituals were probably the most important forces in the development of practices from which mathematics and science were born. Land surveying induced the invention of techniques that, when considered in themselves in a highly idealized form, eventually became geometry. Looking at the skies to find out when to sow and plow and honor the gods gave origin to practices and a body of practical knowledge that in due time developed into theoretical astronomy unconcerned with practical problems. To follow these and similar developments from their beginnings to the establishment of mathematics proper, not historically as a succession of events in time but genetically, in a philosophical sense, as sequences of intentional acts of an intentional subject, is philosophically enlightening. This approach, however, has so far been almost completely ignored by a philosophical tradition that chooses not to maintain close ties with either factual history or intentional genesis preferring to consider mathematics *sub specie aeternitatis*. Little wonder then that it tends to see mathematical objects and facts independently of human action and mathematical investigation as exploration and discovery.

Notions like intentionality, intentional subjectivity and intentional genesis point naturally to phenomenology, the philosophy of Edmund Husserl. I will borrow freely from it here, but I do not offer my reflections as faithful interpretations of Husserl’s philosophy of mathematics. There is such a thing and I have already written about it (see references), but this essay is not concerned with it. I will often align with Husserl with regard to fundamental questions of the philosophy of mathematics, but the bulk of what is to come is not to be found in his writings. At times, I will be in conflict with him, particularly with respect to the central position reserved to intuition in the dynamics of knowledge. Although most of the concepts and the philosophical outlook of this work are Husserlian through and through, my phenomenologically oriented reflections on mathematics are not always Husserlian to the letter, even if it remains Husserlian in spirit. As I see it, mine is a phenomenological, but not strictly Husserlian, philosophy of mathematics. For this reason, this work begins with an account of phenomenology and the clarification of phenomenological notions that are used thoroughly.

All the central figures of the foundational philosophies of mathematics were mathematicians, Frege (logicism), Brouwer (intuitionism), Hilbert (formalism), and Poincaré (predicativism). Husserl was likewise a mathematician, but unlike the just mentioned one with a vastly more accentuated talent for philosophy. It is then natural that one at least peruses his writings for relevant insights on the nature of mathematics. This has only recently become acceptable, but not disseminated yet. Husserl had a doctorate in mathematics (calculus of variations) and acted for some time as an assistant of the great Karl Weierstrass in Vienna. The topic of his *Habilitationsschrift* of 1887 was the philosophy of arithmetic ("On the Concept of Number"), an essay later enlarged to become his first important philosophical work, *Philosophy of Arithmetic* (1891). Logic and mathematics were central to his philosophical interests throughout his career. Husserl had planned a second volume of his philosophy of arithmetic to deal with general arithmetic and generalizations of the concept of finite cardinal number, the subject of the first volume. Failure at carrying out the original project as planned made him turn to foundational logical questions, and then write and publish his opus magnum, *Logical Investigations*. The philosophical problem related to the logical-epistemological justification of "imaginary" entities in mathematics is, I claim, essential to understand Husserl's philosophical development. "Imaginary" entities are objects that not only do not exist, but cannot exist in a given context, like complex numbers in the real numerical field, but which can, nonetheless, be useful for understanding the context where they are, precisely, absurd, like complex analysis for real analysis. How can this be so? This problem will constantly be at the horizon of my efforts here. Husserl saw in the medieval notion of intentionality, brought to psychology by his teacher Brentano as a characteristic trait of mental phenomena, a key concept with which to articulate a theory of "imaginary" objects and empty representations. The problem of imaginary objects, as Husserl called it, played, I believe, an important role in the creation of phenomenology, a fact not always acknowledged.

Husserl also produced a philosophy of geometry, with profound insights about the problem of space that was so candent at the beginning of the twentieth century due to the creation of the theories of relativity. I address the intentional genesis of the geometric representation of space here as a prolegomenon to the more general problem regarding the mathematization of perceptual reality. But more important than Husserl's direct contributions to the philosophy of mathematics is his general philosophy. It includes, in particular, a theory of intuition that extended the notion of intuition of so limited a scope in Kant to the point of making it perfectly adequate for the treatment of mathematical intuition. Husserl's is a complete system of transcendental philosophy, with the notions of intentionality and transcendental subjectivity at its core, besides interesting ramifications such as the idea of transcendental genesis and a transcendental approach to logic. Husserl also called our attention to the role of language and culture in the objectification of intentional constructs, mathematics in particular. But careful, this has nothing to do with post-modern relativism!

With this mathematical pedigree and so rich a plethora of ideas, it is indeed surprising that Husserl and phenomenology did not make into mainstream philosophy of mathematics. Surprising but explainable. From the recalcitrant perspective of empiricism and naturalism, which dominates modern philosophy of mathematics, particularly in analytic philosophical circles, Husserl's ideas are utterly incomprehensible. Husserl taught that naturalism must be overcome so phenomenology can thrive. But as I said, naturalism is the soil where the usual philosophies of mathematics are rooted, with their naturalistic conception of existence and their naturalistic preconceptions regarding subjectivity as nothing but a mind, intentionality as a mental phenomenon, and phenomenology as psychology in disguise. Despite the fact that Husserl completely de-psychologized the concept of intentionality, regardless of its origins in psychology, and produced the most devastating criticism of psychologism in philosophy in print.[6]

Science and mathematics as practiced depend on a series of presuppositions, the most fundamental being logical presuppositions. Both science and mathematics produce chains of reasoning that depend of logical laws and principles, fundamental logical facts and basic rules of inference. The justification of these laws and principles lies outside the scope of either logic or mathematics; one simply takes them for granted. Can they, nonetheless, be justified, and if, how?

Of course, logical principles cannot be *logically* justified, for otherwise they would not be logical *principles*. The successes of science and mathematics cannot count as justifications either, extrinsically justifications, so to speak, for this, besides being circular, has only pragmatic value. Moreover, we are not willing to give up cherished logical principles when scientific theories based on them fail. Dismissing justification altogether and claiming that logical principles are a matter of choice is a-scientific, if not outright anti-scientific, for it amounts to giving up the possibility of knowledge (if we indeed believe that logical principles express some sort of knowledge).

The route philosophers usually take for justifying logic is either epistemological or ontological. Some, like the mathematical intuitionists, favor the epistemological way. By raising questions as to the nature of truth and knowledge, and by seeing logic as the theory of truth, they believe to have found a ground from where to criticize, accept or reject particular logical principles and laws. If logic is the theory of truth, logic depends on our conception of truth and can be criticized from that perspective, or so they claim. Others, maybe unwilling to pay the price that a too restrictive conception of truth might impose on logic prefer simply to endorse the conception of truth that goes along with the logical principles that science and mathematics depend on. This usually takes them to ontology. For a law such as, for example, the principle of bivalence (a meaningful assertion is either determinately true or determinately false, independently of us being in a position to decide which) to be true or justified, that which assertions are about, the domain of reference, must, they reason, have particular ontological features, in particular ontological determinacy, which they think imply ontological independence. In short, philosophers seeking to justify logical principles tend to base them on epistemological and ontological presuppositions taken as established truth: this is what truth and

[6] Prolegomena to the *Logical Investigations*, which marked Husserl's turning of the back to the philo-psychologism of the *Philosophy of Arithmetic* (which, by the way, as already sufficiently shown, owns nothing to the unfair and incompetent review of this work by Frege 1894).

knowledge *are*; this is how the domain of knowledge *is*, *therefore* these are the valid principles of reasoning.

Phenomenology takes a subtly different approach. Instead of asking how reality *actually is*, it asks how reality must be *conceived to be* so logical laws are valid laws for reasoning about reality. The difference is immense. The phenomenological approach avoids outright metaphysical and epistemological presuppositions, showing that logical principles and laws can only be justified or, rather, clarified by unveiling intentional constitution. By approaching the issue in this manner, phenomenology relativizes; different intentional positings may require different logics.[7] But this requires abandoning ontological and epistemological parti-pris and taking intentional experiences *as they are experienced*, without however "cooperating" with them or, as phenomenologists say, taking them "between brackets". A new philosophical "Copernican revolution" that takes intentionality as the irradiating center of being and meaning.

Science underwent a major revolution in the sixteenth and seventeenth centuries; some even claim that what we know by science was in fact invented then. One of the defining characters of the new science was a completely new conception of empirical reality. Nature became a mathematical manifold, structured according to mathematically expressible laws where truth and being are completely determined in themselves, capable in principle of disclosing themselves in perceptual experience, if only approximately. One of the worst possible philosophical misunderstandings, however, which blocks any possibility of understanding the applicability of mathematics in empirical science, is to mistake this *intentional construct* as the *real* world, existing in itself, independently and, consequently, the scientific revolution of the seventeenth century as the revelation of the true essence of reality, not merely, as it was, the discovery of a methodology. So misguided a way of seeing, despite its apparent naturalness, is only metaphysical *parti-pris* passing for philosophical good sense. It leaves all the interesting questions unanswered, besides raising new, unanswerable ones.[8] For example, why is mathematics such a good instrument for knowing nature? If reality "just happens to be a mathematical manifold", then the applicability of mathematics to the empirical science is non-problematic only if the mathematics that is useful in science comes from observing nature. However, what about mathematics that despite being invented independently of the observation of nature, and for completely different purposes, has, nonetheless, applicability in our best theories of nature? Pre-established harmony? The doors to mysticism lay open and some people are quick to cross them.

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[7] Rigorously speaking, logic is *not* content-free. Of course, the truth of logically true assertions does not depend of their particular contents, but depends on the sense of being of the domain to which they refer. In order for, say, either *A* or *not-A* to be valid, no matter which *A*, the domain where *A* is interpreted must be intentionally conceived *in a certain way*. It befalls on phenomenology the task of clarifying *what* this way of being is and *why* conceiving the domain of knowledge thus is justified in the overall schema of knowledge.

[8] "Those intuitions which we call Platonic are seldom scientific, they seldom explain the phenomena or hit upon the actual law of things, but they are often the highest expression of that activity which they fail to make comprehensible" (Santayana 1955, p. 7).

Of course, the answer must lie somewhere else.

If, however, the conception of physical nature of the mathematical science of nature is only a *methodological device*, expressly devised to give mathematics a role in the investigation of empirical reality (by means I will investigate here, although far from exhaustively), then the mystery begins to fade. Mathematics is applicable in empirical science *because* empirical reality *is intentionally constituted as* a mathematical realm. Phenomenologists must be summoned, first to dispel the mist of metaphysical prejudices that clouds the issue and overcome the naïve naturalism of empirical science and empiricist philosophies, and then to uncover the intentional constitution of the domains of both mathematics and the empirical science and explain how they can "talk" with one another.[9]

Mathematics, which some people call classical but that is the only mathematics that we have (intuitionist mathematics being only a form of constructive mathematics, interpretable in "classical" terms), makes free use of "classical" logic, essentially the logic in which *tertium non datur*, or the principle of bivalence, depending on how one looks at it, is valid. As I said before, classical logic has presuppositions. It is again phenomenology, or more particularly, the transcendental phenomenological analysis of logic, transcendental logic for short, that offers the best instruments to bring forth, clarify, and ultimately justify the presuppositions of classical logic. Instead of the epistemological and ontological presuppositions previously mentioned, which a naturalist perspective requires, I show that, in fact, the presuppositions of classical logic have a transcendental character rooted in intentional positing.

As I see it, the task of the philosopher of mathematics is to investigate the sort of knowledge "classical" mathematics provides and the role it plays in the overall scheme of human knowledge, particularly scientific knowledge. This naturally leads the investigation to the applicability of mathematics. As I plan to show here, there is essentially no difference between applying mathematics to itself and applying mathematics to empirical science. Curiously, philosophers seem more puzzled with the latter than with the former, usually taken for granted. The reason is that they feel there is an ontological gap between empirical and mathematical realities. How, then, can the latter have anything to do with the former? As I see the matter, the applicability of mathematics, either to itself or to science, turns out to have the same explanation, for there is no ontological gap between mathematical realms proper and the representation of empirical reality in the mathematical science of nature.

Let us begin at the beginning.

[9] The relevant Husserlian bibliography on the critique of naturalism is varied and very interesting. For example, Husserl 1965, 2006, Chap. II of *Ideas I* (Husserl 1962), or the masterly Chap. II of *The Crisis of European Sciences and Transcendental Phenomenology* (Husserl 1954b, 1970).

Chapter 2
Phenomenology

In this chapter, I introduce the reader to some central notions of phenomenology, which he may not be familiar with, and fix the terminology. All these ideas come from Husserl, particularly from its later, transcendental period. I do not have, however, scholarly intentions; I appropriate rather than explicate. There are exegetical books on Husserl's philosophy that the reader can consult; this is not one of them. I will be as close as possible to the meaning Husserl attached to his ideas, but will allow myself some freedom of interpretation. I avoid as much as I can, and the subject allows, lengthy, erudite explanations. My aim is clarification, hopefully in the simplest and most direct way, and fixing the terminology without unnecessary pedantry. These ideas play a central role in my understanding of the nature of mathematics and its applicability, as will become clear later. There will inevitably be repetitions, but I hope at the service of clarity.

Intentionality This is the most central concept of Husserl's phenomenology, which he borrowed from Brentano, for whom intentionality was the characteristic feature of mental states. Mental states, Brentano thought, are characterized by directness towards something and intentionality is just another name for it. When one thinks, one thinks about something; one's thoughts have a content, *that* which is thought. One always desires, loves, sees, remembers something; the *intentional object* of one's desire, love, remembrance, or visual perception is the thing desired, loved, remembered, or seen, *as they are* desired, loved, remembered, or seen. In the intentional experience, intentional objects are given a sense, a mode of being, and some of its properties are made explicit: the object intended is intended as having these properties, this sense of being, existing in this way. These things together constitute the *intentional meaning* (or sense) attached to the object in the experience.

The object (the nucleus of the intentional experience) plus its meaning make up the *intentional content* of the experience. Intentional content must be neatly separated from immanent content; the former is not a constitutive component of the experience as a *real* event; it is not, unlike the latter, part of the experience, only a correlate of it; it is that which is posited by, and therefore dependent on (but not a

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real part of) the experience. The intentional object of an experience of perception, for instance, is not a mental representation of something belonging to the external world that the experience conjures but the physical object *itself*. To question what in the *real* experience accounts for the directness of the subject towards the object of the experience is obviously a matter of scientific importance, but phenomenology is not and cannot play the role of natural science (and so, cannot be "naturalized"). As Husserl conceived it, phenomenology (resp. *transcendental* phenomenology) is a pure a priori science of the *essential* traits of intentional (resp. transcendental intentional) subjectivity, in a sense analogous to mathematics, also a pure a priori eidetic science.

The directness towards the intentional object in the intentional act is in general mediated by the meaning attached to the object in that act. But there is, I believe, in any act an element of non-intermediated directness, a link connecting intending subject and intended object that is not necessarily expressible as a way of seeing, a perspective or an element of meaning. It takes sometimes the form of an indexical, such as, for example, the "that" and the "there" in "*that* book on *that* table over *there*".[1] Directness can sometimes be expressed simply by silently pointing to the object intended – of course in a context where pointing makes sense and is interpreted as denoting. In terms of the denotation/connotation distinction, intentional meaning is connotation, intentional directness is denotation, but denotation is not supposed to be always intermediated by connotation. The haecceity or quiddity of objects of intentional experiences, then, does not depend ultimately on intentional meaning; there is more to intentional directness than the meaning that goes with it.[2] Husserl refers to this core of pure objecthood as "the determinable *X*", the *quid* in itself indeterminate but determinable through the *quomodo* of intentional meaning (*Ideas I* §131).[3] This strange terminology expresses the fact that the intentional object, if not garbed in intentional meaning, can only be expressed linguistically by a non-logical, non-interpreted constant *X*. Thus, the "essence" of an object of intentional experience may sometimes reduce to a materially empty "something" whose sole purpose is to serve as a nucleus of sedimentation of meaning. Of course, it is always possible to scrutinize the determinable *X* to determine it; this, however, presupposes that an *X* is already, somehow, *given*. Not in complete agreement with Husserl, admittedly, I think there is room for the consciousness of a *completely indeterminate* "something" as a legitimate intentional experience.

Often the *same* object is meant in different ways; for example, the object we call "Napoleon" can also present itself as "the French ruler who lost the battle of Waterloo". It often happens that objects are presented against an open horizon of further possible presentations that are capable in principle, or so we presuppose, of

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[1] Husserl believed that indexicals too had meanings, which he called "essentially occasional", for they depend on the occasion and circumstances of enunciation (see 1st *Logical Investigation*, chap. 3).

[2] The term haecceity translates Duns Scotus' *haecceitas* or "thisness", which translates Aristotle's τὸ τί ἐστι.

[3] Husserl 1962.

harmonizing as presentations of the *same* object. This requires that the series of presentations be presided by an *identifying intention*: different presentations, with possibly different meanings, *seen as* presentations of the same object. The act of identification consists in the consciousness of the unicity of an object through a series of presentations, a "something" differently meant in different acts. The intentional object of an act of identification is an identity, the object of act A_1, O_1 is the *same* as the object of act A_2, O_2, regardless of the meaning attached to it in each act: $A_1 \neq A_2$, but $O_1 = O_2$. Intentionality in general establishes a link, sometimes a *direct* link between consciousness and the object of consciousness; intentional meaning giving *this* object a particular sense, offering a particular perspective of *it*. Intentional directness presided by an identifying intention is a sort of rigid reference, capable of holding firm its object across series of possible changes or superposition of meaning.[4] Therefore, *self-identity* is an intentional production not in general dependable on the preservation of any particular attribute, property or aspect of the self-identical object in different adumbrations of it. Identity does not depend on an inalterable nucleus of essential features (essentialism); rather, it is constituted in an identifying act.[5]

Intentionality is both the characteristic feature of a type of experience (intentional experiences in contrast to experiences that are not intentional, such as non-conscious experiences), of a subject, ego, or I, and a relation between the ego and the object of its experience (the intentional object).[6] The ego *intends* and intends *something*; the thing intended and the ego are related via the directness of intentional experience. Although the intentional experience has a real dimension, it is not necessarily happening "in the mind", (it can also be a cultural process) and the intentional object is not necessarily a mental object (although it may sometimes be; in the conscious experience of pain, for instance, the intentional object, the pain I am conscious of, is itself a mental state). The intentional object can be a real thing of the external world, but it can also be an abstract or ideal entity.[7] When one remembers a beautiful tree in the courtyard of a long-gone childhood, the tree that one remembers, charged with the sentimental undertones of things past, does not belong

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[4] A name refers rigidly if it refers to the same object independently of how it is conceived, i.e. if naming by that name goes along with an identifying intention. In this case, we usually say that the name is used consistently.

[5] Whereas for Frege identity is a relation among connotations, for Husserl, it is the object of a supervening act of keeping the same object (sometimes a mere something) under the intentional focus of acts with different intentional meanings. Whereas, for Frege, identities express the ego-independent fact that the same objects can be denoted by different connotations, for Husserl, identities are correlates of acts of identification, involving the ego and intentional consciousness in an essential manner. Identities are *constituted*, not simply "grasped" in identity-assertions.

[6] Intentional experiences are also called *acts* for the reason that the ego, who undergoes the experience, does not only provide the locus where the experience simply "happens", but is actively involved in making it happen. I will also refer to the ego by an "it" since it is not always an individual person.

[7] "'Real' [*is*] that which exists in space and time" (Husserl 2006, p. 16). Abstract objects are ontologically dependent objects, such as the color of a body as a *real* aspect of it. Ideal objects are non-real objects.

to the remembrance as a real part of it, it is *not* a mental representation of a tree. The tree remembered is (or was) a physical object of the world. The intentional act establishes a relation between the subject and the intentional object, which is not a mental image of a tree that happens to correspond somehow to the real tree of one's childhood; the *intentional object* of the act is this tree *itself*.

The intentional content of the act does not in general reduce to the intentional object simpliciter, but to the object in a certain way, from a certain perspective, with this and that quality, as a particular type of being. The intentional act wraps the intentional object in layers of intentional meaning. The tree remembered belongs to a certain family of trees, possesses a particular type of foliage, a certain coloration, it bears fruits, and so on. It can also be charged with emotional undertones that are also components of how the tree is remembered. As I said, there are in general two distinct components of the intentional content, i.e. the content of an intentional experience, its object and the intentional meaning that gives the object its particular sense of being and its distinguishing particularities.

Objects are often meant as objects of a certain type, inheriting the intentional sense of objects of *this* type. When I remember my tree I remember a *physical object*, and even if I do not explicitly give the tree the qualities of physical objects (materiality, spatiality, temporality, individuality, etc.), it necessarily has these properties, which belong by intentional necessity to the intentional construct "physical object". One might call this "collateral (or implicit) intentionality". Objects of sensorial perception are meant as physical objects, existing in the physical world, independently of being perceived, etc. A perceptual object is an object that necessarily exists in the empirical world, for perception is, *by definition*, the experience in which the subject comes in contact with the *real* world.[8] However, the subject can interpret as perception an experience that is, in actual fact, one of misperception. Misperceptions is always a possibility, but perception and misperception are different acts. If I see, say, what is in fact a coiled rope *as* a snake, the intentional object of the act is a snake, not a rope. "Perceiving" the snake is misperceiving the rope. The intentional object of an act of misperception does not coincide with the intentional object of an act of perception; this is why it is a *mis*perception. Although induced by the same physical stimuli that could have been correctly interpreted as a rope, the act presents instead a snake. Misperception posits a different object, endowed with a different meaning; misperception is always possible, for both the objects of perception and misperception are intentionally constituted from the same basic sensorial data, the *hyletic material*. *Meaning-giving is a sort of interpretation* of *the hyletic given*.

However, misperception is an unstable experience and can cancel itself with further, more careful observations. The subject can come to realize that what it mistook for a snake was in fact a piece of rope. *Sensorial stimuli from the world do not completely determine the object of experience*. Perception and misperception are both active acts of meaning-bestowal upon passive raw sensorial material. The subject

[8] In fact, the real world simply *is* the maximally consistent system of all possible objectively valid perceptions.

cannot know a priori whether its experience is one of perception or misperception; however, the particular sense of being of percepts includes criteria of validation of the perceptual experience that can, in principle, dispel misperception. Objects of perception are stable and can re-present themselves in further acts of perception. If the subject is unsure of the soundness of its perception-like experience, it can always repeat the experience and verify whether the object remains the same; or better, whether the object can be reidentified as the same object in face of this new experience. The same object of perception can present itself, in different acts, from different perspectives, with different qualities, but in legitimate perceptual experiences, these *adumbrations* tend to harmonize. Disharmony indicates misperception and harmony suggests perception proper. Consistency of the system of adumbrations of the same object counts as a *criterion* of correction of the perceptual experience, implying the real existence of the object perceived (since objects of perception *really* exist).[9]

Intentional objects need not be individuals; they can also be states-of-affairs, situations, concepts, domains of objects, ideas, what have you, even mathematical objects. One may miss drinking beer, hope for freedom, or wonder whether 3671 and 3673 are twin primes (they are). Intentional acts mean, intend or posit objects with their characteristic senses of being and existence and their own criteria of validation. Ideal objects, such as, for instance, mathematical objects, do not exist in the same manner as real objects of the empirical world, but they too have a reality of their own. Numbers, for instance, are ideal, not real objects. They do not exist in the physical world, for they are not physical objects, or in the mind, for they are not mental objects. Nonetheless, they are conceived as *objectively existing* objects, i.e. they can present themselves *as the same* for *anyone* who goes through the intentional experience in which they are posited. They are also conceived as *self-subsisting* objects, i.e. objects that exist independently of actual presentations, but that can in principle present themselves whenever conjured in adequate intentional acts. They are also meant as *transcendent* objects, i.e. objects capable of presenting themselves anew from different perspectives, with new properties and aspects. The meaning with which numbers are conceived may mislead the phenomenologically naïve philosopher into thinking of them as a sort of quasi-physical objects living in a quasi-real world that is not, however, to be found anywhere in this world. This, of course, is what Platonists believe.

To ask whether an intentional object *really* exists has sense only *within* the intentional experience itself. If the intentional subject believes, for instance, that it perceives a real object, i.e. an object of the empirical world, the ego has the right to ask "does this object *really* exist?" "Is it a fantasy or a hallucination?" To answer this

[9] The physical object does not exist *because* its adumbrations are consistent; consistency is not a definition of existence. It is existence that implies consistency of adumbrations; therefore, one can take consistency as a reliable *criterion* (or *sign*) of existence (criterion = necessary condition). One can, however, advance the following definition of existence for physical objects: an object of the empirical world exists if, and only if, the *ideal infinite* system of all *ideally* possible perceptions of *it* is consistent. This definition can be generalized. Consequently, the existence of a real object is always *sub judice*, our practical and scientific lives must cope with this fact.

question the ego must be certain, as much as it can, that its experience is one of perception, not misperception. "Really?" is an internal question. Objects of perception *really* exist if (and only if) they exist in the empirical world. A particular number, on the other hand, *really* exists if it exists *as a number*, with the sense of being attached to numbers.[10] In order to answer existential questions the ego must inquire the sense of existence proper to objects of the intentional experience in question. If the object meant fails to satisfy the criteria of existence associated with the intentional positing[11] of objects *of its type*, it fails to exist as an object *of this type*. A mathematical object exists insofar as it consistently *coexists* with the totality of mathematical objects. In "classical" mathematics, the domain of objects is conceived as an ontologically definite, epistemically accessible, maximally consistent domain of being. This implies that meaningful assertions of classical mathematics have intrinsic truth-values, any mathematical problem is in principle solvable, and any mathematical entity exists whose positing is internally and externally consistent, i.e. the positing does not attribute to the object contradictory properties nor conflicts with the overall system of mathematical positings.[12] Asking, however, whether a mathematical object *really* exists with a sense of existence *alien* to that of mathematical objects is a category mistake, not a legitimate philosophical question.[13]

Intentionality is so pervasive a phenomenon that it would be surprising if it were not relevant in science too. "Empirical nature" is also an intentional object, an

[10] "In contraposition to nature, to the world of factual spatial-temporal existence, to the 'empirical' world, there are, as one says, *ideal* worlds, worlds of ideas, which are non-spatial, non-temporal and unreal. And yet, they exist indeed […]" (Husserl 2006, p. 16).

[11] A terminological observation. Husserl uses the term "positing" acts to acts whose objects exist with the sense of being with which they are posited. Perception proper, for example, is in this sense a positing act, whereas misperception is not. They stand in opposition to non-positing acts, such as fantasying or daydreaming (when I entertain the phantasy of, say, an unicorn, *conceived as a physical object*, the unicorn, although existing *in phantasy* does not exist as the *real* object it is meant to be – phantasizing does not really posit its object; it does not confer real existence to it). Also, Husserl calls "objectifying", in opposition to non-objectifying, acts whose objective correlates are objects in a restricted sense of the term, such as naming and judging (for Husserl, names denote individuals and judgments, states-of-affairs). I will not strictly adhere to Husserlian terminology. I use the term "posit" (also mean and intend) here as a generic term for intentional presentation, with its different degrees of "clarity", i.e. intuitiveness, modes and characters.

[12] We must be very careful with the expression "in principle". As I use it here, it does not mean "effectively" or "actually". By saying that a problem is in principle solvable I only mean that it is not a priori, considering only *meaning*, seen as unsolvable.

[13] Mathematical objects, as I will argue below, are (ordinarily or "classically" posited as) abstract (ontologically dependent), ideal (non-real) objects outside space and time. To treat them otherwise, as, for instance, temporal objects, in the manner of intuitionists, is to falsify the experience in which they are posited. Which does *not* mean that experiences of constitution of the intuitionist type are illegitimate; on the contrary, *all* positing intentional experiences are legitimate *on their own terms*. My point is that intuitionism does not coincide with ordinary mathematics; it is a completely different thing. It cannot count as a philosophy of mathematics, only as an *alternative conception* of mathematics. My approach, in short, offers not only a possibility of philosophically clarifying usual, ordinary mathematics, but alternative versions of it too.

intentional construct, and the methods of science, *including mathematization*, can only be understood and justified by a careful analysis of the intentional positing of empirical reality. Intentionality is so complex a phenomenon that Husserl devised a whole – pure, a priori – science to investigate it, namely, phenomenology. By offering the possibility of a first-person approach to mental phenomena – such as perception, for instance –, Husserl's theory of intentionality has known some success among psychologists and cognitive scientists. However, even at this level of application, phenomenology is *not* an empirical science. Husserl's is an a priori, pure theory whose object of inquiry is the *formal* structure of intentional experiences *in genera*l. Like, in a sense, the *a priori* theory of physical space, whose aim is not space as experienced a posteriori, but as experienceable a priori.[14] Husserl is quite clear about this; phenomenology is not psychology, but a pure science, which among other applications can serve as a prolegomena to empirical psychology.[15]

The terms subject, ego, or I, lend themselves easily to misinterpretations. The reading of Husserl as a "psychologist", even though he has produced maybe the most devastating critique known of philosophical psychologism (vol. 1 of *LI*, "Prolegomena to Pure Logic"), derives from these misinterpretations. The fact, I insist, is that the ego is not necessarily, although it can sometimes be, a person or a mind. The subject, ego, or I, is only the locus of intentional experiences, the positing-pole logically required by the posited-pole, the meaning-irradiating center that need

[14] There are many points in common to mathematics and phenomenology, both are eidetic, not factual sciences (sciences of essences, not facts) and both are a priori. In *Ideas I* (Husserl 1962 §§ 71, 72) Husserl raises a question that seems, then, natural: could phenomenology be mathematized? His conclusion is that it cannot. For one, phenomenology, as a *material* eidetic science, does not belong to the *formal* eidetic sciences like formal mathematics. Could, then, phenomenology be put together with material mathematical sciences, such as, for example, geometry? Can phenomenology be developed as a sort of geometry? Still, it cannot, according to Husserl, for geometry is an axiomatized and ideally logically complete (definite, in Husserl's jargon) theory, which proceeds essentially by logical derivation from axioms, i.e. fundamental laws of essence, whereas phenomenology is a *descriptive* eidetic that is not and cannot be axiomatized. By being essentially non-formalizable and incapable of axiomatization, phenomenology is, for Husserl, essentially non-mathematical.

[15] Much has been debated about Husserl "transcendental turn", which happened in between the publication of his *Logical Investigations* (1900–1901) and *Ideas I* (1913), more precisely in courses of the period 1906–1907, and what it means. Intentionality was central to his thought both before and after the turn. As I see it, however, before, in the "realist" period, Husserl's goal was to investigate intentionality as a *natural phenomenon* within a *naturalist* context; after, in the "transcendental" period, he approaches intentionality as a *pure form* in a *transcendentally purified* context (the notion of *epoché* that I will examine soon is fundamental in this transition). Transcendental phenomenology imposes itself the task of investigating the *necessary* features of intentional experiences and intentional consciousness in general. The transcendental intentional subject is absolute, the center from where meaning flows; it is a *function* rather than a *thing*. In the transcendental period, intentionality is no longer seen as "a manner of seeing" things that may exist otherwise with a sense of their own. Nor, on an epistemological key, as the way in which the subject approaches, as knowing subject, the object of knowledge (the intentional object), which exists, with a sense it intrinsically has, independently of being known. Transcendentally considered, no object exists independently of being intentionally meant and no object has a meaning without being given a meaning in an intentional experience.

not be a mind in the usual sense of the term. It can also be a community of individuals working together across space and time as a single entity, building together, with maybe a variable sense of being, a domain of investigation (for example, mathematics). The ego can be the mathematical community in the task of positing mathematical objects and developing their a priori theories.

There is an obvious phenomenological difference between seeing a tree and remembering a tree, even if this happens to be the very same tree. Husserl calls that which accounts for this difference the (*thetic*) *character* of the act. Seeing and remembering are acts that intend the *same* physical object differently; the intentional object of the act of seeing the tree is the tree *as an object of visual perception*; that of remembering the tree is the tree *as an object of remembrance* (in both cases, however, the tree is a physical object). I would proceed differently if I had to verify the adequacy of either experience. In the first case, I would multiply my sensorial perceptions of the tree and see if they harmonize with my first experience (there is no other way, for fundamentally only perceptions validate perceptions). But if I want to verify whether my memory of the tree corresponds to the real tree, I must see the tree, to have, that is, an intentional experience with a different character; it does not suffice, although it helps, to try "to remember it better". The criteria of validation of an act of remembering (does my remembering correspond to reality?) requires acts other than remembering.

Transcendental phenomenology does not eliminate empirical reality or reduce it to a projection of the ego. For Husserl, perception is an intentional act, but sensations are not. Sensations provide the matter (the *hyle*) that is intentionally elaborated into perception proper. The same sensorial matter can, for example, as already discussed, be elaborated as either perception or misperception. Things can get very messy, but we do not have to go into the minute details of the analysis of intentionality here (Husserl himself often despaired with the complexities of the task).

Intentional positings can superpose one another in the positing of an object. Mathematics provides plenty of examples. By "perception" one often understands sensorial perception, the intentional experience in which objects (perception-of) or situations (perception-that) are *immediately presented* to intentional consciousness. Perceptions are experiences with a peculiar character of act, the actual presence of the object perceived, as opposed to its mere representation as, for instance, in descriptions of the object *in absentia*. Obviously, other intentional acts can have the same character; presentification is not an exclusive trait of sensorial perception. In other words, one can generalize the notion of perception. Husserl called "intuition" any act that has the same character of presentification of sensorial perceptions, no matter its object. One example, which will be elaborated further later. According to Husserl,[16] in order to *perceive* or *intuit* the number 2, one must first posit (maybe in imagination) two objects (*a* and *b*); then, these objects in conjunction (*a and b*)[17];

[16] See the detailed analyses in his Husserl 1970.

[17] The mereological sum *a and b* includes, as an *abstract moment*, the syncategorematic *and*, the *categorial* element of the objectual complex.

then, the mereological sum *a and b* as a single object (the collection {*a*, *b*}). Only then, by abstraction, one can intuit the *quantitative form* of the collection, an *instance* of the number 2 (I will deal with abstraction as an intuitive act below). The intuition of the *number* 2 requires the positing of something – the *ideal* 2 – of which this particular aspect is an instance. In short, the intuition of the number 2 requires the superposition of many intentional acts, in which the object of a lower-level act serves as the *matter* of an immediately following act, which, based on the given matter, intuitively posits a different intentional object.

The Intentional Ego Intentional experiences require an active agent who undergoes these experiences, the intentional subject (I or ego). When phenomenology was meant as pure psychology, the intentional ego was simply the mind as the real seat of consciousness. For transcendental phenomenology, however, the ego is an intentional agent in general, which can be differently instantiated, sometimes even as a real mind. For Husserl, the transcendental intentional ego and its experiences have a *necessary* structure, it is the task of transcendental phenomenology to investigate it.[18]

By being de-psychologized, the ego is no longer an individual mind, or even the abstract form of the individual mind, although it can be instantiated as one. It can also be a community of individuals acting cooperatively in the task of intentionally constituting an object, a scientific domain, or a science.[19] By being communally constituted, the intentional construct is *objectively* posited, for phenomenologically clarified the notion, objective is that which is intersubjectively shared. Objective entities are constituted publicly. By being thrown in the intersubjective space, intentional constructs become communal possession. Intersubjective space is the communal space of shared practices of the community of intentional coworkers, from the most fundamental pre-scientific life-world of daily concerns to the scientific world. Systems of communication, linguistic or not, are important elements of articulation of the intersubjective space and play a central role in the constitution, preservation, and communalization of intentional constructs.

The intentional coworkers acting collectively are not in general individually responsible for every step of the constitutive process. Each individual intentional agent is only a link in a chain of shared responsibility. But, although no single agent may be capable of *actually* reenacting the entire constitutive process, it is presupposed that it can, at least *in principle*, do so. Any single individual intentional agent

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[18] Husserl attributes to Descartes the discovery of transcendental philosophy and the transcendental ego, the Cartesian ego cogito. But, according to Husserl, Locke took possession of the transcendental ego and psychologized it. However, he believed, only by being thoroughly de-psychologized the transcendental ego can serve scientific philosophy. Transcendental phenomenology is, for him, such a philosophy.

[19] The problem of intersubjectivity is a central problem in Husserlian phenomenology, to which Husserl dedicated the fifth of his *Cartesian Meditations*, part of *Ideen II* and *Zür Phänomenologie der Intersubjektivität*. The ego is primarily *my* ego; after the transcendental reduction, the primordial ego has the task of constituting the world and other intentional egos. Only after the rights of alter-egos are recognized, the ego can extrapolate the limits of individuality.

must take personal responsibility for the entire constitutive process.[20] The original sense of intentional products, which they received in their positing, can be obliterated or forgotten. In the first case, the sense is still available, but its origins ignored; in the second, it is gone. In cases, such as these, intentional constructs are handled as if in a ritual that one performs without understanding its meaning, by simple going through the moves. Husserl saw either case as a form of "crisis-inducing" "alienation". However, sense can be reactivated, pretty much like the meaning of a coded message. Husserl believed that the phenomenologist should be a sort of decoder, reactivating deactivated senses and thus overcoming alienation and crisis.[21]

One analogy can be useful. When proving a theorem, one may use a previously proved lemma without having proved it, or being capable of proving it. The new theorem, if the proof is valid, will join the stock of mathematics, it will enrich the sense of some intentionally constituted mathematical domain. A sense, however, that neither the mathematician who has proved the new theorem nor anyone else can *fully* grasp without going through the entire chain of constitution of the domain in question from its inaugural inception. *To comprehend* an intentional production means to be in principle able of reenacting the entire process of production, thus fully grasping the meaning the product gets in the process.

Transcendental Phenomenology Husserl gives another sense to the term "transcendental", which in Kant refers to the a priori necessary conditions of possible (sensorial or pure) experience. In phenomenology, transcendental has to do with the necessary aspects of intentional positings, in general and in particular, *taken on their own terms* (by which I mean under the action of the *epoché* – see below). No matter in which sense, however, the term puts off philosophers with empiricist tendencies who feel uncomfortable with the ideas of necessity and aprioricity. Kant, to whom the notion is directly linked, believed that there are necessary preconditions of experience. For instance, "intuitions" (sensorial or "pure") are *necessarily* located in space or time. For Husserl, who considered the issue from a more general perspective, all intentional acts, not only perceptions, have a necessary structure and all intentional objects necessary features, those precisely that are linked, directly or indirectly, to the intentional meaning associated with the positing. If, for example, a domain of being is meant as an *ontologically* (or *objectively*) *complete* domain, i.e. if it is conceived as a domain where every possible situation is determinately a fact or determinately not a fact (in which case the complementary situation is a fact), then it follows *by necessity* that any assertion about this domain is already in itself either determinately true or determinately false (in which case its negation is true). To the extent that the necessity in question depends on the meaning intentionally attached to the domain posited, it has a transcendental character. It also befalls on

[20] The theme of the individual responsibility was dear to Husserl. In his last work, *Crisis of European Science and Transcendental Phenomenology* (Husserl 1954a) this topic is central. The "crisis" alluded to in the title is precisely one of responsibility.

[21] See *Crisis*.

transcendental phenomenology the task of investigating the necessary structure of the intentional ego and its experiences, in particular the correlation intentional ego/intentional object, considered in general or in particular, under the action of the epoché (see below).[22]

The intentional subject considered abstractly in general (therefore, in the singular), is invariably "incarnated" in real agents acting in real time. As any ideal entity, the transcendental ego can be instantiated in manifold ways, like the number 2 in any collection of two things. Sometimes Husserl refers to the transcendental subject as "transcendence in immanence". Any instantiation of the intentional ego must act according to a priori constraints imposed on the pure ego on grounds of necessity. The analogy with mathematics is obvious (one should never forget Husserl's mathematical origins); both are a priori sciences of idealities that, nonetheless, impose necessary constraints on reality.

Epoché The inaugural act of transcendental phenomenology is the (phenomenological) *epoché* or *phenomenological reduction*. Essentially, it means taking intentional experiences on their own terms.[23] Epoché opens up a completely new domain of scientific investigation; a domain the phenomenologist treats more or less like the psychologist treats his patients, with detachment, without passing judgment. Epoché is an attitude of neutrality vis-à-vis intentional experiences.[24] In transcendental phenomenology, nothing exists that does not exist for the ego; the ego, on its turn, exist only as ego cogitans.[25] Intentional ego and intentional object are mutually dependent. The ego, its experiences and the content of these experiences is an interconnected system whose structure is the phenomenologist's task to investigate. Here are some of the typical questions he raises: by which intentional actions do objects, with their particular sense of being, come to be? How does the ego constitute them; what is their *intentional genesis*? What are the criteria of validation, built into their intentional sense, for judgments about these objects? However, it is *not* the phenomenologist's task to raise extrinsic questions such as, for example: do these objects *really* exist as meant? Simply because the notion of existence of an object *as meant* independently of being *so meant* is, in transcendental phenomenology, absurd. Objects exist only as objects-for-the-ego, correlates of intentional experiences. Provided, of course, that they are consistently posited; that is, that they do not crush under the weight of inconsistent intentional meanings.[26]

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[22] "Phenomenology is, as Husserl depicts it in his 1907 lectures, an eidetics of cognition. The method of reduction signifies the critical means of access not to any de facto consciousness but rather to the essential structural correlation of consciousness and objectivities per se intended therein" (Sandmeyer 2009, pp. 75–6).

[23] "Reduction" means, literally, to lead back (*re-ductio*). By using the term (cognate of the German verb "to reduce", *reduzieren*) Husserl probably meant a going back to the intentional phenomenon *as such*.

[24] See Husserl 1960, §8

[25] The title of §8 of Husserl 1960 is precisely "The *ego cogito* as transcendental subjectivity".

[26] In his article for the *Encyclopedia Britannica* (Husserl 1927), Husserl characterizes the "transcendental problem" as "having to do with the being-sense of 'transcendent' relative to conscious-

When in a transcendental phenomenological disposition, the philosopher of mathematics does not deny rights of existence to mathematical objects *exactly as they are intentionally meant*, as nominalists do, or force on them a type of existence that they are not meant to have, as intuitionists and ontological realists do. However, again, intentional experiences may exist that posit objects as mental entities, and some may argue that there are *mathematical* experiences of this type, but it is a category mistake to try to relocate in the mind objects that are not posited as mental under the argument that only thus they can *really* exist. Mathematical objects may be, and usually are posited as objectively complete entities, that is, they either determinately have or determinately do not have any meaningful attribute they can in principle have, even though one may not be able to actually decide which is the case. Objective completeness is an intentional attribute, built into the intentional positing of the object. This, however, does not make the object less dependent of the intentional act that posits it. It is an error to believe that mathematical objects, or any object for that matter, exist independently of any positing only because they are (*meant* as) transcendent or objectively complete objects. Transcendence, i.e. the willingness to present ever-new aspects and perspectives, and objective completeness are *intentional* attributes, they go with *Platonism between brackets*, so to speak (which I often write as "Platonism"). As Husserl claimed, epoché does not change anything; it only brackets contents of experience, cancelling *naturalist* existential commitments. Transcendental epoché is meant to get us out of the "natural attitude" underlying "naïve" philosophical perspectives in which objects, with their rich variety of modes of being (the empirical and the mathematical, for instance) are simply given. Empiricism, recall, is a form of naturalism in which the empirical mode of being imposes itself as the model of being in general. For empiricists, if something exists, it must exist as either an empirical object or "just like" an empirical object, i.e. independently, in and for itself. This, of course, is how Platonism conceives mathematical entities. For a transcendental-phenomenologically oriented philosophy of mathematics, instead, mathematical entities are intentional objects and have an intentional genesis. The task that such a philosophy imposes on itself is that of unveiling the intentional meaning attached to mathematical positings, clarifying and ultimately justifying the modes of reasoning about them.

Intentional Consciousness The expression *intentional consciousness* highlights and emphasizes the essential character of conscious states, namely, directness or intentionality. In Brentano's original psychological approach, intentionality characterized a particular state of the mind, the conscious state; being conscious was, for him, tantamount to being conscious *of something*. Husserl de-psychologized the concept by de-psychologizing consciousness, no longer necessarily a psychological

ness". The "transcendental attitude" is required so the phenomenologist can raise and deal with the "transcendental problem". In the pre-epoché "natural attitude", sense of being is a given, it does not require constitution; it does not have a genesis. At best, constitution has the epistemological sense of the unveiling of the object to the subject. The correlation object-for-the-ego/object-positing-ego avoids a plethora of ontological and epistemological problems originated in the naturalistic separation of subject and object.

entity. In Husserl's philosophy, intentionality means *awareness* in the most general sense, awareness of something, a determinate intentional content, by an ego that no longer necessarily is the individual mind. Consider, for example, the so-called imaginary numbers. There was a time the mathematical community (the intentional ego in this particular case) was not aware of them; they had not been created yet. Platonists would prefer to say that they had not been *discovered*, but I find this a misleading way of speaking that mistakes intentional action for discovery. Gradually, imaginary numbers were admitted as legitimate mathematical entities and given a place and a role in the body of mathematics. In other words, the mathematical creative subject – which is only another name for the mathematical intentional ego –, incarnated in a more or less well-defined community of real mathematicians, in a more or less precise stretch of time, gradually became conscious – or aware – of imaginary numbers.

One can tell the factual history of this development, but one can also tell its transcendental history. Transcendental history reports intentional genesis, of imaginary numbers in our example; it is not factual history and need not coincide with it. For one, factual history depends on who is telling it and the events he chooses to tell. Transcendental history, on the contrary, is not a chronicle of facts but of intentional acts. It is a pure science whose task is to determine by which series of intentional acts the intentional ego has become conscious or aware of something. Transcendental history tells how intentional objects of any given type came to be and the necessary structure of its coming to be. Factual history merely registers the real manifestations of this coming into being.

There is an intentional archeology too, whose task is to uncover the many layers of intentional sense sedimented in habitus and traditions; it allows the reactivation and, consequently, reenactment of intentional genesis. The possibility of reenacting intentional genesis renders the intentional object *available*; the communal sharing of this possibility renders it *objectively* available.[27]

Intentional Act In intentional acts (or experiences) the intentional ego, in which form it may take, becomes conscious of something. For example, seeing a red rose, intuiting the number 2, abstracting the form of a physical body, or inventing complex numbers are all intentional acts. Neither of these experiences is purely passive, even seeing the rose. The senses offer the perceiving ego a manifold of sensations (the *hyle*), a color, a shape, a scent that the subject must elaborate into the *perception* of a red rose taking into consideration, among many things, expectations, memories, and its stock of empirical categories. The intentional subject is an agent; it acts. The ego is the locus of a process, maybe a mental event, maybe a historical development.

Intentional acts in general have a basic structure. There is the event occurring in the ego, which Husserl called the *noetic* aspect of the act, and there is that which the agent becomes, by going through the process, conscious of, the *noematic*

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[27] In the introduction to his translation of Husserl's essay of 1936 "The Origin of Geometry" (Husserl 1954b) Jacques Derrida presents an interesting discussion of this question.

correspondent of the noetic performance. For Husserl, the pair noesis/noema (pl. noeses/noemata) is necessarily present in any intentional act. Noesis is the intentional act as a *real* event, noema is that which is meant in the act. The psychophysical process of perceiving a red rose, for example, is the noesis in which the ego becomes conscious, in the particular form of a perception, of a red rose, the object of the act. However, the object "this red rose" is not the sole component of the noema; in the complete noema the red rose appears *as a physical object* (not, for instance, an idea) and *as a perception* (not, for instance, a memory). There are different aspects of the act that posits the rose as an object of the empirical world, and there is a certain character to the act that makes it different from other acts that might mean the same rose; in this case, the rose is *perceived*, not, for instance, remembered. If the ego is only recalling this particular red rose, the character of the act changes, remembrance, not perception. The *same* object can be meant in thetically different acts.

There is, I recall, an important distinction between the object of the act simpliciter and the way in which it is meant, the intentional meaning attached to the object in this act. By *intentional object* I mean, unless explicitly said to the contrary, the whole package, the object proper, the *intentional nucleus* and the intentional meaning attached to it. By taking co-intentional acts as components of the main act, one adds their meanings to the intentional meaning of the main act. Thus, in our example, the rose is perceived not only as red, fresh, beautiful or any other characters it is perceived to have, but also as a physical object, with all the characters attributable necessarily to physical objects (pertaining to the intentional meaning of the intentional construct "physical object"). There is an obvious resemblance between the notion of intentional and linguistic meanings. Both serve for "grasping" objects. Husserl himself tells that he conceived the notion of intentional meaning as a generalization of that of linguistic meaning.[28]

Let us consider in more details the intuition of the number 2. Let us first make clear what I understand by that. Intuition, as already stated, is a generalized form of perception; intuitive acts are intentional acts that *present*, not merely *represent* their objects, and there are misintuitions just as there are misperceptions. Intuition is presentification, in opposition to empty representation. As already discussed, abstracting the numerical or quantitative form of any set of two things is the first step into intuiting the number 2. Abstraction is the intentional experience whose object is a particular (ontologically dependent) *aspect* of a given object, the color or geometrical shape of a physical body, for example, or, our case, the quantitative form of a set of objects. The terms *concrete* and *abstract* denote types of objects; concrete objects are ontologically independent, abstract objects are not, they depend ontologically on other objects.[29] Acts of abstraction are intuitive acts if based on the intuitive presentation of the object upon which abstraction acts. Sets and collections

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[28] However, whereas for Frege denotation requires connotation, for Husserl, as I interpret him, intentional directness does not depend *necessarily* on any particular attribution of intentional meaning.

[29] Check definitions in the third *Logical Investigation* (Husserl 2001).

are abstract entities, even if their elements are *concreta*, for they depend ontologically on their elements; change the elements and the set or collection changes too. Collections are mereological sums, sets are obtained from collections by unification, i.e. by taking them as themselves collectable individuals (unity in multiplicity). Unification is a particular intentional experience, requiring collections as matter. The intentional correlate of the act of putting the parts of a collections together, the "and" in *a and b*, this particular *moment* of the whole, is abstract, even if the sum is made of concrete parts.[30] Sets have *two* abstract moments, correlated to the acts of collecting its elements and unifying the corresponding collection.[31] A particularly interesting abstract aspect, or moment, of quantitatively determined collections or sets is their quantitative form. Abstracting a particular quantitative form is the first move in the presentification of a particular number to consciousness.

However, "seeing" 2 as the form of an arbitrary collection of two things is not yet intuiting the *number* 2. A form has the same spatial location of the matter that it informs. "Seeing" two objects as 2 can count as the intuition of the number 2 only if this "seeing" is accompanied by the consciousness that *any* collection of two things has this very *same* form. In other words, intuiting 2 in a collection of two things involves abstracting the numerical form of the collection and *ideating* it. Ideating a form is making it into the *idea*, a higher-level species of which all the equivalent abstract forms are specimens. Ideal objects, by opposition to real objects, which are essentially temporal, are non-temporal, not merely omni-temporal. Ideation involves, then, first, the recognition of an equivalence among objects of a determinate type *with respect to some common aspect* (an equality, but not an identity) and, second, that all equivalent entities are instances or realizations of *the same ideal form*. Two quantitative forms are *equivalent* if the collections of which they are forms are equinumerous. The number 2 is then the ideal quantitative form of all collections of pairs.[32]

Suppose that the subject considers, maybe non-intuitively, i.e. without having these objects under the gaze, in a two-rayed intentional act, the Sun and the Moon.

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[30] Husserl calls *parts* the independent components of a whole and *moments* its dependent components.

[31] What is the difference between an object, say a, and the singleton $\{a\}$ whose sole element is a? *Materially*, of course, there is none, but *formally* there is a difference, namely, the *categorial* aspect of $\{a\}$ that a does not have. Husserl calls "categorial" the abstract (ontologically dependent) aspects of objects of higher-order cognitive intentional acts, set-collecting in this case, denoted by $\{\ldots\}$. Another important higher-order *act* is that which posits a *state-of-affairs*, for instance, "*that* the paper is white" based on the *object* "white paper"; "that the paper is white" is a content of *judgement*, not merely perception.

[32] It is worth noticing that not all abstract objects are ideal, although all ideal objects are abstract. *Abstract* objects are ontologically dependent objects, which all ideal objects are, since they are ontologically dependent on their realizations. For example, the number 2 *would not exist* if collections of two things did not exist. Abstract objects, on the other hand, can present themselves as aspects or moments of *real* objects, like the color or the form (the real, not the geometric idealized form) of physical bodies. Hence, abstract objects can be real, although they are never *concrete*, which are ontologically independent objects that can exist independently of the existence of other objects.

The collection 'the Sun *and* the Moon' can now be unified into a single collectable *object* {Sun, Moon}. Both the collection (or mereological sum) 'the Sun *and* the Moon' and the set {Sun, Moon} occupy the same space that the Sun and the Moon jointly occupy, it is a *real* object. The idealized abstract form, the number 2, however, is an ideal object with no temporal or spatial location. Of course, we can also collect ideal objects, for example, {0, 1}, whose elements are the *numbers* 0 and 1. As all quantitatively determined collections, {0, 1} instantiates a quantitative form, in this case, the number 2. However, 2 is *not* equal to {0, 1}, for numbers are *not* sets; {0, 1} can sometimes be chosen to *represent* the ideal object 2 in set theory, nothing more.[33] Numbers are *ideal forms* whose instantiations are *abstract numerical forms*. I will come back to this later, but it is already obvious that this way of understanding numbers and collections can throw light on some ontological issues (for instance, Benacerraf's dilemma) and show the inadequacy of certain "naturalist" ontologies of mathematics.

I have been describing the intentional genesis of a particular object, the number 2, *as an object of intuition*. In other words, the intentional process of *presentification* of the number 2. This could be, at first, only a *subjective* experience. This ideal object, the number 2 as *originally* conceived, could have been confined to the intentional space of the intuiting subject, a particularly imaginative individual who kept his intuitions for himself. There would, then, be no numbers 2 as an *objective* entity. *Objectivation* is an intentional experience performed by a community of egos operating *cooperatively* as intentional subjects. The *Ur-subject* originally responsible for the intuition of the number 2 (of course, I do not want to imply that there was a *real* Ur-subject) must *share* its intuition; it must be presented to the community of subjects and reproduced by them as productions of the *same* object. The Ur-subject must somehow direct the attention of the community to it ("consider *that* which all collections of pairs have in common irrespectively of what these pairs are, provided they are pairs", he could have said). The communitarians could engage in collaboration with the Ur-subject without having actually performed the intuitive act themselves, blindly so to speak, but one always presupposes that they can, in principle, perform it. The relevant thing is that the cooperating subjects must agree that they are referring to the *same thing* when they refer to the number 2. An object is objectively available when it presents itself as the same object for all subjects of some relevant community of subjects who agree that they have the same object, with the same properties, under the intentional gaze. Objectification involves identification and cooperation. Presentifying to oneself the number 2 as an *objective* entity is presentifying it and simultaneously conceiving it as a possible object of intentional

[33] Usually, this is how one defines cardinal numbers. In modern mathematics, where set theory provides a context of materialization (or instantiation) of ideal entities, the number 2 can be defined as the class of all pairs, or a particular set, {0, 1} or {{0}}, indifferently, provided these "avatars" have the same *formal* properties of the number 2 (they have, of course, other properties, but they are arithmetically irrelevant). Set theoretical avatars *represent* ideal objects only to the extent that they offer a particular material basis for abstraction and idealization and thus for intuiting the idealities they represent (abstraction can be understood in this case as the specification of which features of set theoretical representatives are and which are not *arithmetically* relevant).

experience – intuition, ideally – to alter egos (the whole community of intentional egos).[34]

There are also non-intuitive acts, acts that are non-presentational. In non-intuitive acts, the object is meant but not presented. Here is a mathematical example. By varying, in imagination, a given collection of whatever objects, themselves given or only imagined, the subject can intuit the following fact: collections can exist with arbitrarily large quantity of elements. The subject, of course, does not and cannot actually intuit, not even in imagination, each collection of a potentially infinite array of arbitrarily large collections. The subject simply intuits that it *can* go on forever; it intuits a *possibility in principle*, an *ideal* possibility, neither a factuality nor a *real* (actualizable) possibility. Husserl calls this the ideality of the "and so on".[35] To each collection, there corresponds a number; therefore, there are infinitely many different ideally possible numbers. This is an intuitively justifiable *fact*. Of course, not all numbers are *really* intuited individually, but all are *ideally* intuitable. Since *mathematics is the science of ideal possibilities*, mathematics has the right to claim the *existence*, in this special idealized sense of existence, of all the numbers. Mathematics has devised clever ways of denoting non-intuited numbers, for example, the decimal notation. By writing 10^{100}, i.e. by *naming* a number one also *means* it; even if this numbers can never be directly intuited as the number of an intuitively given collection of objects. 10^{100} exceeds the number of atomic particles in the universe, a quantity no subject can contemplate.

By writing 10^{100} with the *intent* of denoting an object, an act Husserl calls *naming*, an object is meant without being presentified. Husserl calls this a purely intentional, in this case, *signic* act. The sign denotes by being *meant* to denote. Now, if there were no ways of denoting in principle all numbers individually, would they still exist? The answer is yes, just like things that exist in the world but are not *actually* named. The existence of infinitely many numbers is *not* a Platonist presupposition but an intentional positing. They exist because they are meant to exist; they exist because they are conceived as *in principle* capable of presenting themselves to consciousness. One could conceive empirical reality in such a way that things only existed in the world if actually observed. This would make empirical science almost impossible for science depends on a series of presuppositions about the world, including that empirical objects can exist without actually being observed. However, they must be, at least on grounds of principle, observable. It follows from this that if an object cannot even in principle, not only actually, be directly or indirectly observed it cannot exist (for instance, objects with logically contradictory properties). Let this stand as a reminder that the foundations of science beg for

[34] Empathy, that is, intending other egos (alter egos) as intentional agents, plays an important role in the process of objective positing. I cannot enter the theme here, but it is an important one when considering the constitution of an "objective world", for instance, physical nature. I will come back to this when discussing the constitution of the modern notion of physical reality.

[35] Among the idealities that play "a universal role for a pure analytics" Husserl mentions "the fundamental form of the and-so-on, "the form of *reiterational* 'infinity'", which has "its subjective correlate in 'one can always again'" (Husserl 1969 § 74).

phenomenological clarifications to counteract the temptation of Platonism and other forms of mysticism ("Platonism", of course, is a different matter altogether).

Intentional Object The term "object" (lat. *obiectum*) means, literally, "that which is put in front". This is how Husserl understands it too, if by "putting in front" is taken as "intending". Intentional acts of intuition, pure intending, or a hybrid of both, always put something in front of the subject, the intentional object, the thing seen, named, described, abstracted, idealized, desired, remembered, missed, etc., etc. The intentional object can be an individual, a class, a moral principle, a color in general, a particular color, a determinate shade of color, a concept, a law, physical nature, etc., etc. Intentional objects, however, are not in general just "somethings", but things with characteristic features and properties, which they are *meant* as having. The features and properties that objects are posited as having constitute, as already mentioned, their intentional meaning. Intentional meaning, I presuppose, is never ineffable; this is the *expressivity thesis*, it states that intentional meanings can always be linguistically expressed in true statements about the object whose meaning they express.

The same intentional objects, I recall, with the same intentional meaning, can be differently meant; the same tree, with the same characteristics, for example, can be either seen or remembered. The character of the act accounts for the difference; it may change without its intentional content – the object (the intentional focus) plus its intentional meaning – changing. However, the intentional content changes either if the object changes or if it is the same, but differently meant. For example, suppose the nominal acts[36] whose contents are, respectively, "the winner of Marengo" and "the winner of Austerlitz". In both acts a person is meant, a military commander supposedly, not prima facie the same. The winner of the battle of Marengo in the first; the winner of the battle of Austerlitz in the second; two (maybe equal, maybe different) objects; each with its characteristic sense; each *meant as* the bearer of its respective qualification.[37]

A further act may intervene, in which these two designations are meant as referring to the same object. The content of this act is, as already discussed, an identity "the winner of Marengo = the winner of Austerlitz", and for this reason it is called an act of *identification*. The intentional object, in this case, is a *fact* (or *state-of-affairs*) expressible by an identity assertion, namely, that the *same* object supports two different attributions. Identification plays a pivotal role in the dynamics of knowledge. For Husserl, knowledge in the fundamental sense of intuitive knowledge is defined as a *synthesis of identification* of the object of an act of mere intending with that of an act of intuition. Let us be more precise about this. The subject may, for instance, emptily (i.e. non-intuitively) posit (become conscious of) a regular polyhedron. The act can have the form "let x be a regular polyhedron, i.e. a

[36] Husserl calls "nominal" the acts of naming and judging, whose objective correlates are, respectively, the thing named and the states-of-affairs asserted.

[37] Husserl developed the distinction between sense and denotation independently of Frege. In fact, both worked within a rich philosophical tradition in which this distinction was, in some form, already present, sometimes more, sometimes less clearly (see Husserl's 1st *Logical Investigation*).

closed figure in three-dimensional space whose faces are congruent regular polygons and whose polyhedral angles are congruent". Is the positing enough to grant the existence of a regular polyhedron? Of course, not. Existence requires that the positing be consistent, i.e. that the features attributed to regular polyhedra do not "cancel one another" due to logical inconsistencies (as, for instance, the meaning "round square"). To be sure that such an object exists, the subject must first verify that the positing is consistent with itself.

But there is more to existence than the *internal* consistency of the intentional meaning. The intended object "regular polyhedron" is meant *as a body in Euclidean space*, and hence must accord with this intentional meaning; it exists only if its existence *as a member of that category of being* is not ruled out. In other words, the positing must also be *externally* consistent. Now, given that the category of Euclidean objects is meant as an *objectively complete* category of being, anything exists therein whose non-existence is logically ruled out (I will have much more to say about this later).[38]

But showing that the object *exists as meant* is not the same as *intuiting* the object as meant. It suffices for a proof of existence to show that the object meant can *ideally*, *in principle*, be intuited, even if the subject cannot put itself in the position of *actually* eliciting the relevant intuitive experience. However, to know that something can ideally be an object of presentation is *already* a form of knowledge. It is, to use a Kantian jargon with a somewhat different meaning, an *anticipation of* (*intuitive*) *experience*.

But the subject can also present, say, a cube to consciousness (by abstracting and idealizing from actually perceived or imagined cubes of the real world) and realize, by examining *this cube* (in imagination or in a physical representation of it, by eliciting the relevant abstractive and ideational acts by which the geometrical object proper emerges from the perception of the representing physical object) that it has the property of regularity. The experience has the form "I *see that* this cube is a regular polyhedron" and counts as an intuitive presentation of the cube *as* a regular polyhedron. This experience *fulfills* the anticipation of experience of the consistent empty positing, providing *intuitive* knowledge. Intuitive knowledge is a form of knowledge, maybe a more desirable form of knowledge, but not the *only* form of knowledge. The existence of regular polyhedra, in the (classical) *mathematical* sense of existence, does not require the intuitive presentation of any regular polyhedron. To believe that it does is confusing two different senses of existence (or disqualifying one of them).

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[38] Let us consider a more illustrative example, the positing of "the largest prime number". The intentional meaning "largest prime number" is consistent with itself, for nothing in the *definition* of prime number rules out that there could be a largest one. However, the *concept* of prime number, once considered more comprehensively in the larger context of mathematics, *requires* that there is no such number. That is, the positing is externally inconsistent. The distinction between internal and external consistency seems to me necessary so conjectures (either true or false) have a place in mathematics. The existence of *meaningful* but *false* conjectures (such as, for example, "there is a largest prime") requires the distinction between internal and external consistency.

The subject can also represent (or emptily mean) the cube *as a regular polygon* (for example, by judging “the cube is a regular polyhedron) without having actually intuited any cube *as* a regular polyhedron. There is no a priori guarantee that this act (the judgment) *actually* posits an object, the cube as a regular polyhedron. To validate the positing, the subject need *not* to conjure a cube in intuition and verify its regularity; it may do so indirectly by verifying, within the context of Euclidean geometry, that the characteristic features of cubes logically imply their regularity. By being thus validated, the experience is a positing experience; the cube *can consistently be conceived* as a regular polyhedron. This experience enriches the subject’s stock of knowledge about the cube, but it is *not* intuitive knowledge. It is, again, an anticipation of experience; the subject knows that *any* cube that can present itself to intuition *can* do so *as a regular polyhedron*, although it can also present itself in other ways that have nothing to do with regularity (for example, *as a hexahedron*), even if no cube *ever* actually presents itself as such to consciousness.

The subject can also validate the positing of *the* cube as a regular polyhedron by actually presenting *a* cube to consciousness and somehow experiencing intuitively its regularity as a necessary property of *this* cube merely *as* a cube. By analyzing *this* experience, the subject can generalize; by becoming conscious that the particular cube it experiences is representative of the category of all cubes with respect to the relevant property, the subject *intuits* the cube (in general) as a regular polyhedron. This complex experience counts as the intuition of a *general* fact: *all* cubes are regular. *The intuitive experience of all is not the experience of every*, but the intuitive experience *of* a specimen *and* the reflexive experience *that* the particularities of the specimen are irrelevant to the property in question. This second, dependent act is called *generalization*.[39] The complete experience counts as the intuitive fulfillment of the validated anticipation “the cube as a regular polyhedron”.

Formal and Material Meaning The distinction between formal and material meaning for *judgments* can be easily drawn. Judgments (or assertions), understood as objective correlates of acts of judging (or asserting), sometimes also called propositions, have both a formal (syntactic) and a material (semantic) content. Consider the assertion (1) “this rose (which I have in my hands now) is red (a red thing)”. The terms “rose” and “red thing” denote different and definite concepts or categories, of flowers and of colored things, respectively. The assertion expresses the fact that a definite object of the first category (*this* rose here) belongs also to the second. By completely abstracting from the meaning of “red things” and “rose” as *particular* categories, i.e. their *material* meaning, and keeping in mind only the categorial nature of the terms, i.e. the fact that they denote categories or concepts, in short, their *formal* meaning, “red thing” and “rose” become category-names without definite denotation and can be substituted in language by symbols of adequate logical types – in our example, first-order concepts or categories of objects, say R_1 and R_2.

[39] Since Kant did not accept intuitions-that, only intuitions-of, he did not explain convincingly how constructions, which are always particulars, can have general validity. How, for example, the construction that brings to light the fact that the internal angles of a *particular* triangle add to two right angles can justify asserting this property of *all* triangles?

The assertion (1), stripped of all material meaning, reads now (2) "this *x*, which is an R_1, is also an R_2", "*x*" being the name of a generic object, a "something"; (2) expresses the formal content of (1). From a logical point of view, the difference between (1) and (2) is essentially one of scope; (1) is true of particular entities, the object in my hands and the categories "red" and "rose", (2) is not in itself true, but can be made true by giving the purely formal, materially empty symbols *x*, R_1 and R_2 adequate arbitrary material contents (interpretations). (2) is true of more situations than (1), but these situations are all expressed by *formally equivalent* assertions, they all have the same *logical form* expressed by (2). Two assertions have, independently of their logical value (true or false), the same logical form if they are *identical* if stripped of material sense. *Formal abstraction*, as Husserl conceives it is not a search for some hidden, supposedly true logical form, such as, for example, Russell's analysis of descriptions. For Husserl, logical form is superficial and appears as soon as the reference of non-logical terms is obliterated.[40]

Although *x*, R_1 and R_2 do not denote anything in particular, they are meant to stand for things of *definite* ontological types; *x* denotes individuals, R_1 and R_2, first-order concepts or, extensionally, categories of individuals. Not any concatenation of symbols, however, expresses a logically valid logical form; to do so, they must accord to a priori syntactic rules of combinations of logical types. A *valid* logical form is a form that obeys the a priori rules of logical grammar, the grammar of logical types (for instance, objects can fall under first-order concepts or belong to first-order categories, but not the converse). An assertion is *formally* (*syntactically*) *meaningful* if its logical form is a valid logical form. However, a formally meaningful assertion may not express a possible fact and, then, have a definite, although maybe unknown, logical value, for example, (3) "this pain is green". (3) and (1) have the same logical form (2), and are, then, formally equivalent, but (1) is, supposedly, true, and (3) meaningless. Assertions must be meaningful in still another sense to qualify as *proper judgments*, judgments capable, that is, of a *definit*e, but maybe unknown truth-value (either true or false).

Husserl calls this the *material* meaning of the assertion. Whereas "red" and "rose" can be attributed to the same object, i.e. they are materially compatible, "pain" and "green" cannot, they are materially incompatible, and this is not a matter of fact, but right. It is *a priori* true, or so Husserl thinks, that "pain" and "green" are incompatible (this is an example of a *phenomenological* synthetic a priori truth). Ontological types are submitted to a priori rules to the same extent that logical types are; judgments that conform to the a priori grammar of ontological types are *materially meaningful*. Judgments are meaningful simpliciter when they are both formally and materially meaningful, and only in this case they possess definite, although maybe unknown and effectively unknowable, truth-values. For Husserl, it befalls on formal logic the task of investigating both logical and *formal*-ontological catego-

[40] It is a task for formal logic, logical grammar in particular, to determine which terms are logical. The main feature of the logical being universality.

ries; a task, respectively, for formal apophantics and formal ontology.[41] *Material* ontological categories, such as “rose” and “red” have their a priori truths too, that *regional* ontologies must disclose. Husserl uses Cartesian terminology in this context to mark an important distinction. As he says in *LI VI* chap. 8 § 63, “the realm of meaning is […] much wider than that of intuition […]”. Although formal meaning prevents nonsense (*Unsinn*), it does not rule out the possibility of counter-sense (*Widersinn*), which only material meaning guarantees. An assertion with both formal and material meaning, exhibiting in Husserl’s terminology the *evidence of distinction*, is *in principle* capable of being true or false. This is the *definition* itself of the notion of possession in principle of a definite, maybe unknown, truth-value. When a truth-value is effectively determined based on an intuitive experience, Husserl says the assertion has been *clarified*. Often, the term “clear” is reserved for *true* in face of confirming evidence. In this sense, in the spirit of Descartes, distinction and clarity are indeed the characteristic notes of intuitive truth.

Now, according to the expressivity thesis, intentional meanings can be linguistically expressed, that is, expressed in judgments. In this sense, positing acts come with a theory, i.e. a collection of assertions that are *true of the object posited*. This “intentional theory”, let us call it so, is posited concomitantly with its object, the object which the theory is true of. The logical consistency of this theory, internal and external, is the necessary and sufficient condition of existence of the object *as posited*. Suppose, for example, that an empirical object is perceived. The perception justifies a series of assertions about the object perceived, including that it is an object of perception (not illusion or hallucination); these assertions constitute the intentional meaning of the perceptual experience linguistically expressed. The object exists as an object of perception, and then as an object of the empirical world, provided these assertions are consistent with one another as well as the ideally complete system of perceptions. Objects of perception exist insofar as their positing remains consistent with the whole system of perceptions. In a formula, *empirical reality is the maximal system of coherent objectively valid perceptions in principle experienceable*. There is no room here for the Kantian distinction between noumenal and phenomenal realities. *The phenomenological noumenal is only the ideal limit of the phenomenal*.

The intentional theory, devoid of material meaning or, in Husserlian terms, *formally abstracted*, expresses the formal meaning attached to the object posited. Consider, for example, the already discussed semi-intuitive positing of the closed domain of finite cardinal numbers. The theory attached to this positing tells *what a finite cardinal number is as posited in the experience*. As we have seen, cardinal numbers are ideal abstract quantitative forms, answers to the question “how many?” However, the mathematician is not in general interested in what numbers are, but in how they *behave operationally*, their operational properties, which are formal in the sense that materially different objects can also display them. From a mathematical perspective, numerical domains are operational domains. The object “finite cardinal number” can be adequately captured *mathematically* by what is called second-order Dedekind-Peano theory. Numbers have both *material* and *formal* aspects; the first

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[41] Formal ontological categories are those that apply to objects merely as such, without further material specifications. See *Logical Investigation VI*, chap. 8.

has to do with numbers as the objects they are, namely, ideal abstract quantitative forms; the second, with numbers merely as bearers of properties or attributes, in particular operational properties, that they share with objects of different natures. It is methodologically relevant that *materially* different things can behave in *formally* identical or similar ways, for this already throws some light on the wide applicability of mathematics. It also explains why mathematical objects can be so easily *reconceptualized* by leaving behind material meaning for which mathematics has no use. For the sake of phenomenological purity, however, one must distinguish what a thing *is* (the material meaning) from how it *behaves* operationally (which is part of its formal meaning), although we may be interested, as in mathematics, only in the properties the object has simply as an object upon which one operates.

The intentional theory associated with the positing of the realm of finite cardinal numbers is an *interpreted* theory to the extent that it expresses aspects of this domain as originally given in the positing experience. The meaning expressed by Dedekind-Peano theory is material only to the extent that it is attached as intentional meaning to a definite entity, *the* natural numbers as intended in the positing experience. An interpreted theory expresses material meaning, or part of the material meaning of the object to which it refers insofar as it *refers* to this object. As soon as the theory is formally abstracted, it no longer refers to anything determinate. Nonetheless, it *still* expresses something, namely, the formal meaning attached to the object to which it originally referred. But not exclusively, since the formally abstracted theory expresses *also* the formal meaning attached to *any* object that happens to have the same formal meaning of the object originally associated with it. Formal meaning materializes as material meaning when associated with a *specific* object either intuitively or only conceptually determined. In the example above, the domain of natural numbers given in semi-intuition.

I will come back to this issue in details later, but an important point can be made here. The fact that different intentional objects can be formally similar in relevant ways justifies a powerful method of mathematical investigation. To the extent that *only* the formal is of interest to mathematics, one may formally explore an object by exploring *another* (let us call it the *avatar*) that happens to share enough formal properties with the original object, and then transferring to this object what was disclosed in the investigation of the avatar, or part of it anyway. The method can be very useful if the avatar is cognitively more accessible than the original object. *Provided we are interested only on the formal aspects of some object* (i.e. we are *not* interested in *this* object particularly, only on its properties regardless of which object presents them) mathematics can be a useful method of investigation of this object by providing avatars with a sufficient degree of formal similarity with it. In this, essentially, resides the reason for the wide applicability of mathematics in daily life and science. Formal truths are materialized by receiving a material content, i.e. by interpretation.

By investigating an object, intuitively if the object is intuitively given or by logically deriving the consequences of the meaning intentionally attached to it, no matter how the object is given, we develop the theory of the object, whose assertions are true, and a fortiori meaningful of the object in question. These assertions are material in the sense that they are properties of a *specific* object, but also *formal* in the sense that they can, by being formally abstracted and reinterpreted (i.e. given another material content), be true of other objects as well. Essentially, material and formal truths are truths displayed by, respectively, interpreted and non-interpreted true assertions. A formal truth is a formally abstracted material truth. A material truth is an interpreted formal truth. We can define formal and material knowledge and formal and material sciences likewise. In short, material involves specificity of denotation; formal does not. Mathematics is a formal science to the extent that it really does not matter what it is talking *about*, only *what* it is saying about this object that is also true of other objects. A formal science does not care for any particular domain of objects, only for the properties that a domain has but that different domains may also have. Material sciences, on the other hand, such as, say, zoology, cannot ever lose sight of the specificity of their object, animals in this case; not out of principle, but because it must constantly return to them for insights. If a domain of the empirical world could offer itself *completely* in original giveness, for example, in intuition, the investigation of this domain could be carried out, as in formal sciences, by logically unpacking the meaning associated with it in the original positing act. If the meaning "animal", for example, could be completely unveiled, zoology could be formalized, i.e. formally abstracted (and eventually axiomatized). Zoologists needed no longer, ever, look at animals; they could confine themselves to their offices, examining all the logical consequences of the supposedly completely disclosed meaning "animal". Better, they could turn to *any* theory that happened to be logically equivalent to the original theory; the conclusions arrived at by investigating this "avatar theory" could be immediately transferred to zoology. Only a change of objective focus would be required.

Intuition This is probably the most misunderstood aspect of Husserl's phenomenology, when it should be one of the easiest to grasp. In a little book entitled *Intuición y Razon,*[42] particularly §2.4, Mario Bunge endorses almost every possible misconception concerning Husserl's notion of intuition, particularly the intuition of essences (*Wesensschau*). Here are some: Husserl is an old-fashioned essentialist (together with Plato and Aristotle); essential intuition requires a special faculty of the *mind*, the intellect must be capable of performing certain "purifying" operations in order to intuit certain types of objects, essences in particular; the knowledge of essences is *independent* of factual knowledge; essential intuition provides synthetic a priori knowledge, which is immune to experience even when referring to the empirical world; intuitive knowledge is apodictic, i.e. necessarily true; intuition is a sort of contemplation. Bunge was not the first to bash Husserl's notion of intuition; Frege in his notoriously unfair critique of *Philosophy of Arithmetic* (Frege 1984) preceded him. Bunge and Frege either have not read Husserl at all or have, but with all sorts of prejudices and preconceptions in mind.

Anyone capable of distinguishing between seeing a person and designating her in absentia by a name, or see the difference between perceiving a thing and becoming

[42] Bunge 1986.

acquainted with it through descriptions, and the difference in epistemic accessibility that these differences make can appreciate Husserl's conception of intuition. It is nothing more nor less than the generalization to *all* intentional acts of the notion of sensorial perception. Suppose one looks at a red flag and sees the *redness of the flag*, or *that* the flag is red. Then, Husserl claims, one has seen, respectively, a *dependent moment* (or aspect) of the flag and a *state-of-affairs* of the world. Both are, obviously, perceptual experiences. One actually *sees* the moment and the state-of-affairs; both are *empirical* entities. But the seeing is not in either case a passive experience, for the sensorial stimulus on which they are grounded, the red flag, are rigorously the same. One can look at a red flag and see a piece of cloth, or a flag, or the redness of the flag, or that the flag is red. Each is a different intentional experience; each elaborates the sensorial stimulus or hyletic data, the red flag merely as a complex of sensations, differently. Now, since the same sensorial matter can elicit *different* perceptual experiences, there must be non-sensorial components in the perception of things like aspects and state-of-affairs. For Husserl, the difference lies in intentional action. The hyletic sensorial material is intentionally elaborated in different ways to produce different perceptual experiences. Perceiving is not a passive experience (for precisely this reason, intentional experiences in general are called *acts*). In a formula, *perceiving is already a form of thinking*. What is true of perception is true for intuitions in general, for intuiting *is* perceiving.

As previously clarified, intentional acts have a typical polarity, the subjective pole and the objective pole; the former undergoes the experience, the latter is the intentional content of the experience. The intentional object, with its characteristic sense of being is not in general a mental entity (unless a mental entity, a longing or a pain, for example, is meant), a copy, or a representation of something "out there". In perception, which is a particular intentional act,[43] the intentional object is *that* which is perceived *itself*, not a mental copy of it. In our examples, the cloth, the flag, the redness of the flag, or the fact that the flag is red, all entities of empirical reality; they occupy a place in space, can be destroyed by fire, had an origin and probably will have an end.

Frege grossly misinterprets Husserl's theory of abstraction, which requires that an intentional act be performed for the intuition of the redness of the flag, along naturalist (in this case psychologist) lines.[44] Frege believes that, for Husserl, abstraction is a mental operation acting on mental representations, the famous "chemistry" he ridicules. This is a serious misunderstanding for abstraction has nothing to do with mental representations. It is instead an adjustment of intentional focus. It is a way of seeing in which an abstract (i.e. non-independent) aspect of a whole, not the whole, occupies the intentional focus. The matter of the intentional action is the whole *itself*, not a mental representation of it; the action is intentional, not real.

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[43] For Husserl, the infra-conscious levels of perception, closer to the sensorial given, are not, for not being fully conscious, strictly speaking intentional. But when higher-level intentional acts such as abstracting or judging are involved, perception is a truly intentional act.

[44] In her "Frege's Attack on Husserl and Cantor" (Hill and Rosado Haddock 2000, pp. 95–107), Claire Hill argues that, in fact, through Husserl, Frege is in fact aiming at Cantor.

Anything that is "bodily" present, anything that the subject is conscious of as standing before it, not indirectly meant or intended, is intuited, for intuition is nothing more than presentification. The intuited object, however, as already stressed, is not immanent in the *real* act; that is, it presents itself, but not as a *real* component of the act. There are different intuitive experiences, and they can be iterated. I have just mentioned abstraction, there is also ideation. One may look at a red flag and intuit in it not simply the redness of this particular flag, but redness in general. "The flag is red", one may say upon looking at a red flag, not meaning that the flag has a particular shade of red but that it is red in a generic sense. In other words, the flag can *presentify*, by a convenient "way of seeing", redness *in general* or the *idea* of red. In this experience, the red flag, or better, the redness of this particular flag stands for the idea "red in general". Redness, of course, is not an empirical, but an *ideal* object, whose presentation requires, first, the *sensorial perception* of a red object, in actual perception or in imagination; then, by *abstraction*, the perception of the particular red of the object; and finally, by *ideation*, the idea of redness in general. Perception, abstraction and ideation in sequence, all forms of intuition; in each act an object is given which serves as the matter that the following act intentionally elaborates into its own object.[45]

Husserl believes that intuitions are the most important form of intentional experiences and that they play a fundamental role in knowledge, including mathematics. But one must carefully distinguish Husserl's from Brouwer's intuitionism. From the phenomenological perspective, Brouwer's intuitionism is a form of psychologism (and then naturalism) that misinterprets intuitions, mathematical intuitions in particular, as mental experiences. For intuitionists, the object of intuition is immanent in the experience of intuition; it is a real component of it, a mental object. For Husserl, contrariwise, the objects of mathematical intuition are *mathematical* objects proper; ideal, non-spatial, non-temporal, objective, transcendent objects, which however require intentional acts such as abstraction and ideation to come into being. Here is an example. One can draw a triangle on the blackboard and see it as a roughly triangular *physical* object. One can also look at it and see its (roughly) triangular *form*. This requires abstraction, whose intentional object is the visible triangular form of the object on the board. Proto-geometrical forms, like the *actual* forms of physical objects, "rough" by comparison with ideal geometrical forms, are abstract objects.

Now, different objects can have the same proto-geometrical form. By "same" form, I mean, in this case, that the objects can be (more or less) exactly superposed. Sameness of forms does not require that one actually moves and superimposes the objects, only that this can in principle be done. Sameness is not, in this case, identity; as moments of physical objects, proto-geometrical forms are different if the objects in which they are instantiated are different. For this reason, I call them

[45] The intuition of the ideal, however, does not require perception necessarily; we could have imagined a red flag instead of perceiving one. Imagination is also a form of presentification and can in the intuition of the ideal substitute perception proper.

proto-geometrical; they are not yet mathematical entities proper; in fact, they are still empirical objects.

Now, by an act of *ideation* the subject can see all equal forms as identical, i.e. mere manifestations of the *same* ideal form. The triangular form of the object on the board is now merely the instantiation of an idea. The idealized object is no longer an object of this world, although its instantiations are. A mathematician would say that the ideal form is the class of equivalence of all equal forms, and would maybe represent it as a set. But this is *only* a representation, nothing more. Seen in phenomenological clarity, the ideal form is an object that has *all and only* those properties that all equal forms have in common (which does not include, for example, spatial location). By attributing to a *particular* form (an instantiation of the ideal form) only the attributes that it shares with *all* forms similar to it, one is treating this particular form *as* the ideal form it instantiates. In a sense, ideation is the act of considering *only* what is generic in the particular.

Plato would say that similar individual forms partake in the ideal form; mathematicians that the former belong to the latter. This is irrelevant, if one sees the essential, that ideation requires intentional action. There would be no problem in saying, with Plato, that ideas exist sub species aeternitatis in a world of their own, but this would be only a way of speaking that would not, under the action of the epoché, have any bearing on reality *outside* the intentional realm. Noetically, ideas are ego-dependent, even if noematically they may be conceived as ego-independent. The meaning "realm of beings existing in and for themselves" is, as any other (for meaning always emanates from subjectivity), an intentional meaning.[46]

However, the ideal form of the triangular figure on the board is not yet a mathematical idea. To get one, the ego must experience the form on the board *as a mathematical triangle*.[47] The rough triangular form must be intentionally *exactified* and *seen* as a triangle proper. This requires a specific intentional act. It may involve, on the noetic dimension, a mental operation, if the ego happens to be a mind, or a certain disposition, a common "way of seeing" shared by the collectivity of individuals that play the role of the intentional ego. A physical triangle can be taken as a mathematical triangle only if it is seen "as if" it really satisfied the mathematical definition of triangle, even though it does it only approximately. Properties that a particular physical triangle possesses only approximately can be taken as properties of mathematical triangles *in general* only if they enjoy a kind of robustness relative to arbitrary triangularity-preserving transformations of the physical triangle. In particular, a certain class of properties of one *particular* triangular figure can be taken as properties of a *particular* mathematical triangle, the idealized (exactified) form of *this*

[46] This has an important consequence; phenomenology can safeguard a Platonist *way of seeing* without embracing Platonism as a *theory*. Some phenomenologists have called this non-naïve Platonism. I call it, as I already did, Platonism between brackets, "Platonism".

[47] One takes an object *as* a mathematical object when one attributes to this object only the properties of the mathematical object one takes it to be (even if it has these properties only approximately). But I am not particularly concerned with this question; I only want to emphasize the fact that whatever "taken as" is, it is an *intentional* act.

physical triangle, if these properties are robust under arbitrary (roughly) *congruent* transformations of the original figure. A property is robust under a series of transformations (triangularity-preserving or congruent transformations) if all the physical objects in the series possess *to a reasonable degree of approximation* the same property.[48] The verification that the properties of a physical triangle are robust does not require that all the transformations be actually performed; it is enough for the ego to *perceive* that pertinent relations among the elements of the triangle on which the property in question depends will *necessarily* be preserved. For example, one may *intuit* that the internal angles of any *mathematical* triangle sum two right angles (*mathematical* intuition) by *perceiving* (*physical* intuition) that this is (approximately) true for a *particular* physical triangle and, by arbitrarily transforming in *imagination* the given triangle into another triangle (a triangularity-preserving transformation), that this will necessarily remain (approximately) true. Many subsidiary acts are involved in this and similar acts of mathematical intuition based on sensorial perception: a (finite) series of sensorial (visual) perceptions and the intuition, based on these experiences, of a *fact*, namely, that the relevant property is necessarily robust along the entire series of transformations.

The mathematical triangle is neither a physical nor a mental object; rather, it is a non-real, transcendent and objectively existing entity. Abstraction, idealization and similar acts are not mental operations on mental *representations*, as Frege thought, but intentional acts, which may involve, on the noetic side, mental operations, but whose objects are extra-mental entities, such as forms and ideas.[49] It is not a task for phenomenology to investigate the *real* dynamics of constituting acts; this belongs to empirical science, psychology or cognitive science. The philosophically relevant fact is that the ego can intuit a geometrical triangle and geometrical properties of this triangle, or triangles in general, on the basis of a roughly triangular figure in actual perception or imagination. This explains why diagrams and other forms of graphic representation are so efficient instruments of mathematical reasoning.

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[48] As is clear, form-preserving (congruent) transformations constitute a proper subclass of triangularity-preservation transformations.

[49] Aristotelian empiricists might approach the matter from a different perspective. Suppose a denotes a triangular figure (a physical triangle) and P a property of a. By definition, P belongs to *a as a mathematical triangle* if for *any physical object x*, if x has P, then x is also triangular. Now, one can define a mathematical triangle as the "equivalence class" of all (roughly) congruent triangular figures. Any P that belongs to an element of the equivalence class as a triangle belongs also to all elements of the class as triangles, for they are all (roughly) congruent. The mathematical triangle that this class represents has all and only the properties that any element of the class has as a triangle. The empiricist can then take the mathematical triangle as only a *façon de parler*. But this would be a falsification of the mathematical experience. Mathematics posits ideal objects *as ideal objects*, not merely as ways of referring generically to physical objects. Neither Plato nor Aristotle are completely right or completely wrong. Geometrical forms are (with Plato) ideal, but (against Plato) they are not ego-independent. Geometrical forms are abstract aspects of actually or possibly existing physical objects (with Aristotle), but (against Aristotle) ideal forms have a sense of existence that is not that of real entities, although they may be or are, as in the geometrical case, constituted in acts whose matter are real objects.

Besides objects, relations among objects can also be intuited. Husserl calls intuitions of the categorial, relational and syntagorematic elements of structured complexes of objects *categorial intuition*. When one sees the book *on* the table, one does not only see the table and see the book, but the book *on* the table. The categorial component of the state-of-affairs, expressed by the preposition "on" belongs, as a component, to the perception. When one perceives a book *and* a table, one does not only perceive a book and perceive a table, but the book *and* the table. The conjunction is also an object of perception. Analogously, one can perceive the book and the table as a *single* object against the background of their environment; this is the perception of the collective {table, book}, a higher-level entity vis-à-vis its elements but still an object of physical reality and visual perception. One can perceive *the book on the table* without perceiving *that* the book is on the table and asserting it, i.e. the higher-level *judicative act* may be missing. By performing it, the ego elevates itself to a higher level of involvement with the world. Judging with *clarity* is judging on the basis of the intuition of the *content* of the judgment; the clear judgment "the book is on the table" requires the perception of the book on the table, but the judgment posits a different object, namely, the *state-of-affairs that* the book is on the table (the ego can also judge without clarity, i.e. without the accompanying intuition of the content of the judgment, but insofar as the judgment is meaningful, it is a *distinct* judgment whose object is a state-of-affairs *in principle intuitable*). In all these acts, the perceptions of the book and the table, the book on the table, the collection {table, book} or the state-of-affairs that the book is on the table, the hyletic, or purely sensorial matter is always the same, a book, a table; the difference is the categorial component, … and ….; … is on …; {…, …}, that (…. is on …), which are also objects of intuition.[50]

Essences can also be intuited. But we must be clear about what essences and essential intuition are to avoid misunderstanding. Essentialism is the metaphysical view for which objects have certain properties, the so-called essential properties, which they must necessarily have in order to be what they are. Transcendental phenomenology operates a change of perspective, from what objects are (in themselves) to how objects are meant to be (for the intentional ego); so, the essence of an object is not in it, *metaphysically* speaking, but in how it is conceived to be – the essence is *phenomenologically* in the object. Transcendental phenomenology cannot inquire science or a supposedly independent reality to know in what the metaphysical essence of an object consists, epoché forbids it. Phenomenology has access only to the phenomenon, the objects *as meant*, and can only ask which properties the object must have to be as it is *meant to be*. To answer this question the phenomenologist has only to inquire the phenomenon itself. In a somewhat risky, but valid parallel, to ask for the *phenomenological essence* of an object is like asking for the meaning of a word. What in the intentional meaning associated to this object as it appears to the ego is *necessarily* required for it to appear as *this* object (or an object of *its type*) in any possible appearing of *it*? Asking for the phenomenological essence of a thing,

[50] For detailed analyses of the relations between judging and experiencing (perceiving, in particular) see Husserl 1973.

in short, is asking for what it means to be this thing.[51] Phenomenology has something to say about *phenomenological* essences, but nothing about *metaphysical* essences. Hence, phenomenology is not essentialist in the traditional metaphysical sense of the term. Those, like Bunge, who believed that Husserl was a metaphysical essentialist, have simply not understood the meaning and scope of phenomenology.

We can understand now how, according to Husserl, essences can be intuited by what he calls *eidetic* or *essential intuition*. The process is called *imaginative variation*. It starts by conjuring the object on whose essence one is interested, either actually or in imagination. It does not matter, for imagination is also a form of presentification. The object will present itself with a certain sense, which we can (expressivity thesis) render as a set of assertions true of the object. Imaginative variation proper starts now; for each assertion *p* true of the object, the ego must *imagine* a presentation of the object such that *not-p*. By so doing, the ego is forcing, so to speak, the object to present itself with a different meaning. If this cannot be done, due not to debilities in the power of imagination of the ego, but to objective impossibility. If the "I cannot" is irremovable, then the object *must* be such that *p* to *present* itself as the object it is. In other words, *p* expresses an aspect of the essential core of meaning of the object. By going through assertions true of the object *in the original presentation*, the ego can eventually disclosure the phenomenological essence of the object. There is nothing particularly mysterious or metaphysically compromising in the process.

In general, the process has a different dynamics. The ego submits the object in imagination to arbitrary variations, looking for lines of tension and resistance. The ego tests, in imagination, different possibilities of variation in the search for the limits of variability. In a sense, the process resembles proofs by contradiction; one tries to conjure a counter-factual presentation of a given object and sees if one succeeds. If one does, that which the counter-factual presentation cancels is not an essential aspect of the object in question. Once the field of variability of the object *as meant* is, so to speak, mapped, its phenomenological essence comes out clearly. In short, essential intuition consists in verifying what in a *particular* presentation of an object has, by necessity, *universal* validity, which must appear in *any* presentation of the object.

For example, imagine a color, any color. The conjured color impression certainly has spatial extension. Must it? Can you make it extensionless in imagination? You can change the impression in imagination in many ways, by changing its hue, intensity or luminosity. All this is possible – which, incidentally, indicates that the *particular* hue, intensity and luminosity of the original color impression are not essential aspects of the *eidos* "color". Colors can come in different hues, intensities and luminosities. But no matter how you try, you will not be able to conjure a color impression

[51] This is why it is so easy to change the question as to the meaning of an *object* into the question as to meaning of the *word* that denotes it, as analytic philosophers do. But whereas phenomenologists only need to inquire the phenomenon, analytic philosophers must step outside the intentional experience and inquire linguistic usage. See da Silva 2016b.

that does *not* occupy a region of space (or, for that matter, has *no* hue, intensity or luminosity). Therefore, extension is a *phenomenological* or *intentional* essential property of color impressions.

The truth "no color without a colored extension", then, imposes itself with apodictic evidence. For Husserl, since truths of this type are not logical laws or instances of logical laws, nor empirical truths, they are, he claims, *synthetic a priori*.[52] Analytic philosophers, however, who due to empiricist prejudices abhor the synthetic a priori, believe that truths of this type are due exclusively to the meaning of words, being, then, analytic. The problem with this approach is that meaning has to do with linguistic usage and, as such, it is factual and does not touch essential matters. A word just happens to have a certain meaning, which can change in time or depend on the context. Necessity, however, cuts deeper than usage; it has to do with intentional constitution. Linguistic practices only acquire a dimension of necessity – for example, one cannot use the word "color" for a property of anything extensionless – if based on relevant essential attributes of things named – the eidos "color" in our example. Otherwise, it is only a matter of contingent convention.

Names are indicators that, according to Husserl, have meaning too, which differentiates them from mere signs.[53] The meaning of a name allows it to refer to an object; the name refers *because* it is meaningful and *by means of* its meaning.[54] Like anything else, names are infused with meaning in intentional acts. However, the meaning of the *name* does not necessarily coincide with the intentional meaning attached to the *object* itself. The meaning of a name has the task of allowing the name to indicate, denote or singularize its object, nothing more; the intentional meaning of an object tells what the object *is* (or is intended to be). Essential intuition, as understood phenomenologically, unveils the phenomenological essence of things, which may have little or nothing to do with the meaning of their names.

Another Husserlian example of a synthetic a priori truth, still related to colors, is the following: (1) "no two different colors can cover the same extension all over simultaneously". Again, this truth expresses an essential necessity unveiled in essential intuition. It is immaterial whether this is so because of the peculiarities of how humans perceive colors. We *could*, after all, *see* distinctively two different colors, one superposing the other, just as we can *hear* all the notes of a chord distinctively. But this does not make (1) an empirical truth, since its justification does not require experience. Analytic philosophers think that (1) is an analytic truth, resting exclusively on the meaning of words; it is, supposedly, a "grammatical rule" concerning the meaning of "color", telling us how to use this word properly. The fact, however, is that we do not experience *impossibility* by trying to use words with meanings different from those they have; we may not be understood, but we *can* do it. We cannot, however, experience, not even in imagination, an extension covered

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52 See da Silva 2016b.

53 See Husserl's 1st *Logical Investigation*, entitled "Expression and Meaning".

54 We must allow also for *signs* that denote directly by convention and, I believe, *indexicals*, such as "this", "that" or "I", which have, for Husserl, a meaning that, however, is only completely determined in a context of use.

all over simultaneously with two different colors. Reducing phenomenological necessity to linguistic usage misses an important aspect of the phenomenon: *necessity*.

The notion of essential intuition has an important place in Husserl's approach to mathematics and it will be relevant in mine as well. An example is in order. I have already shown that numbers are abstract forms that can be intuited by considering given collections of objects, no matter which objects they are or how they present themselves to us. Cardinal numbers, Husserl thinks, in accordance with Plato, are collections of undifferentiated units. As already discussed, intuiting numbers requires abstraction. Conceiving different objects *as units* is not the solvent Frege ridiculed; it does not change them in the least, they are still different objects, only considered *under the concept of* "*something*". In counting, objects are *conceived* as units (things whatsoever) just as people in a plane are conceived as passengers, and treated likewise (only more decently than passengers).

Now, one can vary quantitatively in imagination any given collection of things and, by abstraction, intuit different numbers. But not all; from some point on, when they become sufficiently large, collections can no longer be clearly differentiated and adequately intuited. But, as I have already said, the ego can still intuit *that* it can in principle enlarge quantitatively a collection arbitrarily; Husserl, as I said before, called such "I can in principle go on forever" the idealization of the "and so on".[55] A *generative process* comes then clearly to consciousness, by which numbers can *in principle* be intuited (every time I say "in principle" idealization is at work). The ego can now inquire the process intuitively presented to it. What does it find therein? Many things, for example, that (1) the process has an inferior limit, when the collection undergoing imaginative variation has no elements, no further element can be removed; (2) the minimum quantum of variation is one unit; (3) the process can go on "forever", by augmenting the collection an unit at a time indefinitely. A new object emerges in intuition by reflecting on this generative process, namely, *the series of quantitative forms* (*numbers*). The ego can also reflect on the meaning it associates to "forever". It may become clear to it in reflection that "forever" means "provided one could, from any given point in the series, in finitely many steps, come back to the initial point by subtracting units one at a time; one can undo what one has done". This intentional meaning attached to the object "series of finite cardinal numbers" is faithfully and fully expressed by the well-known second order axioms of Dedekind-Peano (the last remark would come out as the axiom of induction). Or, alternatively, a *relation* among finite cardinal numbers, induced by the process of number generation, is intuited; together with the basic truths about it expressed as a system of axioms.

However, the intuition of either the process of number generation or the relation among numbers is not the intuition of the *concept* of finite cardinal number. To intuit this concept, it suffices to vary a number in imagination and verify that all the legitimate variations are still quantitative forms. Therefore, numbers are *essentially* quantitative forms. An important thing must be kept in mind; *one can intuit a*

[55] See Husserl 1969, §74.

concept without intuiting every one of the objects that fall under it. Intuiting a concept is not tantamount to intuiting its extension, but its intension. Exploring imaginatively the extension is only a form of grabbing intension. As far as the generative process is concerned, one may disclose sufficient intentional meaning by seeing how the process operates on a few instances only. By reflecting on the generative principle (reflection is an intentional act whose matter is another act) the ego can disclose those aspects of the intentional meaning attached to the concept of number that have to do with how numbers relate to one another operationally.

However, and this must also be kept in mind, even when no such generative process is available, imaginative variation and conceptual intuition are still possible. Varying *freely* in imagination an exemplar of a concept may be sufficient for the ego to intuit, without having to go through each and every exemplar, *patterns* according to which the variation *can in principle* go on indefinitely. Reflection of these patterns allows the ego to realize which traits the concept can and which it cannot fail to display, thus bringing to consciousness the concept itself and its essential characteristic notes.

Intermezzo Let us pause for a moment to reflect. Those who subscribe to philosophical naturalism and consequently refuse to take the phenomenological standpoint may mistake my analyses of intuition for an exercise in a priori pure psychology or an attempt at disclosing hidden motifs of actual historical developments. They, however, would be wrong; phenomenological analyses, even when focused on the noetic dimension of the intentional experience, are not psychological, for the ego, the locus of noeses, is not necessarily a mind. Moreover, transcendental history, the chronicle of the intentional acts involved in the constitution of intentional objects, is not factual history. Mathematical intuition is no more mysterious a phenomenon than sensorial perception; to perceive a number or a concept is not essentially different from perceiving beauty in a beautiful flower. Clarifying the concept of mathematical intuition, or mathematical positing experiences in general have obvious philosophical interest; they offer a standpoint from where to consider certain metaphysical theses critically. Phenomenology has here a therapeutic role.

Phenomenological constitutive analyses can be misconstrued in at least two ways. One, by taking the ego as the empirical mind and identifying constitutive acts to psychological processes of genesis of mental entities. Another, by reverting ontological priorities and seeing *constituted* noemata (objects and their meanings) as ego-independent and constitutive processes epistemologically as acts by which the ego grasps ego-independent entities and facts. The first is the error of Brouwerian intuitionism; the second, that of Platonism.

Some phenomenologists, who find the transcendental turn unappealing and prefer the first period of Husserl's philosophy, tend to interpret genetic analyses in epistemological terms. For them, one can accept the realist presupposition that mathematical entities and facts are completely ego-independent, but that in order for the ego to be conscious of them, it must go through "constitutive" experiences; thus the object-out-there becomes object-for-the-ego. Some have called this non-naïve Platonism, presupposing that naïve Platonism does not have at its disposition the phenomenological notion of intuition to account for the ego's access to ego-independent realms of being. These phenomenologists believe that Husserl's notion of intuition offers Platonism a better, more sophisticated notion of intuition.[56]

Gödel, a well-known Platonist influenced by Husserl, probably endorsed so-called non-naïve Platonism or at least has been so interpreted. He famously believed in a sort of conceptual intuition that would give us access to a supposedly independent concept of set and the fundamental truths pertaining to it. He also believed that set-intuition would eventually help us to make our minds up as to the truth of the axiom of choice (which he believed to be true) and the continuum hypothesis (which he thought to be false). He first interpreted this notion naturistically as a superior form of perception having its locus in the brain close to the centers of language. Later, after encountering Husserl's philosophy, he thought that it could provide him a decent philosophical context where to interpret his notion of conceptual intuition. But he was never very clear about details.[57]

Maybe the weakest spot of mathematical realism (or Platonism) is the problem of access – how can one access a supposedly independent mathematical realm of being? I will not go into the details of the access problem here and the many ways it was approached in the philosophy of mathematics; the history is well known. Realists believe that facing this problem is the price to pay for enjoying the benefices of realism. But this is a false dilemma, one can have a philosophically more sophisticated version of Platonism, Platonism between brackets as I have called it, *without* any problem of access. But for this we need *transcendental* phenomenology, in a *naturalistic* version of phenomenology the access problem does not go away and naturalized phenomenology does not seem to fare much better than "naïve Platonism" in dealing with it.

The problem of access can be definitively solved only by recognizing that there is no gap between the object of knowledge and the knowing subject. Rather, the object and the subject are united from the very beginning; in transcendental phenomenology, there is no object that is not an object-for-the-ego and no ego that is not an object-intending-ego. Transcendental phenomenology subtracts *nothing* from the *meaning* of an object that is *conceived as* ego-independent; it however sees ego-independence as an intentional attribute emanating from a meaning-bestowing ego. *In transcendental phenomenology, the problem of access dissolves, giving place to the problem of intentional constitution*.

Although Platonism and intuitionism contain elements of truth, they are one-sided perspectives. Intuitionism only sees the noetic, the real dimension of constitution, incorrectly interpreting intentional objects as objects immanent to the noeses in which they are constituted; Platonism only sees the noematic, wrongly interpreting noemata as ego-independent entities. Only transcendental phenomenology, by seeing both sides and their intimate connection correctly can offer a way of

[56] According to Dagfinn FØllesdal, for example, Husserl is a realist in ontology and an idealist in epistemology. See, for instance, his introduction to Gödel 1961 (Gödel 1995, p. 372).

[57] See da Silva 2005.

surpassing the one-sidedness of naturalist philosophies: noemata are correlate and do not exist independently of noeses, but are not real constituents of them.

Empty Intending Intuitions are, by definition, acts of presentification, but not all positing acts are intuitive, there are also purely intentional acts in which objects are meant but not "bodily" present. Whereas the typical intuitive act has the form "*this x* such that $M(x)$", where *this x* stands for something the ego experiences as actually there, and $M(x)$ expresses the intentional meaning attached to x, the typical purely intentional act has the form "let x be such that $M(x)$", where x stands for something meant but absent. Objects can also be named concomitantly with their intentional positing: "*this a* such that $M(a)$" or "let a be such that $M(a)$", where "a" is the name of the object intended.

Unlike intuitive experiences, in empty intending the object is not *presentified* as existing, only *intended* as existing. Either type of experience leaves open the possibility of doubt and cancelation of the positing. The positing of an intuitively given or emptily intended object can at any moment be either suspended by doubt or nullified by further experiences. Even if the positing act has the *character* of certainty, it, together with its character, can be cancelled or nullified, in total or in part: "I was certain of it, but then I realized that is was not so". The character of certainty is not a psychic epiphenomenon of the act but a modalization of the positing. Certainty, possibility, probability are modes of intentional positing.

As already discussed in the case of perceptions, which are intuitive acts, intuitions can only be checked against other intuitions, being thus validated or not. Further acts of perception, for example, can disclose inconsistencies in the meaning intentionally attached to the object of a previous act of perception; no object exists which has both A and *not-A*, for any property A.[58] Inconsistencies immediately cancel the positing and the intentional object "vanishes". In short, things can come in and out of (intentional) existence. The defining character of a *transcendent* object, such as empirical objects, is that they can *always* present new aspects and so the possibility is constantly open that they can "vanish", no matter how many times their perception was validated. The existence of the posited object, either experienced or only intended, can only be maintained insofar as the intentional meaning remains consistent and the positing valid. This, I claim, is the true sense of Poincare or Hilbert's criterion of existence in mathematics: "to exist is to be free from contradiction". The concept of existence alluded to in this criterion is, of course, that of *intentional existence*, namely, *that which exists by being meant to exist* (but which *really* exists if, and only if, its intentional meaning is consistent, both internally and externally).[59]

[58] Not as a matter of fact, but of principle. Self-consistency is a necessary criterion of existence – nothing exists that can support contradictory attributions; this is part of the *meaning* of existence, any type of existence.

[59] If the positing of either the object a or the object b is consistent, internally and externally, but that of *both a and b* is not, the ego is free to posit either a or b, but not both. The *extension* of the intentional meaning of the domain by the introduction of, say, a into it is valid if consistency is maintained, and only until it is maintained. Should an inconsistency follow from introducing a into the

Intentional existence is directly connected with the possibility *in principle* of *adequate intuition*, where by "adequate" one means intuition in full clarity of the object precisely as meant – the adequate experience cannot be "improved". For example, the memory of an empirical object is a form of intuition, but inadequate vis-à-vis actual perception. The qualification "in principle" requires explanation. To say that the ego can in principle intuit an object means simply that the intuitive experience is not an a priori impossibility, given that no inconsistency in the positing of the object is *manifest*. However, it may happen that inconsistencies are hidden and objects previously meant as capable of manifesting themselves adequately in intuition eventually reveal themselves as incapable of so doing, vanishing consequently out of existence.[60]

Recall that intentional meaning can be consistent or inconsistent in two different ways, internally and externally. If it is internally consistent, no explicit contradiction can be logically derived from the assertions expressing intentional meaning; if it is externally consistent, the positing of the object is consistent with the meaning of the ontological category to which it is meant to belong. The meaning attached to the object must be logically consistent with the meaning attached to objects of the same type. For example, the positing is not valid which posits an ethical value as colored. Goodness, for instance, cannot *ever* present itself to consciousness as being, say, green (and, as already explained, this is not simply a matter of what "green" means).

A valid positing can bring into existence things that did not exist until then. Intentional positing can be, in this sense, creative. One example are *creative definitions* in mathematics. Suppose, for instance, straight lines in geometrical space intuitively presented to consciousness through the series of intentional acts required for this. One can perceive that they stand in different spatial relations with respect to one another, parallelism in particular. It may then occur to the ego that *there is something* that all parallel lines have in common, and call this their common *direction*. A new entity is thus brought into consciousness, something that is in some sense spatial but not *in* space. But what is this something? It is first a binary *relation* among lines in space: two lines either have or do not have *the same direction*; if they have, they are parallel lines, and conversely. But it is also an *object* with properties of its own; for example, two directions can be perpendicular to each other (if one line in one direction is, and then all lines are, perpendicular to one, and then all the lines in the other direction). In projective geometry directions are thought as points at infinity, lines meet at one of these points if they have the same direction (i.e. are parallel).

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domain, then *a* does not belong there, but *b* may. If the domain of objects in question is posited as objectively complete, the fact as to whether *a* belongs or not to the domain is *objectively decided* (but maybe neither *subjectively* nor *logically* decided, in which case the decision stands as an *ideal*).

[60] See for instance *Ideas I* § 142 for the intimate connection between consistency and existence. The issue is related to an important question in transcendental logic, the justification of the principle of non-contradiction, related to the intentional positing of the concepts of *being* and *reality*.

The definition of direction is a creative definition, in the same family with the definition of number in Frege or temperature in thermodynamics (temperature is the quality that all bodies of same temperature have in common; bodies are of same temperature when they are in thermal equilibrium). Set theory is a very convenient context for *representing* these entities, and this accounts for the foundational role set theory enjoys in mathematics.[61] Mathematicians typically identify the new objects with *classes of equivalence* of old objects, a direction in space being the *set* of all parallel lines. However, this representation does not have any serious *ontological* consequence; directions are not *really* classes of parallel lines, classes only provide a manner of *representing* them set-theoretically by sharing with them the same *relevan*t formal properties. From the phenomenological perspective, set-theoretical reductionism is a sort of ontological blindness, a confusion between what a thing is and how it can be *formally* represented.

As already mentioned, the identification of an intentional object as *the same* in different experiences of *it* is also an intentional act. It may be founded on total or partial superposition of intentional meanings, but not necessarily; the object is not experienced as the same in different experiences of it for necessarily presenting the same meaning in all these experiences. Two objects, posited with completely different meanings, can be identified as the *same* object. This requires a further act, properly called identification. Objects of different experiences, with different meanings, can be identified as the same object even if the act of identification is not motivated by total or partial identity in the intentional meanings of these objects. The thread that unifies different experiences as experiences of the same thing is an *intentional* element added to these experiences. Identification can be based on shared aspects, but not necessarily.[62] In a sequence of acts of perception of an object from different perspectives, upon noticing different aspects of it, the ego *can* see the sequence of perceptions as the perception of a *transformation* of the object. Or as a sequence of perceptions of different objects, a sort of *transubstantiation* in which the object *changes* into *other* objects. Or as different *adumbrations* of the *same* unchanged object, provided these adumbrations are not inconsistent with one another. Or still as a combination of change and invariance, the object being partly the same and partly another along the sequence of perceptions.

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[61] Benacerraf's error was to presuppose that if numbers are sets they must be *definite* sets. For one, numbers are *not* sets, they can only be *represented as sets*. Moreover, numbers can be represented set-theoretically in any *convenient* way provided the representation is throughout consistent. In short, 2 can be (interpreted as) either $\{\{0\}\}$ or $\{0, \{0\}\}$, although it is neither. Benacerraf purported to show that since there is no definite way of identifying numbers to sets, numbers are *not* sets, and must then be something else. I agree that numbers are not sets, but not for this reason. Numbers, as we will see later, are an altogether different type of objects, but they are *objects*, which can be *individually* intuited, referred to, named, and conceptually characterized.

[62] When discussing the idealizing presuppositions behind the principle of identity in his *Formal and Transcendental Logic*, Husserl recognizes the "I can always come back to *this* object in future experiences" as the noetic correspondent of the noematic meaning "object that can present it*self* again to me in future experiences". The principle of identity is, for Husserl, as we will see later, rooted in the intentional positing of objects conceived as capable of manifesting themselves as the same in different intentional experiences.

The evolution of conceptions, a phenomenon so present in mathematics, is a phenomenon in general experienced as change coupled with invariance. Our concepts of space and number, for example, have evolved through the history of mathematics preserving some of its aspects, losing some, and acquiring some new ones. The questions "what is a number?" or "what is space?" cannot have definitive answers and must be contextualized. The tendency among mathematicians is to offer the last conceptualization as the definitive one, reinterpreting the previous conceptualizations in terms of the most recent one. Therefore, no conceptualization is definitive; for millennia, numbers were understood as finite until Cantor extended the concept of number into the transfinite.

These processes are also of interest for transcendental history.[63] As I have already emphasized, transcendental history is not the factual chronicle of historical events that marked the origin and evolution of this or that conception, but the investigation of positing acts, both noetically and noematically, in order to identify geneses and follow eventual changes of intentional meaning. Conceptual evolution is often "naively" interpreted as a *revelation* and the new meanings brought to consciousness as *discoveries*. It is often said that Cantor *discovered* transfinite numbers when in fact he only acted as the inductor of a collective process of intentional genesis of *new* entities and a *new* conception of number. Again, here as always phenomenology offers an exhaust valve of Platonist pressures.

Truth and Knowledge In his sixth Logical Investigation (§ 39), Husserl introduces the notion of truth as the correlate of an act of identification, the content of an intuition is identified with that of an empty representation. The act of identification can be either intuitive or purely intentional. In the first case, the "living experience of truth", in the second, the empty representation of truth. Evident truth, i.e. truth as the content of a truth-experience of intuitive fulfilment of an otherwise empty representation, is the most fundamental notion of truth, but not the only one. Truth can also be merely represented as an *ideal*, a *terminus ad quem* towards which the cognizing ego orients its cognizing activity, but which may, nonetheless, elude its best efforts. There are also *partial truths*, posited in imperfect truth-experiences, when intuitive and intentional contents only partially overlap. Husserl accepts both the notions of *ideal* and *partial truths*.

Identity of contents is not an all or nothing matter, there are gradations. Complete fulfillment of an emptily intended content with an intuitive content is truth with the highest degree of *clarity*. One may call such truths *apodictic* truths. Nothing is missing in the apodictic experience of truth; that which is emptily meant manifests itself fully in intuition *precisely as it is meant*. The ego recognizes in the object presented in intuition *that* which he had meant, neither more nor less. Some critics of Husserl, who do not make the effort to perform the epoché, not even as an exercise of understanding if not in seriousness, and measure Husserl's conception of truth with the meter of naturalist prejudices, usually misinterpret apodictic truth as certain and non-revisable truth. The expression of that which simply *is*, not only that which

[63] See Derrida 1989.

appears in full clarity to be. The concept of apodicticity is, for them, a sign of Husserl's epistemological absolutism, the belief that the ego is capable of grasping truth beyond any possibility of doubt. However, the fact is that although truth experiences may present a perfect covering of an intentional content by another, the former merely represented, the later intuited, so that no room is left *in the experience* for doubt, the ego can always cancel the validity of the whole experience, that is, the intuitive positing itself, in the light of further experiences. The ego can *doubt* its intuitions and eventually *cancel* intuitive positing or the validity of a truth experience. Apodicticity belongs to the character of the act and does not have force outside the act, which can always be canceled. Putting it in the most prosaic terms, the ego is entitled to *firmly* believe its intuitions (for example, perceptions) and the (apodictic) truths based on them, without restrictions, until a *force majeure* (for example, further perceptions) forces it to reconsider.

Adequate intuitions are experiences of presentification that cannot be improved. But adequate intuitions are not non-cancelable intuitions, only intuitions *in the mode of* adequateness. They stand in contrast with non-adequate intuitive experience, whose object is only partially presented; partially in the light, partially in the shades. The apodictic experience of truth, however, does *not* require adequate intuitions. For example, as already noted, although one can have only an inadequate intuition of the domain of numbers, one can build on this experience an adequate intuition of the process of number generation, on which to ground apodictic truths about numbers. In fact, the adequate intuitive presentation of a few *exemplars* suffices for adequately bringing to consciousness the numerical generative process, which the usual Dedekind-Peano axioms characterize (at least as to their formal properties).

There is, then, a gradation of intuitive acts, which go from complete non-adequateness to full adequateness. The former is non-intuitiveness, a lower bound to intuitive experiences and the latter is intuitive experience with the highest degree of perfection. Husserl admits degrees of adequateness in intuitive experiences and degrees of perfectness in the mutual covering of contents in truth experiences. There are two types of truth-experiences, the experience of truth as the adequacy of an intuitive content vis-à-vis an empty representation and the experience of conflict between representation and relevant intuitions. The first is the experience of truth, the second, that of falsity; either admits degrees, culminating in the experience of apodictic truth and apodictic falsity, respectively.[64]

By being an identity, truth involves two poles, the object *as intended* and the object *as intuited*; since truth is the *fulfilment* of an intention with an intuition, empty intending seems to be required as a precondition of the experience of truth. The mere act of intuition is not yet a truth-experience if the intuition is not *recognized as fulfilling* an empty intention.[65] This means that empty intending and its

[64] Note that a non-apodictic truth is not necessarily an apodictic falsity.

[65] The intuition of the object is not per se a truth-experience; it is necessary that the presentation of the object, with the sense it has, fulfills or fails to fulfill, partially or completely, explicit or implicit expectations. Of course, the ego can, by reflecting on an intuition, judge with clarity, that is, truth-

products, empty intentions, play a substantial role in the dynamics of knowledge, preparing the ground, so to speak, for the experience of truth.

Knowledge is the possession of truth; to know *A*, at the most fundamental, *intuitive* level, is to experience *that A is true* (preferably apodictically), i.e. to experience an intuitive content as fulfilling (preferably adequately) the content expressed by *A*. There are also non-intuitive forms of knowledge, for example, by deriving *A* by logical means from truths already established. But logical reasoning, if valid, serves only as a channel of transmission of the truth contained in the premises, never more. Hence, non-intuitive knowledge generated by logical reasoning ultimately depend on intuitive knowledge. In general, as just noted, the dynamics of knowledge requires more than truth-experiences, it also involves emptily representations and anticipations. Before trying to elicit the truth-experience that could determine the truth-value of *A*, the subject must know whether *A* has a determinate truth-value attached to it. And this is a task for logic. Moreover, given certain presuppositions (that I will examine next chapter), one may be able to determine which truth-value *A* has by indirect logical means, independently of any truth-experience of *A*. This is also a case of non-intuitive knowledge and can be seen as an *anticipation* of the *intuitive* experience of the truth of *A*.

Let us consider empty judging and the role it plays in the dynamic of knowledge more attentively. As I have already noticed, empty judging is itself a cognitive act, one in which a judgment is expressed that can *in principle* be intuitively fulfilled. Judging without clarity, i.e. without supporting intuitions, is to set oneself a goal, that of verifying the truth of the judgment, preferably in a truth-experience. *Empty intending is, in some sense, a sort of conjecturing and as such plays a pivotal role in knowing*. Empty judging, however, in order to play this role adequately, must satisfy the precondition of truth, namely, *consistency*. Consistency is the necessary and sufficient condition for the a priori possibility of intuitive fulfillment. Hence, consistent empty judgments participate in the dynamics of knowledge by posing problems, possibilities, hypotheses, that can in principle be *directly* (i.e. intuitively) verified. I will come back to these matters when discussing logic; by now I want to stress that Husserl is not committed to a strictly "intuitionist" conception of either truth or knowledge, his phenomenological approach to epistemology is not a "constructivist" one, despite the pivotal role the ego and its experiences play in it.

Language and Validation To the extent that intentional meaning is expressible, a language goes along with the positing. The positing determines both what can be said about the object it posits and in which language; let us call this the *intentional language*. Intentional meaning expressed in the intentional language constitutes, of course, the intentional theory associated with the positing. Another positing, of a different kind of objects, with a different meaning, expressible in a different language, may be such that the first domain and its theory can be *interpreted* in the second. There may be, so to speak, a *translation* from the original to the new

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fully; in such cases, intending and intuiting are concomitant, but intending is still there, as part of the reflexive act.

language such that everything expressible in the original language has a correspondent in the new language. For example, by conceiving numbers as abstract (quantitative) forms that can relate to one another in terms of more and less, one attributes sense to assertion such as $2 \geq 3$ (meaningful but false), but none to those such as $3 \in 2$. Abstract forms do not "belong to" one another (they are not sets), although they may be contained in one another. The theory of numbers, however, can be translated or interpreted in the theory of sets, a correspondence can be established between the old and the new languages; for example, $a < b$ iff $a = a_0 \in a_1 \in \ldots \in a_n = b$. Now it makes sense to assert $2 \in 3$ and interpret it as having the same meaning as the assertion $2 < 3$ in the original language. It is all a matter of context and we must be careful not to attribute meaning to assertions in one context that only make sense in another. Moreover, the translation may not be unique.

One thing, however, must be kept carefully in mind. In general, the possibility of interpreting (re-conceptualizing) objects of one type as objects of another depends on both interpretations sharing the *same* relevant formal properties; i.e. both must satisfy the same formal theory, namely, the formal abstraction of the original intentional theory. One can treat numbers as sets insofar as sets are, from a *formal-operational* perspective indistinguishable from numbers. Numbers, however, I repeat, are *not* sets. However, on a purely formal level, where matter does not matter, we can change material content freely. Formally, all different interpretations are essentially the same. It does not matter whether the number 2 is identified either with $\{0, \{0\}\}$ or $\{\{0\}\}$; from a *formal* perspective 2, $\{0, \{0\}\}$, and $\{\{0\}\}$ are the *same* object.

To be meaningful, assertions about an intentional object must first be expressible in the language validated by the intentional positing and, second, be formally and materially meaningful. Formal meaningfulness, as already discussed, depends only on the grammar of syntactic categories, material meaningfulness, on the particular ontological categories involved in the positing and the semantic laws associated with them. These laws depend, of course, on the meaning intentionally attached to the positing and co-positings that go with it. Now, the important question is this: what should count as *validating* a particular meaningful assertion? What are the grounds for asserting that a meaningful assertion φ is true?

If one understands the concept of truth in the *narrower* sense as intuitive truth – which requires the intuition of the state-of-affairs denoted by φ, φ is validated only provided one *intuitively* experiences *that* φ. However, if one conceives truth in the *broader* sense as that which *cannot be false*, *even if it is not directly experienced as true*, the intuition of the content expressed by φ is no longer required *provided the domain is objectively complete*. It suffices that one establishes that not-φ is incompatible with the intentional meaning of the domain in question, i.e. that not-φ explicitly or implicitly conflicts with this meaning. In other words, the validation of the principle of bivalence (or excluded-middle) can be justified in reasoning about objectively complete domains. I will come back to this in detail the next chapter.

Intentional Existence Let us now investigate the notion of existence in more details. What does it mean to exist? This question has tormented the best (and less so) philosophical minds throughout the ages. Our most basic experience of existence is that of our own selves, self-consciousness is consciousness of ourselves as existents. Hence, self-consciousness is, a fortiori, consciousness of our existence: cogito, ego sum. Equally immediate, in the "natural", pre-philosophical attitude, is the existence of the external world; which present itself as existing out there independently, as a substance, i.e. a self-subsisting thing (*substare* is Latin for "stand firm"). Descartes privileged the experience of existence of the *ego cogitans* as the primordial experience of existence, justifying the existence of the word only indirectly. As mentioned before, the mode of existence of the empirical world – independently, self-subsisting in and by itself – has in some philosophical circles become the model of existence. But as I have already stressed, this is a limitation; there are other modes of existence.

If we inquire a bit further into what precisely in the consciousness of the self or the world justifies attributing them existence, the answer imposes itself that it is the *presence* of the object of experience itself in the experience, not as a real content of it but as a correlate of it. *There is* something *given* (*bodily present*, in Husserl's colorful expression) in these experiences, but *transcendent* to them, to which consciousness is intentionally related. And what is *given* exists. In other words, intuition is the most basic experience of being and existence; intuited objects exist *because* they are intuited. In self-awareness, the ego is presented to itself as existing. Analogously, the presentification of the world to the ego in perception establishes the existence of the world. This is the starting point to understand the phenomenological conception of existence: to exist is, in its most basic mode, to be intuited – *esse est percipi*.

But empty intending, to the extent that it also posits something, even though in the mode of absence, is also a variant of the consciousness of being. But with an important proviso, *the positing must be and remain consistent*. To posit something consistently with itself and the overall system of valid positings must also count as the positing of something *as existing*.[66] Therefore, *esse est* – also – *concipi*. It is presupposed, however, that objects that are *merely conceived* (not intuited) as

[66] A quote from Husserl seems in order here. Talking about the positing of a transcendent, in this case *real* world, he says: "What is transcendent is given through certain empirical connections. Given directly and with increasing completeness through perceptual continua harmoniously developed, and through certain methodic thought-forms grounded in experience, it reaches ever more fully and immediately theoretic determinations of increasing transparency and increasing progressiveness. Let us assume that consciousness with its *experimental content* and its *flux* is really so articulated in itself that the subject of consciousness in the free theoretical play of empirical activity and thought could carry all such connections to completion (it would be necessary to consider the mutual comprehension with other egos and other fluxes of experiences); let us further assume that the proper arrangement for conscious-functioning are in fact satisfied, and that as regards the course of consciousness itself there is nothing lacking which might in any way be required for the appearance of a unitary world and the rational theoretical knowledge of the same. We ask now, presupposing all this, is it still *conceivable*, is it not on the contrary absurd, that the corresponding transcendental world could not *be*?" (*Ideas I*, § 49). These considerations are, mutatis mutandis, valid for positing in general. To the extent that the positing is consistent and remains so in the continuous flux of experiences, the posited object *exists*.

existing (that is, objects enjoying *purely intentional* existence) can in principle present themselves adequately in intuition (i.e. as *intuitively existing* objects). Although intuitive presentation counts as the fundamental mode of validation of existence, the mere possibility *in principle* of intuitive presentation is also a form of existence. The criterion for the *a priori possibility* of intuitive presentation is the absence of *manifest* inconsistencies in the positing experience, with itself and other positing experiences. Any posited object whose positing is internally consistent with itself (i.e. whose intentional meaning is not self-contradictory) and externally consistent with other positings, *to the extent that it remains so*, exists.

The positing of a *transcendent* realm of being deserves special attention. An important observation should be kept in mind; transcendence is, as *all* attributes of intentional objects, an aspect of intentional meaning and it is not the same thing as ontological independence. Transcendence is a character intentionally attached to posited objects that are *conceived as having* attributes that necessarily or contingently belong to them but are not given *originally* in the positing; attributes, however, that can in principle be disclosed in the progressive development of the positing experience, supervened of course by an identifying intention. A transcend object can reserve surprises, which will when disclosed, if the object is to remain in existence, harmonize with the original meaning of the object posited. It belongs to the positing experience of a transcendent object that its intentional meaning is not *fully* given originally, but can be progressively disclosed, *ideally* to full completion. This means that it is possible and desirable that the positing experience develops to the point that no further meaning remains occult. In certain cases, enough intentional meaning is eventually disclosed which is sufficient, when linguistically expressed (expressivity thesis), for answering any relevant question concerning the posited object by strictly logical lines of reasoning. The transcendent object can also be posited as objectively complete, and one may argue that objective completeness belongs to the meaning of transcendence. If this is so, it is part of the *meaning* associated with transcendence that transcendent entities are fully determined in themselves, that is, no attribute that can in principle pertain to a transcendent object can fail to either determinately pertain or determinately not pertain to it. Anything that can be said about the transcendent object is determinately true or determinately false, even if the positing experience does not offer means for the ego to determine which.

Another positing experience deserves special consideration, that of an *objective* being. To be objective means, essentially, to be capable of manifesting itself *as the same* to the individual or collective ego in multiple experiences. The object must be capable of being experienced as the same object in different experiences, which are then experiences *of it*, by the individual ego or any ego of a community of co-positing egos. An objective entity (an individual, a realm of beings, a concept, an idea, and what not) is open, by remaining *the same*, to different experiences, it maintains its individuality throughout open series of experiences, with possibly different intentional meanings (which, however, must consistently harmonize with the originally posited meaning). An objective entity is one that is “out there”, for anyone to experience, repeatedly. Objectivity and transcendence are not exclusively attributable to real (that is, temporal) objects; abstract and ideal entities – mathematical entities, in particular – can and usually are also conceived as objective and transcendent. The error of Platonism consists in believing that if objects are objective and transcendent, they are also ontologically independent.

The intentional meaning originally attached to objects in their intentional positing determines which logical principles are valid for reasoning about them. Logic is a priori in the sense that logical laws impose themselves for reasoning about objects of a type by being validated by the meaning attached to objects of this type. Or contrarily, laws are invalidated if they do not find support therein. The universality of logic must be correctly understood; logic is universal in the sense that its laws and principles are *formal*, i.e. indifferent to the particular *nature* of the objects over which they rule. But not in the sense of being indifferent to the intentional meaning attached to them. For Husserl, one of the tasks of *transcendental logic* is to *clarify*, and then *justify* fundamental principles of formal logic in terms of their hidden presuppositions, that is, in terms of intentional meaning. Transcendental logic must identify the presuppositions on which the validity of logical principles depends, keeping in mind that these presuppositions have transcendental, not hypothetical nature.[67] I will address this issue later; by now, it suffices to point out that the validity of the principles of non-contradiction, identity and bivalence depends on presuppositions that can only be validated by the intentional meaning attached to the domains where these laws are valid.

Anything that exists, exits somewhere; any existent has a *locus*. Empirical objects exist in the empirical world and may or may not be meant as ontologically dependent on the ego; that is, they may or may not be conceived as existing by themselves. A tree in the woods exists in space and time; it has a determinate location in space and occupies a stretch of time, the duration of its existence. This tree is conceived to remain in existence even if not directly perceived, and always capable of being, at least in principle if not actually, perceived. This is the mode of existence of empirical objects of the *exterior* world, an objective and transcendent realm of being. But there are also subjective empirical objects, such as psychological states and qualia of various types (the smell of a rose, the pain in my arm). These things exist in time but not in space, in the sense that they do not stand in spatial relations with objects of the external world, even though they can be said to roughly occupy the same space of my body – although my body occupies space, the sensations in my body and the states of my mind do not. Empirical objects of the internal world exist in time but not in space; the internal world is not an objective realm of being open to external inspection (unless, maybe, indirectly through their objective manifestations). Empirical objects, in short, no matter of the internal or the external world, exist in time, they are *real* objects; some are (conceived to be) *concrete*, that is, ontologically independent, others *abstract*, that is, ontologically dependent.

A pain, for example, understood *stricto sensu* as a personal experience, cannot exist without someone who *feels* it. Moreover, *my* pain cannot ever be the *same* as *your* pain, although their respective intensities can be somehow *objectified*, maybe

[67] See *FTL* II Chap. 3.

at the price of falsifying the original experience to some extent, and compared. These things are not *empirical* statements, even though they refer to empirical objects, but transcendental truths validated by intentional positing. The color of a body, on the other hand, considered as an *aspect* of this body, despite being also an abstract object, since the vanishing of the body implies it vanishing as well, i.e. since it is an ontologically dependent object, is an object of the external world just as the body itself, occupying the same position in space. This color *as such*, that is, as an ideal entity, a species, on the other hand, is not an empirical object and does not exist in space or time. But it exists all the same and has its locus too.

Ideal objects exist by being consistently posited, by being referred to, by being subjects of true statements, and thus in the intentional context in which they are posited, named and investigated. Intentional action leaves traces, real traces. In scientific contexts, this usually takes the form of theories, expressed as articulated systems of assertions in convenient languages, within particular logical frames, subject to established criteria of validation, whose formulation and continuous support may engage an entire community of cooperating intentional agents, separated maybe in time and space. Theories are consigned to books, which people read and learn from, engaging consequently in the communal activity of constituting theories and their objects. These supports, material and cultural, a communal language, written documents, communal memory, traditions, schools, provide the locus where idealities exist; without them they cannot exist. Ideal objects, in short, exist in the space of *culture*.

This last sentence is bound to be misinterpreted; it can be read as an endorsement of cultural relativism or a variant of psychologism where the community takes the place of the individual, unacceptable in any case to an honest objectivist who believes in the objective, human-independent (in particular, mind-independent) nature of ideal objects, such as, for example, numbers and numerical truths. For him, $2 + 2 = 4$ is true independently of the vicissitudes of human culture and history, or human consciousness, it was true before man was conscious of it and will remain true after man and human culture cease to exist. Phenomenology does not deny the objectivity and a-temporality of arithmetical judgments; it only refuses to give these attributes metaphysical value. In other words, from a phenomenological perspective, $2 + 2 = 4$, like all arithmetical truths, is a *necessary* truth *of a conception*, but one cannot give, without changing perspective, the conception itself metaphysical reality, epoché forbids. In short, numbers would not exist if they were not *intentionally posited*, but given that they exist, for their positing is not manifestly inconsistent, truths about numbers will necessarily impose themselves upon us as truths referring to an objective, transcendent, ideal (and a fortiori non temporal) realm of being. Arithmetical truths are not a matter of convention or themselves cultural products; culture is only where the positing of numbers, and their *objective* existence, are anchored. The *content* of arithmetical truths are in no way culture-dependent. Although chess is an invented game, the truths related to the game are necessary truths, not a matter of convention, valid no matter where the game is played.[68]

As I said before, individual numbers, such as 2 and 4, can be intuitively given through abstraction and ideation. But they can also be given indirectly by means of characterizing properties contained or derived logically from the intentional meaning attached to numbers (for example, 2 as *the successor of* 1). Given, however, not as products of imagination, but objectively existing entities that can present themselves, intuitively or not, as the same in different experiences of any of the cooperating egos collectively engaged in the task of intentionally constituting numbers and the science of numbers. Numbers are transcendent entities capable of presenting different, sometimes new aspects in different presentations. All these things justify the communal engagement with a science of numbers. They (are conceived to) exist objectively as non-temporal entities even though their constitution is a temporal process. The constituting act is, at the noetic pole, temporal, but a-temporal, or non-temporal, at the noematic one. In short, none of the beliefs the objectivist (in particular, the Platonist) cherishes concerning numbers is denied – numbers, and numerical truths, are mind-independent, objective, a-temporal. Transcendental phenomenology, however, blocks the rod from this to a metaphysical taking of position, Platonism in particular. For the phenomenologist, numbers can be conceived as the Platonist believes them to be (although they can *also* be differently conceived, such as, for example, in the manner of the intuitionists, as creatures of the mind; but this is *not* how they are conceived in "classical" mathematics, the Platonist and the intuitionist positings are incompatible), but he does not go any further, refraining from endorsing any particular metaphysical thesis. Phenomenological epoché so imposes.

Phenomenologists believe that numbers did not always exist and can cease to exist. Since the noesis of the number-positing act is a temporal phenomenon, numbers had an origin, and may have an end if the positing can, for some reason, no longer be sustained, either with the annihilation of the cogitating ego(s) or with the "explosion" of the posited noema due to inconsistencies. When man disappears, so will all the products of man's cogitations (even if the products of his *actions* on the word may persist, for a while at least). If arithmetic proves to be inconsistent, the particular manner of conceiving numbers that our arithmetic expresses, i.e. the intentional meaning associated with the mathematical conception of number, will cancel itself. Although the concept may be posited anew by being differently meant. By considering the noema of the number-positing experience, the phenomenologist sees no reason to contradict the Platonist as to the *meaning* attached to the concept, but by considering the noesis of the act and the noetic-noematic correlation, he sees no reason for endorsing the Platonist metaphysical edifice that goes with it.[69]

[68] One must be very careful here not to shock those who are incapable of abandoning the natural for the phenomenological attitude and can easily misinterpret the whole thing in naturalist terms. Numbers do not exist unless they are consistently posited and rational beings can very well exist who do not posit them, but as long as, for us, numbers exist, they exist as they are *meant to exist*.

[69] In his "The Origins of Geometry" (Husserl 1954b) Husserl offers a model analysis of the constitution of geometrical objects and geometry from both the noetic and the noematic perspectives.

Formal Objects Formal objects are objects of formal theories. Some prefer to call them abstract objects, but this should be avoided, for although formal objects are indeed abstract not all abstract objects are formal. Some philosophers think that philosophical questions concerning formal objects can be settled by giving the notion an axiomatic characterization, usually within a modal language.[70] Formalism, however, is a poor substitute for philosophy; only a properly philosophical treatment can clarify what these objects are.[71] Husserl thought that they are forms of objects (or object-forms). To characterize formal objects, the distinction formal/material that I have already discussed will be useful. More explicitly, I will approach the issue from the perspective of Husserl's conception of logic, even at the cost of repeating myself to some extent. I hope this will help clarifying the cluster of ideas to which the notion of formal object belongs.

The category of objects, understood in in the largest possible sense, includes all the things about which something meaningful can be said, all the things we can refer to.[72] Objects fall into well-determined ontological categories, Individual (or Object in a restrict sense), Relation, Function, Manifold, Concept, etc., which Husserl called *formal-ontological* categories.[73] *Formal ontology* is the formal-logical discipline concerned with formal-ontological categories. Including formal ontology in formal logic is justified, Husserl thinks, given the generality of formal ontological concepts. To the extent that all sciences involve objects, states-of-affairs, concepts and the like, all sciences involve formal-ontological categories, and since logic is, for Husserl, the a priori theory of science, it befalls on logic the task of investigating formal-ontological categories and the a priori laws related to them.[74]

The exposition of the series of noetic acts, with their noematic correlates, which may or may not be discernible in the factual history of geometry, constitutes what Husserl calls the "transcendental history" of geometry. Factual history records the traces (or a selection of them) that transcendental history leaves in culture.

70 See for instance Nodelman and Zalta 2014.

71 In the preface of his *Das Kontinuum* (1918), Weyl says that "it is not the purpose of his work to cover the 'firm rock' on which the house of analysis is founded with a fake wooden structure of formalism – a structure which can fool the reader and, ultimately, the author in believing that it is the true foundations" (Weyl 1994, p. 1). I too believe that formalism cannot account for the true philosophical foundations of anything.

72 Husserl characterizes object in the sense of formal logic as "any possible subject of true predicative judgments" (*Ideas I* §3). Husserl's characterization is equivalent to defining object as a subject of a meaningful assertion.

73 Husserl gives as examples of *objective categories* those of Object, States-of-Affairs, Relation, Connection, among others (Husserl 2001, vol. 1, *Prolegomena to Pure Logic* §67).

74 For Husserl, formal logic contains also, parallel to formal ontology, the discipline of formal apophantic logic, concerned, according to him, with syntactic categories to the same extent that formal ontology is concerned with ontological categories. For Husserl, the most fundamental task of apophantic logic is to investigate the a priori laws of meaningful combination of syntactic types. There is, for Husserl, a strict parallelism between syntactic and ontological types (the syntactic type Subject corresponding to the ontological type Object, Predicate to Property, and so on). Logical-grammatical laws determines the boundaries of formal meaningfulness for assertions, and correspond, on the ontological side, to the a priori laws regulating ontological types. Husserl's logical-grammatical laws are Carnap's "laws of formation" or the "rules of formation" of modern

Formal ontological categories admit proper subcategories, the category of Object, for example, admits the subcategory of Physical Object, the category of State-of-Affairs that of Physical State-of-Affairs, etc. There is, so to speak, a surplus of meaning that accounts for the specificity of physical objects within the category of Object or a physical manifold within the category of Manifold. *Proper* subcategories of formal-ontological categories, like that of physical objects, constitute what Husserl calls *material-ontological* categories. The *matter* of material categories is that which makes them the particular categories they are, the extra meaning that goes into them. In short, for Husserl, formal ontological categories are the most general ontological categories and material ontological categories are particular ontological categories; the former are formal-logical, the latter are not. A priori laws related to formal ontological categories are analytic; those related to material categories are synthetic.

The category of Number is, I believe, also a material category, since by "number" one means *something more* than merely an object. The extra meaning that characterizes numbers in the domain of objects extrapolates the boundaries of the strictly formal-ontological since one cannot express what numbers are with formal ontological categories only (*although* one can with them express *formal* properties of numbers, which, however, *any* objects, not only numbers, can in principle display). Husserl thought differently. For him, although the category of number was a subcategory of the category of objects in general, it was not a *material*-ontological category, but still a *formal*-ontological one. Same thing with sets. The reason is that both numbers and sets are *forms* that can in-form any collection of objects. For Husserl, since sets and numbers are forms – a view with which I agree – Number and Set are formal-ontological categories – a view with which I do not agree.

Loosely characterized, *formal objects* are objects determined as to form but indeterminate as to matter. More specifically, an object is a *material* or *materially determined* object if it falls into a determinate material ontological category, it is a *formal* object, or an object determined only as to form, if it is determined *only* as to its formal-ontological category. The material content of the number 2 *merely as a number*, for example, distinguishes it from other objects, but not from other numbers, which can only be accomplished by the properties of the number 2 *as the particular number it is*. The form or formal content of an object is simply the logical type to which it belongs. The form of the number 2 is its objecthood, the form of the concept of number, its concepthood, etc. *Formal properties* are the properties that objects have that any entities of their logical type *can in principle* also have. For example, the number 2 can stand in an anti-symmetric binary relation with respect to the number 3 simply by being, both, objects; this is a formal property they enjoy.

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logic. Unlike modern logic, however, Husserl saw an ontological correlate to syntax that, however, is not still semantic in the modern sense. (In *Formal and Transcendental Logic*, Husserl introduces semantic notions, such as truth, among others, in apophantic logic; this clearly indicates a distinction between syntactic and semantic notions at the interior of apophantic logic itself, but nothing of the sort of Tarskian semantics.)

The assertion “there are *objects* x and y and a binary anti-symmetric *relation* R such that xRy” expresses a formal property of the formal entities denoted by x, y, and R.

Formal abstraction is a higher-level intentional act that takes as matter a materially determinate object and considers it merely as an object of its formal-ontological category. Contrary to Frege’s infamous criticism of Husserl’ theory of abstraction, abstraction, of whatever nature, formal or not, is not a chemistry of mental representations. In formal abstraction, the object does not change, only its categorization does. Material properties of materially determined objects are those that require material categories to be expressed; for example, the *number* 2 is the *successor* of the *successor* of the *number* 0, where “successor” denotes a particular, (materially determined) *numerical* function. By formal abstraction, “0” and “2” are reduced to generic object-denoting names and can be substituted by non-logical constants $\underline{0}$ and $\underline{2}$ respectively; likewise with “successor”, the name of a particular numerical function, substitutable by a function-name S. The formal abstraction of the materially filled assertion “the *number* 2 is the *successor* of the *successor* of the *number* 0” can then be expressed by “$\underline{2} = SS\underline{0}$”, which is the formal content of the original assertion. The formal expression “$\underline{2} = SS\underline{0}$” expresses a *possible* formal property of objects considered in utmost generality. *Any* objects whatsoever can in principle, given adequate interpretations, satisfy “$\underline{2} = SS\underline{0}$”. Since numbers, with the usual interpretations, have this property, one says that “$\underline{2} = SS\underline{0}$” expresses a *formal property* of numbers. It is important to keep in mind that formal properties of materially determined objects can be shared by materially different objects.

A language L is *materially determined*, or a *material* language, if the non-logical symbols of the language denote materially determined entities. One also says that material languages are *interpreted* languages. Otherwise, the language is materially indeterminate. The symbols of materially indeterminate languages are determinate only as to their logical types; i.e. their referents are *formal objects*, nominal terms are generic object-denoting terms, conceptual terms, concept-denoting terms, and so on. For this reason, materially indeterminate languages are also called *formal* languages. To divest the symbols of a language, and thus assertions and collections of assertions of this language of their material content whereas preserving their formal content is the act we call formal abstraction – one can also use the term *des-interpretation* for it.

The laws of the logical grammar of syntactical types is in strict correspondence with the logical grammar of formal-ontological types. But when material ontological types are involved, formal meaning is not enough for determining meaning. Material meaning is determined by a priori laws of compatibility and incompatibility of material-ontological categories. These laws are a priori but, according to Husserl, material, i.e. synthetic; their task is to establish the a priori conditions for material meaning. Assertions of a material language are meaningful insofar as they are both formally and materially meaningful, in which case one says that they are distinct (or, in Husserl’s terminology, have the evidence of distinction). A material assertion is formally correct insofar as its formal abstraction obeys the a priori laws of logical grammar, and materially correct insofar as it respects a priori compatibilities and incompatibilities of material ontological types.

Consider the following intentional positing: *let N* be a domain of objects, 0 a particular non-specified object in *N*, and *S* a unary function in *N* (S: $N \rightarrow N$) such that (1) *S* is 1–1, (2) there is no *x* such that $0 = Sx$, and (3) *N* is the smallest domain closed under *S*. The positing is entirely contained in this "let". But notice, the positing does *not* determine the domain *N* and the objects in it materially. In fact, only their logical types and some of their formal properties are determined. One does not know *what* these things are, i.e. what their material ontological types are, only that *N* is an objectual domain whose objects are somehow related to one another by some indeterminate function *S* only formally characterized in the positing. To an object a name is given, 0. In fact, only the logical types of *S*, *N* and the entities in *N* are determined, respectively Function, Manifold, and Object. Further *names* can be defined recursively: $1 = S0$; $2 = S1$; etc., which are, given the positing, names of objects in *N*. These symbols are not given any particular material content (reference). Insofar as the positing is concerned, the objects in *N* (let us call them "numbers") do *not* exist in isolation, although they can be "interpreted" by objects that do (for instance, numbers proper); "numbers" are, so to speak, necessarily gregarious objects; the original intentional act posits them *collectively*.

As mentioned before, a language comes along with the positing, the minimal or simplest language in which the intentional meaning associated with the positing, i.e. (1)–(3), can be completely expressed; let us call it *L*(*N*); *L*(*N*) can be enlarged by definitions. In this example, *L*(*N*) is the second-order language required for expressing Dedekind-Peano axioms (second-order is needed to express (3)).

Now, an important question faces us: is any syntactically meaningful assertion *A* in *L*(*N*) (possibly enlarged with defined symbols) a meaningful assertion *about* "numbers"? The answer seems to be positive if *A* is *in principle decidable* on the basis of the axiomatic "numerical" truths (1)–(3) expressed in *L*(*N*) (the intentional theory). In other words, if one could, *in principle*, either prove or disprove *A* assuming only axiomatic truths about "numbers". Three questions now arise: (*a*) what does "in principle" mean? (*b*) What is the underlying logic? (*c*) What if *A* is logically independent of the axioms?

I will deal with these questions in a general context later, but a few things can be advanced here. Suppose that the problem concerning the underlying logic is solved and that the intentional theory – let us call it **N** – is *logically* (or *syntactically*) *complete*, i.e. any (syntactically) meaningful assertion in *L*(*N*) is decidable in the underlying logical context on the basis of the axioms. In this case, any such assertion is a meaningful assertion about "numbers" and the possibility (*c*) is ruled out.

Now, to determine which logical principles are valid in the logic underlying the intentional theory one must turn to the inaugural positing act. I will be more explicit about this in the next chapter, but for the moment I just want to remark that no logical principle or law is context-free in the sense of being valid for reasoning about no matter which domain of entities. Logical laws and principles depend on the sense of being attributed to the domain over which they rule, which is an aspect of the intentional meaning attached to it. If *N* is meant as a domain where any syntactically meaningful assertion in *L*(*N*) either expresses a formal property of *N* or its negation does, then *N* is, of course, objectively complete. Notice that objective completeness is neither a metaphysical presupposition nor a hypothesis open to verification. Rather, it is a *transcendental presupposition* that is *part of the positing itself*.

If N is meant as an objectively complete domain, then any syntactically meaningful assertion is a meaningful assertion about "numbers", *regardless of the logical completeness of the theory*. Objective completeness of the domain implies that any meaningful assertion in the language of the domain, which as we know expresses a *possible fact* in the domain, is ipso facto *decided in itself*, i.e. it has a truth-value *objectively* attached to it, and thus it is *in principle* decidable, not necessarily in the theory of the domain, whereas logical completeness of the *theory* of the domain implies that each meaningful assertion is *effectively* decidable in the theory as to its "truth" or "falsity". The fact that any meaningful assertion referring to an objectively complete domain has an *intrinsic truth-value*, which may be unknown but not in principle unknowable, is another way of characterizing objective completeness. If a meaningful assertion A expresses a formal property of N, then I say that it is true in N; if its negation does, it is false in N. The objective completeness of N implies that *tertium non datur* is valid for truths in N. I will have more to say about these matters later.

Let T be the intentional theory in a language L of the formal domain D and A an assertion in L. I say that A is T-*true* (resp. T-*false*) in D if A is a theorem (resp. not-A is a theorem) of T. It is clear that an assertion that is not T-true is not necessarily T-false, unless the intentional theory is syntactically complete. We have then two notions of truth (resp. falsity), namely, true (resp. false) and T-true (resp. T-false) in D. The principle of bivalence is valid for the first notion if D is objectively complete and for the second if the theory of D is logically complete. Of course, logical completeness implies objective completeness, but not the converse, and T-true implies true but not the converse. The point of this distinction is to call the attention to the fact that a formal theory can posit a formal domain but not be the sole responsible for disclosing the formal properties of the domain it posits.

I will call the intentional theory **N** "arithmetic" for the same reason I called "numbers" the formal objects in N. Numbers proper, however, are something else; namely, quantitative forms associated with quantitatively well-defined collections of objects, and arithmetic proper is the science of *these* forms. One must be careful not to invite confusion. However, one can easily verify that there is a 1–1 correspondence between numbers and "numbers", where the number 0 corresponds to the "number" *0* and such that if n corresponds to x, then $n + 1$ corresponds to Sx. In short, there is what mathematicians call an *isomorphism* between the material domain of numbers and the formal domain of "numbers". This, however, does not erase the important differences between numbers and "numbers". Numbers exist individually, they can be individually intuited; "numbers" do not and cannot. However, the *domain* of "numbers" can be intuited, although inadequately, by formally abstracting the (also inadequately) intuited domain of numbers. Remember, to formally abstract a domain of objects is to consider each object in the domain merely as an object, each relation merely as a relation, and so on for all entities of the domain with respect to their logical types. Linguistically, this corresponds to divesting the symbols of the language that have referents in the domain of their material content whereas preserving their formal content.

The domain of numbers and numerical relations proper provide an *interpretation* for the domain of "numbers" and the formal relations defined therein. Interpreting is the converse operation of formally abstracting; whereas the latter subtracts material content, the former adds it. When two domains are isomorphic, as the domain of "numbers" and numbers, I say that they are *formally equivalent*. Formal equivalence is *not* identity, but equality under an aspect, form. Formal "numbers" admit other interpretations; for example, consider the universe of sets, make *0* correspond to { } (the empty set) and *Sx* to $s\cup\{s\}$, where *s* is the set that corresponds to *x* and ∪ denotes set-theoretical union. It can be shown in set theory that *N* is also isomorphic to a subdomain of the universe of sets where *S0* is interpreted as {{ }} (the singleton of the empty set). Notice that *S0* is *not* {{}} to the same extent that it is not the number 1; both are only *interpretations* of *S0*.

The domain of "numbers" is a structured domain of formal objects that is univocally determined by the theory **N**. By this I mean that **N** is a *categorical* theory, that is, all interpretations of *N* are isomorphic; their formal abstracts can be identified, and *N* is the ideal formal domain they all instantiate. In view of this, I say that *N* (which I also call the ω-sequence) is the *formal structure* of the domain of numbers or any of its isomorphic copies. The abstract formal structure of a *structured domain* of entities (a structured domain is essentially a domain of objects in relation) is this very domain where, however, every entity in it (objects and relations) is considered merely as an entity of its logical type. Abstract structures of isomorphic domains instantiate the *same ideal* structure. Since obviously the domain of "numbers" is isomorphic to itself, the ω-sequence is the formal structure of the domain of "numbers" too.

Husserl thought that *any* formal or non-interpreted theory determines as a correlate a formal domain, even if this domain is not completely characterized by the theory. I will come back to the notion of formal domain later, by now it is enough to say what formal domains are *not*: (*a*) they are not sets, (*b*) they are not materially determined, (*c*) they may not be objectively quantitatively determined; i.e. they may not *have* a definite cardinality, (*d*) they may not be objectively complete. Suppose, for example, the formal group theory **G**: *let G* be a domain of objects with a binary operation + such that (1) there is an object, denoted by *0*, such that, for any object *x*, $x + 0 = 0 + x$, (2) + is associative, (3) for any object *x*, there is an object *y* such that $x + y = y + x = 0$. Again, **G** is a formal theory and *G* its formal domain, but **G** does not completely characterize *G*, for interpretations of *G* are not all isomorphic. Nonetheless, *G* has some well-determined formal properties, namely, those that are true in *all* groups, i.e., true in all interpretations of *G*; those are, of course, the logical consequences of **G**. *G* is not completely determined, but not completely undetermined either.[75]

[75] Since **G** is a first-order theory, the formal truths valid for all groups (i.e. the formal properties of *G*) are theorems of **G**.

The concepts of formal object and formal domain (of formal objects) seem to impose themselves on grounds of phenomenological honesty. However, the development of set-theoretical semantics created a problem for it, since it is not always possible to interpret the notion of formal domain set-theoretically. Sets are well-determinate collections of entities, formal domains are not always so (*G*, for example, is not). Formal objects are not usual objects either; like "numbers" they do not exist independently of a totality of co-posited formal objects, other "numbers" in this case; moreover, formal objects are always, in some sense, dependent entities, either on materially filled objects whose forms they are or on formal theories that posit them. In some cases, not even identity among formal objects is a determinate relation. Given two (formal) terms, t_1 and t_2, which can also be definite descriptions, the assertion $t_1 = t_2$ *is* true if, and only if (*i*) it is a *logical consequence* of the formal theory characterizing the formal domain, or (*ii*) the domain is objectively complete and its negation is false in the domain. In case (*ii*) every assertion about the domain is either true or false in it; but if the domain is not meant as objectively complete and the intentional theory is not logically complete, identity statements may exist that are undecided. These things show what a peculiar type of objects formal objects are.

The theoretical determination of formal domains, then, to be complete require both categoricity and logical completeness of the positing formal theory[76] (or objective completeness of the domain, if the theory is not logically complete). Otherwise, they will lack some of the most basic ontological features of material domains of being. Situations in the domain may be possible that are not determinate as to their facticity; there might be relevant questions about it that are not answerable, not even in principle. All these things may have spoken against the ontological credentials of formal objects, explaining why the concept did not survive in modern logic. Husserl, however, may have presupposed that something like categoricity and logical completeness were attainable ideals.[77]

Formal and Material Truths, Theories, and Knowledge Material theories are interpreted theories, referring to materially determinate domains of being; formal theories, on the contrary, refer directly only to their formal domains and can be given different materializations or interpretations. To interpret a formal theory is to give it a domain of reference or, which is the same, to provide the theory – better, its formal domain – with a material content. Assertions, truths, and knowledge, then, can be either material or formal, depending on whether they refer to or are about materially filled domains or, contrarily, materially empty domains characterized only in terms of formal-ontological categories. A mathematical example of a material theory is physical geometry, whose domain is physical space, a mathematical idealization of abstract aspects of perceptual space. Examples of formal theories are "arithmetic" and abstract group theory, concerned respectively with the formal-ontological categories of Object and Operation. Given, however, that formal and material theories

[76] As we know, these notions are logically independent.

[77] The exegesis of Husserl's approach to this problem is difficult and occupied many researchers. See da Silva 2016a and the bibliography therein.

are obtainable ones from the others by, respectively, interpretation (or material instantiation) or des-interpretation (or formal abstraction) the only real difference between material and formal knowledge is that the former is the latter instantiated. In other words, material knowledge is formal knowledge *in concreto* (particularized) and formal knowledge is material knowledge *in abstracto* (generalized).

Material and formal sciences differ essentially on their methodological strategies. Obviously, by having their domains constantly under the eyes, material theories can always turn to them for insights. In material sciences, intuition plays a major role as a truth-provider.[78] Formal theories, on the other hand, cannot rely on the exclusive information any of its interpretations can intuitively provide, except in a few particular cases (for example, if the theory has essentially only one interpretation, i.e. all its interpretations are formally identical in the strongest sense of being isomorphic to each other). Formal sciences must in general rely almost exclusively on logic, and this is why axiomatization plays such an important role in them. Theories, however, formal and material, can often profit from other theories. Algebra, for example, is useful for geometry and geometry is methodologically effective in physics. How this is possible and how it can be logically or methodologically justified should have attracted more extensive logical-epistemological and philosophical investigations. Unfortunately, philosophers of science and mathematics do not seem very interested on these questions. I will address them later, by now a few examples should suffice. A material theory like arithmetic can, for instance, after being formally abstracted into "arithmetic", be formally extended into larger formal theories where maybe truths can be derived that yield truths about numbers proper. It is the task of logicians and epistemologists to investigate under which circumstances a formal extension *T'* of the theory *T* formally abstracted from a material theory T_D whose domain is *D* (as mentioned before *T* and T_D are formally equivalent) can be effective in proving truths about *D* (even when *T'* is not a conservative extension of *T* nor *D* an interpretation of *T'*).[79]

The Life-World I would like to close this chapter with a notion that became very important for Husserl at the end of his philosophical career, the concept of *life-world* or *Lebenswelt*. Present in Husserl's thought from the early days, it became increasingly important in his philosophizing as it matured. Essentially, the life-world is the pre-scientific world where the ego carries out its routine, habitual tasks. In *Ideas I* § 28, Husserl says that it is "this world where I find myself, which surrounds me, towards which are directed the spontaneous activities of consciousness". In *Crisis* §36 Husserl refers to the life-world as a world of "habitualities", or still, as the spatial-temporal world of things. There are many hints throughout Husserl's writings as to what he thought the life-world might be, but no unambiguous

[78] This is what Husserl meant by saying that phenomenology is a descriptive material a priori science; phenomenology cannot do without direct intuition on its domain of investigation, the phenomenon of intentionality. But some material sciences, such as, for example, physical geometry are able to obtain a strong enough basis of intuitive truths to afford giving up intuition altogether.

[79] Essentially, this is what Husserl called the problem of "imaginaries" in mathematics. See da Silva 2010.

characterization of it. Some things, however, seem clear; an important one is that the life-world is not the scientific world. Even though scientific activity is an activity of the life-world and the productions of science can be incorporated in the life-world in the form of "habitualities", the idealized worlds *produced* by science (for example, the world of geometrical idealities) are not themselves part of the life-world.

However, and this is relevant, the worlds of science are *rooted* in the life-world. Against the *metaphysical* thesis of Platonism (not simply "Platonism"; i.e. Platonism as an intentional construct), phenomenology presupposes that the worlds of science are intentional productions that are, ultimately, rooted in the life-world.[80] The road that leads from the life-world to the worlds of science is paved with intentionality. It happens that intentional productions are often incorporated as *habitus* in the life-world and intentional products mistaken by things of the life-world; for example, physical space of *perception* taken, however, as a properly *geometrical* manifold. Physical-geometrical space is not perceptual space, although the former is intentionally constituted from the latter. For Husserl, it is the phenomenologist's task to trace the *intentional genesis* of the productions of science from their beginnings in pre-scientific practices of the life-world. Transcendental history, as already emphasized, is an a priori investigation of intentional production. Husserl himself gave us a marvelous example of how it should be practiced in his "The Origins of Geometry" (Husserl 1954b). In this essay, Husserl follows the intentional production of mathematical-physical space as an objective ideality constituted from perceptual space, and physical-geometry as a communal activity of a priori investigation of mathematical-physical space devised as a methodological tool for investigating perceptual-space.[81] I will take up the problem of the intentional genesis of geometrical space, numbers, and sets, in later chapters.

Concluding In the following chapters, I will approach anew, from the phenomenological perspective sketched here, an array of problems in the philosophy of pure and applied mathematics. In particular, I will present a structuralist philosophy of mathematics able, I think, of avoiding the embarrassments of more popular structuralist (or structural) accounts of mathematics. My approach can, or so I hope, fare better than previous versions of structuralism particularly with regard to the often-neglected problem of the "unreasonable" effectiveness of mathematics in the empirical sciences and mathematics itself. But I must first settle some logical questions.

[80] "Knowledge of the objective-scientific world 'is founded' in the evidence of the life-world" (*Crisis* §34).

[81] As just said, this is a recurrent interest of Husserl's; in *Experience and Judgment*, for example, he traces judgments back to perceptions, and in *Crisis* he carries out a detailed analysis of the intentional production of mathematized physical nature from perceptual empirical nature.

Chapter 3
Logic

Logical principles are fundamental laws of reasoning and who says reason says argumentation and justification. But how logical principles *themselves* can be justified, and what would such a justification look like? Of course, principles cannot be justified in the same context of reasoning whose principles they are, since any chain of reasoning in this context presupposes the principles on which the reasoning depends. In other words, logical principles cannot, short of a circle, be justified logically.

Could logical principles be instead generalizations of experience? Is, for instance, the principle of non-contradiction – no assertion can be simultaneously true and false – a generalization for all assertions of the fact that no assertion has *actually* been verified to be both true and false? Could the principle of *excluded middle* – or, in semantic formulation, *bivalence*: a proposition is either true or false*, tertium non datur* – be only an optimistic generalization to the totality of propositions of our rather limited success in verifying propositions (when the opposite seems more reasonable)?

To take logical principles as very general empirical laws misses one of their most essential aspects: necessity. Despite empiricists' discomfort, the notion of necessity is so associated with logical principles that ignoring it stands to reason. No assertion has so far been verified to be both true *and* false *because* no assertion can *ever* be so verified. This is not a contingent, but a necessary fact. For this reason, logical principles cannot be generalizations of experience. For analogous reasons, they cannot be conventions chosen out of convenience either. We do not impose logical principles upon us, logical principles impose themselves. Considering all this, I take for granted that logical principles are necessary and a priori, although in a peculiar sense of the terms to be clarified here. Therefore, any serious attempt at justifying them must identify the sources of their necessity. However, there is no necessity that is not somehow self-imposed and if we want to uncover the ultimate sources of logical necessity, we must investigate how we manage to impinge it upon us. This will direct us to meaning bestowing and intentional action. As I will argue for here, the validity of logical principles depend, *subjectively*, on certain idealizing

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presuppositions concerning experiences available to the ego, truth-experiences, i.e. experiences in which truth and falsity are decided, and *objectively* on presuppositions concerning the domain of experience itself. Such presuppositions, I argue, can only be ultimately justified as aspects of the intentional meaning attached to the domain of experience. In short, as I plan to show, the validation of logical principles depends essentially on the a priori delimitation of the field of experiences available to the ego on grounds of principle. Not the experiences the ego can "reasonably" expect to effectively undergo, but those whose possibility can be determined completely a priori on lines of principle. This, I claim, can be accomplished only by examining the intentional meaning attached to the domain of experience. Grounding the validity of logical principles on intentional positing gives them, consequently, a *transcendental* dimension.

Ruling out that logical principles follow by inductive generalization from experience (against radical empiricism) is not to say that experience plays no role in logic.[1] In fact, according to Husserl, the foundations of logic demand a theory of experience. Some, like D. Lohmar,[2] think that this involves an analysis of what course of actions or cognitions are "reasonably motivated", allowing idealizations and generalizations. I believe otherwise, that the theory of experience Husserl demands does not have to do with *actual* experiences and "reasonable motivated" generalizations based on them, but with experiences that are *possible merely on grounds of principle*. In short, the justification of logical principles requires a transcendental investigation of what constitutes an experience possible in principle, which, I claim, depends essentially on how domains of experience are meant, i.e. on their characteristic sense of being.

One can take logical principles, as Husserl says in *Formal and Transcendental Logic,*[3] as axiomatic explications of the concept of truth. Husserl has in mind here what he calls *truth-logic*, i.e. the domain of formal apophantic logic concerned with assertions as truth-bearers.[4] The principle of bivalence, for example, attaches a definite truth-value to a proposition independently of any verification, and so the concept of truth to which the principle implicitly refers cannot be truth as a lived experience of conformity (or non-conformity in case of falsity) of what is asserted with what is directly experienced (i.e. intuited). Intuitive truth, however, is not the only conception of truth; truth can also be conceived in terms of *idealized verifiability*, and, obviously, this is the notion of truth the principle of bivalence refers to. An assertion has a definite yet maybe unknown intrinsic truth-value attached to it if *ideally* it can be verified. Of course, a justification of the principle of bivalence must

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[1] The relation of logic to experience constitutes an important problem to which Husserl dedicated a reasonable amount of attention. See, for example, his *Experience and Judgment*, Husserl 1973.

[2] Lohmar 2002.

[3] Husserl 1969, §76.

[4] In truth-logic "the judgments are thought of from the very beginning, not as mere judgments, but as judgments pervaded by a dominant *cognitional strive*, as meanings that have to become *fulfilled*, that are not objects by themselves, like the data arising from mere distinctness, but passages to the "truths" themselves that are to be attained" [Husserl 1969, p. 65].

spell out what "ideally" means here and the extent to which "can ideally" or "can in principle" differ from a simple "can" or a qualified "can actually or effectively". Idealizations are clearly at work here; presuppositions too, which we must delimit as to their scope and give a justification.

To turn away from these questions, as logicians do (and as they must as mere practitioners of logic), belies nonetheless the radicalism of philosophical inquiry; philosophy must uncover the hidden sources of validity of logical principles to clarify their meaning and limit their scope, if necessary.[5] This is what I call a *justification* of logical principles. Principles cannot be *verified*, for they are not *factual laws*, they cannot be proved because any such proof would be circular, but they can be justified. This chapter will be mainly concerned with the *transcendental* justification of logical principles along the lines traced by Husserl in his *Formal and Transcendental Logic*.[6] My specific goal is to answer the following question, which has consequences for the logic of mathematical reasoning: what sense of being justifies the idealizing presuppositions required for the validity of the traditional logical principles of reasoning of ("classical") mathematics?

The justification of logical principles touches a question that has always intrigued and challenged Husserl: how to fill the apparently insurmountable gap that separates subjectivity from objectivity. Take bivalence, for example, how can it be an *objective* fact that truth or falsity, which can only be properly attached to a proposition by means of a *subjective* evidential experience of adequacy or inadequacy with the facts, belongs to any proposition *independently* of any *actual* evidence? In other words, how can it be that propositions have *intrinsic* truth-value, or, equivalently, on which grounds can we claim that evidential experiences are at our disposition?

Meaningful assertions with empirical content, say, those of physics (for example, pure water at sea level boils at 100 °C), can only be said to be *determinately* true or *determinately* false by being *actually* empirically verified. We, however, *presuppose* that this assertion has a determinate *intrinsic* truth-value independently of any such verification. How can we? Of course, this presupposition must be indifferent to how things actually are in the world, but *not* to how things are *in principle* conceived to be and, correlatively, which experiences of validation are in principle available to us. In short, the principle of bivalence is valid for meaningful empirical assertions in general *because* empirical reality is conceived to be such that any meaningful empirical assertion is *in principle* experienceable. Physics tells how the world is, but

[5] For Husserl, *transcendental* logic "intends to bring to life the system of *transcendental principles* [*my emphasis*] that gives to sciences the possible sense of genuine sciences" [Husserl 1969, p. 16]. For him, genuine sciences are those that have overcome their naïve positivity and self-sufficiency by means of philosophical criticism.

[6] But – let us make it clear from the start – a justification of bivalence along the lines I will follow here cannot safeguard it against its critics, most notably the intuitionists, *on their own terms*, for what intuitionists actually contest is the intentional meaning of mathematical reality as posited in classical mathematics. As it will be made clear below, the intuitionist's denial of bivalence involves a refusal of the thesis of verifiability in principle. They do not accept it as a transcendental principle (which does not mean that they do not endorse transcendental principles of their own, as we will see).

the logic of physical assertions reveals how the world is intentionally conceived to be. Analogously for any science; so, in this sense, logic lies indeed at the foundation of any science.

There are three basic logical principles in truth-logic, identity, non-contradiction and bivalence. In one formulation, the principle of identity states that truth-values are stable, i.e. once a definite truth-value is attached to an assertion it remains attached to it; non-contradiction, that no assertion can be simultaneously true and false; and bivalence, that any assertion has a definite truth-value attached to it, independently of us knowing which.[7] Since truth involves a correspondence between saying and being, these principles tell us something about the "world" over which they rule. Identity, that the facts of the world are stable, i.e. determined once and for all (provided, of course, they do not involve indexicals). Non-contradiction, that no possible situation in the world is both a fact (i.e. a realized possibility) and not a fact (a non-realized possibility). Bivalence, that any possible situation is determinately either a fact (a realized possibility) or not a fact (an unrealized possibility) – a possibility not being realized is the same as the possibility complementary to it being realized, the possibility complementary to the possibility expressed by the assertion *A* being that expressed by not-*A*, and conversely.[8] Any science that so conceives the facts of the reality it investigates must necessarily endorse the above mentioned three principles of reasoning. The presuppositions on which they rest, I claim, are *not* empirical hypotheses but *transcendental presuppositions* that go with the intentional positing of the domain of reference, i.e. *a priori* truths concerning it. Let us now get down to the details.

Meaningfulness No assertion can be true that is not meaningful; our first task, then, is to determine what the criteria of meaningfulness are.[9] Remember that for Husserl formal logic is the a priori theory of science. Science is made of theories and theories are collections of assertions about particular domains of objects. Therefore, logic, the theory of science, has two domains of investigation: assertions and objects merely as such. The first is the domain of formal apophantic, the second, that of formal ontology. Assertions, however, fall under the scope of formal apophantic merely as unities of meaning and, consequently, possible conveyers of truths. Objects, on their turn, concern formal ontology only as "things" about which one can meaningfully assert something. For Husserl, one task of formal logic is to

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[7] This is the *objective* formulation of the principles, their *subjective* formulation refer to truth-experiences instead of simply truths. For Husserl, the justification of the objective versions of the principles depends on the justification of the subjective versions, which involves idealizing presuppositions regarding experiences in principle available to the intentional ego.

[8] In general, the situation complementary to that expressed by *A* is not-*A*. Therefore, the complementary of not-*A* is not-(not-*A*). Bivalence, however, identifies not-(not-*A*) and *A* as expressing the same situation.

[9] Of course, there is more to truth than meaningfulness; the attribution of a definite truth-value to a meaningful assertion requires a truth-experience. Whether this truth-experience must be actually lived or can be only idealized depends on the notion of truth involved. I will argue here that different conceptions of truth are related to different intentional senses of being.

determine the basic syntactic categories involved in making assertions and the a priori rules for their combinations in (syntactically) meaningful assertions and, correlatively, the basic formal ontological categories and their a priori compatibilities and incompatibilities. Assertions with material content are (materially) meaningful if in addition to formal meaning they respect a priori compatibilities and incompatibilities of *material* ontological (not only *formal* ontological) types.

Another relevant phenomenological dichotomy is that between the knowing subject and the object of knowledge, the act of asserting and what is asserted. Consequently, every logical principle has, for him, both a subjective and an objective version. In Husserl's logic, the subjective and the objective are as much correlated as the apophantic and the ontological. However, the phenomenological perspective on the relation between object of knowledge and knowing subject is radically different from the naturalist way of seeing. For the latter, the object of knowledge, in case it is objectively given, is independent of the knowing subject (who may interfere with the object of knowledge altering it to some extent, but not determining it in any essential way). Provided the object is objectively given, it exists independently of the subject, whose task is simply to investigate the given. The phenomenologist sees this relation differently. No object is given that is not given in an intentional act, including perceptual objects of the real world, and no intentional object is posited that is not immersed in a web of meaning, the intentional meaning attached to the object. The meaning of the object determines, in particular, what sort of object it is, how it can be intuitively presentified, and what are the criteria of validation for assertions about it. For example, objects of the physical world are meant as objectively existing (i.e. the same for all), transcendent (i.e. opening up to a "horizon" of hitherto hidden adumbrations that can disclose themselves in potentially infinite sequences of perceptions), and *objectively complete* (i.e. any possible situation in the physical world is objectively determinate as to its facticity and one, and only one of two complementary possibilities is a fact). Perception is the fundamental way of intuitively grasping physical objects, and assertions about them can only be *directly* validated perceptually (although never definitively, given the transcendent nature of physical reality). Validation can also be indirect, by means of logical inferences, but this is a derived and dependent form of validation. As this shows, the intentional object "physical world", although constituted from sensorial data, involves aspects that go beyond what is or can actually be object of direct perception.

The first task of apophantic logic is to identify the basic syntactic categories of enunciation (subject, predicate, relation, variable, quantifier, etc.) and the a priori laws to which they must abide to produce, by concatenation or combination, formally (or syntactically) meaningful assertions. This is the domain of logical grammar. Consider, for example, the "judgment" "John is and". It lacks sense in an obvious way, since it does not convey any thought, and no truth-value can be attached to it. The reason is that this "judgment" is formally (or syntactically) ill formed, since the open sentence "John is –" can only be filled in by a property-name (an adjectival expression), like "brave" or "a friend of mine", or by a verbal form, like "running" or "seated". It cannot be filled in, as in our example, by a conjunction.

In short, "John is and" lacks sense for it does not conform to the a priori grammar of syntactic types. According to Husserl, the first task of formal logic is to develop such a grammar. The system of all the relevant syntactic categories and the *a priori* laws of their combination, a pure universal logical grammar, is the first stage of the edifice of logic, and its task is to avoid nonsense of the type of our example (*Unsinn*).

But formal meaning is not enough for assertions to be meaningful and qualify as bona fide assertions. Consider now the "judgment" "the number 2 is green". It conforms to logical grammar, since "the number 2" is a name (denoting an object) and "green" an adjective (denoting a property of objects). But it lacks sense all the same. So, pure logical grammar alone is not capable of guaranteeing meaningfulness. It must be complemented by a grammar of contents, so to speak. Contents of judgments fall into categories whose *a priori* laws must also be investigated if we want to avoid senseless combinations like the one above. I will call *semantic rules* the laws of this grammar (Husserl does not, as far as I know). These rules have to do with the objective harmony or the conflict of material types. *Ideal* objects, for instance, like the number two, cannot *as a matter of principle*, not merely fact, be objects of visual perception, which solely can be colored. For this reason, our purported "judgment" is meaningless. That ideal objects cannot be seen is an *a priori* truth pertaining to the "region" of ideal objects. Ideal objects are, by definition, non-real, and so, in particular, non-physical objects. Since objects of visual perception, which only can possibly be green, belong to the region of physical objects, it is a priori true that ideal objects, which are non-physical, cannot be green. The counter-sense of "the number 2 is green" originates from attributing an *exclusive* property of a subclass of *real* objects to an *ideal* object. The task of semantic rules is to avoid *material counter-sense* (*Widersinn*), as opposed to merely formal nonsense (*Unsinn*).[10] Assertions that have both formal and material sense are the meaningful assertions, which only can purport (maybe unsuccessfully) to express facts.

This characterization of meaningfulness is purely formal, depending only on linguistic rules, both syntactic and semantic. Given, however, the close connections between saying (asserting) and being (that to which the assertion refers), meaningfulness plays the important role of predetermining the *a priori possibility of being*. I will come back to this soon.

Logic and Experience Before raising the question as to the relations between logic and experience, one must first define the terms. By "experience" I mean an *intuitive* act of presentification according to the notion of presentification of the domain in

[10] Although the investigation of particular ontological types does not belong to formal ontology, they too, of course, must obey general formal laws whose investigation do fall under the scope of formal ontology. The disjunction of the realms of real and ideal objects cannot be accomplished in formal ontology, nor the inclusion of physical objects into the category of real objects. But once these inclusions and exclusions are established it follows from a general formal ontological law that physical objects are not ideal. This law can be expressed thus: if A, B and C are ontological categories (regions), C a subcategory of B, A and B disjoint categories, then A and C are also disjoint. Mereology is an important chapter in formal ontology, which Husserl treated axiomatically in his third *Logical Investigation* (for Husserl, set theory, arithmetic and all *formal* mathematical theories are *formal ontological* sciences).

question (for physical bodies, for instance, it is sensorial perception; for mathematical objects, mathematical intuition, which requires higher-order intentional acts such as abstraction, idealization and ideation). Truth-experiences are, of course, a particular type of experiences. What, then, have logical laws and principles to do with experience? If one takes for granted that logical principles are *a priori* (one may not, if one believes that logical laws only reflect very general a posteriori features of experience), one believes, contrarily to empiricists, that instead of following from experience, logic imposes itself *a priori* and *unconditionally* over all experiences that are possible in a given domain of experience. One has here a hint where to look for the sources of validity of logical principles, namely, the realm of possible experiences, the experiences that one can expect in principle of actually experiencing in a given domain (the "in principle" here is essential). Experiences are possible if they can in principle, not necessarily effectively, be experienced. *Determining whether an experience is in principle possible has nothing to do with determining how to produce it effectively, i.e. with actualizable or "rationally justifiable" courses of actions*.

Logical principles, as we have decided, do not express general empirical facts, hypotheses or presuppositions but, as I said, transcendental hypotheses. Hypotheses are transcendental to the extent that they are rooted in intentional constitution; transcendental hypotheses essentially spell out *intentional meaning*. Otherwise put, the validity of logical principles rests on validating idealizing presuppositions, whose validity, on their turn, depends on specific modes of being. Since modes of being are rooted in intentional constitution, logical principles are ultimately expressions of intentional meaning. Hence, logical principles can only be clarified and justified via transcendental-phenomenological analyses of meaning and intentional constitution. From a subjective perspective, such a justification depends essentially on which experiences the ego can determine a priori as available to it (some would say, which course of actions it can "reasonably" expect to be able to undertake, but as I have already said, I believe this is a watering down of the real task facing the ego). The *a priori* determination of which experiences, truth-experiences in particular, are possible in principle consists, I claim, in the determination of which experiences *are not ruled out a priori* according to the constituted sense of being attached to the domain of experience. This characterization seems to impose itself naturally for the following reason. *A priori* determination of what counts as a possible experience cannot depend on what is *actually* experienceable, otherwise the determination would not be a priori. Hence, it must depend only on what can, but only *on grounds of principle* be experienced, which must depend solely on how the realm of experience in question is *intentionally constituted*, for this only is primarily given. In short, what can or cannot be *in principle* experienced in a domain of experience depends on the meaning and sense of being intentionally attached to this domain. Summarizing, the a priori justification of logical principles depends on which experiences are meant to be possible in principle, which depends on how the domain of experience is intentionally meant to be, that is, the intentional meaning attached to it.

Some consequences immediately follow. One is that logical principles have a scope, restricted to the domain of experience on whose sense of being they depend and where, consequently, they rule. Another is that logical principles are universal only relatively to the domain whose posited sense validates them. Logical principles apply to *all* the *meaningful* assertions referring to a domain, regardless of their actual material content, i.e. that which they actually express, but only insofar as they refer to *this* domain (or any other sharing the same relevant intentional features). Contradictory as this seems, *logical necessity and universality are not absolute, but relative to a way of conceiving*.

Logical Principles Here, I will be mainly concerned with the justification of the three basic logical principles, the principle of identity, the principle of non-contradiction and the principle of excluded-middle or bivalence. The justification of the principles of "positive" logic (to use an expression dear to Husserl) constitutes a sort of vindication of the fundamental ("realist") tenets of "positive" science, namely, an objective, stable, coherent, and completely determined world exists "out there" that is in principle knowable and whose truths science can in principle *completely* disclose. Therefore, a transcendental-phenomenological justification of "positive" logic must rest ultimately on the phenomenological clarification of the intentional construct "world" as understood in positive science, not, as is usual, on more or less well-disguised metaphysical presuppositions about the world passing for established facts.

Two notions are central in Husserl's conception of logic, distinction and clarity as attributes of thoughts, judgments, assertions or whatever truth-bearers. *Distinct* assertions are assertions that are both syntactically and semantically meaningful; *clear* assertions are those with an *intuitively determined* truth-value (either the true or the false) – *clear* judgments are judgments *enlightened* by intuitions. Husserl identifies two provinces within formal apophantic logic, the logic of consequence and truth-logic; the central notion of the former is distinctiveness, that of the latter, clarity. The task of consequence-logic is to determine the laws by which distinctiveness is achieved in combinations of elements of thought and preserved in combinations of thoughts and chains of reasoning; that of truth-logic, correlatively, is to determine the laws by which truth-values are preserved or change in combinations of thought and chains of reasoning. As, for Husserl, judging is both an act and the product of that act, both asserting and that which is asserted, he believes that the three logical laws mentioned above have each two versions, one related to the act itself, another to the assertion as product. In short, for Husserl each logical principle has a formulation in the logic of consequence and another in truth-logic, but also a subjective and an objective version. Here, I will restrict my analyses to logical principles as principles of truth-logic, that province of formal logic that has to do with assertions insofar as they have a truth-value attached to them.

Assertion can only admit *intrinsic* truth-values, i.e. truth-values that belong to them independently of actual verifications, if they are meaningful, or, more precisely, a *necessary* condition for assertions to have definite truth-values attached to them, independently of verifications, is that they are meaningful, both syntactically and semantically (whether the condition is also *sufficient* will be discussed later). Whereas true assertions denote *effective* facts, meaningful assertions denote *possible* facts. The question “what is a possible fact?” has, then, a straightforward answer, it is a fact represented by a meaningful assertion. In other words, assertions are meaningful if, and only if, they represent possible facts.[11] The notions of “meaningful assertion” and “situation a priori possible” are correlate notions, a meaningful assertion expresses *that* a certain situation is an a priori possibility in the domain to which the assertion refers, which is effectively the case (a fact) if, and only if, the assertion is true (or *experienced* to be true in the most fundamental conception of truth; i.e. intuitive truth). This is what to be meaningful *means*. Meaningful assertions, *precisely by being meaningful*, express situations that are in some sense possible; meaningful assertions are true if the a priori possibilities they express are actualized. By *asserting* a meaningful assertion, the judging subject commits himself, objectively, to the factuality of the possibility the assertion expresses (its truth) and, subjectively, to the possibility in principle of clarifying the assertion in an intuitive experience. But how can he commit himself to these things if not by presupposing them? And where to ground such presuppositions if not in the intentional meaning attached to the domain of experience?

The Principle of Identity The principle of identity can be formulated thus: $A = A$, where the variable “A” stands for a name. It is implicit in the formulation that each “A” occurs in a different context, corresponding to a different act of naming. One could read the principle as saying that no matter which name, provided it is used non-ambiguously, always denotes the same object in a given context of reference. Identities are, as we know, objective correlates of acts of identification, even this supposedly “vacuous” and “obvious” identity. In this case, the objects that receive the same name are identified as the same object. However, one cannot identify objects of different experiences as the same object if these objects are not *capable* of being the same in different experiences. For example, if objects of experience were *immanent* to experiences and consequently vanished with them. The principle of identity, then, is saying something about the objects of the domain of reference and, correlatively, the domain of experiences available to the subject. *Objectively*, the principle states that objects in the domain of reference (where the principle is valid), to which one can refer by naming, preserve their identity in the flux of experiences. *Subjectively*, that different experiences can be experiences of the same object. One would not be justified in supposing that names are “rigid designators” across experiences (adapting an expression of Kripke’s) if objects could not manifest themselves as the same objects in different experiences. The validity of the principle of identity, then, depends on presuppositions, essentially, that objects of experience can subsist as the objects they are in the flux of experience.

The interesting question is what sort of presupposition this is. Realists have no difficulty in accepting that “one can always come back to the same object” *because* “the same object” exists and persists out there independently of any experience and

[11] This equivalence highlights the close connections between apophantic and formal-ontology. Determining meaningfulness on the apophantic side is tantamount to determining possibility of being on the ontological side.

always available to experiences. Since from a transcendental phenomenological perspective all objects are objects-for-the-ego, a presupposition is needed to secure the persistence of objects in being independent of particular experiences. However, the realist and the phenomenological presuppositions have entirely different characters. For realists, the subjective presupposition that one can always come back to the same object rests on the objective "fact" that the object is always "out there". However, short of vicious-circularity, this "fact" cannot be a fact of experience; it can only be a metaphysical presupposition. The phenomenologist must also presuppose the persistence in being of the object of experience, but in his case, the presupposition has a transcendental character – the object is *meant* that way.

$A = A$ can be read now as stating the "fact" that the objects of the domain in question are meant as capable of appearing as the same objects in noetically different experiences, or still, that each time the ego uses the name "*A*" unambiguously, it is referring to the same object. Meaning objects as capable of being re-experienced as the same objects in different experiences *justifies* acts of identification. Indeed, we cannot identify objects of different experiences as the same objects if they are not conceived as capable of preserving their identities in the flux of consciousness. If the intentional ego is a communal ego, the principle of identity guarantees the objectivity of objects of experience, i.e. that they can appear as *the same for different subjects* in different experiences. This is not simply a "fact", as for the realist, but a true *transcendental* presupposition rooted in intentional constitution. Phenomenologically understood, the principle of identity is a transcendental principle expressing the intentional *presupposition* that objects *persist in being*, independently of being actual objects of consciousness.

One can also approach the issue from the perspective of truth. According to Husserl,[12] the principle of identity in truth-logic states a *fact* about truth (since logical principles explicate the concept of truth). It says, objectively, that truth-values are stable, i.e. that once a truth-value is attached to an assertion it remains attached to it and, subjectively, that once an assertion is judged to be true (resp. false) it must, in any further judgment-act, be also judged to be true (resp. false). In short, truths can be stocked. For this to be justified idealizing presuppositions are required.[13] Subjectively, that different judgments can refer to the *same* state-of-affairs and, objectively, that states-of-affairs persist in being. In other words, no matter the angle of approach, the principle of identity of formal logic states that objects or states-of-affairs persist in being and can re-present themselves in different experiences (unless, of course, they cease to exist). This can only be so if the *domain* of experience in question, that is, the system of objects and states-of-affairs is *meant* as an objectively existing domain where objects and facts can resurface as the same in the flux of experience. In short, f*lux of consciousness does not imply flux of being*.

It is not difficult to see the extent to which science depends on the principle of identity. In fact, as Husserl claimed, this and all other logical principles are

[12] Husserl 1969, §77.

[13] Husserl 1969, chap. 3.

preconditions of possibility of objective sciences. However, the stability and objectivity of scientific domains (or domains in general, to which one can meaningfully refer) is neither a metaphysical presupposition nor a hypothesis that could somehow be verified and possibly dismissed as false. Rather, the presupposition that the "world" exists "out there" objectively and stably is a *transcendental presupposition*, that is, part of the intentional meaning attached to the world, any world, in intentional positing. Whereas from a subjective perspective identity concerns the domain of experiences possible in principle in the world: any object of experience can be re-identified in further experiences, objectively it concerns the world itself. If *A* is the domain of meaningful assertions referring to *D*, the principle of identity is true in *A* if, and only if, objectively, *D* is a stable and objectively given domain of experience and, subjectively, the ego can undergo different experiences in *D*, intuitive or not, with the *same* content.

The Principle of Non-contradiction In subjective version, the principle of non-contradiction states that no experience of verification can be simultaneously an experience of confirmation *and* disconfirmation of any given meaningful assertion, in objective version that no assertion can be simultaneously true *and* false, in agreement *and* in disagreement with the facts. Contradiction, either subjective or objective, either in saying or in being is never a possibility. And this is not a contingent but a necessary truth.

Despite dialetheism, i.e. the view that there are true contradictions, and the existence of formal systems of logic in which contradictions are admitted (paraconsistent logic), the principle of non-contradiction lays its roots deeper into our conception of reality and rationality than the principles of identity and bivalence. The principle of *ex contradictione quodlibet*, which allows the inference of *any* assertion from a contradiction is the expression itself of the collapse of rationality in the face of contradiction.[14] Paraconsistent logic can be consistently interpreted by weakening the notions of negation or truth, and examples of objective contradictions are not very convincing. Whereas the principle of bivalence can be denied by refusing to idealize truth-experiences and by sticking to the notion of truth-experiences as *intuitive experiences*, the principle of non-contradiction cannot be dismissed without profound effects on our very conception of experience or a world given to experience, if, of course, our notions of negation and denial remain unaltered. An experience of *dis*confirmation cannot harmonize with an experience of confirmation of the same content because the world *must* be coherent with itself. In this sense, the principle of non-contradiction is not at the same level of the other two that I consider here, identity and bivalence. Whereas the latter principles rest on transcendental presuppositions concerning particular domains of experience, non-contradiction rests on presuppositions concerning *no matter which* domain of experience. It lays its roots in our very conception of a domain of experience simply as such.

[14] ECQ is not necessarily a consequence of the principle of non-contradiction in non-classical logic; in fact, paraconsistent logic admits a form of the principle of non-contradiction (*not*(*not-A&A*) is a theorem) but does not admit ECQ. This rule has not always been seen as a valid rule of inference in the history of formal logic.

Non-contradiction states that, for any meaningful assertion *A*, there cannot be an experience of confirmation of *both A* and not-*A* (but it says nothing about whether for any *A* there must be either an experience of confirmation of *A or* one of not-*A*). Therefore, the confirmation of *A* (resp. not-*A*) implies the disconfirmation of not-*A* (resp. *A*). Now, an experience of disconfirmation of *A* counts as an experience of confirmation of *not-A*, but the disconfirmation of not-*A* is *not* tantamount to the confirmation of *A*.[15] Non-contradiction joins the principles of identity and bivalence as a priori determinations of the domain of experience; the difference is that whereas non-contradiction refers to *any* domain of experience, the other two refer to particular domains of experience.

The world, any world, is a coherent totality of facts and therefore no pair of contradictory assertions regarding the world, any world, can ever be validated in experience. This is the sense of being of reality – any reality – that goes with the principle of non-contradiction. The denial of the principle of non-contradiction entails the denial of *this* conception of reality.

The Principle of Bivalence This principle states, subjectively, that *any* meaningful judgment can *ideally*, *in principle* be verified, i.e. confirmed or disconfirmed in a truth-experience, i.e. an experience of conformity or conflict of the content of the judgment with relevant facts, and, objectively, that meaningful assertions have *intrinsic* truth-value, the true or the false, attached to them, independently of any *actual* verification. How can it be that truth or falsehood, which can only be properly attached to assertions by means of *subjective* evidential experiences of adequacy or inadequacy of the content asserted with the facts, belong to assertions *independently* of such experiences being *actually* carried out? Or still, how can any assertion be either true or false *in itself*, *intrinsically*?

For Husserl, only an idealizing presupposition can close the gap. Although the assertion has not yet been verified, it *can in principle* be; the actual verification stands as an *ideal*. If the ideal is actualized in an actual verification, as it in principle can, a definite truth-value is attached to the assertion and, by the principle of identity, this value has *always* been attached to the assertion (if its meaning did not change). So, by *presupposing* the *ideal* possibility of verification of meaningful assertions, it follows that meaningful assertions have definite truth-values attached to them independently of actual verifications.

One can think of *ideal* verifications as limit points at the infinite, *foci imaginarii*, the paths to which are cut as we proceed but that may be forever beyond reach, very much like Kantian regulative ideas. In metaphorical terms, they organize the field of experiences as vanishing points organize the pictorial perspective space. The complete experience of truth may not actually belong to the ego's field of experiences – as vanishing points do not belong to the pictorial space – but as an imaginary focus this ideal unifies all the partial and limited experiences of truth into an integrated whole. This is already enough to see how problematic it is to read Husserl's notion

[15] An experience of disconfirmation of not-*A*, which is an experience of confirmation of not-(not-*A*) is not necessarily an experience of confirmation of *A*, unless either *A* or not-*A* must be true.

of verifiability in terms of actual decidability, which, if anything, can only be thought metaphorically in terms of points at finite distances actually reachable by well determined pre-established paths.

If we interpret Husserl, as I do, as grounding the truth-in-itself on the presupposition of verifiability, and interpret *this* notion, as S. Bachelard does, in terms of the existence of effective decision procedures, then some results of modern logic (Gödel's incompleteness theorems) would immediately turn Husserl's analysis into an argument *against bivalence*, and thus an argument for intuitionism.[16] But Husserl never considered trimming mathematics along intuitionist lines and never suggested restricting *bivalence* to a proper subclass of the class of all well-formed assertions.

One could say, of course, that since Husserl was not familiar with some of the most relevant results of modern logic, he could be excused of making logic rest on impossible presuppositions. But this excuse is not available to us. We seem to have only two choices in this matter if we interpret the condition of verifiability in terms of effective decision procedures, either to abandon Husserl's analysis of the ultimate grounding of the principle of bivalence or embrace intuitionism. But I believe that, in this case, there is a *tertium*. We can simply not give the notion of *ideal* verifiability involved in the subjective version of the principle of bivalence the sense of *effective* decidability. I will soon show how to interpret this notion in a way both to make sense of Husserl's analysis and avoid his defeat by modern logic.

But, after all, even if Husserl had meant his notion of verifiability in terms of *actualizable* procedures of decidability (which he did not), it would *not* follow that he must surrender to Gödel, since incompleteness theorems do not imply the *absolute* undecidability of any assertion (that is, undecidability no matter which context of reasoning), not even that of the consistency of arithmetic (which, incidentally, was shown with all the necessary mathematical rigor by Gentzen, even though this proof cannot be formalized in a way so as to fall under the scope of Gödel's theorem).

Anyway, regardless of Husserl, what sense can we make of the notion of ideal verifiability? If *bivalence* is, as I believe, an a priori principle, ideal verifiability cannot depend on matters of fact, such as the actual existence or not of decision procedures. It cannot depend on how we happen in fact to think either; the validity of a

[16] There have been some misinterpretations of the thesis of decidability in the literature. Suzanne Bachelard's classic *A Study of Husserl's "Formal and Transcendental Logic"* (Bachelard 1968) contains one. For her, decidability means *effective* decidability. Jacques Derrida in his introduction to Husserl's "Origin of Geometry" has an interesting discussion on how Husserl's notion of decidability, which is closely related to his idea of a *complete* – or *definite* – theory, as Derrida correctly remarks, need not be read as effective decidability. According to Derrida, the notion of decidability, correctly understood, must be conceived in terms of the horizon of science in general, and mathematics in particular, thus saving Husserl's idea of a nomological mathematical theory from the limitations imposed by Gödel's theorem. Derrida correctly notices that "[g]eometrical determinability in the broad sense [as opposed to decidability in strict sense, that is *effective* decidability – *my note*] would only be the regional and abstract form of an infinite determinability of being in general, which Husserl so often called the ultimate horizon for every theoretical attitude and for all philosophy." (Derrida 1989, p. 55, n. 51).

logical principle is not a fact of psychology. The validity of any principle can only be a matter of principle, and then a problem for transcendental philosophy to investigate. Logical principles, bivalence in particular, as principles of apophantic logic, i.e. the logic of assertions, presuppose a world to which assertions refer; there must then be a connection between the validity of logical principles and the meaning attached to the domain of reference, i.e. how this domain is conceived to be. The validation of a logical principle demands, then, that we investigate the intentional meaning attached to the domain to which assertions submitted to the principle refer. In case of bivalence, what is important is how the intentional ego predetermines a priori the domain of experiences in principle possible in the relevant domain. That is, how it determines completely *a priori* which experiences it can in principle expect to be capable of actualizing. But as I have already shown, the experiences the ego can determine a priori as actualizable in principle are precisely those that he cannot rule out *a priori* by considering only the meaning intentionally attached to the domain of experience. Summarizing, the validity of the principle of bivalence depends ultimately on the a priori determination of which experiences are possible in principle, which depends on intentional determinations of meaning.

Spelling this out. What kind of "world" does the presupposition of ideal verifiability require? Better, what is the sense of being that a domain must be invested with to validate the "idealizing presupposition" of ideal verifiability? The answer, of course, is that such a presupposition requires that the domain of experience be objectively complete. Or, in other words, that any *possible situation* in the domain must be definitely either the case (a fact) or not the case (not a fact) and one out of two complementary possible situations, expressed by an assertion and its negation, must be the case (a fact). The existence of an objectively complete world to be exhaustively investigated under the guiding *ideal* of a definite (i.e. syntactically complete) theory is a *precondition* of *objective* science and the principle of bivalence is there to *guarantee* it.[17]

The clarification of the scope and meaning of the principle of bivalence and its justification necessarily requires, then, that the *ontological* presupposition of objective completeness be itself clarified and justified: why is it licit to attribute objective completeness to the "world" (any world)? How can we presuppose *a priori* that one (and only one) of two complementary but incompatible possible situations is a fact and that any possible situation can in principle be checked against the facts (the presupposition of ideal verifiability)? The answers to these questions lie in the sense

[17] A science is *positive* if it presupposes the existence in itself of its domain of interest as an objectively complete and in principle completely determinable domain of knowledge; the role of the principle of bivalence is to grant "positivity" to the science that embraces it. This is true even for objective logic itself: "…logic, by its relation to a real world, presupposes not only a real world's being-in-itself but also the possibility, existing "in itself", of acquiring cognition of a world as genuine knowledge, genuine science, either empirically or a priori. This implies: Just as the realities belonging to the world are what they are, in and for themselves, so also they are substrates for truths that are validated in themselves – "truths in themselves"" [Husserl 1969, p. 225]. "Thus logic, as Objective in this new sense, as the *formal logic of a possible world*, finds a place for itself in the multiplicity of "positive" sciences …" [Husserl 1969, p. 227].

of being attached to the "world"; therefore, to justify the principle of bivalence is to justify the presuppositions it involves by showing that they are imbedded in the intentional meaning attributed to the "world" over which the principle rules.

The meaning intentionally attached to a domain of experience, a "world", plays such an essential role in the justification of the presupposition of objective completeness because it *determines the sense of being* of the "world". To the extent that the intentional positing posits an objectively complete world, we are justifying in attributing a definite, although maybe indeterminate truth-value to any meaningful assertion about this world. Assertions are meaningful, as already said, if and only if they represent possible situations in the world and possible situations can in principle be checked against the facts if and only if the world is objectively complete, i.e. a maximally consistent domain of facts.

The meaning attributed to a domain of reference also plays a role in determining which assertions about the world are meaningful. Indeed, as already said, an assertion is meaningful if and only if it has both syntactic and semantic senses. Semantic meaning depends, as I have already discussed, on semantic rules, which, on their turn, depend on the sense attributed to the many categories of being of the world. A world contains many different ontological regions, objects of different types, and no assertion about this world is materially meaningful that does not respect a priori compatibilities and incompatibilities of ontological types. But a priori ontological compatibilities and incompatibilities are also aspects of the intentional meaning attached to the world. Once assertions respect syntactic and semantic rules of formation they are meaningful, that is, able to represent possible situations of the world. The following, then, are equivalent notions for assertions: having syntactic and semantic sense, being meaningful, representing possible situations. The objective completeness of the world to which assertions refer guarantees, moreover, that they possess intrinsic truth-value by being in principle, given the completeness of the world, capable of being verified.

Let us see now how Husserl himself approached the issue of justifying the principle of bivalence. First, he believed that it was an *a priori*, universally valid logical principle. In his own words[18]: "the following pertains *a priori* to every proposition each one is true or false". But, as he recognizes, since truth-values are not constitutive components of assertions, the fact that a *definite* albeit maybe unknown truth-value can be attached to any assertion *independently of verification* is, he says, a "very remarkable" fact. Indeed, since it is a fundamental tenet of Husserl's epistemology that the truth or falsity of assertions (in the most basic sense of the term, evident truth and evident falsity) depends, respectively, on assertions being adequately fulfilled by intuitions or the evidence that such a fulfillment is impossible, how can it be that assertions have truth-value attached to them independently of truth-experiences?

Subjectively, the principle of bivalence guarantees that any meaningful assertion is *decidable*, in the sense of being already, in itself, decided, i.e. it is in itself true or in itself false. The meaning and scope of this *thesis of decidability*, its "presuppositions

[18] Husserl 1973, p. 297.

and necessary limitations" must then be investigated and clarified. In Husserl's words[19]: "many judgments are *for us* undecided with respect to their legitimacy, and for us the majority of judgments that are *possible in general* [my emphasis] are, therefore, not "decidable" *de facto*, but are so *in themselves*. *Any judgment is "decided" in itself*"". This quote raises many questions: (1) what does it mean to say that a judgment is *possible in general*? (2) What does it mean to say that a judgment is *decided in itself*? (3) How does intrinsic decidability relate to verifiability, does it presuppose the existence of means of verification that may, nonetheless, be inaccessible to us; how can *this* be presupposed *a priori*, regardless of the content of propositions (i.e. what they express) and the means available for the ego to carry out the required verification in the relevant context[20]? (4) What are the *limitations* of the thesis of decidability to which Husserl refers?

Recall that, for Husserl, assertions proper, that is, meaningful assertions, are those that have definite truth-values but that he makes a distinction between intuitive truth and intrinsic truth (for which the experience of truth is only an ideal). Assertions with intrinsic truth-values are, he says, "decided" in themselves (intrinsically decided), which means that actual decision stand as an ideal. Assertions with intuitive truth-values are those that are effectively verifiable. The precondition for assertions to have intrinsic truth-values is *distinctiveness*, i.e. syntactic *and* semantic meaningfulness; that for having intuitive truth-values is *clarity*, i.e. effective verifiability. The thesis of decidability implicit in the principle of bivalence, however, does not refer to effective decidability but intrinsic decidability. Bivalence, then, collapses two different notions, meaningfulness and intrinsic decidability. On what grounds, we may ask.

The *a priori* character of the thesis of decidability requires, of course, that decidability be understood as intrinsic decidability, since no principle can determine *a priori* which propositions can *actually* be verified, only which propositions can *in principle* be verified, and this is a big difference. *Actual* or *effective* decisions are posed only as regulative ideals. Since the *actual* verification of all meaningful propositions amounts to a *complete* experience of reality (that is, the totality of facts), the thesis of intrinsic decidability translates into a thesis of intuitive accessibility of *any* aspect of reality *as a matter of principle*.

From a "naïve", that is, non-phenomenological perspective, this could be read as a *metaphysical* presupposition concerning the accessibility of reality, or, correlatively, a *psychological* presupposition concerning our powers of intuition. But, for

[19] Husserl 1969, §79.

[20] Husserl himself was bewildered by this presupposition: "[judgments are supposed to be decided 'in themselves']. That surely signifies: *by a 'method'*, by a course of *cognitive thinking*, a course existing in itself and intrinsically pursuable, which leads immediately or mediately to an adequation, a making evident of either the truth or the falsity of any judgment. All this imputes an astonishing a priori to every subject of possible judgment and therefore to every actual or conceivable human being – astonishing: for how can we know *a priori* that courses of thinking with certain final results 'exist in themselves'; paths that can be, but never have been, trod; actions of thinking that have unknown subjective forms and that can be, though never have been, carried out?" [*FTL*, p. 175/197–98].

Husserl, the thesis of decidability has a *transcendental* character. Consider the following question: how can we be sure that *any* proposition can be confronted with the facts without endorsing metaphysical presuppositions about reality and our power to access reality in intuitive experiences? The answer Husserl gives is straightforward: by a transcendental hypothesis. By respecting the rules of syntactic and semantic meaning, the ego determines completely *a priori* the scope of the domain of possible situations – precisely those expressed by meaningful propositions – which are, then, *hypothesized* to be ideally verifiable.

Let me reinforce this: a meaningful proposition represents a possible aspect of reality in principle accessible to experience, that is, a possible content of evidence, *as a matter of principle*. In fact, since, as we have already noticed, we cannot determine based on our *de facto* experiences which experiences are reasonable to expect in the future, we need a principle to accomplish this task, a transcendental principle in fact, whose task is to delimit the boundaries of the domain of *possible* experiences, that is, the boundaries of reality. The thesis of verifiability does precisely this; it *identifies* on grounds of principle meaningfulness with the possibility of evidence.

Let me consider now some problems that the identification of meaningfulness with the possibility of experience seems to pose (Husserl does not deal with them, as far as I know). Since the negation of a meaningful assertion is also meaningful, possible situations come in pairs, but only one is an actual fact. Although the truth-value attached to a meaningful assertion is, say, the truth, it *could* have been the false. This is fine with contingent assertions, but what about necessary ones? Consider the following proper judgment: "a sequence of seven 7's appears in the decimal expansion of π". It has formal and material sense, since any sequence of digits can in principle appear in any decimal expansion. But if this statement is true it is *necessarily* true, and if it is false it is *necessarily* false. Suppose that, as a matter of fact, it is false. In what sense, then, can we still say that the situation it denotes is a *possible* evidentiable situation? This apparent problem dissolves when we notice that semantic rules involve *types* only, not instances of types. The type of sequences of digits is materially compatible with the type of decimal expansions, even though some *instances* of each may be necessarily incompatible as a matter of fact.

Saying that a meaningful judgment represents its content as the content of a possible evidence *only means* that the terms of the judgment fall into materially compatible types. In a way, the notion of a content of possible evidence is a negative one: a situation can *in principle* be experienced if we can see *a priori*, *considering material categories only*, that it cannot be excluded from being a content of experience (that is, a fact). Syntactic and semantic rules have then the power of determining what is and what is not possible in principle, but only in so far as *types* are considered.

In our example, even if the assertion about the decimal expansion of π were false, and then necessarily false, there must be a sense of possibility attached to its negation, given that the assertion itself is not evident and could have been *conjectured* to be true *before* a proof of its falsehood became available. In short, there must be a sense of possibility for meaningful mathematical assertions in general for the distinction between a proven and a conjectured mathematical fact to make sense. To say that a mathematical assertion represents a possible situation, even if it is false, is simply to say that it is a *distinct* assertion, formed in accordance with the sense of being attached *a priori* (i.e. before *clarification*) to the mathematical domain to which it refers. The sense of possibility attached to meaningful mathematical assertions is a priori and purely formal.

One way of interpreting the principle of bivalence is as the completion of the domain of effectively verifiable truths. The domain of *actually verified* truths (which describes the aspects of reality we have already experienced) is obviously incomplete, but ever growing; it opens up to a horizon of not-yet verified, but effectively verifiable truths (we *know how* to bring about the required experience of evidence). But this is still an incomplete totality. So, reason and science (or logic *at the service* of science) compel us to postulate its completion by a domain of truths we *still* do not know how to bring to evidence but that must exist to provide a full picture of reality; these are the truths-in-themselves. Transcendental principles are required now to determine the scope of this domain; the principles of non-contradiction and bivalence work together to carry out this task. The former decrees the *consistency* of the domain of truths-in-themselves, the latter its *maximality* within the domain of all formally possible truths. These principles together determine that reality, as a *complete* totality, is a consistent and *maximal* domain (this is the reason why, for Husserl, logically complete theories are the ideal of science). Hence, to say that a proposition is either true-in-itself or false-in-itself (in which case the negation of the given proposition is true-in-itself) means only that *either* this proposition *or* its negation must describe an aspect of a *complete* reality.

Now, does the identification of meaningful propositions with propositions with definite truth-values accomplished by the principle of bivalence imply the *necessity* of all truths, as already observed by Aristotle with respect to the contingent future? Let me say a few words about this question. Consider the proposition p = there will be a naval battle tomorrow; is to say that p has a definite truth-value tantamount to saying that the existence of a battle *tomorrow* is already decided *today*? Saying that p has a fixed, yet unknown truth-value today could only mean that in a *complete* reality standing at the end of time or, if you wish, in the mind of an omniscient god, the issue of a naval battle tomorrow would have been settled *sub specie aeternitates*, *not* that this issue is already settled *today* – simply because today reality is not yet complete: the issue concerning the occurrence of a naval battle tomorrow is not settled. But, of course, p is still *decidable*; the decision procedure is simply "wait and see". But we must wait until tomorrow. The principle of bivalence does *not* imply fatalism because it does not predetermine the facts; it only determines that at *any given point in time* a temporally changing reality must be objectively complete, that is, a maximal consistent subdomain of the domain of all (formally) possible situations *at that time*. Empirical reality, immersed and changing in time, is supposed to be objectively complete *at any point in time* – all assertions referring to *present* reality are in themselves decided *now* –, but by being changeable in time, reality at time t is not supposed to be objectively complete at any time prior to t – assertions about the future are, in general, *not* decided in themselves *now*. The puzzle with future contingents derives from collapsing different moments of a temporal process, supposing that the reality of today is the same as the reality of tomorrow. They are not.

Platonism and Intuitionism The conflict between Platonism (realism) and intuitionism rests essentially on the validity of the principle of bivalence. *Ontological* Platonists believe that since mathematical realms are *ontologically independent* they are also, *consequently*, *objectively complete* and that all possible mathematical facts are already determined in themselves. *Truth-value* Platonists, on the other hand, believe that all meaningful mathematical assertions have definite, although maybe unknown truth-values. Obviously, ontological Platonism entails truth-value Platonism, but as my analyses here show, the converse is *not* true; one can consistently claim that meaningful assertions always have definite albeit maybe unknown truth-values without endorsing ontological Platonism. A point however deserves attention; whereas truth-value realism is a *metaphysical hypothesis*, its bracketed version is a *transcendental presupposition*. The difference is that metaphysical hypotheses have to do with how reality *is*, whereas transcendental presuppositions on how reality *is conceived to be* or how it *must be* given how it is conceived to be.

Intuitionism denies both claims; for it, mathematical realms are temporal domains in the making, so to speak, and it can happen that certain possible situations are not or not *yet* determined as to their facticity. Therefore, mathematical reality is neither ontologically independent nor objectively complete. Also, since for the intuitionist nothing can be true or false that is not *experienced* as such in a truth-experience, that is, "truth" has for him only the sense of *intuitive* truth, he cannot endorse truth-value realism either. It may be that the fact to be verified is not yet determined or it is, but one does not have the means of actually verifying it. Platonist and intuitionist presuppositions have the consequence that Platonists endorse the principle of bivalence and intuitionists deny it.

The intuitionist refuses to accept the presuppositions underlying Platonist claims. It refuses both the metaphysical claims that mathematical realms exist and are already completely determined in themselves and that mathematical assertions have intrinsic truth-values or, equivalently, that they are decided in themselves; for him, decidability always means *effective* decidability. One cannot claim that an assertion is verifiable if one does not know how to verify it, even if the verification is not actually carried out.

But intuitionism is not completely free of presuppositions. By accepting the principle of identity, which guarantees that truth and falsity are stable, intuitionists accept that once a verification has been carried out, the verified statement, provided it preserves its meaning, is a permanent acquisition, which one can stock and eventually comeback to; verifications need not to be reenacted. However, there is no guarantee that the coming into being of mathematical domains, which is a temporal process, does not change the meaning of already verified assertions, which would require new verifications. But one could always say that in these cases the assertions are no longer the *same* assertions.

Intuitionists also accept the principle of non-contradiction, but regardless of the fact that contradictory assertions have never been found to be both true, there is no a priori guarantee that this will *never* happen independently of fixing a priori that it *cannot* happen. Indeed, in intuitionism, to truth-verify *not-A means* that one is in possession of a process that produces a truth-verification of an absurdity from a supposed truth-verification of *A*. Hence, once *A* is truth-verified, *not-A* cannot ever be, because – and this *is* a presupposition – absurdities cannot ever be true. Platonists think of consistency in terms of objective facts. To presuppose a priori that the members of a pair of conflicting possible situations cannot be both facts, that is, that reality is *objectively consistent* is to presuppose an objective fact. Intuitionists, on the other hand, think in terms of subjective procedures of verification, presupposing instead that truth-verifications are *subjectively consistent*. Nonetheless, despite the different formulations, both Platonists and intuitionists endorse the principle of non-contradiction, although for different reasons.

Bivalence is really where both philosophies disagree. It would be interesting to see where each "school" thinks, from its own perspective, the opposing one goes wrong. The intuitionist dismissal of Platonist metaphysical hypotheses is a direct consequence of its *own* presuppositions concerning the nature of mathematical reality and mathematical truth. And conversely, for the Platonist the intuitionist is simply wrong concerning these things. The conflict is dogmatic and thus unsurmountable. The phenomenological-transcendental approach, being non-dogmatic, can *understand* each side's perspective, corresponding as they are to different conceptions of reality, each acceptable on its own terms under the scope of the *epoché*. From the phenomenological perspective, Platonists and intuitionists are simply not talking about the same thing, since they do not have the same intentional conception of mathematical reality.

It is *because* mathematical realms are constituted as objectively complete that meaningful assertions are conceived as having intrinsic, although maybe unknown truth-values. Truth-value realism no longer comes out as a metaphysical hypothesis, but as a transcendental presupposition. One that one can either embrace (Platonism) or refuse (intuitionism). It is not a matter of right or wrong, but differences in intentional constitution. Transcendental phenomenology cannot endorse the Platonist ontological *thesis* that mathematical realms exist in themselves. Only "Platonism" between brackets can be phenomenologically vindicated; for it, ontological independence is an intentional feature of the object "mathematical realm of objects", constitutive of its intentional meaning, analogously to "empirical reality", also conceived as an ontologically independent realm of being causally responsible for sensorial perceptions (sensorial perception being *meant* as means of accessing an independent reality).

The metaphysical hypothesis behind truth-value realism is, from the perspective of intuitionism, unacceptable, because given the intuitionist meaning of "true" and "false" the realist hypothesis amounts to presupposing that any meaningful mathematical assertion is effectively verifiable and, consequently, that any well-posed mathematical problem is effectively solvable. Although truth-value realism, formulated as a realist thesis, does not imply the *effective* version of *epistemological* *optimism*, its intuitionist reformulation does have this consequence. The phenomenological perspective endorses truth-value realism (rather, "realism") but, again, not as a metaphysical hypothesis but a transcendental presupposition. It also endorses a sort of epistemological optimism, but still, not in terms of effectiveness and not as a hypothesis, but as also a transcendental idealizing presupposition. In short, Platonism and truth-value realism can be vindicated, but only as "Platonism" and truth-value "realism", which, I claim, are philosophically more sophisticated views than their original pre-epoché, ontologically and metaphysically charged versions. Although "Platonism" and "intuitionism" are irreconcilable within a single intentional perspective, they have, both, a right to exist as different intentional meaning-formations.

Conclusions I believe that most philosophers of logic agree that logical principles are a priori and then necessary and universal, for what is validated prior to any experience must be valid for all experiences. The conventionalist, of course, disagrees; for him, one chooses a logical system, encompassing the basic principles one prefers, like one may choose a geometry to structure one's experience of physical space, by convenience rather than truth. But this is not a very popular view; it is difficult to accept that the most basic tenets of rationality are chosen out of convenience. Different logics are different ways of reasoning and there must be strong reasons for why to reason in a way rather than in another. The questions, then, impose themselves: if logical principles are indeed valid a priori, what does a validation of logical principles look like? How can a priori laws on which rational justification ultimately rests be themselves rationally justified? What does it depend on? Moreover, is such a validation acceptable no matter what, universally?

As approached here, logical principles come out as indeed a priori and universal, but in a particular sense of the terms. Usually, a priori means independent of experience, particularly perceptual experience, and universal means simply everywhere, no matter where. In the sense given to these terms here, these characterizations must be qualified. First, a priori is still independent of experience, that is, prior to any *actual* experience, but not independent of the a priori determination of which experiences are possible in principle. Before actually undergoing experiences in any domain whatsoever (the empirical world, mathematical contexts, or any other) – where by experience I mean, of course, intuitive experience – one must *first* determine which experiences are in principle possible. This determination has a transcendental character and depends on the intentional meaning attached to the domain in question. In short, logical principles are a priori, but not empty formal principles indifferent to intentional meaning. By positing a domain of being as objective, stable, coherent, and objectively complete (maximally coherent), the ego thereby *validates* the three classical logical principles in reasoning about *this* domain.

But validation in one domain is not exportable to *all* domains. The validity of logical principles of reasoning is confined to the domains whose sense of being validates them. Hence, in a sense, logic is *not* universal, or is, but only within the limits of a given intentional positing. Different ways of conceiving a domain of being – for example, the domain of real numbers classically and intuitionistically conceived – may require different ways of reasoning about the objects of this domain.

One can also approach the issue, with Husserl, from a different perspective, logical principles as axioms of a theory of truth. However, still with Husserl, axioms, at least in material sciences, are not conventions, but assertions true in that material domain. A theory of truth is a conceptual, and then material theory whose subject matter is some *concept* of truth. Therefore, the axioms of a theory of truth are truth-of-a-conception. But there is no single conception of truth; Husserl mentions at least two: intrinsic truth and effectively experienceable truth. Both presuppose that there are two truth-values, the true and the false but, unlike the former, the latter conception claims that truth-value can only be attached to assertions in intuitive experiences of either confirmation or disconfirmation. Experienceable truth requires experience and, consequently, clarification. This is the fundamental conception of truth, the ideal that our cognitive activities pursue. However, to be qualified to pursue such an ideal, theoretical activity must satisfy certain preconditions. First and foremost, distinctness; i.e. theories must necessarily involve only distinct assertions, that is, assertions with formal and material sense or, equivalently, a *content*. Only such assertions can be expected to be eventually clarified. *If* a distinct assertion is clarified, and *when* it is clarified, supposing that it can in principle be clarified, it *will* receive one, and only one, of the two truth-values. In a theory of intuitive, experienced truth, the mere possibility of future clarification does not qualify as a truth-experience. Not even if a *determinate* truth-value imposes itself by logical necessity. Truth must be *experienced*. One exception is admitted, when the ego has at its disposition the means to elicit the required truth-experience, even if it does not actually carry it out.

However, as we have seen, under certain presuppositions and being careful about temporal indexing, we can admit that *all* meaningful assertions are somehow predetermined as to their truth-values. This is another conception of truth. An assertion is *true-in-itself* (resp. *false-in-itself*) if it is meaningful, both syntactically and semantically; i.e. if it is a distinct assertion that conveys a thought, and *would* manifest itself as being true (resp. false) *if* clarified in intuition. But with an important proviso, assertions must refer to domains of being that are already fully determined in themselves, domains that I called objectively complete. Only under this presupposition, all meaningful assertions can ideally be clarified. Otherwise, meaningful assertions could exist describing objectively undecided situations and it would not be justifiable to attach any truth-value to them.

From the perspective of the conception of truth as intrinsic truth, the principle of bivalence, which has always been a bone of contention between classicists and intuitionists, is true; for the conception of effectively experienceable truth it is not. Classicists and intuitionists simply do not have the same conception of truth. But, again, can a conception of truth be only a matter of choice? The answer, of course, is that it cannot, at least not from a truly scientific perspective. How, then, is the choice of a conception of truth to be justified? Both conceptions mentioned before allude to "attaching a truth-value to an assertion", differing on the circumstances in which this can be done. But what does attaching a truth-value to an assertion mean?

Of course, that there is a relation between the content of an assertion – what it says – and relevant states-of-affairs in the domain to which the assertion refers. An assertion is true (resp. false), or has the value true (resp. false) attached to it if it expresses correctly (resp. incorrectly) how things stand in the domain to which it refers. The difference between intrinsic truth and experience truth can now be recast in terms of the relation between saying and being, and ultimately on different conceptions of being. One or another conception of truth will impose itself depending on how the domain of reference is *meant to be*, that is, on the intentional meaning attached to it. If a domain is meant as objectively complete, an assertion is intrinsically true (resp. false) if it expresses correctly (resp. incorrectly) how things stand in a domain where all possible facts are determined once and for all so as to make this a maximally consistent domain of being. In this context, the notion of experienced truth still makes sense, of course, but it does not rule out that of intrinsic truth. If, on the contrary, a domain is conceived as objectively incomplete, depending for its completion on the action of a subject, there is no place for the notion of intrinsic truth. Truth is either experienced or not at all.

If truth is basically a correspondence – an assertion is true if it asserts that things stand as they indeed do or that things do not stand as they in fact do not, and false if it asserts that things stand in a way they do not or that they do not stand in a way they in fact do – then a notion of intrinsic truth can only be validated if the domain in question is posited as an objectively complete totality. For the idealized conception of truth, bivalence can only be validated in *intentional meaning*.

Although not intuitionistically valid, the notion of intrinsic truth has still an epistemological significance in intuitionistic contexts. Indeed, one of the equivalents of the principle of bivalence is the principle of double negation, if the negation of the negation of an assertion is true, then the assertion itself is true. One of the most interesting uses of this principle is in proofs by contradiction: to prove an assertion it is enough to prove that its negation is false. If the negation is false, then the negation of this negation is true, and by double negation, the assertion itself must be true. Of course, given the essential use of bivalence, true here means intrinsically true. Of course, the assertion thereby "proved" is not evidently true, but even the intuitionist should not disregard it. Although, for him, this assertion has *not* been proved true, he at least knows that it cannot be *intuitively* false if its double negation has been shown to be *intuitively* true. So, the falsity of the assertion ceases to be a possibility, which is already a form of *intuitive* knowledge.

Husserl believed that a conception of reality in harmony with the notion of intrinsic truth and the validity of the principle of bivalence is a necessary precondition of *objective* science (the opposite, I suppose, of *subjective* science, if the notion makes sense). For him, positive objective science, such as empirical science or (classical) mathematics itself, are necessarily "Platonist" simply because the world they presuppose *is conceived* as an objective, stable, coherent, objectively complete world existing in itself (recall that "Platonism" is phenomenologically reduced Platonism; i.e. Platonism minus the metaphysical compromise). The positive scientist *takes for granted* that the "world" he investigates exists completely determined in itself ("Platonism"), out there for anyone to explore under the supposition that any meaningful question he may ask about this "world" is intrinsically decided ("truth-value realism") and will eventually be answered, even though one may not have the slightest clue on how to answer it at a given moment (epistemological optimism as a matter of principle, not fact).[21]

Objective, positive science demands a conception of "reality" (empirical, mathematical, or any other) for which "reality" exists "out there" already fully determined as a stable and maximally consistent domain of facts in principle accessible to the (individual or collective) ego, whose effective disclosure, however, will depend on the ego's effective capabilities and means. Nonetheless, the complete disclosure of the totality of the facts of "reality" remains a guiding *ideal*. Objective positive science requires "reality" to be so, or better, this is how it *conceives* "reality" to be. Classical logical principles of reasoning follow necessarily from this conception of reality.

The presuppositions of positive objective science may appear at first as unjustified metaphysical presuppositions, but as soon as one abandons the naïveté of positive science by means of phenomenological clarification one immediately realizes that their source of validity is intentionality. Objective logic, as itself an objective science, admits from the start an objective world already given (if only as a possibility), so that the truths relating to it are also "in themselves". Consequently, it admits also the possibility of obtaining complete knowledge about this world.[22]

This, Husserl notices, makes the notion of an objective world a logical notion that demands philosophical investigation[23]:

> If the anything-whatever of formal logic, taken as Objective logic, ultimately involves the sense, worldly being, then this sense is precisely one of logic's fundamental concepts, one of those determining the whole sense of logic.

According to Husserl, a critical approach to logic leads to a critical approach to experience and thus to transcendental phenomenology[24]:

> …transcendental criticism [*of logic*] …criticism of the intentional life that constitutes both province and theory.

Some intuitionists (like Heyting) correctly count Husserl among those denying the validity of the *law* of bivalence for the *logic of experience in general*. Indeed, Husserl is clear about this, that a proof of impossibility of an evidential experience of inadequacy does not in general count as a proof of the possibility of an experience of adequacy with respect to the same content. But Husserl is here referring *exclusively* to the living experience of truth, not simply truth-in-itself, which is all that the principle of bivalence involves. This principle operates on a different level, namely, on an *a priori* (i.e. prior to experience) characterization of the field of all possible experiences. It is a way, if you like, of downsizing the set of candidates for

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[21] The classical expression of this is Hilbert's famous "wir müssen wissen, wir werden wissen!" uttered in a conference in Bologna in 1928.

[22] See note 9.

[23] Husserl 1969, p. 229.

[24] Husserl 1969, p. 173.

the living experience of truth, not an indication of what can be effectively experienced.

The transcendental criticism of logic does not deprive logic of its objective presuppositions; it only clarifies them.[25] The transcendental approach to logic shows how to bypass metaphysical presuppositions without in the least giving up the principles that apparently depend on them. Nothing is lost, a complete and accessible reality, the power of language to represent possible facts, the intrinsic truth-values of meaningful propositions, but we are not asked to pay the price of endorsing metaphysical theses concerning a *subject-independent* objectively complete reality in order to guarantee the intrinsic character of propositional truth. The naïve realist presuppositions of positive logic and their true meaning are finally clarified in transcendental logic and thus safeguarded.[26]

It is time now that we move into mathematical domains proper.

[25] For Husserl, transcendental logic, "intends to bring to life the system of transcendental principles that gives to sciences the possible sense of genuine sciences" [Husserl 1969, p. 16]. A genuine science is, for him, one that overcomes, by means of philosophical criticism, its naïve positivity and self-sufficiency.

[26] This is why Husserl can say that "no ordinary 'realist' has ever been as realist and as concrete as I, the phenomenological 'idealist'" [*letter to Abbé Baudin*, 1934].

Chapter 4
Numbers

Scientific subject-matter and procedures grow out of the direct problems and methods of the common sense.

John Dewey, Logic, the Theory of Inquiry

What are numbers and why were they invented? My goal here is to answer these questions, not to present a clever reconstruction of the concept of number to fit preconceived ideas of what numbers should be. I will, of course, approach the issue from a phenomenological perspective.

As is known, Husserl's first published philosophical work was a philosophy of arithmetic (*Philosophie der Arithmetik*, 1891, henceforth *PA*).[1] However, although taking into consideration and incorporating most of Husserl's conclusions, I take distance from him insofar as his approach had a "psychologist" penchant that mine has not.[2] Although touching the important problem of the role of symbols and symbolic knowledge in arithmetic, which will be at the center of my concerns here, he failed to see with the required clarity the extent of symbolization in arithmetic and the consequences it has for the nature of arithmetical knowledge. Here, I take to its natural consequences the truth that symbolization is central in arithmetic and for that matter the whole of mathematics, thus extending Husserl's approach to domains he only explored superficially, in particular, the applicability of arithmetic.

The arithmetic of natural (i.e. finite cardinal) numbers (numbers, for short) is my first concern, but I will also consider extensions of the concept of number. There are many problems luring here that philosophies of arithmetic usually ignore. Husserl had planned to deal with extensions of the concept of natural number in a second volume of *PA*, a book, however, that never saw the light of day. Here is one of the problems he had intended to tackle: what do general concepts of number, negative

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[1] Husserl 1970.

[2] But, as Husserl himself repeatedly emphasized, psychological investigations are not without phenomenological interest, to the extent that a priori psychological inquiries into the mental life of an ego considered simply as such, which Husserl called pure phenomenological psychology, are translatable by a change of perspective into a priori phenomenological investigations of the intentional life of the transcendental ego.

J.J. da Silva, *Mathematics and Its Applications*, Synthese Library 385,
DOI 10.1007/978-3-319-63073-1_4

and imaginary, in particular, which mathematicians of the XVI century called "impossible", have to do with the notion of quantity? Another question, to which Husserl paid some attention but answered in a way that I believe inadequate, is the apparent "unreasonable effectiveness" of extending for the sake of theoretical investigations the numerical domain by adding new elements to it that have nothing to do with the notion of quantity. How can the knowledge of objects that are not numbers play any role in our knowledge of numbers? This question, as we know, touches the core of the puzzle concerning the applicability of mathematics in general, to mathematics itself, empirical sciences and daily life. By examining this problem in the limited numerical context, I intend to prepare the ground for approaching it in general.

If you ask a mathematician, and probably also a philosopher of mathematics, what a negative number is, they will probably say that negative numbers are particular classes of ordered pairs of natural numbers. Mathematicians often believe, and philosophers follow behind, that mathematical entities are as the latest mathematical theories interpret them to be; however, as new theories are constantly being invented, we can be sure that answers of today will tomorrow be dismissed as incomplete, inadequate or just plain wrong. Of course, negative numbers are not sets of any sort; this is only how they can be *interpreted* in set theory. Negative and complex numbers, as I will show here, are purely *formal objects* that share with numbers proper only certain formal-operational properties. To interpret them is to give these objects material content in some materially determined domain (for example, the universe of sets), but what they are is independent of how they are interpreted, for interpretations have to do with matter, which is alien to their essence of materially empty formal objects.

Negative numbers made their entrance into arithmetic as "impossible" solutions of arithmetical operations; things that did not exist but that, surprisingly, could be treated as if they did. Mathematicians were quick to realize that they could profitably treat empty numerical symbols as symbols denoting numbers of a different species, the nature of which, however, they were not quite certain. They also realized that they could use these new "numbers" to investigate numbers proper. How is this possible? Answering this question will obviously throw light on the nature of arithmetical and mathematical knowledge in general.

Philosophy of Arithmetic: Questions and Answers Many philosophical questions have been raised about numbers and arithmetic, here are some[3]: do numbers exist, and in case they do, what type of existence do they have (since things can exist in many different ways)? Are numbers objects or concepts? If numbers do not exist, what is the subject matter of arithmetic? Is arithmetic a priori (independent of experience) or a posteriori (a very general empirical science)? In case arithmetic is a priori, how can we account for its applicability? What type of knowledge, if knowledge it is, arithmetic provides? One can group these problems in three classes: (1)

[3]The *infinite* numbers introduced by Cantor, which will not concern me in this chapter, pose intriguing questions of their own.

ontological problems, concerning the nature of numbers; (2) epistemological problems, concerning arithmetical knowledge; and (3) pragmatic problems, concerning the applicability of arithmetic. Any philosophy of arithmetic must face these and related questions. I will confront them here, making free use of phenomenological concepts and ideas, without however intending my efforts as an interpretation of Husserl's philosophy of arithmetic.[4]

A widely popular view on the nature of numbers is *ontological realism*. According to it, although abstract, numbers are *independent* objects. Plato and Frege are among the proponents of this view, and, some claim, "philosophically naïve" practicing mathematicians in general too. It is, some claim, the "natural" view. However, natural as it may be, it raises many difficult questions. For example, how can abstract entities, which are ontologically dependent, have independent existence? Numbers do not have a locus in space and time but, supposedly, exist objectively and independently; so, where precisely are they (the "Platonic heaven" problem)? Not being real objects, numbers are a fortiori causally inert; if, as some believe, our knowledge of (or even our ability to refer to) objects depends somehow on us being causally related to them, how can we know anything about (or even refer to) numbers? This, of course, is a problem only for those who endorse a causal theory of knowledge and reference; philosophers who do not embrace empiricism or naturalism are not very impressed by it. There are, however, those like Gödel, who although not subscribers of the causal theory believe in a form of "perception" or intuition that plays with respect to numbers the same role sensorial perception plays with respect to physical objects. They also see this parallelism as an indication that numbers, like real objects, exist independently.

I do not share this point of view. As I understand it, intuition is not in general a way of perceiving supposedly independently existing entities, but merely an act of presentification, no matter the ontological status of the object presented. The object *O* is intuited if the consciousness of *O* is the consciousness of its *presence in the act as its intentional correlate*, *O* is "bodily present" in the experience, not as a real part of it, but as something to which the subject is intentionally related. Nothing in this characterization implies that *O* exists independently of the act in which it manifests its presence. Phenomenological epoché completely cancels the metaphysical thesis that numbers are independently existing entities; so, from a transcendental-phenomenological perspective, one cannot equate intuition with the naturalist notion of perception. Intuition is presentification, not a way of "grabbing"

[4] Although *PA* is anterior to his phenomenological period (from the *Logical Investigations* of 1900–1901 onward) Husserl never disowned it; rather, he often quoted approvingly from it, implying that it contained a correct phenomenology of arithmetic, if its more "naturalistic" aspects were given a proper phenomenological reading. I think Husserl would approve most of the conclusions I draw here, but maybe not all of them; in particular, I do not think that he would agree with my interpretation of *contentual* arithmetic, that is, the arithmetic of numbers proper, not things that behave only formally like numbers. For Husserl, contentual arithmetic is formal because its objects are forms; for me, albeit numbers are indeed forms, arithmetic is material in the sense that its object-matter is a particular ontological region (that of numbers) and formal in the sense that only formal properties of numbers are of concern to it.

supposedly ontologically independent objects. As I will show below, one can intuit some (sufficiently small) numbers (intuition *of* or objectual intuition) and through them, by imaginary variation, the concept of number (conceptual intuition). By inspecting the concept of number (reflection as an intentional act) one can intuit the more salient aspects of the domain of numbers; in particular, its subjacent structure and some basic facts concerning it (intuition *that* or factual intuition). I will say something more about numerical intuition from a phenomenological perspective later. For the moment, it is important to remember and keep in mind that to intuit something does *not* imply that this thing exists independently of the act of intuition.[5]

Some philosophers interpret the denial that numbers exist independently as the claim that numbers are, ipso facto, *mental* objects. For these philosophers (usually with strong empiricist tendencies), any attempt to subtract self-subsistence from numbers is an open flirtation with psychologism. I claim instead that, although perfectly objective, and certainly not mental, numbers are nonetheless, in more than one sense, *non-independent* entities. Numbers are ontologically dependent on a "support" from where they are (or can ideally be) abstracted (those supports being themselves higher-level entities ontologically dependent on other objects) and on intentional activities of an ego who posits them as objective entities or can in principle do so.[6] Neither the external world of physical objects nor the internal world of "representations" are the natural habitat of numbers.

Numbers, although abstract, exist *objectively* (I am not yet telling what type of entities they are). However, despite their objectiveness, or, which is the same, their being "out there" in principle for everyone to become conscious of them, numbers come into existence by a process of constitution.[7] Individual numbers, the domain

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[5] Recall that Husserl's theory of intuition is far richer than Kant's. Whereas for the latter an intuition is an individual representation, for the former just about anything can be intuited, real or ideal, abstract or concrete objects, concepts, essences, forms, you name it. Intuition means presentification to consciousness. Intuitive contents are not "representations" in the sense of "copies" of things, but things *themselves* – when the ego intuits something, it is this thing *itself* that comes before the ego's consciousness, not a copy of it.

[6] Abstraction, I recall, is a sort of "refocusing" of consciousness. However, we must be careful to not read these terms on a psychological key (as is often the case). As already explained, *to abstract* is an operation (an *act*) by means of which the ego becomes aware of an *aspect* of an object (color, shape, quantitative form, etc.) based on its awareness of the object itself (an important form of abstraction is *formal abstraction*, in which form becomes salient and matter is obliterated). In abstractive acts we do not "separate" the aspect we focus on from its support (the object it belongs), either in reality, which would be absurd, or "in our heads", that is, mentally. We simply make this aspect *itself*, which exists together with its support, the focal point of consciousness (and thus refer to it, theorize about it, etc.).

[7] To constitute or to posit are "acts" or "experiences" in which objects (or, to use a Husserlian neologism, "objectualities"), with their sense of being, come into the sphere of consciousness of the ego, that is, come into being. Constitutive acts can be iterated, the object of one serving as the matter for another. Numbers can be either intuited or merely intended, but still conceived as in principle capable of being intuited. Intuitability is equivalent to existence, but each domain of being has its own characteristic sense of intuition (for example, sensorial perception for the empirical world, mathematical intuition for the mathematical realm). Whole domains of objects can also

of numbers, the numerical structure (the abstract structure of the domain of numbers), or the concept of number are all *objective and subsisting* entities in the sense that the communal ego can re-identify them as the *same* in different intentional experiences. By being re-identifiable in noetically distinct acts of a communal ego, numbers become a communal possession. Both particular numerical truths, such as, for instance, $2 \leq 3$, and *general* truths, such as, for example, for each number there is a number strictly larger than it, are intuitable truths, although the process of intuiting them are different.[8] Maybe the most serious error of certain brands of intuitionism (Kant, Brower) is the belief that only the particular is intuitable. Husserl showed that this is a mistake, that other types of entities are also intuitable, concepts in particular (by means of free variation).[9]

Individual numbers cannot all be given intuitively; those that cannot, however, can still be posited emptily by intentional acts such as naming. For example, by writing 10^{10}, which only makes sense in a system of notation, a number is intended that is not intuited. Another way of singling out numbers non-intuitively is by means of definite descriptions such as, for example, "let *n* be the number such that …", provided one can prove in the system of arithmetical conceptual truths that such a number exists as meant and is unique (a well-defined object consistently posited). The communal availability of numbers individually, the concept of number and truths about them is enough to grant these things *objective* existence, at least insofar as their positing remains consistent.[10] Objectivity does not mean self-subsistence,

be constituted, albeit non-intuitively, simply as extensions of concepts, without the ego having to constitute each of these objects *individually*, for example, the domain of real numbers. And remember, domains of being can only acquire a sense of being in intentional consciousness.

[8] Truth means, in this case, truth *according to the concept*, i.e. conceptual truth. It can be intuitive or non-intuitive in case it is merely a logical consequence of intuitive truths. The phenomenological theory of truth is still the correspondence theory. In the most basic sense of truth, an assertion is *intuitively* or *evidently true* if it is adequately filled by a correspondent *intuition*. Since there are various degrees of adequacy, there are various degrees of intuitiveness of truth. Notice that the correspondence alluded to in the phenomenological conception of intuitive truth is *entirely confined to* the *intentional field*. An assertion can also be true, although not intuitively, if it follows from other truths by logical derivation; truth can flow down logical chains of reasoning (and falsity up), but not intuitiveness.

[9] Conceptual intuition does not necessarily give us *adequate* access to a concept; one can have incomplete or imperfect intuitions. One cannot always expect to obtain a *complete* set of intuitive truths on the basis of which all conceptual questions are *logically* decidable, sometimes because our intuitions are inadequate, sometimes because the positing itself is incomplete.

[10] Positing also grounds reference. A symbol is given objective directness by the ego's *intention* of using it as a name of something. Taking a symbol as a *denoting* symbol of an object is intending this object *and*, simultaneously, the symbol *as a sign* of it. One can use a denoting symbol without being conscious of what it means or denotes (meaning can become "fossilized"), but one can always recover both meaning and object meant by reenacting the original positing act (in which the object appears with the meanings it has). The very *possibility* a priori of reactivating a positing act is enough to grant the object posited objective existence. However, one can still use symbols meaningfully without knowing what they mean by abiding to objectively available *rules of use* originally rooted in intentional meaning. In causal theory of reference, causal chains guarantee reference; in the phenomenological theory, the possibility a priori of reactivation grants reference,

i.e. existence independently of other objects or the ego, but communal availability, being "out there", cast in the public arena (metaphorically speaking) and "graspable" in principle by anyone who plays *the role of the ego*.

One could object that meaning something does not necessarily imply that this thing exists. After all, one can intend an object like Peter Pan, but Peter Pan almost certainly does not exist (although we cannot be sure). However, the positing of Peter Pan is, from its onset, a self-defeating act. One intends Peter Pan as a *physical* object, therefore subjected to the precondition of empirical existence, that is, perceptibility. However, there is no evidence that this object can indeed be perceived, and therefore exist in the physical world.[11] It is possible to refer to objects as objects of a conception even if these objects do not exist *as conceived* (Peter Pan). Intentionality is a relation between the ego and *something* that appears to it (with a certain sense of being, having certain properties, seen from a certain perspective, immersed in a horizon of further possible perspectives, etc.). Numbers exist insofar as the meaning with which they are posited "holds together". If any inconsistency should manifest in the conception of numbers, numbers would immediately vanish out of existence. This is precisely the problem with Peter Pan, posited as a *real* entity he apparently does not conform to the standards of real existence.

Whereas some small numbers exist intuitively, others exist only as pure intentions (but nonetheless still in principle intuitable). Symbolic and conceptual systems for referring to numbers individually (thus discerning them from other numbers) can, however, open a lane to the land where big numbers "live" as potentialities, maybe only dimly intuited, bringing them to consciousness, sometimes as only purely intentional entities. Symbols such as 10^{10}, for example, posit by naming, presupposing the consistency of the entire system of arithmetical concepts on which the symbolic system depends.[12]

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even when symbols are used "blindly", provided they are used *correctly*. One of the most interesting phenomena in mathematics is the subtle changes of meaning mathematical terms usually undergo as the ego continues to use the same terms to denote different, sometimes subtly different, although usually related intentional objects (for instance, number, space, etc.).

[11] Peter Pan is portrayed in the story as a (slightly weird) *real* being, and real entities must conform to determinate conditions of existence; they must, for instance, occupy a position in space and time, be able to participate in causal chains ending in the stimulation of my nervous system, and the like. Peter Pan, however, does *not* satisfy these conditions and thus the internal consistency of the positing act is not sustainable and the act, consequently, loses its object-positing quality: Peter Pan is not a real entity. He is an object of fantasy *because* the intentional act that posits it imposes conditions of existence that it does not satisfy. Numbers, on the other hand, have different conditions of existence, namely, intuitability and *consistent* intentional positing.

[12] The objectivity of mathematical truths is reinforced by the adoption of classical logic, in which the principle of bivalence (an assertion is either true or false, in which case its negation is true) is valid. By assuming that a mathematical assertion has a definite, although maybe unknown truth-value, one assumes that mathematical "facts" are determined in themselves, although not independently as realists claim. In fact, bivalence only means that we assume as a *principle* (in the strong sense of an *unconditioned* fact) that any *properly phrased* (i.e. meaningful) assertion has an intrinsic truth-value. This *only* means that, from the point of view of the a priori laws regulating the combination of the syntactic and semantic categories, *nothing stands in the way* of this assertion being confronted with intuitions of the appropriate type.

As already emphasized, the fact that numbers are intentional constructs does not preclude them from objective existence, or truths about them from being objective. Truths about numbers are either extracted directly from the intentional meaning attached to them (axiomatic numerical truths) or logically derivable from these fundamental truths. The arithmetical ego, i.e. the community of arithmeticians and users of arithmetic, secures the objectiveness of arithmetic by displaying numbers and numerical truths in the public arena as objectualities that can in principle resurface as the *sam*e in different intentional acts. There are also cultural practices and products involved in objectification; for instance, written documents (books, papers, and essays) where truths are preserved and passed on to others in the present and the future and the tricks of the trade are taught; courses, symposiums, meetings, all sorts of communication inter pares where consensus is built. In short, objectiveness is a joint production of the multiplicity of individual subjects who *together* constitute the mathematical ego. The ever-open possibility of reenacting positing intentional acts constitutes, to use a Husserlian expression, the *living body* of idealities. The objectivity of the numerical domain and numerical truths depend essentially on the possibility of them being posited anew, even if this possibility is not actualized.

Philosophies of Arithmetic The history of philosophy is not short of answers for philosophical questions about numbers. Plato and his disciple Aristotle established the frames that still confine, to this day, the debates on the matter. They had distinct views on the issue, concerning not only numbers but also Forms in general. To the former, Forms – of which numerical ideas constitute a species – exist independently in a domain of their own not of this world, which we can only access by reason, independently of the senses. Arithmetical truths are, Plato believed, a priori truths of reason. For the latter, Forms, numbers in particular, are only aspects of the world, which we can "isolate" by abstraction and investigate by rational means. Arithmetical truths were, for Aristotle, still a priori, for although Forms are aspects of the world, they obey a priori laws of essence for simply being the sort of entities they are. Only reason can completely bring these laws to light in a systematic manner (the senses, however, are not completely absent of the scene, since abstraction requires that the physical support of Forms, the physical world, be somehow brought to consciousness).[13]

[13] We can use, I believe, Husserl's distinction between nomological and ontological sciences to clarify the difference between Plato and Aristotle's philosophies of arithmetic. A nomological science (such as arithmetic) is a science whose unity is given by fundamental laws of essence. Ontological sciences (such as geology, for example), on the other hand, are those whose unity depends exclusively on the unity of their domains. Concepts first or objects first. The access to the domains of nomological sciences is intermediated by the concepts that unify them, those of ontological theories, on the contrary, requires objectual intuition and inductive generalization. If no essential legality presides over a domain, our knowledge of it cannot go beyond what the intuition of the domain provides, what follows logically from it, or, at best, what can be coherently built on top of it. I think that for both Plato and Aristotle arithmetic is a nomological science whose object is the intuitable concept of number. The difference is where each believe this concept should be located. For Aristotle, it is fundamentally in the empirical world, numbers exist as (possible or

Kant introduced the transcendental perspective in the debate. He thought that contrary to geometry, arithmetic did not have a basis of axiomatic a priori *general* truths; the intuitive truths of arithmetic, he thought, are particular. For him, both types of truths, geometrical and arithmetical, require constructions – computations in the case of arithmetic – that necessarily involve the a priori form of time (conceptual analyses only are not, he thought, sufficient). Contrary to a common misinterpretation, arithmetical truths are not, for Kant, truths about time in the same way that geometrical truths are truths about space; time is required only for accessing arithmetical truths, which are necessary relations among numerical concepts that the analysis of concepts alone cannot disclose. Arithmetical truths are a priori conceptual truths that require time in order to be *known*, but are not *about* time. Geometrical truths, on the other hand, also a priori conceptual truths that require temporal constructions to come out in the open, are a priori truths about space.[14] Numbers are, for Kant, schemata of a concept of understanding, that of quantity, whose applicability to our experience of the world requires one of the a priori forms of intuition, time (counting, for example, is a temporal process). With Kant, mathematical facts moved from objective realms of existence (the empirical world or the Platonic heaven) to the transcendental domain.

The next turning point in the philosophy of arithmetic was the logicism of Frege in the nineteenth century, which I will treat separately the next section vis-à-vis Husserl's philosophy.

Dedekind introduced a novelty, instead of telling what numbers are he preferred to bring to attention the *structure* of the numerical domain: the smallest linearly ordered chain with a first element, where each element has a single immediate successor different from it and no two elements have the same successor, the so-called ω-sequence. By spelling this out as a set of axioms, Peano established the axiomatic basis of arithmetic, a theory however that, when formally abstracted, i.e. divested of its content and reduced to pure form, admits materially distinct interpretations that share with the numerical domain the same structure. This change of perspective marks a turning point in mathematics; it did not originate with Dedekind, but he certainly reinforced it: mathematics becomes interested in objects *only insofar* as they instantiate mathematically interesting structures, which different domains of objects also instantiate. Instead of "*What* are numbers?" Dedekind asked, "*How* are numbers (how do they relate to one another)?"

As I just said, contentual or material arithmetic, the science of numbers proper, can be formally abstracted as formal arithmetic. There are two different perspectives on the nature of formal theories; one is that they are theories with multiple

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actual) aspects of things whereas for Plato, the extension of the concept of number is a realm of being not of this world. For both Plato and Aristotle arithmetical truths are a priori, referring however, for the latter, to actual or possible abstract aspects of the world and, for the former, to ideal Forms.

[14] Conceptual truths are truths about concepts; accessing such truths requires either conceptual analyses (in which case they are analytic, in Kant's conception of analyticity) or intuitions (in which case they are synthetic). If the synthesis necessarily involves empirical intuitions, the truths are a posteriori; in case they involve only pure intuitions, they are a priori.

material domains, all those that satisfy the formal stipulations of the theory (multiple reference theory), the other is that they are theories of a single *formal* domain (single reference theory). As we have seen before, for Husserl, formal theories are theories of formal domains, i.e. domains of materially indeterminate objects standing in relations with one another whose formal properties the theory expresses. The change of emphasis from what numbers (or, for that matter, mathematical entities in general) are (classes, Forms, abstract aspects?) to which formal properties the domain of numbers have becomes established tendency with Hilbert. Instead of defining *real* numbers genetically, in the manner of Dedekind or Cantor, for instance, Hilbert preferred to characterize them, as he said, "implicitly". This is a problematic notion, since no axiomatic system can by itself single out a unique materially determinate interpretation (selecting one interpretation as the intended interpretation of an axiomatic system is always an *extrinsic* determination). Axiomatic systems considered independently of any particular interpretation (i.e. as non-interpreted systems) can characterize one (in the best possible case) or a family of formal structures and express their formal properties. So, Hilbert-like axiomatic systems for arithmetic can only hope to grasp those formal properties the numerical domain has that identically structured (isomorphic) domains also have. In the best possible case, i.e. when they are *categorical* (all interpretations are isomorphic, such as second-order Peano arithmetic), formal systems succeed in singling out a definite formal domain, i.e. the abstract form of a family of isomorphic materially determined domains. However, from a strictly mathematical perspective (be it theoretical or practical, i.e. considering applications of the theory) nothing more should be required of formal systems of axioms.

Instead of formal domains, some prefer to talk of structures; they are in fact germane notions. The ontological status of structures is a matter of contention among modern day structuralists. Some believe that structures are a kind of Platonic Forms (*ante rem* structuralists). Others, closer to Aristotle, that structures are only aspects of structured domains (*in re* structuralists). Others still that structures do not exist at all, only structural descriptions of actual or possible structured material domains; for them, systems of structural descriptions (formal theories) are not descriptions of anything over and above physically real actual or possible structured domains. Places in structures (formal objects) also pose some ontological problems: are they identifiable across structures? Are they objects in their own right (places-as-objects perspective) or only empty vacancies (places-as-offices perspective)? I will say much more about structures and the structuralist approach to the philosophy of mathematics later. Although written well before this debate took form, Husserl's writings provide some interesting approaches to these questions. In particular, his typology of logic, which reserved a domain for the study of formal domains and logical relations among them side by side with the logical investigation of formal theories and their mutual relations (formal ontology vis-à-vis the metatheory of formal theories), seems adequate to accommodate a structuralist perspective on mathematics. More about this later.

Husserl was, as far as I can tell, the first philosopher to realize that one cannot answer questions concerning the nature of numbers and arithmetical knowledge uniformly. Numbers, he realized, come in different guises; some present themselves as actual aspects of the world, others as only ideal possibilities (and so are, in a sense, Platonic Forms), others still as imaginary entities that only exist because they are given names that one can manipulate as names of numbers (numerals) in a symbolic numerical calculus. Arithmetical truths also come in different formats; some are intuitive, some are devoid of intuitive content (but capable in principle of intuitive filling), and some still are purely symbolic (having only formal-operational, not material meaning – according to Husserl, they can play only a pragmatic but inessential role vis-à-vis arithmetical knowledge proper).[15]

Husserl and Frege on Arithmetic Clearly, Kant's treatment of geometry is far more persuasive than his ideas on arithmetic, a science whose scope, generality and coerciveness far surpass that of geometry. One could embrace Kant's theory of space and be suspicious of his account of arithmetical knowledge.[16] This was precisely the way Frege went; although geometry was, for him, synthetic a priori, arithmetic was analytic, where by analytic he meant essentially reducible to logic. As the passionate thinker he was, Frege nurtured many intellectual rivalries, or rather enmities; mainly with those who wanted to reduce arithmetical laws to laws of nature, not unlike those of physics or psychology, only supported by more robust evidence. Psychologism, for which numbers are ideas in the mind and arithmetical facts, facts of psychology, and empiricism, for which arithmetical truths are contingent, revisable empirical truths (1 + 1, some argued, can sometimes make 1; for doesn't one drop of water added to another drop of water makes only one drop of water, only larger?) were Freges favorite targets. Psychologism, I believe, must have seemed a decent "naturalist" alternative to Kantian transcendentalism for those who did not feel attracted to the latter. Frege combated both by pushing the issue in another direction altogether, that is, by trying to establish that arithmetical knowledge is in fact analytic in his peculiar interpretation of this notion, i.e. that arithmetic is only a branch of logic.

The natural way of showing this, one may think, would be to show that numbers were concepts and arithmetical truths a priori logical relations among numerical concepts – logic has always been conceived, after all, as a theory of relations among concepts (and assertions). However, based on his analysis of numerical identities Frege insisted that numbers were a particular type of objects, logical objects. His ontology imposed that objects must be kept sharply separated from concepts (never lose this distinction from sight, he told us in his epoch-making *Foundations of Arithmetic*; although, we may argue, his ill-fated basic law V – essential for the derivation of Russell's paradox – practically eliminated this distinction by short-circuiting both domains). Numbers, Frege told, are extensions of concepts and what

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[15] See da Silva 2010.

[16] It is conceivable that space is not Euclidean (in fact, modern physics endows the space-time continuum with a non-Euclidian structure), but, or so it seems, 2 + 2 can only be 4, provided the meaning of numerical terms and operations is not perversely distorted to force a different conclusion. One can call into question the a priori character of geometry, but arithmetic does not seem open to this possibility.

makes them logical is not only the fact that extension of concepts is a traditional notion of logical, but that arithmetical truths are derivable from basic logical principles after being conveniently translated in terms of logical notions. Of course, Frege's project of providing arithmetic with logical foundations collapsed completely in face of logical contradiction (Russell's paradox).

Husserl also believed that numbers are objects and that it befell on him as a philosopher the task of clarifying the sort of objects numbers are. For this reason, I find it interesting to present Husserl's views in contrast with Frege's, considering moreover that in *PA* Husserl makes explicit references to Frege, criticizing him, and that Frege has subsequently written an acid review of *PA*.[17]

Despite its positive aspects, even without taking into consideration its failure, I believe that the logicist project as conceived by Frege has some serious shortcomings. First, it does not take into consideration the genesis of numbers and the concept of number. For him, numbers are mind-independent objects, they just are "out there"; constitutive analyses, he thought, are essentially psychological in nature and irrelevant for the establishment of the nature of numbers. As I believe, however, and as Husserl argued for, although mind-independent, numbers are not ego-independent and historical considerations, not of a factual but *transcendental* nature, are essential for the establishment of what numbers are. Numbers are as objective but as fabricated as hammers (and in a sense as instrumental). To find out what they are, it is essential to inquiry how they came to be, i.e. how they entered the life-world, the purposes they serve therein, and how the theoretical concept of number and theoretical arithmetic originated from the pre-theoretical notion of number and practices of the life-world.

Secondly, although capturing an important aspect of the essence of number, namely, the relation of number to quantity, Frege's reduction of numbers to classes of equinumerous concepts is an unnecessary artifice devised exclusively to satisfy logicist parti-pris (that this caused the doom of his project indicates the error of the choice). As Husserl sees it, numbers are objects of essentially the same type of sets, namely, reified *forms*. Quantitative forms[18] are abstract, i.e. non-independent aspects of quantitatively determined multiplicities of objects; these forms intermediate between the concept of quantity and collections of objects quantitatively considered, i.e. to have a determinate quantity is to have a determinate quantitative form. Frege also saw numbers as expressing quantity, more specifically, the quantitative aspect of concepts (and only indirectly of multiplicities), which he preferred, however, to characterize in terms of the *mathematical* notion of abstraction (abstract moments defined as *classes* of things having that moment). By choosing to define numbers by mathematical abstraction, but using an inconsistent theory of classes, Frege condemned his entire project to failure. Numbers, however, if one does not try to disguise their true nature, are *not* Fregean classes; at best, numbers are only *formally similar* to them. Similarly, set-theoretical "definitions" of number such as von Neumann or Russell's offer, at best, only arbitrary formal equivalents of numbers,

[17] Frege 1894.

[18] The terminology is mine, not Husserl's.

not proper definitions in the sense of clarifications of the *essence* of number. Therefore, contrary to Benacerraf, I do not think that the fact that numbers are reducible to sets in arbitrary ways implies that numbers are not objects; they just are not that kind of objects. For Frege, numbers "belong" to concepts (in fact, in his theory, the opposite is true, for it is concepts that belong, in the set-theoretical sense, to numbers) and the number of a concept expresses the quantity of elements that fall under the concept, i.e. that belong to the extension of the concept. To associate numbers to concepts is somewhat artificial and serves only to make numbers logical objects.

In fact, numbers are more naturally attachable directly to (quantitatively determined) multiplicities, whether or not they are extensions of concepts (although there are artificial ways for making a multiplicity the extension of a concept). Numbers, Husserl thought, are ideations of quantitative forms. Two quantitative forms, which are different if they are forms of different multiplicities, are *equinumerous* if there is a one-to-one correspondence between the collections whose forms they are. A number is the *ideal* form that each member of a class of equinumerous quantitative forms indifferently instantiates. Quantitative forms are non-independent aspects of the collections whose form they are; they do not exist independently of these collections and occupy the same spatial locations their collections do, if they are empirical collections. Numbers are non-temporal, non-spatial ideal entities posited by ideation from abstract quantitative forms. Two numbers are the same if they are instantiable as equinumerical quantitative forms.

A quantitatively determined multiplicity is one that is subsumable under the concept of quantity; it is a multiplicity with a determinate quantity of elements, expressed by a number. Numbers intermediate between multiplicities regardless the nature of their elements and the concept of quantity (something analogous to Kant's schema of the category of quantity). The number of a multiplicity does not depend on the nature of its elements, only on their quantity; as far as the notion of quantity is concerned, the objects of a multiplicity are undifferentiated units.

Different elements become undifferentiated units by abstraction. A *unit* is simply the abstract form of an object (recalling, an object is anything about which one can meaningfully say something); i.e. the object considered simply as such, in other words, an instance of the notion of object, a pure *something*. Frege ridiculed the Cantor-Husserlian theory of number and the abstraction it involves; his criticism, however, relies on a series of misinterpretations. He thought that, for Cantor and Husserl, the number of a collection of objects is a *mental representation* of this same collection where one substitutes *in the mind* the mental representation of each object in the collection by the *same* (mental) *thing*, their common *abstraction*. Frege misrepresents the Cantor-Husserl operation of abstraction as a mental operation on (mental) representations, a *real* phenomenon, a sort of mental chemistry. Contrary to Frege's gross misunderstanding, as Husserl *explicitly* says (and Cantor implicitly presupposes), abstraction is an *intentional* operation on objects *themselves* (not mental representatives of them) whose objective *correlates* are abstract (i.e. non-independent) aspects of these objects. Considering the objects of a multiplicity as units to attach a number to the multiplicity has an analogous in Frege's theory of numbers, namely, subsuming objects under concepts to count them.

For Frege, each element of the extension of a concept *C* is, for numerical purposes, only *a C*. To conceive objects as units (that is, as ones) is no more mysterious than conceiving men as humans regardless of their specificities. Together with the many properties and attributes objects may have, and by means of which they are individuated, objects are also *things* and can be treated merely as such. To conceive objects as units is simply to take them as things, equal (but not identical) to other things in their common thinghood, despite their many differing properties, like taking people as citizens for the purpose of a demographic census. In counting, things are considered exclusively as *things*, but with individualizing properties that make them *different* things. Meaning objects as units is an intentional operation that consists in taking objects, the objects *themselves*, not mental representatives of them, with all their characteristic properties, merely as unqualified *things*. In short, taking objects as units is to take them exclusively as objects. For Frege, to count the objects falling under the concept *C* one counts each of them as merely *a C*; for Husserl, to count the elements of a collection *A* one counts each of them as a thing (a unit) of *A*.

A collection of *individual* shoes is not the same collection of objects as this collection taken now as a collection of *pairs* of shoes, and does not have the same quantity of objects. One can determine the unit of counting by means of a concept; a unit being a singular instance of the concept, individual shoe, pair of shoes, moon of Jupiter, etc. This is probably one of the reasons why Frege preferred to attach numbers to concepts instead of directly to collections; a concept determines *both* what the units of the collection are *and* extensionally the collection itself. But, of course, both the collection and what counts as a unit for the purpose of quantitative determination can be fixed by other means.

PA was Husserl's attempt at giving arithmetic psychological and logical foundations. The psychological part dealt, basically, with the intuition of collections and numbers, and the logical essentially with the logical-epistemological justification of symbolic arithmetic. Husserl had plans for a second volume in which he would deal with general concepts of number; but in trying to bring this task to completion, he realized that a new start, a much more encompassing investigation of the foundations of logic and epistemology was required. Instead of the second volume of his philosophy of arithmetic he then wrote his opus magnum, the *Logical Investigations* (1900–1901).

This work contains Husserl's mature philosophy of mathematics, but despite the newly found disposition to oppose all attempts to reduce mathematical and logical necessity to contingent laws of psychology, he did not completely reject his previous account of arithmetical knowledge. The reason is that his first theory of number intuition, given in terms of *psychological* processes of "presentification", can easily be rewritten in terms of concepts and ideas pertaining to the new science of phenomenology he had just created.

Husserl's philosophy was to undergo still another major change, the transcendental turn, in which phenomenology, from a theory of knowledge, became the a priori science of the transcendental ego (an "egology" as he himself described). This turn was completed with the publication of *Ideas* (1913). But Husserl had already by then extrapolated the restricted scope of his original philosophical interests, and no explicit and sustained treatment of the philosophical problems of mathematics, arithmetic in particular, is to be found in his *published* works after the *Logical Investigations*. Of course, mathematics was always present, but superseded by logic and, naturally, theoretical (or pure) phenomenology. I want here to fill in this gap; I want to present my version of how a philosophy of arithmetic developed in the spirit of *transcendental* phenomenology might look like. I do not claim this is *precisely* what Husserl had in mind; in fact, I think many aspects of my account would not get Husserl's approval (for one thing, he did not give formal mathematics the preeminence I accord it here, nor did he, unlike me, give the problem of the applicability of mathematics the importance I think it deserves). For this reason, I do not offer the present account as a reconstruction or interpretation of Husserl's thoughts and intentions, but as a possible way of coping, in the context of transcendental phenomenology, with the ontological, epistemological and pragmatic problems posed by arithmetic.

Numbers as Quantitative Forms Multiplicities (also classes, collections or ensembles) are *objective* correlates of intentional acts of collecting in which many objects are co-intended as "belonging together" as a single object, many in one. A collecting intention presides over the multi-rayed consciousness of the elements of a multiplicity, constituting the multiplicity as a new object, composed of and ontologically dependent on its elements. The elements of a multiplicity can be given in many different ways, intuitively or only emptily intended, remembered or fantasized, etc. For example, a collection can contain the objects *A* and *B*, with *A* originally given in intuition but *B* only recalled in memory. The objects of a collection can also be given indirectly via a common characteristic or a unifying concept; for example, the collection of all the planets in the solar system or all red things. The connective "*and*" expresses the formal component of the act of collecting. To intend the multiplicity whose members are *A*, *B*, *C* … is to intend *A and B and C and* …(to intend the *set* {*A*, *B*, *C* …}, however, is a different act altogether, the set is the collection intended as *itself* a collectable object – more about that later).

The quantitative form of a collection of things, supposing that one can subsume this collection under the notion of quantity, that is, supposing that it is a quantitatively determined collection, is this collection *itself* where, however, one conceives each of its objects as a unit. The quantitative form of a multiplicity is *not yet* its number, but an abstract aspect of it, occupying the same spatial extension that the collection occupies, if this is an empirical collection, coming in and out of existence with the collection. Numbers, on the other hand, are *ideal* objects that do not occupy any portion of space or time. Numbers are to quantitative forms what colors in specie (for example, redness in general) are to moments of color (*this* redness-moment here). The act that makes quantitative forms into numbers, by *identifying* all equinumerous quantitative forms into a single ideal entity, is *ideation*.[19] Numbers are ideal forms (one could say Ideas, with Plato) which equinumerous quantitative forms indifferently instantiate (or, to stay with Plato, into which they participate). Equinumerosity is defined in terms of one-to-one correspondences. Numbers, however, are *not* classes of equinumerous quantitative forms but something ideated from them, a particular intentional object; equinumerosity is only a *criterion* of sameness of number: two quantitative forms correspond to the same numbers provided they are equinumerous.[20]

Numbers are, then, ideal collections of units. This characterization is originally due to Plato; Husserl only spells it out phenomenologically. To consider a multiplicity of objects *quantitatively* is to consider it as a multiplicity of objects taken as units, that is, to consider it *as a quantitative form*, and then, derivatively, as a *number*. Numbers are ideal quantitative forms in the sense of being ideations of abstract quantitative aspects of collections of objects: by intending abstractly the elements of given collections as undifferentiated units, one is ipso facto intending this collection *abstractly* as a quantitative form, and *ideally* as a number. As idealities, numbers can be *identically* instantiated in materially different collections: two collections have the *same* number if their quantitative forms instantiate the same number or, which is materially but not definitionally equivalent, if they are equinumerous.

As ideal forms, numbers offer themselves naturally as objects of inquiry of an a priori formal science (where by formal science I mean here a science whose objects are forms), namely, arithmetic.[21] One can conceive arithmetic either as a philosophical science, concerned with ontological and epistemological questions about

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[19] Although closely related, there are differences between ideation and idealization. The latter is like taking the limit of a sequence, for example, idealizing the roughly spherical form of a ball as a mathematical sphere. By realizing the possibility in principle of making, in a sequence of acts, the form of the ball approach that of a perfect sphere, the ego idealizes by closing the series, it "sees" the form of the ball as a perfect sphere. Ideation is the act in which Ideas or Forms are posited; in mathematical terms, it is the equivalent of taking the quotient by an equivalence relation; i.e. seeing as the same what is only equal under a certain aspect. From the intuitive fact that each number has a successor one can idealize by positing a limit number of the series of successors, the first transfinite number; from the equivalence of equinumerous quantitative forms one can ideate the ideal entity all these forms instantiate. Idealization and ideation are creative, object-positing acts that are essential in mathematics.

[20] Husserl criticizes Frege for inverting this order of priority. For Frege, as we know, equinumerosity is defined in terms of one-to-one correspondence and numbers as classes of equinumerous concepts. For Husserl, this is phenomenologically inadequate for, he thinks, the expressions "*A* and *B* have the same number" and "*A* and *B* are equinumerous" are co-extensional but not synonymous, in the sense of having the same meaning; therefore, one cannot *define* number in terms of equinumerosity. The former must be defined (that is, characterized as to its essence) independently of the latter, which is only a *criterion* of identity of number. See this discussion, which includes Husserl's critique of Frege, in chapter VII of *PA*.

[21] There is another meaning of "formal science", namely, a non-interpreted science concerned exclusively with *abstract* or *ideal formal domains* (or *structures*), not particular domains of materially determined objects. This, however, is not the meaning I give to the notion here; arithmetic, considered as a science of *particular* ideal forms, is *eo ipso* concerned with a *particular* domain of entities.

numbers or, as arithmetic is usually considered and as I will consider it here, a mathematical science, concerned with *relations* among numbers.[22]

Collections of objects, in particular objects with which we are ordinarily involved with in our daily life, to the extent that they are *quantitatively* considered, must necessarily obey the a priori relations that numbers by their very essence satisfy. Thus, arithmetic becomes applied. 5 coins plus 7 pencils is equal to 12 objects for $5x$'s + $7x$'s = $12x$'s, no matter what x stands for, which is just another way of saying that 5 + 7 = 12. This is essentially the same explanation that Frege offered for the wide applicability of arithmetic; from his perspective, arithmetic applies to things because concepts apply to things and numbers are collections of concepts. From the perspective I offer here, arithmetic applies to things because things can be collected and collections of things can, *by a change of intentional focus*, be intended as numbers. Both Husserl and Frege believed that arithmetic belonged to formal logic, but for different reasons. For Frege, numbers, as classes, are logical objects, since for him the theory of classes belonged to logic; for Husserl, numbers, as the ideal forms that they are, apply to any collections of objects whatsoever, what qualifies arithmetic as a chapter of *formal ontology*, the formal-logical science of objects considered merely as such, which is a province of formal logic.

The Genesis or the Transcendental History of Numbers There was a time when numbers did not exist, they exist now, we use and study them. There was a time then, during which numbers came into being. Let me be clear about what I mean. Numbers are perfectly objective entities, that is, they exist out there, in the public arena, anchored in the space of culture, with the sense they have, being what they are, for anyone to use them or investigate their essential properties entirely a priori. However, numbers do not exist *by themselves*, independently of intentional positing; numbers have a genesis, an *intentional* genesis, and since the transcendental ego is, in this case, extended over time numbers have consequently a history, a transcendental history.

I have already shown how numbers come into existence by means of intentional acts, which, as also emphasized, are *not* psychological acts by means of which the ego simply becomes conscious or concocts representations of numbers that exist independently of its actions. In adequate intentional acts, of intuition or empty intending, numbers in fact come into existence. One of the ways of bringing numbers into existence is by naming them, since names are endowed with intentional directness. If not intuitively given in the manner already discussed, numbers are nonetheless intended as being *in principle* capable of presenting themselves intuitively to consciousness. The name, for example, 10^{10}, denotes a number that one does not expect, due to factual limitations, to experience intuitively as the number of some collection given properly to consciousness. It nonetheless exists as the correlate of its name, endowed as this name is with intentional directness; it exists

[22] Plato distinguished between ideal numbers, objects of *philosophical* inquiries, and mathematical numbers (collections of undifferentiated units), objects of *mathematical* investigation. See Klein 1968 and the definitions of unit and number in Euclid's *Elements*, Book VII, Def. 1 and 2.

objectively and is in principle, although not effectively, intuitable. I will come back to this soon.

The differing characters of number-positing acts explains why numbers have not come into existence all at the same time. The intentional ego incarnated historically in a community of intentional co-workers spread through space and time, i.e. the mathematical community, operates in stages. Initially, driven by practical necessities of the life-world, the ego devises techniques for tallying, reckoning and ordering that involves a primitive notion of (cardinal and ordinal) number and numbering. By gradually moving to ever-higher levels of abstraction and ideation, and eventually theoretical interest, the ego finally comes to posit an infinite domain of ideal entities as an objective realm of being open to theoretical investigation. To identify the moments of this development is to follow the intentional genesis of numbers and their science, arithmetic. Transcendental history is the history of this genesis; it is, I recall, an a priori variety of history, whose task is to identify the necessary steps in the constitution of, in this case, numbers and a science of numbers. The factual history of numbers, on the other hand, is the chronic of the sequence of historical events where the intentional genesis of numbers manifests itself; the former happens at the surface, involves choices and depends on perspectives, the latter runs deeper and can only manifest itself in phenomenological reflection; its moments need not always be detectable in factual history. Although the genesis of numbers is not to be confused with the historical development of the number concept, there are evident resonances of the first in the second; in a sense the former "incarnates" in the latter. Therefore, one can have a glimpse of the intentional genesis of numbers by following the historical development or the "cultural genesis" of numbers.

The notions of quantity, sameness of quantities and differences in quantity have arguably appeared very early in the development of man and human civilization, since even some animals manifest awareness of small quantities and differences in quantity. The earliest testimonies of the practice of tallying, however, dates from 35,000 BC.[23] It is clear that our ancestors mastered a method of systematic tallying, possibly for keeping track of animals, that consisted in making engravings in bones, each engraving representing a thing or perhaps also groups of things (which, if true, would be a primitive use of a basis for counting). An intentional operation necessary for the genesis of numbers is already detectable in this method, namely, abstracting concrete things – maybe animals – as units. It is also noticeable in it a primitive notion of equinumerosity, that is, sameness of quantity, in terms of one-to-one correspondences – a concrete thing corresponding to a mark and vice-versa.

When men finally invented names for collections of things in function of their quantity, they sometimes depended on the nature of things counted. Barrow mentions Indian tribes in Canada that used seven different but related groups of ten words to count from one to ten depending on the things counted, from flat objects, men, and canoes to measures.[24] This indicates that the nature of the objects being counted influences how their quantitative forms are named, which shows that

[23] See Barrow 1993, chap. 2.

[24] Barrow 1993, p. 39.

sameness of quantity does not imply strict identity of quantitative forms, supposing that if forms were seen as strictly identical they would have the same names. These Canadian Indians were, or so it seems, naming *quantitative forms, not numbers*. The dependence of quantity-terms on things counted survives even in modern languages such as Portuguese, which has *um* and *uma*, *dois* and *duas* for groups of respectively one and two things whose grammatical genders are either, respectively, masculine (or masculine and feminine) and feminine. From three on the names are common for both genders, which indicates that names for groups of three objects were invented later. The same phenomenon occurs in other Latin languages for at least the first cardinal number.

Cardinal numbers, or at least some of them, were fully constituted when it was recognized that groups of equinumerous things had the same ideal quantitative form. Initially, only a few small numbers received a name, with larger groups of things being generically denoted as *many*. The similitude of the French words for *three* (*trois*), *much* (*très*) and *a lot* (*trop*) seems to indicate that for primitive folks three was already plenty. In the earlier stages of the development of human culture, only small intuitable numbers existed; the idea of large numbers, non-intuitable in practice and, even more so, an infinite array of numbers with a systematic way of naming them were still in the future.

Arithmetic too has a transcendental history. Arguably, earlier cultures, even without a clear idea of numbers as ideal entities and the infinite array of systematically generated numbers, have developed a technology for operating with small numbers. It would not require great ingenuity to realize that one can divide a group of four cows in two groups of two cows and that this worked not only for cows. An indication that this was so is the fact that some cultures have used composite words for denoting some relatively larger numbers. According to Barrow,[25] some Australian aborigines have a system for naming numbers beyond two by combining words for one and two (four, for example, is two–two). Obviously, this requires the conscience of some numerical relations (in our example, 4 = 2.2). The French quatre-vingt-dix for 90 (90 = 4.20 + 10) is a dramatic example of the same principle in a modern culture. A technology for operating with numbers for practical purposes was evidently a first step into a science of numbers. Technology, however, is not yet science, for it lacks the theoretical interest that drives the latter.

Science requires a particular state of consciousness, a theoretical disposition in which intentional productions such as numbers, originally devised as instruments for practical ends, become themselves objects of interest, *theoretical* interest, not merely practical interest. The intentional focus of the theoretically oriented ego turns to objects as subjects of true assertions; theoretical interest problematizes what is simply given, but not only what is *given*. The theoretical ego is productive. Being more specific. At some point in the development of human culture, men devised a systematic way for denoting numbers. Some more efficient than others. Roman and Greek number systems, for example, were very clumsy and limited in their expressive powers; they were not systematic in the sense of encompassing a

[25] Barrow 1993, p. 34.

system of rules for systematically generating numerals, that is, number-names, for arbitrarily large numbers. Babylonian system was much better and systematic, although allowing for some ambiguities. The creation of the positional system with a symbol for zero made it possible to denote in principle any number unambiguously.

However, the idea that a system of numerical denotation produces names of *numbers*, not merely an array of empty symbols, requires the intentional disposition to give names the power of positing objects, the objects they name. However, this is not yet enough; that a system to produce symbols can generate a potentially infinite array of number-names, and therefore posit a potentially infinite array of *numbers*, requires an idealizing presupposition, namely, that in principle one can operate the system indefinitely.[26] 10^{10} would not be individuated indirectly to consciousness if one could not denote it; the numbers of the infinite succession of numbers would not, *each one of them*, be in principle capable of being individuated if they did not exist, i.e. if one did not *presuppose* that a name for each one of them could *in principle* be *actually written*. Idealizing presuppositions, however, are not hypotheses or suppositions, but elements of intentional production. By merely presupposing that one can in principle go on forever generating different numbers, one constitutes an infinite domain of non-individualized but in principle individualizable numbers (that this is an *actually* infinite collection requires further intentional acts). Numbers do not live in Platonic heaven waiting for men to discover them, they are invented, firstly as tools, and then as objects of theoretical interest. The invention of numbers and the creation of a science of numbers are intentional processes; one must investigate them to know what numbers and arithmetic are and how they can be useful. Husserl has, to some extent, followed along this path, but not far enough.

Numbers in Presence and Absence[27] Let us recall how finite cardinal numbers are brought into being as intentional objects.[28] One can intuit collections of objects of any nature directly by directly intuiting each and every one of the objects of the collection and taking them *as belonging together*, or indirectly, via concepts under which the objects of the collection fall (maybe not all of them directly intuited). One can consider multiplicities of objects from many different perspectives, quantitatively in particular (regardless of whether quantity is a pure or empirical category of understanding). Collections can be compared as to quantity, two collections have the *same quantity* when they are equinumerous, i.e. there is a 1–1 correspondence between them. A collection being given intuitively, an act of abstraction (which is a positing act, i.e. an intentional act in which an object is given) places us *face to face* with its *quantitative form*, or, upon ideation, its *number* (which is the same for every

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[26] Husserl calls this the idealization of the "and so on". See Husserl 1969, chap. 3 § 74.

[27] See Miller 1982.

[28] Again, by intentional constitution I do not mean the psychological process of formation of representations or the epistemological process of "grasping" something that exist "out there" independently of the subject.

collection equinumerous to it).[29] This is how numbers are constituted intuitively. However, *intuitive numbers*, or numbers in presence, are few; one can intuit collections, that is, presentify them to consciousness *in extension* only if they are reasonably small (collections given intensionally via concepts are *not* intuitively given, but emptily represented; so, their numbers are not intuitively given either).

Constituting, or better, intentionally positing a domain of numbers is like building a city, adapting a metaphor due to Wittgenstein. Once the innermost parts of the city are in place, one can *imagine* further extensions by simply following the city plan, beyond maybe the limits where constructions are *actually possible*. Some neighborhoods actually exist while others, although capable of being, at least in principle, actually brought into existence, are presently only drawings on a piece of paper. Similarly, once a few numbers are actually intuited, one can see how further extend the numerical domain. The city plan, in this case, is the generative number process "add a new unit" indefinitely iterated. The numbers that only exist by being meant (sometimes by being named) are the numbers in absence. The totality of numbers can be intended as either an open or a closed totality, but, in any case, only inadequately, since only a few of its elements are intuitively given, the others being only emptily meant. This fact deserves attention; totalities can be constituted, and even partially intuited, without its elements being *all individually* constituted, either in presence or in absence (the positing act can have the form: let the domain be of all *C*'s, where *C* is a concept).

Fortunately, we have managed, not without great efforts, to develop a systematic way of representing symbolically and *individually* all numbers that can in principle be generated – our usual decimal system is an example –, the system can in principle produce a name for each number that can be conceived. Not only one has the plan for building the still unbuilt neighborhoods of the city, one also knows their names. If our city planner is not so much worried with bringing the entire city actually into existence, but only maybe with studying its internal structure, designing the sewage system or the circulation network, the blueprint is all that he needs. Likewise for the arithmetician, he too needs only his conception of number – namely, that which the productive system generates considered exclusively in their mutual relations – to find out the properties numbers *must* have, giving how they are conceived. Notice that the arithmetician does not have to care about what numbers actually are; his approach is formal-operational, not material.

Now, to the extent that numbers that *actually* exist and are actually instantiable as abstract aspects of reality must a fortiori share the properties of numbers in general, the arithmetical laws the arithmetician discloses a priori are valid for them too, and thus arithmetic is applicable to reality. What is *necessarily true* of possibilities is a fortiori necessarily true of actualized possibilities. In this resides the epistemological relevance of consistently extending reality, mathematical or not, into domains that may not be real in the same sense of the domains they extend.

[29] Definitions in which common aspects of a multiplicity of objects are made into ideal objects are common in mathematics, where they are known as *creative definitions*. For example, spatial direction as that which all parallel lines have in common.

The fact that each number of the infinite array of numbers is symbolically representable allows us to represent numerical operations, addition, multiplication, etc. symbolically as well. On the one hand, numbers and proper numerical operations – for example, the successor operation as the adjunction of a new unit to a number, addition as disjoint union, etc. –; on the other, symbolic number-names upon which one can operate symbolically – for example, the usual algorithms for adding, multiplying, etc., in the decimal system. It is a trivial matter to verify that both systems of objects and operations are *isomorphic*. Consequently, whatever operations one performs with numbers are translatable into operations with symbols and, more importantly, *vice-versa*, any symbolic operation with numerical symbols is in principle realizable with numbers proper. This isomorphism guarantees that number-names resulting from symbolic operations correctly performed are indeed names of numerical outputs of the correspondent operations on numbers proper. One can reason through symbols and as far as computations are concerned, the presentification of numbers in absence is *unnecessary*.

Summing up, we can conceive numbers systematically and indefinitely and represent them individually in a symbolic system so that we can operate indirectly with numbers by operating (isomorphically) with their symbolic representations. On the one hand, numbers proper, either *intuitively* or *conceptually given* (i.e. numbers emptily intended consistently with the intentional meaning attached to the concept of number) and, on the other, *numerical symbols* (for instance, 10^{10}) whose intentional correlates are numbers proper. Remember, numbers are emptily posited as objective correlates of numerals insofar as these symbols are *meant* to denote. In the case of finite cardinal numbers, there is a strict correspondence between numbers, in presence or in absence, and numerals; to generate a finite cardinal number, directly or indirectly, goes together with representing it symbolically.

All numbers, without exception, exist as intentional correlates, which means that they either are or can be in principle intuited (by performing the appropriate intuitive acts). This has important consequences. One is that, against ontological Platonism, numbers do not exist independently of intentional experiences; they are not self-subsisting entities (although they can be *meant* as such). In addition, against nominalism, numbers are (conceived as) transcendent entities, differing from the numerical symbols that denote them. Moreover, against psychologism, by standing on the noematic side of positing acts, not the real noetic one, numbers are not real and, in particular, not mental entities either. Nonetheless, *because* the numerical domain is (conceived to be) an objectively complete realm of being, one is justified in endorsing "Platonism", that is, presuppose that numbers exist in themselves and that each well-formed arithmetical assertion has an intrinsic truth-value and can in principle be clarified in intuition (epistemological optimism). Of course, as already explained, this presupposition does not have a hypothetical character.

Imaginary Numbers Going back to our metaphor. Once the city plan is available the city planner may find it difficult to resist the impulse of extending in imagination the city further and further, even when no such extension is actually possible or feasible, for example, into a fourth spatial dimension. Immersing the city that exists or can exist in a city that does not and cannot exist may help him to understand better the former. However, while city planners are constrained by topography, budget restrictions, or reality, mathematicians are not so troubled and can let their imagination fly.

The system of symbolic representations of finite cardinal numbers and operations with them allows symbols to be concocted that may not play any representational role, such as, for instance, 2–3 or $\sqrt{3}$. From the perspective of a theory of numbers *as quantitative forms*, both symbols are utter non-sense. One can subtract two units of 3, but not the converse. Square rooting is also a limited operation, $\sqrt{a}$ denotes the number b such that $b^2 = a$, and for many numbers a, for example 3, there is no such a b (provided, of course, that operations are confined to the domain of numbers, i.e. non-negative integers). This, however, is not the sort of things that put mathematicians off. They reason thus: numerical symbols are intentionally loaded names, they always denote something; I can then take even meaningless symbols as denoting something, not numbers in the strict sense, but things that in a purely formal sense behave *formal-operationally like* numbers, and add these things to the domain of numbers proper, thus *formally* enlarging my conception of number. If this large context is *formally* consistent and can help me to operate with numbers proper or better understand their *formal* properties, then my procedure is justified.

"Absurd" symbols can be taken as bona fide numerical representations provided we no longer take numbers as ideal quantitative forms. Negative integers and complex numbers are "imaginary" numbers in the sense that they are not answer to the question "how many?" There are two different acts at work here, one of formal abstraction, by which the numerical domain is taken simply as an operational domain and, another, of formal generalization, by which the formalized domain is consistently extended by the addition of further formal elements to it.[30] The extension is usually naturally required for the domain of meaningful operations to be formally closed. It is as if, once in place, symbolic systems acquire a life of their own.[31]

Symbolic systems, conceived as systems of representation, often play in mathematics a creative role as well, by allowing symbolic extensions of mathematical domains. These systems are initially designed to represent entities that have some sort of reality, but they end up producing, by imposing their own internal dynamics, "imaginary" entities that can, nonetheless, be very useful, theoretically and practically. It is not difficult to understand why. To the extent that we are exclusively concerned with formal properties of numbers (or mathematical entities in general), it is immaterial in which context these properties are instantiated, the material domain of numbers proper or the purely formal domain of numerals (extendable for practical reason by symbols for number-like entities that are not numbers). If one insists on a realist perspective concerning mathematical ontology, one may find the

[30] Since material content is no longer a concern, formal consistency is all that is required.

[31] See Zellini 1997. The "discovery" (in fact, invention) of imaginary numbers by the Italian algebraists of the Renaissance constitutes a classical example of this process.

utility of imaginaries intriguing and end up believing that we have a mysterious talent for designing symbolic systems that help us understand reality.[32]

Besides the notion of discrete quantity of units, numerically quantifiable by (natural) numbers, we have also the notion of relation among quantities, either discrete or continuous, quantifiable by rational and real numbers respectively. Let us examine these more general conceptions of number.

Rational Numbers Rational numbers are not numbers properly speaking, but pairs of number with which one can express *quantitative relations* among quantitatively determinate collections.

Provided collections A has n elements and B has m, and that, say, all units of A are also units of B ($n < m$), then one can say that A has n *of the* m units of B. Another way of expressing this is by means of the *fraction* n/m, which tells *how smaller* A is with respect to B, insofar as A is a part of B. If, however, A is a collection of, say, 3 books and B one of 5 pens, one can say that A has 3 things and B has 5, but not that A has 3 *of the* 5 things in B, unless, of course, one considers all objects of A and B as undifferentiated units. In this case, we can always compare A qualitatively with B, for any collections A and B. In general, if two numbers n and m are different, then, either n is a part of m, or vice-versa (another way of saying this is that either $n < m$ or $m < n$). In any case, the fraction n/m expresses a quantitative relation between n and m. *Any* number n stands in the relation n/m to *any* number m.

Suppose now that A and B, C and D have respectively n, m, r, and s units. In what sense is the relation of A to B quantitatively *the same* as the relation of C to D? Suppose that A, B, C, and D are such that the following numerical relations are meaningful; we can generalize by going through cases. Suppose that one can change the notion of unit in C and D such that each new unit is a collection of k old units and C and D have, respectively, n and m of the *new* units. If this is possible, we say that, quantitatively, C is to D as A is to B, or, in other words, $n/m = r/s$. Now, since C has n new units, then it has $k.n$ old units, i.e. (*) $r = k.n$, analogously with D, (**) $s = k.m$. To generalize, one must remove k; multiplying (*) by m and (**) by n and comparing, one has $r.m = s.n$, which is then a necessary condition for $n/m = r/s$. It is not difficult to see that it is also a sufficient condition. We have then the answer to our question: quantitatively, A is to B as C is to D if, and only if, $n/m = r/s$, i.e. if $r.m = s.n$.

A *rational* number is the ideal form of all equal fractions, i.e. that which they all have in common or express. The *equality* $n/m = r/s$ becomes an *identity* if we think of either fraction as representing a rational number. Since any fraction instantiating a rational number is a representative of it, one can indifferently use the symbol n/m to denote either a fraction or the rational number that it instantiates.[33] Any (finite cardinal) number n is in the relation $n/1$ to 1, which is *the same* as saying that n has

[32] M. Steiner (1998) and E. Wigner (1960) endorsed similar views with respect to the problem of the applicability of freely invented mathematical theories in the physical sciences.

[33] The expression $n/m = r/s$ is then ambiguous; if the expressions on both sides of the equation denote fractions the symbol = denotes equality, otherwise, if they denote rational numbers, = denotes identity.

n units. Then, one can identify *n* and *n*/1 and take any number as a fraction and a rational number. Thus, one *immerges* the collection of numbers (henceforth **N**) in the collection of rational numbers (henceforth **Q**)[34]; an *immersion* is an isomorphism between a collection and a sub-collection of another collection (i.e. an *injection*).[35]

To immerge **N** into **Q** may have interesting consequences for the theory of **N**. Suppose, for instance, that the theory of **Q** allows one to derive a truth valid for all rational numbers. This assertion is a fortiori true of all numbers in **N**, independently of whether it is derivable in the theory of **N**. Investigating a domain of entities by investigating another, provided they are convenient related formally is a very powerful theoretical strategy and it is widely used in mathematics. Depending on the logical form of the assertion, proving it in the theory of **Q** entails that it is true in **N**.[36] In any case, suppose that there is a formula $\varphi(x)$ that defines **N** in **Q** (i.e. the theory of **Q** proves $\varphi(x)$ if and only if x is in **N**. Let Ψ be a sentence of the language of the theory of **Q**; if this theory proves Ψ restricted to the domain of φ then Ψ is true in **N**. I obviously cannot investigate here all the logical strategies mathematics applies to study a domain by studying another (I will come back to this later). I only want to point to the *philosophically relevant fact* that such strategies *exist*.

Both natural and rational numbers are ways of measuring quantities; respectively, how many objects a collection has and how big (small) a collection is with respect to another. *Positive* natural and rational numbers are answers to the question "how many?" and "how big (small)?", respectively (I leave *negative* numbers, which are "imaginary" numbers, out of account here). However, besides "how many?" one can also ask "how much?"

Real Numbers Quantity is the discrete *quantitates*, but also the continuous *quanta*. Any continuous magnitude, a continuous stretch of time or space, for example, or a continuous gradation of intensity in temperature, sound, or color is measurable by a (finite cardinal) number provided a continuous part of it is taken as unit. For example, any continuous is n times its n^{th} part. This, however, is not very interesting. The important question is whether all continua of the *same species*, i.e. all stretches of time, all lengths, etc., are measurable by the *same* unit, the same stretch of time, the same length, etc. If this were so, any continuum of a given species would be in a rational relation with any other continuum of the same species. In other words, they would be *commeasurable*. Possibly, the greatest achievement of early Greek mathematics (the Pythagoreans) was to show that this is not so.[37] There are incommensurable continua, for example, the lengths of the side and the diagonal of a square.

[34] It is irrelevant whether we take **N** and **Q** as sets or as collections.

[35] The operations on numbers as natural numbers correspond isomorphically to operations on numbers as rational numbers.

[36] We must be careful here, an existential assertion, for example, even if all its constants are in **N**, may be true in **Q** but not in **N**. For example, "there is x such that $n.x = m$", n and m natural numbers.

[37] See the definitions in Euclid's *Elements* Book X.

By varying the unit conveniently (e.g. by halving it successively), one can express the quantitative relation between two continua of same species in rational terms with arbitrary degree of precision, i.e. with smaller and smaller "error". One can always make a continuum *A* stand in a relation *n*/*m* with respect to a continuum of the same species *B* incommensurable with it "by ignoring" an arbitrarily small part of *A* or *B*. Intentional action takes the lead now and *posits by idealization* a non-rational number *r* that expresses the quantitative relation between *A* and *B exactly* for no matter which *A* and *B* (if *A*, *B* are commensurable continua, this *r* is a rational number). Thus, the ego constitutes a *new* class of numbers that contains all the rational numbers as particular cases. With these new numbers, we can express quantitative relations between *any* two continua of the same species. Let us call these new numbers the (positive) *real numbers*.

The first to come up with the idea of taking arbitrary ratios of comparable magnitudes as themselves magnitudes was Eudoxus of *Cnidos* (sec. IV BCE), who also devised clever ways of comparing such entities in terms of magnitude. It follows from what I said above that any real number is arbitrarily approachable by a sequence of rational number. By idealizing, i.e. by taking the limit of the sequence, a real number is conceived that "closes the sequence". Real numbers are indeed, in the proper sense, idealizations. Obviously, the approaching sequence is not unique; there are others that "converge" to the same real number. This fact plays a role in defining real numbers set-theoretically, for example, as sets of equivalent Cauchy sequences. However, defining numbers as sets is an *interpretation*, not in any way essential for securing their existence. Real numbers exist as intentional correlates of acts of idealization, independently of any interpretation, set-theoretical or otherwise. However, unlike the natural and rational numbers, there is no single system of notation capable of giving each real number a name. Therefore, in a sense, real numbers exist only as instantiations possible in principle of the *idea* of an exact quantitative relation between *any* two continua of a given species. Equivalently, despite of whether some real numbers are individualizable symbolically or definitionally, real numbers in general exist as possible instantiations of the intuitive concept of real number.

Now, two questions impose themselves: is *any* real number the measure of the quantitative relation between *some* continuous extension and an arbitrary *fixed* unit of the same species? Is the system of real numbers, which express quantitative relations among continua of a *given* species, also capable of expressing quantitative relations among continua of *any* species? We can answer the second question positively by simply reflecting on our previous considerations. The nature of the continuum in question played no role, only the fact that it was a continuum of some indeterminate species. Although one assumes that the first question has also a positive answer, this is not *a priori* true for it is not justified by relevant intuition. Hermann Weyl makes this point admirably in his *The Continuum*.[38] To answer the question positively, one must assume that continua are constituted of "points" tightly put together so that no "holes" remain in it. One's *immediate* intuition of the

[38] Weyl 1994.

continuum, however, is incapable of founding such a presupposition. As given in intuition, the continuum is a whole whose parts are also continua; there are no atoms in the continuum. To atomize it as a collection of "points" is an idealization. Such an explicit "violence" to intuition, however, is necessary for the intuitive continuum to be mathematically representable in terms of the arithmetical continuum of the real numbers, probably the most fundamental prerequisite for the mathematization of our perception of empirical reality.

One has, then, two idealizations; on the one hand, continua as ensembles of ideal "points", on the other, the arithmetical continuum as a collection of ideal numbers. For Weyl, the fact that one can establish a one-to-one correspondence between both is an axiom, or rather, a *presupposition*, since it does not rest, as true axioms, on one's *immediate* intuition. However, without this presupposition there will be no mathematical theories of natural phenomena. It would not be possible to represent quantitative relations among continua of given determinate species in terms of relations among numbers of the arithmetical continuum. The usual notion of measurement of continuum magnitudes would have no place in our theorizing about the world. Our immediate perception of the continuum, however, does *no*t justify the presupposition that continua are collections of tightly packed punctual moments; this is an intentional idealization.

Pseudo or Formal Numbers Let us take a second, closer look at imaginary numbers. Natural, i.e. finite cardinal, and *non-negative* rational and real numbers are related in one way or another to the notion of quantity; by their means, one establishes quantitative determinations. The notions of negative and complex numbers, on the other hand, have nothing to do with quantity, and came into being in the purely formal-symbolic contexts of algebra. Created by Arab mathematicians but introduced in European mathematics by the Italian algebraists of the Renaissance such as Bombelli and Cardano, and a bit later by the French algebraist Viète, algebra is a sort of abstract arithmetic. Given a problem, in arithmetic or geometry, whose solution requires, respectively, arithmetic operations and geometrical constructions, one can suppose the problem solved and, by analysis, verify how the solution relates geometrically or arithmetically to the parameters of the problem. This shows which operations or constructions one must perform to obtain the desired solution from the given data; then, by synthesis, which reverts the process of analysis, one can actually solve the problem. This heuristic procedure was not strange to ancient mathematicians, but Arab and Hindu mathematicians improved the method substantially with the creation of adequate symbolic systems of representation. The Babylonians already possessed techniques for solving algebraic equations, at least the simplest ones. An equation is the result of a process of analysis, to solve it is the correspondent synthesis. However, with the creation and perfecting of rule-governed symbolic systems of representation (symbolic calculi) the task of solving equations was in a sense mechanized; i.e. algorithms were discovered for solving them. The synthesis could be carried out *symbolically* and once the symbol for the solution was available, the problem was solved; it sufficed to know what it represented. Symbolic algebraic systems, however, were supposed to have a content, a meaning; all the symbols of the system were supposed to represent something, a number, a magnitude; and valid symbolic operations were supposed to stand for valid operations or constructions with numbers or magnitudes.

However, as already stressed, symbolic operations can sometimes extrapolate the limits of meaningfulness. For example, to subtract a number n from a number m meaningfully, n must be at most as big as m; otherwise, the operation is meaningless. Nonetheless, one can always form the meaningless expression $m - n$. More, this expression can pop up right at the middle of the symbolic manipulations for, for example, solving an equation. The question that mathematicians had eventually to face, in particular, dramatically, the Italian algebraists already mentioned, was what to do with it. The symbolic method worked fine when it made sense, but what if it did not? The advisable attitude would be, of course, simply to ignore non-sense and only rely on what makes sense. History has shown that this is not by far the best way of dealing with this penchant for nonsensical creativity built into symbolic systems. The Italian algebraists were quick to realize that if they manipulated nonsensical symbols as if they had a meaning not only this did not pose any problem but could actually be useful. How can this be so?

They knew negative and complex numbers, but they did not welcome them. The refusal of acknowledging negative numbers, for example, greatly complicated the classification of equations. They also tended to dismiss complex solutions when they showed up on operating the symbolic machinery for solving equations. The situation quickly changed when visionaries like Cardano and Bombelli decided to put qualms aside and take what probably was one of the most daring steps forward in science. By "forgetting" what symbols meant and simply operating with them according to rules, that is, by leaving material meaning aside and retaining from the symbols only their formal operational meaning, these mathematicians managed to obtain true, materially meaningful solutions of equations – meaningfulness by way of meaninglessness. They did it but did not know how and why it worked.

The answer is clear to us now. Material meaning does not matter because calculations are purely formal. By consistently introducing pseudo or purely formal numbers, such as negative and complex numbers, in a formal calculus extending formal arithmetic, the formal abstraction of contentual arithmetic, one obtains a more convenient formal context where to carry out formal calculations more easily. Solving equations involves exploring formal relations among numbers, which are preserved by leaving material content aside whereas preserving formal-operational content. Now, if we can carry out symbolic manipulations more conveniently in formal extensions of formal abstractions of materially meaningful calculi, and the results obtained by operating in the extended symbolic context are *true* when they are *materially* meaningful in the standard interpretation of the narrower calculus we can use the former to investigate the latter. It is a matter for logicians to find out what the formal-logical conditions for this strategy to work are.

Not only calculations can benefit from purely formal extensions; they can also be very useful theoretically. A simple example will clarify this point. Suppose one wants to prove the cancelation law for addition of (proper) numbers; i.e. $n + k = m + k$ implies $n = m$. One can do that in the arithmetic of natural numbers by induction on

k. However, suppose that one introduces new “numbers” in the game; for each number n a negative number $-n$ such that $n + (-n) = 0$ and formal properties such as commutativity and associativity being preserved. Let us suppose that one has already shown the converse of the cancelation law in the extended calculus. Hence, the following is true: (*) $n + k = m + k \rightarrow (n + k) + (-k) = (m + k) + (-k) \rightarrow n + (k + (-k)) = m + (k + (-k)) \rightarrow n + 0 = m + 0 \rightarrow n = m$. In short, one can prove a truth of arithmetic by resorting to formal pseudo-numbers, avoiding resorting to methods that may be more complicated. How can one do such a thing?

In fact, one has *not* proved a truth of arithmetic proper, but a truth of extended formal arithmetic. Despite our tendentious use of n, m and k as symbols purportedly denoting natural numbers, they in fact denote numbers in the extended sense, i.e. they can be either positive or negative. Therefore, the truth proven is valid for *all* numbers in the extended sense; it is a universal assertion. It is then, in *particular*, true for the natural numbers. The logical justification for accepting the conclusion is simply that what is true for all is a fortiori true for some. But attention, one has only proven a *formal* truth; nowhere the material meaning of the symbols played any role. However, since the law of cancelation is a formal law, this is all one needs. One does not have to confine oneself to the original formal context, one can move to extensions of this context provided in the end one can come back safely to the original context.

It took mathematicians some time to realize that *formal* reasoning could disregard *material* content. Despite one being now capable of interpreting every formal entity of mathematics in set theory, and convincing oneself that by so doing, i.e. by means of a *translation* of a formal context into another, one is ipso facto adding material meaning to them, mathematics has no use for material meaning.[39] Mathematics is *essentially* a formal science, which *explain*s its methodological flexibility and wide applicability.

Kant was one of the thinkers who struggled to give negative numbers a meaning that was not purely formal.[40] His suggestion was very intuitive; a negative sign did not denote a non-existing quantity, but a real quantity that somehow cancels another real quantity, like walking in one direction and then back. Others conceived negative quantities in terms of default, quantities that could nullify existing quantities; like positrons and electrons. For example, a 1 m hole in the ground is the negative of a 1 m pile of earth. But, as one knows, and Cardano[41] also did, the product of negative quantities is a positive quantity, for example, $(-2)(-1) = 2$. This fact, however, cannot be explained so easily in the context of the interpretations proposed. The rule of signs follows necessarily upon imposing the usual properties of multiplication with natural numbers to negative numbers. $(-1)(-1) = (-1)(-1) + 0 = (-1)(-1) + ((-1) + 1) = [(-1)(-1) + (-1)] + 1 = [(-1)(-1) + (-1)1] + 1 = [(-1)((-1) + 1)] + 1 = (-1).0 + 1 = 1$. It is a purely formal necessity. Kant and all those who

[39] Set-theoretical reductionism can, of course, provide a uniform logical-conceptual context of translation, in this resides its utility, not in telling what mathematical entities “really” are.

[40] “Versuch den Begriff der negativen Grössen in die Weltweisheit einzuführen” (1763).

[41] Cardano 1993, p.9.

struggled to give negative numbers a material meaning thought that it had to be related somehow to the notion of quantity, since negative numbers are counterparts of numbers that did represent quantity. As late as late XIX century, Husserl still thought that such a thing was possible. He believed that he could show this in the second part of *PA*, which he never published upon realizing that the task was not accomplishable. He eventually realized that "imaginary" numbers (negative and complex numbers) are purely formal entities that could only be adjoined to numbers proper after these were devoid of material meaning. Thus, a new problem faced him, namely, how to account for the efficacy of formal means of knowledge. I have dealt with his solution of this problem elsewhere,[42] where I show that Husserl's treatment of the problem was somehow conservative.[43]

By far a more significant advancement was the invention and introduction in mathematics of complex numbers by, among others, R. Bombelli. Tied as he was to the idea that numbers must relate to quantity, Kant believed that complex numbers were absurdities that had no room in mathematics. He reasoned thus: $\sqrt{a}$ is the real number b such that $1/b = b/a$. Now, if a is negative, b can be neither positive nor negative, both possibilities implied a contradiction with the rule of signs. Since, for him, a number must be either negative or positive, the square root of a negative quantity was an absurd notion.[44]

Complex numbers appeared naturally in the context of symbolic methods for solving equations. They showed their utility as means of obtaining *real* solutions of cubic equations. This is the natural place for complex numbers to arise, for although Baskara's formula for solving quadratic equations opened the possibility for complex roots, one could dismiss them, since the quadratic equation with two complex roots has no real root. However, cubic equations are different; they *always* have real solutions, which one can find by operating with complex numbers.

How this can be so is not essentially different from the situation that I have examined a little earlier. By moving to the complex domain, one's usual notion of equation is generalized; now, equations take coefficients and accept solutions in the complex field. One can then show that the usual formula for solving a cubic equation by radicals, whose paternity was bitterly disputed by the sixteenth century Italian mathematicians S. del Ferro, N. Tartaglia and G. Cardano, does in fact always produce the required solutions. All the computations are carried out in the complex field. It happens, however, that when the coefficients of the cubic are complex numbers of a *particular* type, i.e. complex numbers of the form $a + 0i$, where $i = \sqrt{-1}$ is the complex unit, then one of the solutions is *always* of the same form. There is, however, an *isomorphism* between the field of real numbers with real operations and the subfield of complex number of this form and complex operations. So, if $z = a + 0i$ is a solution of $z_3x^3 + z_2x^2 + z_1x + z_0 = 0$, where $z_n = a_n + 0i$, then a is a real solution of $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$.

[42] See da Silva 2010, 2012a.

[43] See in particular da Silva 2016a and the bibliography therein.

[44] See Kant 1986.

The complex domain turned out to be more efficient a context to deal with complex equations than the real domain for dealing with real equations. Consequently, it pays off to treat real equations as a particular type of complex equations; if a real equation has a real solution, this solution will naturally appear in the form that real numbers take in the complex field. The "trick" depends logically on the fact that the complex field has the real field as a subfield, and identities that are true in the real domain *as a subdomain of the complex field*, are true in the real field *ignoring its complex extension*. It is as if to see something one took a certain distance from it, with better conditions of observation, which one could abandon once one had seen what one wanted to see.

One might feel tempted to say that immerging the real field into the complex field is a sort of reconceptualization. But it is not. In fact, one completely *abandons* the concept of real number as quantitative measures, preserving only the formal-operational concept of number as "things" one operates upon according to certain explicit rules. Only then one can consistently enlarge the domain of "real numbers" into the purely formal-operational domain of complex "numbers". This allows one to transfer formal-operational problems such as solving equations by symbolic means from the real to the complex domain, and reintroduce material meaning whenever possible by taking certain symbols as denoting real numbers proper.

After the introduction of complex numbers in mathematics, against fortunately Kant's "better" judgment, there were many attempts at giving them a material meaning. The British mathematician John Wallis attempted but failed to interpret complex numbers geometrically. More than a century later, the Norwegian Caspar Wesel succeeded in interpreting them as two-dimensional (plane) numbers. However, one usually attributes to Gauss the usual interpretations of complex numbers as vectors or displacements on the plane. These interpretations have the great merit of showing that the purely formal extension of real into complex numbers is at least as consistent as our geometry, which is formally consistent. This eliminated the doubts as to whether the whole thing even made formal sense that bothered the early promoters of this leap of faith. Nonetheless, it does *not* "explain" what complex numbers "really" are, in the sense of giving them their due material content. The possibility of interpreting complex numbers as vectors only shows that the domain of vectors in the plane and vector operations is *formally identical* (isomorphic) to the purely formal domain of complex numbers and operations with them. In themselves, complex and negative numbers are purely formal entities. Set-theoretical reductionism does not fare better; numbers are *not* sets. However, the numerical relations that numbers establish among themselves is also instantiable among sets of particular types standing with one another in particular set-theoretical relations.

The creation of complex analysis has proven to be a wonderful invention for the formal investigation of the domains of complex numbers and complex-valued functions. It provides strong methods of investigation of the formal properties of these domains, and consequently their real subdomains. The use of residues to compute real integrals much more easily is an instance of this phenomenon that students of mathematics much appreciate. Once the benefits of the *method* of extending given domains of entities in purely formal ways or simply inventing formal domains anew, for their own sake or for practical and theoretical applications, became obvious, mathematicians were not shy in using it in all possible contexts. In arithmetic, with the creation of quaternions, a generalization of complex numbers, in geometry, with the formal extension of Euclidean geometry into *n*-dimensional Euclidean manifolds, or the introduction of a purely formal notion of space and geometry. The mathematical imagination was set free and there were no limits to it. For this reason, it does not seem farfetched to say that the Italian algebraists of the Renaissance were among the first to reveal the *true* formal nature of mathematics.

Arithmetical Structuralism The possibility of divesting numbers of their original material meaning as quantitative forms while preserving their formal meaning as things upon which one operates in certain ways, extending for mathematical purposes the domain of numbers with purely formal objects, shows that from a strictly mathematical perspective it does not matter what numbers are, only how they behave formally. This comes out clearly in Dedekind's formalist approach to arithmetic. He does not bother to tell *what* numbers are, only how they dispose themselves in a certain pattern. This contrasts vividly with approaches that are more traditional, such as, for example, Kant's, whose only concern was with the nature of numbers. Dedekind's way of seeing is justified, however, considering that arithmetic is essentially a science of the relational, non-material properties of numbers, and when the focus are relations, one may ignore the nature of the *relata*. They can be just anything and the "successor" relation defined as one pleases, provided it satisfies the formal properties of the successor relation proper. Dedekind's, in short, was a structuralist approach to arithmetic. The object of arithmetic, he tells, are not the numbers themselves, but the pattern they display irrespective of what the relata are or how the "successor" relation is defined provided it has the "right" formal properties. Dedekind does not say that a number succeeds another if it contains exactly one unit more than its antecessor; he only says that the successor is a bijective function of the domain of numbers $\mathbf{N}$ onto $\mathbf{N} - \{0\}$. This is a purely formal property satisfied by the numerical successor function proper, but also by many others that have nothing to do with numbers.

Husserl approached arithmetic from both sides, the material and the formal. He wanted to know what numbers are and made it clear that, for him, arithmetic, in the strictest sense, that is, contentual arithmetic, is the science of numbers proper. But he also realized that arithmetic could be abstracted from its material content and transformed into a theory of materially indeterminate objects only formally determined, which could be materially filled in any way consistent with the formal stipulations of the theory. Husserl also saw the important fact that insofar as we are interested only on the formal properties of numbers proper we can carry out our investigations in formal arithmetic. More, he also saw that formally consistent extensions of formal arithmetic could provide means that are more efficient for the formal investigation of numbers. He then paused to consider the logical-epistemological problem that this posed: can we formally investigate numbers proper in a formal context that *extends* formal arithmetic? How can we be sure that formal properties derived therein, concerning the objects of the extended formal domain, are *true* of numbers proper?

For Husserl, formal mathematics in general is a chapter of formal ontology, whose task is to investigate a priori, through formal theories, the formal properties of possible domains of being. But Husserl saw some potential problems if formal theories are used for the investigation of the formal properties of materially determined domains that are *not* interpretations of the theory. It was possible, according to him, that formal properties of the material domain derived in the auxiliary formal theory are not justified in the theory of the domain itself, unless this theory was syntactically complete with respect to the assertions that "refer to its domain", a property Husserl called *relative definiteness*.[45]

A related epistemologically relevant question to which Husserl did not pay due attention is how formal theories can be theoretically relevant for other *formal* theories. Of course, he clearly saw the possibility, for he reserved a whole extract of formal logic, the third level of formal apophantics, for the investigation of the logical relations among formal theories, a sort of formal metamathematics. However, he never actually conducted investigations of this nature, maybe because he did not see them as philosophically relevant, leaving them for the logicians. This, however, is an important question with philosophical consequences. One can view arithmetic (or mathematics in general) simply as the science of arithmetical (mathematical) structures (or forms), regardless of their instantiations. What makes mathematics interesting and *scientifically relevant* is that mathematical structures and their theories *can help us understand other structures, in particular those that are actually discernible in empirical reality and other domains of scientific or mathematical interest*.[46]

From the structuralist perspective, to know what numbers are is not relevant for arithmetic seen as a theory of the formal properties of numbers. This, of course, is true, but it does not imply that numbers do not exist or have an "intrinsic" structure and a nature of their own.[47] However, moving the focus from numbers to the numerical structure does not, of course, solve ontological and epistemological questions. We can raise the same questions about structures that we raise about numbers. There are philosophers who believe that structures exist in-themselves as ideal entities, others that they are only ontologically dependent aspects of actually or possibly existing structured domains of objects, and others still that they only exist as *façons de parler*. The old naturalist misconceptions about the nature of mathematical being and truth persists. The phenomenological outlook opens new perspectives. I will

[45] See da Silva 2016a for details.

[46] Explaining the applicability of mathematics to the empirical sciences, then, boils down to explaining the formal relations between formal structures discernible in experience, or idealized from it, and purely intentional mathematical structures, and under which conditions structural properties of mathematical structures can be transferred to empirical structures.

[47] Structuralists who deny the existence of numbers and their relevance for arithmetic must nonetheless explain why arithmetical operations are defined the way they are. If adding numbers, for instance, has nothing to do with collecting units, why is it that we have chosen to define it as if it had?

come back to these issues when discussing structuralism in mathematics in general (Chap. 7).

Final Considerations Platonists are right in believing that numbers exist objectively but wrong in thinking that they are ontologically independent objects. Constructivists are correct in believing that the numbers exist due to human action but wrong in thinking that they are, consequently, mental objects. Nominalists and conceptualists are right in pointing to naming and conceptualization as means of bringing numbers into existence but wrong in believing that, because of that, they are nothing beyond names and concepts.

Some numbers manifest themselves intuitively, and those that do not can at least in principle do so. Most numbers appear to consciousness either as non-specified instances of the concept of number or as intentional correlates of names and definite descriptions. In any case, the numerical domain is intended as an objectively complete domain of being.

The concept of natural number is itself an intuitable concept. By freely varying in imagination any arbitrarily (intuitively) given natural number, the correspondent concept and its characteristic features present themselves clearly to consciousness. Together with it, a domain of objects, the numbers, is posited, only partially intuited but *in principle* fully intuitable. The formal structure of this domain as determined by the principle of generation of numbers (the ω-sequence) is also intuitively displayed. A language offers itself in the experience with which to describe the intuited structure in a set of sentences, the so-called second-order Dedekind-Peano arithmetic (DPA). This description is adequate in the sense that it can singularize the *abstract form* of the domain, the ω-sequence (which domains isomorphic to the numerical domain have in common with it). Dedekind-Peano arithmetic is an interpreted theory, a theory of natural numbers proper, which, however, can only "grab" the *abstract structure of the numerical domain* as determined by the successor relation. By formal abstraction, DPA becomes the theory of the *ideal* ω-sequence. In short, the *interpreted* theory of numbers is essentially the theory of an aspect of the numerical domain, its abstract structure, whereas the formal abstraction of number theory is the theory of the *ideal* structure instantiated in this particular aspect.

Given any formula in one free-variable $\varphi(x)$ of the language of DPA such that DPA proves, maybe with the assistance of tertium non datur, that there is an a (not necessarily uniquely) such that $\varphi(a)$, then such a number (or numbers) *indeed* exists satisfying such a property. The meaning intentionally attached to the numerical domain, in particular that it is objectively complete (thus the validity of tertium non-datur) requires this number (or numbers) to exist; in a manner of speaking, it enjoys derived or dependent intentional existence. If a number cannot not exist on purely *conceptual* grounds, then it does exist, *independently of actual intuitive presentation*, considering that the domain of numbers is (meant as) objectively complete. This number is the intentional correlate of the non-vacuous description encapsulated in $\varphi(x)$.

By *idealizing* the systematic process of number generation disclosed in the intuition of the concept of natural number, *all* numbers come into existence as outputs of this process. One can also devise a system for generating number-names in a lawfully manner that goes alongside the process of number generation, so that each conceivable symbolic formation in the system denotes a single number, and vice-versa. The idealization associated with the process of systematic number-generation guarantees that each name generated in the symbolic system of number-name-generation does denote a number, and conversely. Names intentionally posit numbers whose names (in the system) they are (but that may have different names in different naming systems). One can introduce symbolic operations in the system that represent numerical operations in the sense that the system of number-names and symbolic operations is isomorphic to that of numbers and numerical operations proper. Such an isomorphism allows us to operate with numbers indirectly by operating with symbols (this logically justifies arithmetical logistic, the technique for operating *correctly* with numerical symbols).

Whereas natural numbers provide quantitative determinations (measure) of finite collections of objects, whatever they are, (positive) rational numbers measure quantitative relations among such collections; i.e. how bigger or smaller they are with respect to one another. There is a systematic process of generating rational numbers and naming them that is parasitic on the correspondent system for generating natural numbers and number-names. As we saw earlier, to any two natural numbers n and $m \neq 0$, there corresponds a unique rational number, denoted by n/m, and $n/m = r/s$ if and only if $n.s = m.r$. The concept of rational number is, as that of natural number, intuitively given and if n, m are intuitive so is n/m, and conversely. One can introduce operations in the domain of rational numbers in terms of operations with natural numbers and represent them symbolically. It is also possible to operate with rational numbers symbolically.

There is a relevant difference between natural and rational numbers. Unlike natural numbers, rational numbers do not display a natural structuring. There is no natural process of generation of rational numbers and a natural ordering that goes with it. Any structure this domain may acquire comes from structuring operations one defines in the domain. The favorite one is the structure the domain obtains when negative rational numbers are introduced in it and addition and multiplication are defined in the formally extended domain, namely, the structure of a field of characteristic zero. Each such structuring brings with it its theory, which as always one can formally abstract as the theory of an ideal structure with many different possible (in general non-isomorphic) materializations.

Exact quantitative relations among continuous magnitudes of any given species is an *idealization*; (positive) real numbers are idealized entities conceived to express such relations. In a way, real numbers are generalizations of rational numbers. The concept of real number is also an intuitive concept, but there is no systematic way of generating real numbers or naming them in any single symbolic system. Despite the intuitiveness of the concept, the domain of real numbers (the extension of the concept) is only imperfectly intuited. Properties of real numbers are disclosed in most cases only indirectly by reflecting on the correspondent concept (what makes the theory of real numbers an *intuitive conceptual* theory). Only very few real numbers are intuitively accessible or have names in some symbolic system, the infinite "names" produced by decimal expansion are in fact no names at all, but in general disguised descriptions. Like the rational numbers, there is no natural structure in the real domain, and the ones it receives (complete ordered field of zero characteristic, for example) depend on structuring operations unrelated to generative processes.

Unlike natural, positive rational and positive real numbers, negative numbers such as negative integers and negative real numbers, besides the complex numbers, are not numbers proper, but number-like purely formal entities that have nothing to do with the notion of quantity. The concepts of a negative "quantity", a negative relation between quantities, or a complex "quantity", have no intuitive content. Negative and complex "numbers" are purely formal "things" upon which one operates in determinate ways. These "numbers" have only formal content.

However, the invention of pseudo-numbers revealed the true nature of mathematics. The only *consistent* way of introducing negative or complex "numbers" into numerical domains proper is by relieving the latter of their material meaning (or content), preserving only their formal meaning. By so doing, however, one obtains richer *formal* contexts where formal operations are performable that may have consequences for the original domains. One may operate with meaningful symbols referring to numbers proper indirectly by operating with symbols that do not refer to anything materially determinate, or prove true assertions about numbers by proving formal truths about pseudo-numbers.

Suppose, for example, that φ is an assertion about natural numbers and T is usual arithmetic. Let φ' be the assertion φ formally abstracted and read as an assertion concerning the domain of numbers enriched with new formal numbers, whose theory is T'. I suppose also that if φ is a theorem of T, φ' is a theorem of T'. Suppose now that we can *prove* φ' in T'. Can we conclude that φ is true of *numbers* proper? Can we use purely formal extensions of materially meaningful domains to draw conclusions about them? Husserl thought that we could not, unless the theory of the original domains were syntactically complete. Indeed, suppose that T proves either φ or not-φ. If it proved not-φ, then T' would prove (not-φ)' = not-φ'. But if T' is a *consistent* extension of T, this cannot be. Therefore, T must prove φ, and we know that even if no proof of φ in T is *known*. As we will see later, this is *not* the only condition that would allow one to conclude the truth of φ without a proof of it in T, provided we suppose the original domain to be objectively complete (either φ or not-φ *must* be true in it). In short, *formally extending the formal abstractions of materially filled domains can provide powerful formal means of investigation of these domains*. In another, briefer, formulation, one can investigate formal aspects of the real by passing through the ideal. The applicability of mathematics in science relies heavily on this possibility.

Material arithmetic, the theory of numbers proper (i.e. idealized abstract quantitative forms) is applicable in practical affairs involving the notion of quantity thus. From "I have only two dimes and three quarters in my pocket" and "2 + 3 = 5" I can conclude "I have only five coins in my pocket" because 2 + 3 = 5 expresses a *necessary* property of quantitative forms. Frege thought numbers apply to the world because numbers are classes of concepts and concepts apply to the world. My approach is essentially the same: arithmetical truths, seen as necessary truths about numbers, are applicable because quantitative forms can "in-form" collections of objects irrespectively of their nature.

A striking feature of the theory of numbers is the amount of extra structure that one must sometimes add to the numerical domain for proving arithmetical theorems. The proof of the so-called Fermat's last theorem (the equation $x^n + y^n = z^n$ does not have solutions in the numerical domain for $n > 2$), for example, which eluded the best mathematical minds for circa three centuries, requires the numerical domain to be vastly enlarged and formally enriched so as to allow the import of finer methods of formal analyses.

This epitomizes mathematical methodology. A fact of utmost importance (to which, however, philosophers pay little attention) is that *any* mathematical theory, to the extent that it expresses itself linguistically, cannot determine a privileged material interpretation. All theories of materially different but formally identical domain are formally abstracted into a single formal theory. Another way of saying this is that mathematics only touches the formal-abstract surface of its domains. Consequently, it often happens that the mathematical investigation of a given domain can be carried out more successfully by investigating other, often richer domains. This is so due to relevant formal relations between both domains and the fact that mathematics only cares for the formal.

One would not have this liberty if the mathematical ego had not made the choice of restricting its interest to formal features of the domains it investigates. Standard mathematical procedures of proof would make no sense otherwise. Mathematical objects, no matter how the ego intentionally constitutes them, enter the picture only as bearers of formal properties. One can then characterize *pure mathematics* as *the a priori study of abstract forms*, *either instantiated in intuitively given or merely intended materially filled domains or posited as intentional correlates of freely created formal theories*.

Although mathematical forms can present themselves as correlates of formal theories or formal calculi, systems of symbolic manipulations according to rules, *forms* interest mathematics, not symbolic manipulation per se, contra the usual (mis)reading of Hilbertian formalism as "mathematics is a game with signs". This reading fails to acknowledge the fact that the mathematician is *never* interested in symbolic systems for their own sake; symbolism is a means, not an end in itself. Forms can present themselves either *in concreto*, that is, as forms of materially determined domains, or *in abstracto*, via formal-symbolic descriptions. The ego brings to evidence the formal properties of the ω-structure, for example, by reflecting, via the intuitively given concept of number, on how numbers relate to one another with respect to succession. Symbols interest mathematicians only insofar as they are intentionally related to objects and objects interest them only, or mostly, as supports of abstract forms.

Mathematical forms are interesting either in themselves or as means for investigating other forms. Pure mathematics is nothing beyond the study of abstract forms and relations among them, and applying mathematics consists essentially in using mathematical forms and our knowledge of them to investigate, from a purely formal perspective, domains of scientific or practical interest. As Hilbert emphasized, nothing can restrict the mathematician's freedom to create, that is, intentionally posit formal domains for whatever reasons he may have, practical or theoretical.

As long as mathematicians remain focused exclusively on formal properties of numbers, they can "forget" they are referring to numbers proper. This allows them a wide range of proof strategies. For instance, they can carry out their investigations of the numerical domain in any context structurally identical to it, even if the elements of this new domain are of an altogether different nature. They can also enlarge and structurally enrich the numerical domain in any way they find convenient. For example, by introducing new formal objects into the domain, provided they do so in a *formally consistent way* (consistency is the formal precondition of existence). Some formal properties may be lost, but some will be preserved (in the case of extending the finite cardinal numbers into the integers, or, in formal terms, the ω-sequence into the Z-chain, the existence of a first element is lost, but identities and inequalities are preserved).

To conclude with a moral. Old metaphysical ideas and controversies contaminated the age-old debate on the nature of mathematical objects, numbers in particular, and our access to them. By giving preeminence to this debate in detriment of mathematics *as practiced*, its methodological strategies and wide applicability, philosophy of mathematics failed to provide a consensual account of the nature of mathematical objects and mathematical knowledge. By "going back to the things themselves", i.e. mathematics as it is done, phenomenology seems better equipped to fulfill this task.

Having established that numbers are a particular kind of objects, forms of a certain type, but that arithmetic is not *particularly* interested in numbers as the objects they are (emphasis on "particularly"), only in the abstract form (structure) the domain of numbers instantiates, I will now direct my attention to another class of objects that are in many senses analogous to numbers, sets. The foundational role set theory came to play in mathematics may suggest that sets are the most fundamental, maybe the only true mathematical objects. As will be clear in the sequence I do not agree with this; to my view, sets only serve as a convenient context of materialization of theories whose formal character does not in fact require any materialization.

Chapter 5
Sets

Que serions nous, sans le secours de ce qui n'existe pas?

Paul Valéry

From its creation in the nineteenth century, set theory quickly advanced to occupy a central position in mathematics; it is, in Hilbert's famous words, the "paradise" created by Cantor from where we shall not be expelled. The reason of such prominence is essentially foundational. We can translate essentially all mathematical notions in set-theoretical terms. This led a number of philosophers to believe that sets are the most fundamental, if not the only, mathematical objects. The ontology of set theory has, consequently, attracted a lot of attention. Which sets exist? Russell has eloquently shown that the unrestricted axiom of comprehension leads to contradictions; it is not true that any well-formed formula of the language of set theory in one free variable defines a set. Which formulas, then, can? Predicativists believed that set-defining formulas cannot contain "vicious circles". The axiomatization of Cantor's "naïve" set theory by Zermelo and Fraenkel seems to have solved this problem by keeping under control the generation of sets. But there are still "problematic" axioms, such as the axiom of choice or the continuum axiom. Can they be accepted? On what grounds?

How can set theoretical axioms in general be justified? This can only be answered, one may assume, by somehow accessing the realm of sets, directly or via the concept of set. But if these are independently existing objective entities, how can this be done? Gödel believed that set theory was a conceptual theory and that the axioms of the theory are justifiable by conceptual intuition. He claimed to have got this idea from Husserl, but Gödel's Husserl is not the real Husserl. Gödel was a realist and believed in the independent existence of the concept of set; Husserl, contrarily, was not and, given the epoché, could not have been a realist in Gödel's sense. Of course, Husserl believed in the possibility of intuiting the concept of set, thus justifying at least some of the axioms of the theory, but the concept of set was, for him, an intentional construct, not an independent entity, as Gödel believed.

In this chapter, I want to analyze the intentional constitution of the concept of set, similarly to what I have done with the concept of number. By so doing, I hope to be

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able to see which axioms the concept so constituted supports. The domain of sets is, of course, subsumed under the concept of set and a set exists if its existence can be *justified* on purely conceptual grounds.[1] Sets can exist intuitively, but also only intentionally; not all existing sets are effectively intuitable, but they are all in principle intuitable. As I have stressed in previous chapters, intentional existence carries with it the possibility of intuitive presentation, in the sense that nothing stands, as a matter of principle, i.e. given our understanding of the essential nature of the intentional object in question, in the way of its effective presentification to consciousness. As we will see, the concept of set presupposes a set-constituting agent – a set is necessarily constituted by a set-constituting agent who assembles its elements; but being themselves collectable items, sets naturally dispose themselves in levels of constitution if, of course, the set-constituting ego is supposed to act whenever it can (everything that can be collected will be collected). As we will see, this can be a somehow idealized human agent or a purely formal set-constituting process that preserves nonetheless some traits of a human agent. A set can be in principle intuited if it exists and it exists if it is eventually constituted (not necessarily intuitively) by the set-constituting agent. The domain of sets is essentially a domain of highly idealized possibilities in principle actualizable.

As I will discuss later, being a convenient context of interpretation does not give the domain of sets any ontological preeminence. However, by now, my interest lies on the justification of set theoretical axioms from the perspective of the concept or, better, the intentional meaning attached to it.

It is a common belief that the mathematical theory of sets was born from the head of Cantor fully dressed and fully armed, like Athena from the head of Zeus. Unlike geometry or arithmetic, set theory is usually believed to be an utterly original creation, not the scientific refinement of notions and concepts of the life-world and the empirical sciences. This, of course, is not true, despite the originality of Cantor's *mathematical* notion of set and its related theory. First and foremost, Cantor's was a theory of transfinite (ordinal and cardinal) numbers, and although the mathematical tradition had no clue of the transfinite, the notion of number, cardinal and ordinal, could not be more firmly established. Cantor's numbers, despite their brilliant originality, were extensions of well-known concepts. But set theory is also, of course, a theory of pure sets, and the notion of set obviously belongs to the family of concepts denoted by the terms "collection", "ensemble", "family", "manifold", and "class" (perhaps with more or less important differences of meaning). Set theoretical constructions (like assembling a set by selecting elements from sets of a given ensemble of sets), moreover, derive from our pre-scientific handling of collections.

[1] As I argue below, a set-constituting intentional subject must act in certain ways for sets to come into existence – this much is required by the *concept* of set. But the concept itself does not prejudge the constituting powers of the subject. However, as I also argue here, to consider the concept in full generality, as an a priori conceptual science requires, we must give the set-constituting subject maximal freedom consistent with the *idea* of a set-constituting agent. The existence of a set a (for example, and empty set, an infinite set or the choice set in general) can then be established *axiomatically* provided the existence of a accords better with the acts of a very liberal set-constituting subject than the non-existence of a.

Last, but not least, the notion of class as the extension of a predicate (the so-called logical conception of class) was common currency by Cantor's time.

The crucial difference between collections (classes, manifolds, etc.) and sets is that the latter are *objects* whereas the former are not; collections are *parts*, but not *elements* of larger collections. (However, in mathematics, even before Cantor, collections were already taken as *elements* of other collections – in Dedekind or Cauchy's theory of real numbers, for example, collections and sequences of rational numbers were for all purposes treated as *objects*, and in no way ordinary ones, but *actually infinite* objects.) On the one hand, mereology, the theory of collections from the perspective of the relation of part to whole, on the other, set theory, the theory of *individuals* of a particular kind standing in a different kind of relationship, that of membership. Apart from taking collections as collectable *objects*, Cantor literally blew beyond all limits the process of set-formation by collecting sets as *elements*, the set-theoretical counterpart of assembling collections as *parts* of larger collections (in scientific taxonomy, for example). Although the concept of class was in Cantor's time, and still is of some utility for empirical science (again, scientific taxonomy) and, obviously, to a much greater extent, mathematics, neither mathematics nor the empirical science had shown any interest in developing a theory of sets or classes – pure or impure – before Cantor.

In the work on the convergence of trigonometric series that led to his groundbreaking ideas, Cantor was involved with collections of real numbers and the operation of deriving collections of numbers from given collections of numbers. The fact that the operation of derivation could sometimes be *indefinitely* iterated opened the doors for the stroke of genius, the idea that "indefinitely" could mean more than "arbitrary finitely many". He saw that the formation of derived collections could sometimes be continued not only beyond *any* finite limit, but beyond *all* finite limits; i.e., that it could be *transfinitely* iterated. Transfinite ordinals were then created, first as *indexes* of derived collections, measuring the "number of times" the operation of derivation was iterated, and so as generalizations of finite ordinals. The need for a theory of numbers that incorporates transfinite numbers does not per se call for a theory of pure sets, but the idea that numbers are abstract aspects of sets of objects did naturally lead to the development of transfinite arithmetic in the context of a theory of sets. And so set theory, a point of convergence of concepts and ideas coming from different directions, came into being.

Upon considering Cantor's creation, we cannot avoid being struck by its *naturalness*. Why had mathematicians not thought of it before? It may be that transfinite numbers only appear natural to us because we, unlike those before Cantor, have been raised under their shadow; we have been educated into accepting them *naturally*. But maybe the true reason why mathematicians had not come up with transfinite numbers before Cantor is that mathematics did not have much use for them. One of the reasons why they *still* do not have use for them is the more or less trivial character of transfinite arithmetic; another, the undecidable character of some of the questions the theory was *devised* to answer (the most important being the cardinality of the continuum – Cantor certainly saw the elusiveness of the solution to this problem, which he and others, such as Hilbert, tried hard to find as a shortcoming of the theory). "Cardinality arguments", based on the fact that infinite sets can have different cardinalities (or, more precisely, on Cantor's theorem: for any set A, card $(2^A) >$ card (A)), which allows for non-constructive proofs of existence, is probably one of the few *mathematical* techniques that originated with Cantor's creation. It is, however, precisely the abundance of techniques exportable to *other* branches of mathematics that mathematicians usually rely on as a criterion of relevance in the field. Because of this and despite its immense significance for the *foundation* of mathematics, set theory was never considered a very relevant theory from the point of view of the proverbial working mathematician (what is often called "set theory" in mathematical journals and departments today is the *meta*mathematic*s* of set theory).

There is, of course, the issue of the axiom of choice. After it was "discovered" by Zermelo it became clear how much classical mathematics depended on it in its many variants, and how often mathematicians had relied on it without taking notice of the fact. But to what extent is the axiom of choice actually *justified* and *secured* by set theory in its present axiomatic form? It would no doubt be if it had been *proved* from indisputably intuitive set-theoretical truths, but it has not. So, we cannot say that axiomatized set theory *justified* the axiom of choice, in a logical sense of justification. In fact, it has been proved that this axiom is *independent* of more elementary, seemingly obvious set-theoretical propositions. We are only logically justified in believing that the axiom of choice is consistent with (but independent of) more intuitive propositions, which is definitely a good thing to know but not a guarantee of truth. Moreover, as many have argued, the axiom of choice *is* obvious in a combinatorial conception of set, and that is why it had been so widely used uncritically before. So, although set theory as a *mathematical* theory is not to be thanked for it, the clarification of the conception of set that the theory allowed (a task more philosophical and metamathematical than mathematical) did indeed serve mathematical purposes. But still, this is not central core mathematics as we understand it.

Set theory was born, and despite open questions remains still, from a mathematical, not foundational perspective, concerned mostly with the arithmetical structure of the continuum. Philosophers, however, have always tended to see it on grander terms. The reason is obvious; on the one hand it provides a foundation for classical mathematics (despite the fact that practicing mathematicians see no need for foundations in mathematics); on the other, and this is what most directly touches philosophical sensibilities, the theory clarifies the notion of infinite, which for so long (from Zeno to Kant and beyond) plagued philosophy, providing even a subtle distinction between relative and absolute infinites. Cantor himself saw in this such an important contribution that he counted heavily on it as a means for making the theory acceptable (not least to theologians).

But my purpose here is not to discuss the *mathematical* (as opposed to foundational) relevance of set theory, or to point to its *historical* connections with prior theoretical and practical notions. As for the former, I tend to be less enthusiastic than most philosophers, recognizing all the same that the theory, while maybe not the paradise that Hilbert believed it to be, nevertheless does play an important role in mathematics, although a more foundational one. As for the latter, I leave the matter for historians; my interest is of a more purely philosophical nature.

I want here to trace the concept of set to its origins, that is, to follow its intentional genesis or transcendental history. Factually, I believe, the Cantorian concept of set derived from concepts that had never before been submitted to adequate mathematical scrutiny, but were freely used in mathematics, boldly generalized. The history I want to tell is a different one, the transcendental history not of facts but of constituting experiences of the transcendental ego.[2] By ascending from its most basic perceptual experiences to the predicative level of involvement with the empirical environment, the ego constitutes new empirical objects, *empirical* sets, and then, by moving to an even higher level of consciousness, the theoretical one, it constitutes the domain of all possibly conceivable empirical sets offered to *a priori* theoretical investigation. And eventually, once the concept is liberated from its empirical origins and taken in its purity as required by the very idea of a priori theoretical inquiry, the ego constitutes the intentional domain of *mathematical* sets and its theory.

As already noticed, transcendental history is not always easily identifiable with factual history; the ego often works at levels undetected by factual history, performing its acts behind the scene where historical drama unfolds. In *Crisis*,[3] Husserl gives us one example of the type of transcendental analyses I carry out here, focused on the constitution of the modern scientific concept of nature and the intentional presuppositions it harbors, which usually pass unnoticed when the history of modern science is told.[4] Transcendental history has the task of revealing the silted over layers of intentional constitution on top of which sits factual history.

This investigation has a foundational character, but in a *particular* philosophical, not mathematical sense. As an exercise in genetic phenomenology, I want to drive mathematical set theory back to the *Lebenswelt*, the life-world, by showing that the positing of mathematical sets lies are at the end of a series of intentional acts that ultimately rest on sense perception. This sort of investigation was advocated by Husserl in *Crisis* as a means for overcoming the “alienation” of modern physical science (but not in the least for “correcting” its methods or disqualifying its accomplishments), and was carried out with respect to geometry in the essay *The Origin*

[2] Remember, I take the concept of ego as Husserl understood it; in few words, “an intentional center of sense giving” (Moran and Cohen 2012, p. 90). One can think of the ego as an “intentional consciousness” generically and abstractly considered, which can materialize in an individual or a community of individuals, in a moment of time or throughout history. As a “center of sense giving” the ego creates its own “worlds”, inhabited by “intentional objects” or beings-for-the-ego that have the sense the ego endows them with. The “transcendental history” I plan to tell here is the chronicle of the acts, or experiences, that go into the constitution of realms of sets, mathematical and empirical, as intentional objects, and their correspondent theories.

[3] Husserl 1954a (English translation in Husserl 1970). Henceforth cited as *Crisis* with reference to paragraph number; English versions are mine

[4] See Chap 8.

of Geometry.[5] *Experience and Judgmen*t (Husserl 1973) contains a brief analysis of the first moments of the transcendental history I will tell here.

Remember that one of the fundamental tenets of transcendental phenomenology, one that distinguishes it sharply from empiricism and other forms of "naturalism", is that any domain of experience or cogitation, the empirical world or any of the many domains of science, empirical or mathematical, are not simply "given" but constituted, and consequently, that their "ultimate sense of being" emanates from constituting transcendental subjectivity.[6] But, I must insist, by bringing in subjectivity and constituting acts of the ego phenomenology does not fall, ipso facto, into psychologism; transcendental subjectivity is not a psychological concept. On the contrary, only phenomenological analysis can show the error of empiricist philosophies of mathematics, such as psychologism, along with the many brands of constructivism and "naïve" realism.

Let us recall some fundamental phenomenological concepts (for a more detailed discussion see Chap 2). To posit a domain of beings is not to concoct them "in the mind" as real objects, in language or what have you. Objects are "intentional correlates". Intentional objects can be *meant* to exist on their own, independently of being intended, such as the real objects of the empirical world, but this is only a *way of meaning*. Intentional objects, no matter which, always lie at the center of a web of meaning emanating from subjectivity. No object, including sensorial percepts, can be given (to the ego) without being meant (by the ego). Sometimes objects are intentionally posited as effectively *present*, not merely *represented*; these are the intuited objects; intuitively meant objects, to use a term dear to Husserl, are "bodily" present. Sometimes objects are meant simply as *that which* satisfies certain conditions. The typical "empty" intentional positing has the form: *consider a thing (or a realm of things) such that…*, where the dots are filled in by what is considered as an adequate expression of the sense these things are intended to have. Intentional directness is there, as well as intentional meaning, but the intentional objects, things individually or whole realms of things, although clearly meant, are not intuitively present to the positing consciousness.

The intentional meaning can or cannot be intuitively fulfilled; in case it cannot *as a matter of principle* (for instance, when it involves conflicting characters such as "a round square"), the intentional object does not exist *at all*. *Intentional* existence, i.e., existence *for the ego*, can be granted only insofar as no inconsistency manifests itself at the core of the intentional meaning. As soon as an inconsistency appears, the intentional object "vanishes", it ceases to exist. This is the *minimal* sense of existence and non-existence in phenomenology: an object (or domain of objects) exists if the sense with which it is posited is consistent and *only as long as* it remains

[5] Husserl 1954b. With respect to the intentional constitution of the predicative judgment see Husserl 1973 (henceforth cited as *EJ* with reference to paragraph number).

[6] "It is not the being of the world in its unquestioned evidence that is primary, and it does not suffice to ask simply what belongs to it objectively; on the contrary, what is primary in itself is subjectivity, which pre-gives naïvely the sense of the world, and then rationalizes it or, which is the same, objectifies it" (*Crisis* §14).

consistent. This is true even for empirical objects. Since, for example, it is part of the sense of being of empirical objects that they can *in principle* be perceived (this is true even for subatomic particles) no empirical object *can* exist that is not in principle perceivable. As is immediately clear, the questions "but does the intentional object *really* exist?" and "is there something in *objective* reality that answers to our positing intentions?" are phenomenological nonsense. Of course, an object only "merely" intended can also manifest itself intuitively, fulfilling the intentional meaning associated with it, but this, intuitive existence is only *another way of existing*; there is purely intentional and there is intuitive existence. A theory whose objects exist in part intuitively and in part only intentionally or a theory of purely intentional objects is no less a theory of perfectly legitimate objects than a theory of intuitively given objects. Contrary to constructivist conceptions of existence, as long as the integrity of the intentional meaning is preserved, the object it "frames" exists and can be theoretically investigated a priori (by, for example, spelling out this intentional meaning and deriving its logical consequences). This is the correct reading, I believe, of Hilbert's and Poincare's conception of existence in mathematics: to exist is to be free from contradiction. The phenomenological conception of existence is the virtuous mean between the constructivist and the realist conceptions. Phenomenology allows mathematics to be "realist", but not realist.

A domain of objects can (but need not) be posited as an ontologically and epistemologically complete realm, in the sense that any conceivable situation in the domain and any meaningful question about it, that is, any situation and any question in conformity with the intentional meaning attached to the domain, is either a fact or not a fact (no possible situation is *in principle* indeterminate as to its factuality) and has an answer (no meaningful problem is *in principle* unsolvable). Positing a realm of being under these conditions is the condition sine qua non for the principle of bivalence (and classical logic) to be valid in reasoning about it.[7] Mathematical domains in particular are so *conceived*, as domains that although *posited by the ego* are nonetheless ontologically (objectively) complete (the theory of the domain, i.e. the collection of all sentences of the language of the domain that are true in it is, then, logically complete).[8] This is *not* a presupposition that can be somehow disproved, or a hypothesis that can be put to test, but an intentional trait that goes along with the *sense of being* of these domains. The error of intuitionism is to reduce the positing ego to an empirical mind (idealized to some extent, but still immersed in the flux of time; the ideal mathematical subject of intuitionists is still a human being, no matter how much improved his cognitive abilities are) striving for clarity (intuitiveness), and intentional positing to mental experiences of *actualization*. By so doing intuitionism cannot help but reduce mathematics to an investigation of the life of the mind, inexorably trapped in the flow of time.

[7] See Chap. 3.

[8] An objectively complete domain of being, recall, is a domain where every possible state-of-affairs is determinately either a fact or not a fact, in which case the complementary state-of-affairs is a fact.

Phenomenological analysis also highlights the errors of ontological realism, which confuses objectivity with ontological independence. The domain of sets as conceived in "classical" mathematics is a domain *objectively* given, *intentionally meant* as "out there" for me and anyone else, now and in the open infinite future, about which any meaningful question is *in itself* decided, but it is not an *independently* existing domain. Sets are objectively real but not self-subsisting; sets exist objectively but not self-sufficiently. The exhaustiveness of the classical trichotomy, independent existence (Platonism), intuitive existence (constructivism) or non-existence (nominalism), depends on a too narrow a sense of existence that does not take into consideration purely intentional existence and the *power* of intentional positing. Phenomenology shows the possibility of another conception of existence with all the benefits of realism without its metaphysical burden (this is why Husserl once claimed, on being "accused" of idealism, that nobody was more realist than him, the transcendental idealist).

The investigations that I carry out here show, I think, that contrary to certain views about set-theoretical axioms,[9] the fact is that, as already pointed out by, for instance, Joseph Shoenfield[10] and Gödel, these axioms, or at least some of them "impose themselves upon us". Of course, nothing can impose itself on us that is not *intended by us*. So, the justification of naturally looking axioms can only be carried out in a phenomenological investigation of set-positing experiences and the sense they bestow on the things they posit. The phenomenological clarification of the concept of set is the task I impose upon myself in this chapter.

Given the proven incompleteness of Zermelo and Fraenkel's axiomatization of set theory, new axioms are necessary for settling the many questions the theory leaves unanswered. Since we presuppose that any meaningful question raised about sets has in principle an answer, we could interpret logical incompleteness as a failure in our *grasping* of the concept, rather than in the conception itself. However, this might justify the belief that the concept lies beyond its conception, and thus set theoretical realism of the Gödel sort. However, we can remain faithful to the transcendental-phenomenological thesis that there is no concept beyond the conception, and still account for incompleteness, by viewing the conception as positing a *transcendent entity* whose sense of being the conception itself does not fully express. Now, as transcendent entities in general, the concept of set is meant as always capable of disclosing new aspects, particularly by interacting with other mathematical concepts, not only by reflection on the intentional meaning originally attached to the concept, as Gödel wanted. If we understand such an interaction as a form of indirect intuition, we can hope to justify set-theoretical axioms that are not directly justifiable in the original positing of the concept. This is a form of the so-called extrinsic justification. By being immersed in the system of mathematical concepts, the concept of set is, so to speak, given a chance to reveal aspects that, although consistent with the original positing, are not derived from it, but somehow required given the overall system of concepts. Mathematical concepts can clarify

[9] See, for instance, Maddy 1980.

[10] See Shoenfield 1967, chapter 9.

one another and letting a concept interact with other concepts may reveal hitherto hidden aspects of it. However, my concern here are with axioms that are justified by clarifying the concept as originally posited, leaving more elaborate forms of justification out of consideration.

Empirical Sets The beginning of the history I want to tell is better told by Husserl, who in his *Experience and Judgment* (*EJ*, Husserl 1973) dealt, if only *en passant*, with the problem of the intentional constitution of the concept of set.

As far as empirical knowledge is concerned, Husserl recognized two distinct levels of intentional activity of the ego: the level of receptivity and the predicative level.[11] At the pre-predicative level of "receptivity as the lowest level of activity of the ego" (*EJ* §17) sets do not exist, but collections may. On their basis, however, empirical sets can be constituted at the predicative level. Collecting is an act in which *many* objects are simultaneously intended in a multi-rayed experience (or *polythetic operation*, in the words of Husserl); a *multiplicity of* objects, a collection or aggregate, which however is not yet a set, is thus intended. A set is one object, a collection many; the constitution of sets requires an act of *unification* in which *many* objects simultaneously intended are meant as *one object* with many elements. Consequently, sets have *material* components, their elements, and *formal* components, corresponding to the intentional acts that go into constituting them (collecting and unifying, to name them). Empirical sets are new objects of the empirical domain, which is enriched by their coming into existence. The insertion of these higher-level objectualities (I use this word here to translate Husserl's technical use of *Gegenständen*) into the world is concomitant with the ego's *predicative* involvement with it. The predicative level is also the level in which states-of-affairs, the objective correlates of judgments are constituted in judging, which is why Husserl calls sets (*many* objects *as one*) and states-of-affairs (objects *in relation* to each other) syntactic objectivities or objectivities of understanding. Both types of entities have formal components; neither is reducible to their material aspects. Nonetheless, empirical sets (like empirical states-of-affairs) are still *empirical* objectualities and cannot, unlike mathematical sets, a different sort of entities, have properties that are a priori not allowed to such objects (such as infiniteness).

So, for Husserl, sets are not simply given, they are products of a disposition of the ego. Unavailable at the level of pure receptivity (which, it is important to remark, is not for Husserl one of pure passivity, for constituting activity is already at work at this level), sets appear, together with other higher-level objectivities, only at the predicative level of experience. By constituting sets, the ego is neither concocting "representations" to take the place of things independently existing "out there" (representationalism), nor creating a subjective realm of mental objects, figments of imagination with which to entertain itself (psychologism). On the contrary, the ego is shaping reality *itself*, objectively available for anyone willing to reenact the

[11] For Husserl empirical sets are objectualities of the understanding. For this reason, they belong together with states-of-affairs to the predicative level of involvement of the ego with empirical reality.

relevant constitutive experiences (i.e. anyone willing to play the part of the ego). In order for sets *to be*, the subject must constitute them, performing the required intentional acts. Therefore, the sets that *actually* exist in empirical reality depend on which sets the subject has *actually* brought into being, which depends materially on the availability of their elements and formally on the subject *actually* carrying out the required intentional acts.

Being new objects of *empirical* reality, sets must accord with the meaning intentionally attached to the empirical domain; otherwise, they cannot exist *as empirical objects*. The set of eggs in the basket, being a physical object, has all the properties a priori granted to physical objects (locality, temporality, etc…), and is *materially* identical to the mereological sum of its elements. Its formal aspect, however, that which makes it a set, not simply the sum of its elements, is a categorial, non-independent, abstract component of it,[12] similar to that which makes a physical state-of-affairs *more* than the ensemble of its objects. Nonetheless, empirical sets are still empirical objects; hence, there *cannot be* a transfinite set of eggs (and since they are not purely mathematical objects, sets of eggs are of no interest for pure mathematics). Sets of objects satisfying determinate properties, moreover, can only be constituted if these properties befit on lines of principle the objects of the domain from where the elements are selected. To give a mathematical example, we cannot properly constitute the set of happy integers, but we can constitute the – empty[13] – set of proper divisors of 17.

In order to "come into being" sets depend on acts of collection and unification, the former a necessary condition for the latter (so, which sets exist depend, first, on which collections can be assembled and, secondly, on which collections can be unified, that is, taken themselves as collectable objects).[14] Sets must be constituted, and to constitute sets, according to the most basic sense of constitution, means to *intuit* them, which requires that their elements be, each individually, distinctively presented to consciousness; only then can set-constitution acts be properly performed and the newly constituted sets clearly presented intuitively to consciousness. That set intuition is the most basic form of set constitution is in line with Husserl's understanding that the intuitive judging is the most basic form of judging. As he explained in *EJ*, the fundamental form of judging from which other forms derive is judging about things that are "under our eyes" (with *clarity*).

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[12] See Husserl 2001 for the precise meaning of these terms. Although the set of eggs in the basket is a *physical* object occupying the same space of the eggs, and existing as long as they exist, the set cannot be *identified* with the eggs in the basket for there is an irreducible *intentional* element that goes into making the set. The eggs must be *taken* as a collectio*n*, which, on its turn, be *seen* as a single object (Maddy's "realist" approach to empirical sets is tangled in confusion for not seeing this).

[13] It is not obvious that this set is empty; we could as well say it does not exist, that the constituting intention is frustrated. Many mathematicians and philosophers (including Husserl himself), at the time these questions were first being discussed, did not accept the existence of empty sets. I deal with this issue below.

[14] The question, then, whether there is an empty *set* will depend on whether, *first*, there is an empty *collection* and, *second*, whether this collection can be unified.

When theoretically inclined, however, the ego opens its intentional focus to encompass empirical sets *in general*, in which case intuitive presentation is no longer required. The focus is now directed to the *concept* of empirical set *itself*, whose essential traits, as Husserl taught us, are as much discernable in intuited instances as in merely represented (purely intentional) ones. This new theoretical approach considers all the empirical sets that *can* be intuited, in some sense of "can" (and it is my task here to clarify which sense this is). As empirical objects empirical sets are immersed in time; so, to consider *all sets* that can be constituted is the same as considering all sets that can be constituted *at some point in time*. Of course, these sets will never coexist at *any* point in time for time never ceases to flow and with the flow of time set-constituting acts can forever be reenacted.

To consider what can be intuitively constituted in some domain of theoretical interest, in accordance with the sense attached to it independently of *actual* intuitive presentations is the modus operandi of *a priori* sciences, which involve only matters of principle, not fact. It is required of any *a priori* science of objects of a certain type, according to its very sense, that all the typical objects that can *conceivably* exist are taken under consideration. The *type* is the focus of theoretical interest; the tokens are means to it. The interest shifts from factuality to possibility, from objects that are given to the positing intention that can *in principle* give them. In a priori sciences, the ego is concerned only with clarifying its own intentions. Only intentional existence completely free of any factuality counts. A priori theorizing about empirical sets, then, must consider all empirical sets that can *in principle*, but only *in principle*, maybe not actually, be brought into being. To exist here, as Husserl has noted with respect to mathematical existence in general and indeed any a priori science, must be modally affected, not simply "there is" but "it is in principle possible that there is".[15] Intuition is no longer the criterion of existence, the possibility in principle of intuition, i.e., the mere consistent intentional positing, takes its place.[16]

Having ascended to the predicative level of involvement with its environment, the ego is in a theoretical disposition. Being driven by theoretical interests to be satisfied a priori, the ego is no longer interested only in sets that can *effectively* be constituted, but sets that can *in principle* be constituted. A field of possibilities opens up to the theoretically concerned ego, which it sets out to investigate. The

[15] A priori possibility of existence is, for Husserl, as far as mathematics is concerned, existence: "all mathematical propositions of existence have this modified sense […] not simply a 'there is' but rather: *it is possible a priori that there is*. […] All existential judgments of mathematics, as *a priori* existential judgments, are in truth judgments of existence about possibilities […]" [*EJ* § 96].

[16] In the following quotes, Hermann Weyl calls our attention to the transition from actuality to possibility, and further, to the absolute positing of the possible, *beyond the possibility of actual intuition*, that characterizes theoretical sciences: "[...] the transition to theoretical cognition proper: the transition from the a posteriori description of the actually given to the a priori construction of the possible" and further, the conversion of "the possible [...] into transcendental and absolute being, in its totality naturally inaccessible to our intuition" [Weyl 2009a, 69–70]. Also: "It is typical of the mathematizing sciences (in contradistinction to the descriptive ones) that they pass from the classification of the given examples [...] to the ideal, constructive generation of the possible" [Weyl 2009a, 56–7]. Phenomenological analysis, such as the one I carry out here, aims at revealing the "hypothetical" nature of the absoluteness "naively" (i.e. uncritically) attached to scientific domains.

theoretical focus is on empirical sets that can *in principle* be constituted, *no factuality* considered, and what is *a priori* true of them. The resulting theory is an a priori theory of empirical sets which, however, is not yet *mathematical*, but that can become one if we divest empirical sets of their material content. Once removed the empirical constraint (although not completely, as we shall see), the mathematical theory of empirical sets fully blossoms into Cantorian set theory.[17] Of course, by leaving aside the concept of set itself, i.e. by formally abstracting either theory, formal theories are posited which are only equiform with the material set theories. Formal set theories, however, are not of my concern here.

Of course, the joint availability of collectable objects, by which I mean their availability at a determinate point in time, is a necessary condition for their collection to come into being (if the objects to be collected are never jointly available the ego can never collect them, not even in principle, and their collection would not, even in principle, come into being). The condition would also be sufficient if at *any* point in time the ego would be willing and capable of collecting *any* multiplicity of objects available at that time. It is at this point that the fact that we are dealing with possibilities, not actualities, must be seriously considered. Now, since the possibilities in question are possibilities *in principle*, the ego must be considered willing and capable of selecting *whatever collection of objects among the objects that are available for collecting*. There cannot be any a priori restriction on the ego's possibilities of collecting; this is what "whatever" means here. In other words, I claim that *from the point of view of the a priori theory of empirical sets* the fulfillment of *material* conditions of set existence is both a necessary and sufficient condition of set existence, independently of the fulfillment of *formal* conditions of existence.

Some may wonder whether principles of selection (which are the form that formal conditions of set existence can take) should not be somehow available to the ego, on which it could rely as guides for set-constitution. The sets that could be so assembled, some may think, although not actually assembled, would nonetheless exist *in potentia* in the rules by which they could be actualized. Some would go further and require that rules for set collecting must necessarily be expressible in some language. But I ask: why should these intentions be linguistically expressible? And even if they should, why in a *single* language, and, moreover, *which* language? Is there a natural language for referring to empirical objects? In *pure* set theory, and only there, one may argue, the language of mathematical set theory is such a language, but here there seems to be no natural choice.

[17] The mathematical theory of empirical sets, although never actually developed, bears similarities with physical geometry, which also involves abstraction and idealizations. Physical geometry does not consider the actual spatial experiences of a human being in particular, but the experiences in principle available to a human being in general, which is why physical space is endowed with properties (like unboundedness and continuity) that are not actually experienceable by real human beings. Room must be made, however, for all experiences that are possible *in principle*. Like physical geometry, empirical set theory can be abstracted and generalized into a purely mathematical theory. Empirical set theory never actually saw the light of day probably because scientists never saw any use for it.

However, even if no language is predetermined, the requirement that no set can in principle be collected if no rule for collecting the set is available restricts the arbitrariness implicit in the expression "in principle". The set-collecting subject must not be constrained to provide the means for actually implementing, at any given time, the possibilities in principle available to it at that time. Provided a multiplicity of objects is jointly available, the subject can in principle collect them; this is what "can in principle" *means*. I believe this is the only way of not restricting a priori collecting experiences available to the ego. So, the mere *possibility* of collecting granted by the joint availability of collectable objects is also a *sufficient* (besides necessary) condition for empirical collections and, consequently, empirical sets to exist. In short, *as far as a priori theorizing is concerned, empirical sets exist, in the phenomenological sense of the term (i.e. they can in principle be intuitively presented to consciousness), provided their elements are jointly available (i.e. they are all available at some point in time).*

The a priori science of empirical sets must necessarily consider the full domain of all that can *in principle* be. Now, it is clear that infinite empirical sets cannot even in principle come into being (the sense with which the empirical world is posited so demands), but could *all* finite empirical sets be in principle constituted, or is there a *finite* limit beyond which the ego cannot go? The latter alternative is obviously not consistent with the idealizations a priori theorizing requires; the set-constituting ego remains a temporal being, but unhindered by time limitations; it has all the features of empirical egos in general, temporality in particular, but with limitless allowance of time.[18]

Now, some may insist, even though no temporal limits are established, that not *all* finite sets can, even in principle, be constituted. This clearly depends on *all* the collecting acts the ego can *in principle* carry out and, some may argue, it is far from obvious how this can be determined a priori. But this is precisely the point; the only necessary and sufficient condition for a set to come into being that can be established *purely a priori* is the joint availability of its elements. *Any other conditionality involves factuality*.

Some philosophically biased mathematicians may shy away from such an understanding, at least as mathematical sets are concerned. For Weyl, for instance, in *The Continuum,*[19] sets (or, at least, sets of integers) exist only if arithmetically definable. Collecting intentions must, for him, be clearly expressible in the arithmetical language. The constructible universe of pure mathematical sets of Gödel is also an expression of the same *pathos*. But, of course, in these cases no longer mere possibilities in principle are under consideration but *actualizable* possibilities instead (in some sense of "actualizable").

The "classical", *a priori* approach requires that the ego be completely free of the compromise of *actually* carrying, at any given time, collecting intentions; in fact, it is not even required actually *to have* any such intentions. Driven by theoretical

[18] I believe, however, that allowing the empirical ego to perform infinite tasks stresses too much the notion of an empirical ego.

[19] Weyl 1994.

interests that must be satisfied a priori, the ego *maximizes* its *possibilities*. The domain of empirical sets, as far as *a priori* theorizing is concerned, is *nothing but* a domain of *possibilities determined on lines of principle*.[20] The joint availability of objects to be collected is then the *single* pre-condition for collecting acts to be in principle performable, provided the resulting sets accord with the sense of the domain into which they are introduced; i.e. empirical sets are empirical objects. Idealizations are involved here, but again, at the service of the generality a priori science requires. A certain amount of vagueness is consequently unavoidably introduced in the conception of empirical set, which can be minimized by a practical rule: no condition on the existence of a set can in any event require anything beyond the (joint) availability of its elements.

The requirement of joint availability imposes the stratification of the universe of empirical sets in levels (indexed, in this case, by points in time). A set only becomes available (as a possibility) at a determinate level, and remains available for all subsequent levels. Given the (finite) empirical domain of non-sets (E_0), whose objects are available from the beginning, a (still finite) domain (E_1) containing E_0 ($E_0 \subseteq E_1$) plus *all* the sets whose elements belong to E_0 is constituted, whose objects are then available for further set-constituting acts, and so on *indefinitely*, where by "indefinitely" I mean as long as there are non-negative integers to index the levels (this is the *empirical* notion of "forever").[21] A set belongs to E_{n+1} if, and only if, all its elements belong to E_n. The ego is capable of making each level into a set, for it can jointly intend everything that is available to it ($E_n \in E_{n+1}$). This is the hierarchy of *finitely hereditary impure* sets, the *ideal* picture of the universe of empirical sets. Although the set-constituting abilities of the ego are the amplest possible, the sets it constitutes are *empirical* sets. So, chains of the type $x_n \in x_{n-1} \in \ldots \in x_0$ must necessarily be finite and can only be extended until x_n is an empirical object that is not a set. That is, empirical sets are necessarily well-founded. Despite the idealizations, it is presupposed that each empirical set can *in principle* be properly intuited, although no one will probably ever encounter in real life an empirical set of level higher that 5 or 6. But arithmetic also presupposes infinitely many numbers, although no one will ever count, say, 10^{100} objects in the world. *A priori theorizing maximizes relevant possibilities*.

Mathematical Sets The theory of empirical sets, which despite being a priori is not yet mathematical, can be turned into a mathematical theory by reducing objects that are not themselves sets, the urelements, to undifferentiated but numerically distinct units, i.e. instances of the "something whatsoever" (Husserl's *etwas überhaupt*, the

[20] About the notion of *a priori* Husserl says: "*a priori* […] means *by reason of their validity, preceding all factuality*, all determinations arising from experience. Every actuality given in experience, and judged by the thinking founded on experience, is subject, insofar as the correctness of such judgments is concerned, to the unconditional norm that it must first comply with the *a priori* 'conditions of possible experience' and the possible thinking of such experience: that is, with the conditions of its pure possibility, its representability and positability as the objectivity of a uniform identical sense." [*EJ* § 90]

[21] The empirical infinite is the limit of a series of finites.

empty form of the object as such).[22] By so doing, the ego constitutes a realm of pure sets and, concomitantly, its theory, which is now a properly *mathematical* theory.[23]

But as soon as a proper mathematical theory is constituted that is equiform with the a priori theory of empirical sets, an inner tension appears. Although the empirical world falls out of the picture, the available possibilities of set constitution are still empirically constrained, set-constitution acts are still supposed to take place in *real time* and set-constitution iterations can go forever only in an empirical sense of "forever". This tension is similar, but with reversed polarity, to that between "constructible" and "non-constructible" in Gödel's constructible universe, where there are as many stages as ordinals (non-constructible iteration), but only constructible sets in each level. In the formal mathematical counterpart of the a priori theory of empirical sets, on the other hand, there is only an enumerable quantity of levels (only as many as natural numbers) but each level contains *all* the sets that it can possibly contain ("non-constructible" levels).

This tension can only be eased by eliminating all vestiges of empirical time as the medium where the iterative process of set-constitution takes place, i.e., by allowing the process, which, however, *is still taken to be well-ordered*, to go on unobtrusively through all steps of the most general well-ordering conceivable. The set-constituting ego, no longer the ideal form of the *human* ego immersed in time, gives way to a formal-abstract supra-temporal generative process that can leap into the transfinite; not only beyond *any* finite limit, but *all* finite limits, and farther, beyond *all infinite* limits *of any given size*. This bold leap into infinity materialized historically in Cantor's innovative contribution, in his conception of a set-generating iterative process that could go as far as conceivably possible, through all the stages of the most general well-ordering possible, the well-ordering of all well-orderings. The idealized human ego disappears from Cantorian set-theory, but leaves behind its shadow, so to speak, an iterative process that impinges on the universe of sets its characteristic hierarchical structure. The iterative character of the (noetic) process of set constitution corresponds, at the noematic level, to a relation of ontological dependence among sets and, concomitantly, a relation of order among the levels of the hierarchy of sets.

Not any longer the survey of possibilities in principle available to an ego operating in *real* time, the universe of mathematical sets covers now the full extension of possibilities available to an "ego-like" generative process considered *in abstracto*. The theoretical ego is no longer reflecting upon itself, but upon a *mathematical substitute* of itself. The sense of "forever" changes radically, the process of set-constitution can no longer be iterated only as far as the sequence of natural numbers goes, but as far as the well-ordering of all well-orderings allows; this is the

[22] As I have already stressed, Frege did not understand this abstractive act.

[23] Contentual mathematical theories are in this sense also formal, given that its objects are forms. Since set theory is in this sense formal, Husserl places it within formal ontology, a domain of formal logic.

mathematical sense of “forever”.[24] The theorizing ego has now reached its highest, most idealized level of involvement with sets.

Those who oppose the idea that mathematical sets are ego-generated argue that a mathematical set-generator placed outside time does not make sense. The reason M. Potter has for not accepting a non-temporal ego as responsible for the classical universe of sets is that it is not clear for him what it would be like for a non-temporal being *to think*.[25] I find this a very lame argument. Why should a process of collecting, combinatorial in nature, which can be conceived as almost mechanical, involve thinking in any robust sense? At this level of idealization the set-generator can no longer be conceived as an empirical ego who *thinks*, only as a formal abstract, supra-temporal mathematical residuum of the human ego, able to constitute, at any stage of the progressive enrichment of the universe of sets, all the sets whose elements are available at that stage, and capable of iterating set-constituting acts indefinitely *forever* (this is Husserl’s ideality of the “and so on” taken to its limit). Moving to this level of abstraction and generality is *required* to satisfy completely the theoretical ego’s interest on sets. *The mathematical theory of sets devised by Cantor is the natural extension of the mathematical theory abstracted from the a priori theory of empirical sets*, even though historically this was not how it came to be (but this is how adepts of the iterative conception of set see it, even if not always with full clarity).

The joint availability of collectable objects (the *material* precondition of set existence) remains as the *sole* precondition of set existence. Being given any domain (which, by our presuppositions, is necessarily a set) V_0 of non-sets (formal-abstract urelements or “points”),[26] a new domain V_1 is constituted, containing all objects from V_0 plus all sets whose elements belong to V_0. Once the sequence $V_0, \ldots, V_n, \ldots$ is available a limit domain V_ω containing all objects thus far available (whose constitution involves, of course, idealization) is the basis for further acts of set-formation, in a non-stop process running through the whole scale of ordinals.[27] The classical universe of sets is a highly idealized field of possibilities available on lines

[24] Well-ordered sequences of *arbitrary* length are formal generalizations of sequences of finite length that enumerate the acts of the temporal ego. These more general well-orderings enumerate the acts of an ideal set-constituting agent that can always perform an ever new act after no matter how many it has already performed, even if “how many” means transfinitely many, no matter which transfinite power.

[25] Potter 2004, p. 38.

[26] V_0 can even be empty, in which case the first object appears only at level V_1 and is the empty set (supposing it exists).

[27] *Ordinals* are abstract aspects of order-equivalent well-ordered *sets* made into objects. This is a phenomenologically complex process in which particular abstract *aspects* of members of a family of order-isomorphic well-ordered sets is by *ideation* made into *a universal* that is instantiated in any member of the family. In set theory, a set is chosen to *represent* this entity. Since the sequence of all the levels of the hierarchy of sets that precede a given level is a well-ordered *set*, for the set-constituting agent can, I will suppose, keep track of acts it has already performed and consider them collectively, each level of the hierarchy can be labeled by the ordinal representing the sequence preceding it.

of principle to a formal abstract agent partially modeled on the temporal ego, where coming into *actual* being and *effective* existence have very, very little room.[28]

The existence of any *particular* set depends on whether its elements are all available at some level of the hierarchy of sets, and this may not be easy to determine. We often want to know whether there is a set *A* whose elements satisfy some meaningful property $\varphi(x)$ (that is, one that makes sense for the elements of the hierarchy). This is tantamount to asking whether the collection of objects satisfying the property in question can be gathered *at a certain level of the hierarchy*. If φ gathers elements at arbitrarily high levels, *A* obviously does not exist (the objects gathered are never *jointly* available). Questions of the type "is there a set A such that Ψ(A)?", where Ψ is a set-theoretical property, such as "is a strongly inaccessible cardinal", for example, depend on whether certain collections A can be gathered at certain levels of the hierarchy satisfying, as sets, the required properties. This can be done either by a proof from established axioms or by refining the concept with the adjunction of new axioms.

Certain possibilities seem to belong by right to the set-collecting ego; for example, if it can collect a set, it can go through all of its elements once in an orderly fashion, one by one; or, in other words, the ego can well-order any set it has brought into being. This axiom (well-ordering) follows from the fact that *any* sequence of acts of the ego is necessarily well-ordered.[29] It has been argued that Cantor's belief in the well-ordering axiom derived from his *finitism*, that is, the belief that infinite sets behave "just like" finite ones (whatever this is supposed to mean). However, whether or not Cantor saw the matter thus, it in fact derives from assuming that the set-constituting mathematical agent retains from its model, the empirical ego, the ability of going through each object once of a collection of objects it managed to gather. So, *Cantor's "finitism" has something to do with the fact that the mathematical set-collecting agent was modeled on the empirical ego*.

We can now go quickly through the axioms of ZFC to see whether they follow from this notion of set as an object in principle available to a set-collecting agent that can collect objects into sets as soon as these objects are jointly available.[30] The constitution of subsets of arbitrary sets and their power sets, i.e., the sets of all their subsets, replacement sets and choice sets are well within the power of this agent, which can also, obviously, constitute an infinite set (justifications are similar to

[28] Therefore, the notion of existence at work here is neither the realist (not even the modal realist) nor the constructivist. But it is not the nominalist denial of existence either.

[29] A sort of converse seems also to be true, as indicated above; any sequence of acts the ego has already performed can be gathered in a well-ordered set, for the ego can keep track of its acts and take them collectively.

[30] It is part of "traditional wisdom" that only set theoretical realism can give us the universe of sets in its full splendor, for, so the view goes, once an ego comes into the picture, no matter in what shape or form, the universe of sets must to some extent be trimmed (for example, Gödel's partial universe *L*). I do not see how realism can have this power independently of some suspicious metaphysical principle of plenitude of reality, like Potter's "if a set can exist, it does exist" or Maddy's "maximize". The principle of plenitude I subscribe to here, however, is an empty tautology: if a set can exist, then it does exist, for "to exist" only means "can in principle exist".

those Shoenfield gives). Now, the existence of a null set, extensionality and regularity deserve closer attention.

Empty Sets Given our understanding of sets as collections in the first place, the idea of a set that is not, in a *proper* sense, a collection seems contradictory. In his *Philosophy of Arithmetic* and minor works of the period, Husserl is adamant in not accepting empty sets.[31] Symbols for them are taken as improper, empty symbols (ironically). But in later works, *EJ* more specifically, Husserl is willing to admit the *intuitive* existence of empty sets. Consider the following. Collections can be intended as extensions of predicates, for example, the collection of all proper divisors of 17. Any attempt at actually collecting the elements of this collection ends up collecting nothing, the collecting-intention is frustrated. Now, in analogy with the *intuitive* presentation of negative facts, i.e. facts that correspond to true negative assertions, in intentional experiences of *frustration*, Husserl sees the frustration in collecting the divisors of 17 as the *intuitive* presentation of the *empty* collection of divisors of 17.[32] So, empty collections exist. Since the unification of empty collections is coherent with the conception of set, since indeed *all* the elements of null collections are jointly available, empty sets exist. This is different from the case of the universal set, the set of all sets. Although the collection of all sets exists, its unification is inconsistent with the conception of set, since not all sets are *jointly* available. So, a universal set cannot exist. There are then empty sets, but whether there is only one or as many as collecting intentions that are frustrated depends on the validity of extensionality.

However, the existence or not of empty sets is not a very serious matter. We may as well build the entire hierarchy of pure sets starting with a single urelement, a single formal object, a "something" which is not a set (or *two*, if the idea of a singleton is not appealing either) at level V_0. The singleton of this something (or the unordered pair of the two basic "things"), the first set, appears at the level V_1 as its sole member and will for all purposes do the same job of the empty set – be the first set.

The Axiom of Regularity Our concept of set requires that sets are constituted, and that the constitution of sets depends on the availability of their elements, which immediately implies that a relation of *ontological dependence* subsists between the former and the latter. A set can only be if their elements are; the existence of a set (even if only purely intentional or merely possible, when, remember, maybe only the *material* preconditions of its existence are satisfied) *necessarily* requires the existence of its elements (for, precisely, the *material* conditions of existence must be satisfied). More precisely, the elements of the set must belong to a level in the hierarchy prior to the level to which the set itself belongs. Without the elements there

[31] See Husserl 1970.

[32] Although the denomination may mislead one into thinking that an experience of frustration is a psychological experience, this is *not* how Husserl understands it. As an *intentional* experience (for example, of not-*A*), frustration discloses an *objective impossibility*, namely, that a certain intention (*A*) *cannot* be fulfilled intuitively, that any attempt at bringing something (that-*A*) to consciousness faces unsurmountable *objective* restrictions.

would be no collection and, without it, no set. If by writing $x_i \in x$, or $x = \{x_i\}$, we mean that the elements x_i belong to the set x, then assertions of the types $x \in x$ and $x \in x_1 \in \ldots \in x$ are a priori devoid of meaning.[33] Sets can neither be members of themselves nor figure in closed $\in$-chains.[34]

It is however conceivable, or so it seems, that infinite descending chains are possible, that there may be sets $x_1, x_2, \ldots, x_n, \ldots$ such that $\ldots x_{n+1} \in x_n \in \ldots \in x_3 \in x_2 \in x_1$. Such a chain would not subvert relations of ontological dependence. For Husserl, however, these chains are *a priori* impossible. This is what he says: "In retrocession, however, every plurality leads ultimately to absolute unities" (*EJ* §29). "[E]very set, preconstituted in intuition, leads to ultimate constituents, to particularities which are no longer sets" (*EJ* §61). *"[E]very set must be conceived a priori as capable of being reduced to ultimate constituents*, therefore to constituents *which are themselves no longer sets*" (*EJ* §61). "In actual experience there is no division in infinitum, and above all there is no experienceable plurality which, in the progress of experience […] is resolved into ever new pluralities in infinitum" (*EJ* §29).

What he seems to have in mind is the following: for a set to be there must be a set-collecting agent. A set does not exist if it cannot (in principle) be brought into being. But *any* set-constituting agent, be it the temporal ego or the idealized mathematical agent, operates in well-ordered sequences of acts (in time or a formal analogue of time); there is, then, no way "constitution sequences" can exist without a first element. Since the agent, mathematical or not, must always begin with *urelements* and since sets are never simply *given*, they must be constituted, sets can always be decomposed into urelements; it suffices to run the constituting sequence backwards. In short, *well-foundedness is something mathematical sets inherit from empirical sets* (and this is why some may consider it a limitation vis-à-vis a *purely formal* conception of set).

What sort of truth is this, that sets are *necessarily* well-founded? It is an a priori truth, but more than the concept of set as a unified totality, it involves a certain conception of the nature of set-collecting agents. It is, I daresay, a transcendental truth, if by this big word we only mean what necessarily follows *objectively*, regarding the domain of sets, from necessary *subjective* conditions of set constitution. No matter how idealized the mathematical subject is, its acts, if coming in succession, necessarily fall in *well-ordered* sequences. This is why sets necessarily have the property of well-foundedness, which, then, is *the most distinctive trait mathematical sets inherited from their "ancestors", the empirical sets*.

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[33] Are these assertions devoid of meaning or just plainly false? Considering *only* that besides having elements sets can also be elements, sets can appear at both sides of $\in$, and assertions of the type $x \in x$ are apparently meaningful. However, considering that set-constituting acts *presuppose* the (joint) *availability* of the elements of the set, assertions of this type obviously lack sense: a set is not available if it is not yet constituted as a collectable object.

[34] Of course, in formal set theories, in which the relation of "belonging" is only formally equivalent to belonging proper, and "sets" can be anything whatsoever, "sets" may very well be conceived as "belonging" to themselves.

Purely formal conceptions of set, however, can be devised in which "set" stands for whatever objects and the relation of belonging for whatever relation between "sets", and set-constituting agents are no longer involved, where infinite $\in$-chains are tolerable. Formal set theories, however, have nothing to do with our concept of set; they are free creations with no compromise to spell out our understanding of concepts that are independent of their theories (rather, their job is to create new – formal – concepts). For this reason, they can mimic set theory proper if they want, and *only* to the extent they want.

The Axiom of Extensionality Is the collecting intention a characteristic trait of the *collection*; are the collection of all prime numbers smaller than 3 and the collection of all even prime numbers the same collection, despite of being differently intended? Let us consider this question more carefully. We have in fact two different experiences; one directed at the collection from a *particular* perspective, and another, at the collection independently of any *particular* perspective, opening the possibility for two different approaches; one, *extensional*, the latter; another, *intensional*, the former. There is no a priori reason for preferring one to the other (theoretical simplicity can, of course, hardly count). It is then curious that some authors have claimed that if there is an *analytic* truth regarding sets, this is extensionality. I can only agree if it is the *extensional* conception one has in mind. However, one can argue that a set theory that takes the ego and its set-constituting experiences more seriously must necessarily be an intensional theory.

Of course, nothing guarantees that set theory is consistent, for the conception of set it spells out could very well be inconsistent. Or that the theory is complete, for the *original* conception may be incomplete (in fact, we know it is). There is no way we can improve the theory, as Gödel thought, by improving the conception vis-à-vis an independently existing concept, completely determined *in itself* and completely determinable *for us*, for, against Gödel's belief, there is no such a thing. But there is no harm in supposing, and it may be necessary to suppose, as I discussed in the beginning, for classical reasoning to be justified, that it belongs intentionally to the mathematical conception of the realm of sets the property of being *intrinsically determined*, or objectively complete, but not necessarily *extrinsically* determinable, i.e., *determinable* to us or by us, or *independently* existing. One can, of course, be more specific as to the set-constituting possibilities available to the agent, but not from the point of view of the most general a priori theory of sets. So, the only route open for refining the conception is, I believe, by the adjunction of new presuppositions whose justification can only be extrinsic to the original conception; presuppositions that may be required by the applicability of the theory, in particular in mathematics. In other words, the road leading to (inaccessible) completion can be only a matter of theoretical convenience. This however is not true of the more elementary axioms (all those of ZFC *excluding* extensionality), which, as we have seen, are indeed true-of-the-conception.

Genetic phenomenology presents an alternative to the misleading and, I think, utterly preposterous view that the mathematical realm of sets has an *independent* existence, just like the empirical world. In fact, neither has, because both, as objects for the ego, require complex experiences of intentional constitution that give them the *sense* they have *for us*. The constitution of empirical reality, at least, unlike that of sets, rests on *given* formless hyletic material. *Mathematical* sets, on the other hand, are *pure forms*, existing only as intentional correlates (most of them not intuitively).

The Foundational Role of Set Theory in Mathematics The intentional realm of sets structured by the relation of belonging came to play such an important role in mathematics simply because mathematical structures can be *represented* therein (in this consists its "paradisiacal" aspect). Supposing that arithmetic, for example, is essentially the study of a certain type of formal structure (the ω-structure) whose "points" ("numbers") have no internal structure, no properties and no existence *of their own* (i.e. independently of their relations with each other), we can represent such "numbers" by convenient arbitrary sets to reproduce set-theoretically the *same* structure. That is, we can interpret "numbers" (i.e. give them a material content) in set theory in such a way that the resulting interpretation has the *same formal properties* of the numerical domain proper (*among others we conveniently ignore*). The formal properties that both the numerical and the number-like set theoretical domain isomorphic to it have in common are the only properties of mathematical interest. Analogously for all mathematical structures, set theory offers a context of *representation*. But, and this can hardly be overemphasized, the domain of sets, or for that matter, any mathematical domain, does *not* have to exist in a metaphysically serious sense (i.e., independently, in and for itself, Platonically) to play its representational role (and this is *precisely* why empirical reality can be mathematically represented and investigated without this having ontologically serious consequences for mathematics).

My next concern is the problem of space. In the following chapter I investigate how the mathematical concept of space (rather, spaces) is (are) intentionally derived from a proto-intentional construct of the life-world, the space of sensorial perception, subjective space first, and then objectified space. I will be particularly interested in the following questions: is perceptual space the same as physical space, the space of the scientific theories of empirical reality? Does the mathematical theory of physical space, physical geometry, discloses a supposedly inner core of mathematical structure in empirical reality? Of course, these questions are relevant to the more general problem concerning the many uses of mathematics in natural science that I will address later.

Chapter 6
Space

Nowhere do mathematics, natural sciences and philosophy permeate one another so intimately as in the problem of space

Hermann Weyl, Philosophy of Mathematics and Natural Science

The nature of space, no less than that of time, has been for scientists and philosophers, from Plato and Aristotle to our days, a perennial unsolved problem. Aristotle thought that space is a thing of some sort that exists over and above the things that exist in space. In moving, bodies change place, but place remains in place, so to speak. Leibniz, on the contrary, thought that space was nothing beyond the system of relations among coexisting bodies. Insofar as bodies are so related they are, we say, "in space". I believe modern mathematics and science have vindicated Leibniz views on the nature of space. In Einstein's general relativity, space is a system of relations induced on the system of bodies of the world by how substance distributes in the world. Substance creates space that in return acts on substance. Inert nothingness is not space. One often refers to this system of relations as the *structure* of space. This is a bit misleading for it may suggest that there is space *and*, as a distinct entity, the structure of space, when in fact the structure of space *is* space. However, speaking of the structure *of* space is not completely wrong. One can think of space in general as a *generic* system of relations and its different realizations as different ways of structuring space.

In Kantian terms, space is a form of the *phenomenal* world; it structures the system of coexisting things exterior to us that we experience with our senses. Independently of whether the noumenal world has an intrinsic spatial structure, our experience of it has one, which need not coincide with the spatial structure of noumenal reality, if there is such a thing. But, contrary to what Kant thought, the structure of phenomenal space is not discernible a priori, or not completely a priori; one must experience the world to find out what its structure is.

Nonetheless, it is conceivable that at least *some* features of space, noumenal or phenomenal, are *a priori*. In other words, it is possible that the mere *idea* of a system of relations among coexisting things (sensorial impressions, percepts, physical

A few paragraphs in this chapter have already appeared in da Silva (2012b)

J.J. da Silva, *Mathematics and Its Applications*, Synthese Library 385,
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bodies) has necessary features that, consequently, all particularizations of this idea, i.e. all possible spatial structures, must display. Kant was certainly wrong in believing that our physical space is Euclidean out of *necessity*, but it is still conceivable that it might necessarily have a metric structure, for example. If not noumenal transcendent, at least phenomenal space. Moreover, it is reasonable to suppose that the fact that space is a form imposed on *our* perception of the transcendent world may have some consequence for the structure of space. In other words, the structure of phenomenal space may have some aspects for simply being the space of *our* perceptions, which are inextricably connected with *our* ways of perceiving. These, as I call them, are the transcendentally necessary aspects of (perceptual) space.

One cannot answer these questions generically, for there are many species of space, each with its peculiar structure. Here, I will approach the problem from a genetic perspective. My focus is the intentional genesis of our many representations of space, from the immediate space of sensations and perceptions to the idealized spaces of the mathematical science of empirical reality. They all have their characteristic structures, but not all of them are, properly speaking, mathematical. Here, another important question faces us: when, how, and for what purpose does space become mathematical? By answering this question, I hope to throw some light on the difficult problem of the mathematization of empirical reality that will occupy us in a later chapter. By studying the genesis of our many representations of space from a phenomenological-constitutive perspective, I hope to be able to discern how mathematics enters the game and the role it plays therein, that is, where in the constitution series of our spatial representations space becomes a mathematical manifold and, more importantly, why. In other words, I want to investigate the role mathematical idealizations play in our understanding of phenomenal reality (which is *not* in itself mathematical, at least not to the extent that the mathematical science of reality demands). Husserl's analyses of the genesis of what he calls in *Crisis* the "Galilean" approach to empirical science, that is, the mathematical approach to natural science that characterized the sixteenth and seventeenth centuries, will be my guiding model. Our representation of space has a genesis and happens in stages; my goal here is to uncover the intentional actions at work at each stage. Mathematization is a *peculiar* intentional action; I want to understand how and for what purposes the ego performs it.

Since space is only a system of relations, one does not have a direct, intuitive experience of space, only of things in space. These things can be sensations, perceptions, or still physical bodies (chunks of substance). On a most basic level, one interacts with things in space through one's relevant sensorial systems, seeing, hearing, touching and kinesthetic sensations associated with the movement of one's body. It is a task for physiologists to explain how by interacting with other bodies our body constitutes a representation of space useful for surviving in the world. However, regardless of the particularities of the process, the fact remains that one *constitutes* a coherent *perception* of space out of sensorial impressions. It befalls on the physiologist the task of elucidating how sensorial input becomes perception (a process, however, that is not yet properly intentional since it is not fully conscious). Other important questions concern built-in psychophysical schemes for organizing raw sensorial data into perceptions proper. Are their modes of operation fixed forever or have they some degree of plasticity vis-à-vis the sensorial input? Can we *perceive* (not merely *conceive*) space differently or are we constrained to perceive it the way we do? How do the different sensorial systems cooperate in the constitution of a single coherent perception of space?

Regardless of how these questions are answered, from a philosophical perspective one thing stands out. The raw material of sensations, the *hyle*, is simply *given* in one's interaction with the external world. However, the sensorial hyle is not yet perception; sensations must be "processed" to become *perceptions*. The question, then, imposes itself: to what extent is the spatial character of our *perception* of the external world a contribution of perceptual systems themselves? How faithfully does one's perception of space and its structure correspond to real, transcendent space, supposedly causally responsible for our spatial sensations? Although a transcendental phenomenologist should not worry about metaphysical questions such as these, I will admit the existence of a (metaphysically) transcendent space, if only as a limit idea.

Another point worth attention is that one's sensorial-perceptual representation of space is purely *subjective* and can be objectified only communally in shared experiences of perception. The intentional constitution of a *common*, *objective physical* space happens first in the life-world, at a *prescientific* level, involving practices of the life-world and cooperating agents. In interaction with one another and with non-sentient bodies of the external world, we constitute a notion of physical space that allows us to live in the world and interact successfully with it in cooperation with other people. In subjective sensorial-perceptual space, the subject is the center and spatial determinations usually refer to this center. Bodies are either far from me or close to me, moving towards me or away from me, to my left or to my right, augmenting as they approach, shrinking as they recede. Relative position, size, and form of bodies with respect to one another, as the subject perceives them, are determined from the subject's privileged position. Reality is what appears to the subject *as* it appears to him. Subjective sensorial-perceptual space is good enough for a hunter, but not for a social being. Men in community must share a communal notion of perceptual space; they must live in a communal *physical space*.

In community, in ideal conditions, people must agree on which of two bodies is bigger and how far they are apart from each other. Certain needs, land surveying, for example, induced practices such as measuring that had to abide to externally verifiable criteria of correctness. Standards of measure had to be objectively available, invariable in its dimensions, and movable, so that one could compare bodies separated in space with respect to shape and size. Parts of the human body, the foot, hand, palm, or actions performable with the body, such as taking steps, for instance, were natural candidates. These practices structured physical space in a particularly important way; they introduced in it an *objective* notion of distance, i.e. a metric. The metric structure of space, then, depends largely on the practice of measuring by means of easily available objective metric standards. The fact that standards of measurement are *rigid*, that is, that their form and dimensions do not alter by being displaced through space, is of utmost importance and consequence – the metric structure of physical space will necessarily depend on the *factual* spatial behavior of bodies.

Physical space, however, is *not* a mathematical manifold in the proper sense. Spatial determinations in physical space are not *exact*; localization with respect to other bodies, relative distances, size, and form lack mathematical exactness and precision. The ego must then intervene to constitute, by idealization, i.e. exactification, a mathematical representation of physical space. Idealization, a higher-order intentional act, is required so a mathematical manifold can be constituted as a suitable representation of the abstract spatial structure of the empirical world. However, *mathematical-physical* space is not merely an idealization of *perceptual* space, but an extension of it. Often, mathematical manifolds with which we represent reality incorporate elements that do not play a representational role.[1]

Subjective sensorial-perceptual Ur-space and objective physical space have both a material content, the basic hyletic material provided by the senses, in case of sensorial-perceptual space, or physical bodies, in case of physical space, and a form imposed on them by perceptual systems and practices such as measuring. By abstraction, one redirects intentional consciousness to spatial form, ignoring content. Abstractly considered, space is a system of "empty" parts, positions or places, and relations among them. Material contents that eventually occupy positions in space will inherit spatial determinations already established among the positions they occupy. For example, if position *A* of space lies *between* positions *B* and *C*, or if *A* is closer to *B* than *C*, then whatever material contents occupying positions *A*, *B* and *C* will stand in the same relations.

Parts of either perceptual or physical space are still space and so have parts too; neither sensorial-perceptual nor physical space have ultimate parts. However, one can conceive parts that are for all *practical* purposes indivisible; these are the "points" of space. From an abstract perspective, sensorial-perceptual or physical spaces are structured systems of "points", but they are not *mathematically* structured *mathematical* manifolds of *mathematical* points. At least, not before they are mathematically *idealized* as such. I will call the mathematical idealization of physical space *mathematical-physical* space; it is the physical space of the mathematical sciences of nature. To take it for the *real* physical space and sensorial-perceptual space as only a rough approximation to it, the only our "imperfect" perceptual system is capable of, however, is a falsification that stands in the way of the correct understanding of the role mathematics plays in empirical science. As Husserl argued in *Crisis*, our senses are not in the least imperfect; they are what they are, and only *appear* "imperfect" from the perspective of a mathematized space enthroned as real space.

I am only marginally concerned with the sensorial-perceptual space (or spaces, if one consider sensorial systems in isolation); my focus of interest are physical space and its mathematical idealizations. Spaces in general can be continuous or discontinuous, connected in many ways, bounded or unbounded, finite of infinite, *n*-dimensional for any natural number *n*, flat or curved, Euclidean or non-Euclidean.

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[1] It is an important philosophical question to ask whether and under which conditions "imaginary" formal extensions of mathematical representatives of empirical reality can be given an empirical content. This question is related to the *heuristic* uses of mathematics in science.

My task is to identify what the properties of physical and mathematical-physical space are and where they come from. There are three main candidates, the *concept* of space, the *experience* of space, or the *transcendental a priori* related to the experience of space. The first is the logical-conceptual (analytic) a priori, the second the a posteriori and the third the intentional-constitutive (synthetic) a priori.

The mathematical-physical notion of space, as already emphasized, requires idealization. As Husserl argued, idealizations are founded on practices of the life-world such as polishing, straighten up, smoothing up taken to their limits.[2] *Because*, he claims, we can smoothen up the surface of a body we can idealize it as *perfectly* smooth. *Because* we can straighten up a line, we can idealize it as a *straight* line, and so on. This, however, is not essential. What is important is that idealization, regardless of its roots in practices of the life-world, is required for the constitution of our mathematical-physical conception of space, which puts it at a higher level of intentional constitution than physical space. Finally, by submitting the mathematical-physical representation of space to a sort of formal *imaginative variation*, whose possibility historical developments such as the discovery of non-Euclidean geometries indicate, more general purely formal conceptions of space become available. These "imaginary" spaces play in the transcendental history of geometry approximately the same role "imaginary" numbers play in arithmetic. Two questions can be raised concerning generalized mathematical notions of space; one is whether they can represent space more faithfully than the Euclidean conception of space, and on which grounds; another, which role they can play in our understanding of physical space even if they cannot be taken to represent it. Physical space, on its turn, is an intentional derivation of sensorial-perceptual space, which is itself constituted, albeit non-intentionally from a hyletic residuum of purely sensorial impressions whose origin lies outside the intentional or proto-intentional sphere of the ego, supposedly transcendent space. One has then delineated the genetic progression of spaces we must investigate: transcendent space, *subjective* sensorial-perceptual space, *objective* physical space, *ideal* mathematical-physical space, and purely *formal* mathematical spaces.

Kant's Errors For Kant, space is not a concept, but an intuition; it is with time one of the two a priori forms of sensibility, presenting itself intuitively in pure sensibility fully clothed in Euclidean garb. This characterization is wrong on many accounts. For one, space is a concept, and there are many different empirical instances of the concept of space. Conceptually, it is simply the structure subjacent to multiplicities of coexisting things considered as such, i.e. thing-forms. Our familiar physical space is a non-independent abstract aspect of the manifold of physical bodies. In a manner of speaking, which is, as I have already noted, not perfectly adequate, but harmless nonetheless, bodies coexist, move, change, and interact *in* space, and by being *in* space, they stand in a system of (spatial) relations with other bodies. In fact, as already stressed, there is not much to space than this system of spatial relations

[2] See Husserl 1989.

itself, and it is more correct to say that space just *is* the system of all possible spatial relations.

There are other spaces in nature. A particularly important one is the space of time. Time is a continuous manifold of moments or, at the ideal limit, instants. It is one-dimensional, that is, the removal of a single moment of the continuum of time divides it into two separate parts. Time admits a metric and the determination of a metric for time requires that one fix a length of time (a duration) that does not vary with time. One usually does that by choosing as the unit of time the period of a *seemingly* uniform periodical process. Of course, the choice is arbitrary and the temporal constancy of the time unit is to some extent a presupposition. It may happen that our perception of temporal processes does not operate uniformly in time and other sentient beings perceive period alterations in periodic processes that we perceive as periodically constant. One does not have to bring Einstein's relativity theory into the discussion to convince ourselves that the tick-tacking of time is relative. In any case, once a community has chosen and agreed upon a unit of time intersubjectively, it has objective validity in the context of that community. The concept of congruence is essentially the same in physical space as in time: two segments of space (resp. time) are congruent if they coincide perfectly if superposed. It is a presupposition associated with our notions of space and time that two congruent segments of space or time remain forever congruent (space or time do not deform themselves spontaneously).

One also conceives time as infinite. This, of course, does not have a basis in perception; one can perceive only finite chunks of time. Time is only an aspect of physical processes; more specifically, the *flux* of time is the abstract form of the succession of states of a generic process. Time is the formal condition for *difference* within the *same*. Our *physical* experience tells us that the states of a process follow one another in a linear ordering, along a direction, and that there are processes whose states never repeat. We, consequently, attribute a direction to time and represent it as being infinite in either way along this direction (although, by conceiving a beginning of the world and the annihilation of it and everything in it one can conceive a beginning and an end to time).[3]

Kant had a more subjective conception of time. For him, the flux of time was not an aspect of the flux of the world, but of consciousness. Time, he thought, was the form of the internal sense. Consequently, if the entire world vanished leaving behind only my consciousness, time would still exist; and as the world filters through my senses and become sensations, which are states of consciousness, the world becomes temporal.

There are other instances of natural spaces. The space of colors, for example. Any color sensation is determinable in terms of its relative position in three independent color continua: black-white, red-green and blue-yellow. One can think

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[3] This is a difficult exercise because the annihilation of consciousness is a necessary condition for the annihilation of the world. One has the illusion of eternal time because, regardless of the annihilation of the word in consciousness, one preserves one's consciousness and the succession of its states floating above nothingness.

of this as the color analogous of the usual spatial dimensions. The manifold of colors, then, is a continuous tridimensional space. The manifold of tonal sensations is another. Like time, it is a one-dimensional metric space, the distance between two tonal sensations given by a musical interval, quantifiable in terms of the logarithm of the frequencies.

In short, against Kant, space is a concept abstracted from empirical experience, of which the empirical world contains many instances. Although correct as to the formal character of space and the fact that space is intuitable, Kant was wrong as to the a priori character of the intuition of space. One intuits space, by which I mean here physical space, by abstracting the spatial mold (essentially the system of spatial relations) present in our perceptual experience of the physical world, themselves constituted from sensorial impressions. Kant misinterpreted this intentional abstractive process as the non-sensorial donation of a pure intuition. By mistaking the abstract form of already fully constituted physical space, Euclidean structure and all, by a pure intuition, Kant ended up believing that space was *necessarily* Euclidean. Kant's a priori constructions in pure space, which convinced him that space, a formal mold necessarily attached to our experiences of the external world, was intrinsically Euclidean, were indeed only *physical* constructions *in imagination* and would not be possible had he not been previously familiar with similar constructions in *actual* physical space already endowed with a Euclidean structure.

Transcendent Space We constitute our representation of space from our perception of space; therefore, we cannot know how space is in-itself, independently of sensorial perception. Any space that is coherent with our representation of space is a good candidate for transcendent space. Since empirical science is our best way of organizing our experience of the world and trying to figure out what could be causally responsible for it, it is also our best instrument for trying to figure out the structure of transcendent space. It may be discontinuous but represented as continuous due to the way our senses operate; it may have more than three dimensions but our senses only detect three; it may be multiply disconnected; it may be finite. Our ignorance of the true nature of transcendent space gives science room for speculation. How *real*, transcendent space must be so a world as ours, with people like us, having the sensations we have can exist is a question empirical science must answer.[4] For a phenomenologist, however, faithful to his compromise with the epoché, transcendent space is only the source of our hyletic spatial impressions, about which he refrains from saying anything. The absolute *given* is the sensorial hyle.

Sensorial-Perceptual Space One's most basic notion of space is the resultant of sensorial impressions (visual, tactile, kinesthetic) and *innate* psychophysical operations whose task is to accommodate these impressions into a coherent spatial mold. Our senses offer the raw material that built-in systems (selected during our biological history) put in spatial form. Our retinas, for example, offer two slightly different

[4] In a paper of 1955, his last year of life, "Why is the World Four-Dimensional" (Weyl 2012, pp. 203–216), Weyl points out directions that he thought could lead to a scientific and mathematical explanation for the four-dimensionality of space-time.

two-dimensional projections of the multiplicity of outer things (the difference being the binocular parallax); our perceptual system, which usually delivers *more* than what it gets from the senses, eliminates double images by creating a perceptual "depth". I will not go into these matters here; it is enough to keep in mind that our sensorial-perceptual representation of space is *neither* completely prior to sensorial experience *nor* entirely abstracted from it; the subject has an active role in constituting space from raw spatial sensations (by occurring at sub-conscious levels, however, the process is not yet properly intentional).[5]

Our perceptual system organizes the complex of sensorial impressions coming through the outer senses as *bodies in space*. One perceives space by perceiving bodies in space and relations among them. One cannot see space, only bodies in space; when we see, for example, body *B between* bodies *A* and *C*, we see this as an *aspect* of the spatial complex formed by *A*, *B* and *C*. The subject lies at the center of his sensorial-perceptual space, which is completely subjective. Things are in space as they appear to be to the perceiving subject, and they appear to him as a unified structured system of bodies. Any given body in space relates spatially to any other body – in other words, space is one. Spatial bodies (even if perceived through different sensorial systems, for example, a thing touched but not seen and a thing seen but bot touched) must stand in some spatial relation to one another simply because they all belong to the same unique space.[6] Any spatial extension, as Husserl claimed, is in space as a part or a limit (points, lines, surfaces) of it, or in relation with parts of it (for instance, the extension of space *between* spatial extensions).

Sensorial-perceptual space is also *continuous*, for sensorial impressions associated with, for example, hearing a tune, sliding the finger over a surface or looking around are themselves continua. We also represent space perceptually as *tri-*

[5] There is in Husserl a clear distinction between pure sensorial data (the hyletic data) and percepts. In his Lectures of 1907 (Husserl 1997), he presents a minute description of the constitution of rigid spatial bodies and physical space where he constantly reminds us of this distinction. Husserl believes there are essentially two systems of sensorial data involved in spatial perception, the visual and the tactile (although the visual appears with by far more relevance), which are molded into spatial percepts by a series of *intentionally motivated* kinesthetic systems working in isolation and cooperatively. There are essentially four of these systems (some terms are Husserl's some are mine). (1) The oculomotor system, by means of which a non-homogeneous 2-dimensional flat finite space is constituted. (2) The restricted cephalomotor system, by which a non-homogeneous 2-dimensional *curved* space is constituted, limited "above" and "below" by closed lines, like the section of the earth's surface between the tropics. (3) The full cephalomotor system, by which a 2-dimensional spherical space is constituted (which Husserl calls Riemannian space). Finally, (4) the (full) somatomotor system, by which a tri-dimensional space is constituted. It is worth noticing that, for Husserl, binocularity does originate depth, but depth is not yet, by and in itself, a third dimension comparable to breadth and height; tridimensionality requires the subject to be able to move *freely* towards, away from and around the body, and would be constituted independently of binocularity.

[6] It follows from the contemporary scientific image of space as an abstract system of relations induced by physical interaction among physical entities that if these entities were separated in physically unrelated clusters, space could very well be conceived, for scientific purposes, as disconnected into isolated "multiverses", bearing no *spatial* relations with one another (logical relations such as that of difference, would, of course, still hold).

dimensional. As Husserl argued, tri-dimensionality requires the capacity of moving and changing perspectives. Binocularity is not enough; it may create depth, but this is not yet a dimension. Only by moving, and thus bringing about different perspectives intentionally unified as perspectives of the *same* body, one constitutes a third dimension proper. Of course, *transcendent* space might be discontinuous or have more than three dimensions, and it could be scientifically interesting to represent it thus. But this is not how one *perceives* space. Nonetheless, discontinuities in space or dimensions other than three are not conceptual impossibilities. In fact, mathematicians conceive formal spaces that are discrete and *n*-dimensional for *n* different from three. It is also possible that beings exist that represent space as bi-dimensional (for example, those incapable of motion).

The centrality of the perceiving subject in his space, where all other perceiving subjects are only bodies like other bodies (only by taking other perceiving subjects as co-workers in the constitution of space an objective *physical* space can be constituted), makes sensorial-perceptual space *non-homogeneous*. Sensorial-perceptual space has a center, even if it is a mobile center. The free mobility in space *experienced* by the perceiving subject and the unboundedness of the domain of his possible sensorial experiences justifies him to represent space perceptually as indeed *unbounded*. Infiniteness is an idealization of unboundedness and thus not perception-based.

The Euclidean character of sensorial-perceptual space, if only approximately, is a matter of some dispute. Are all sensorial-perceptual spaces (approximately) Euclidean, for example, the purely visual space? Is Euclid only a "solution of compromise" among different sensorial-perceptual spaces (visual, tactile, auditory, and kinesthetic)?[7] Be as it may, sensorial-perceptual space seems in fact to be Euclidean, or better, Euclidean within our capacity of sensorial-perceptual discrimination. This, however, is a consequence of the relevance we attribute to rigid bodies and their displacements in space.[8] A (perceptually) rigid body is one whose shape and dimensions (as perceived) are for all practical purposes invariant over time. There are plenty of them around us; in particular, parts of our bodies (the foot, the arm, etc.) Since one does not perceive either the form or the dimensions of rigid bodies to change in displacement, one feels safe to take any of them as a standard of measurement. One can object that we *seem* to know plenty of bodies that are for all practical purposes rigid, but that one cannot be *sure* that they are indeed rigid. Bodies may alter shape or size in motion without one being able to notice it. If bodies apparently rigid, including our bodies and standard meters deformed in motion in the same proportion, one would not notice any deformation by strictly geometri-

[7] In his interesting study of space-perception, Patrick A. Heelan (1983) convincingly argues for the non-Euclidian character of visual space.

[8] "There is no doubt that the conviction which Euclidean geometry carries for us is essentially due to our familiarity with the handling of that sort of bodies which we call rigid and of which it can be said that they remain the same under varying conditions" (Weyl 2009b, p. 78). "Fundatur igitur Geometria in praxi Mechanica, & nihil aliud est. quam Mechanicæ universalis pars illa quæ artem mensurandi accurate proponit ac demonstrate" (I. Newton. Introduction to his *Philosophiae Naturalis Principia Mathematica*).

cal means. One, however, does not have to be *sure* that apparently rigid bodies are indeed rigid; one's sensorial-perceptual representation of space does not require sound knowledge, being as it is a pre-reflexive response to experience. One does not even raise the question; if bodies *appear* rigid, rigid they are. One simply *experiences* rigidity, even if this experience may not correspond to anything transcendently real.[9]

By fixing two points of a rigid body in a purely translational displacement, one can visualize two parallel lines. Our *practical* dealings with rigid bodies suffices to convince us that the space associated with the system of our perceptions satisfy Euclidian properties. By imagining (imagination, in this case, is *reproductive* imagination) a rigid straight line (rather, a line perceived as rigid) rotating around a fixed point on the plane determined by this point and a second straight line one "sees" the intersection point of the two lines moving farther and farther to one side until it reappears on the opposite side moving closer and closer. Our natural tendency to see this as a *continuous* movement imposes upon us the belief that *there is* a moment, and *only one*, when the two lines are *parallel*. This is Euclid's fifth postulate, and it has a solid basis in our dealings with rigid bodies.

We know – because *geometry*, not direct experience teaches us – that the free mobility of rigid bodies does not necessarily imply that space is Euclidian; it only tells us that it has *constant* curvature, zero, positive or negative (Helmholtz-Lie theorem). However, *if* space had, in fact, a positive or negative curvature *noticeable within the range of immediate perceptual experience*, our sensorial-perceptual experience of space would have been different from what it is (the difference being a function of the degree of curvature). In an extreme case, we would be able to scratch our backs by stretching our arms in front of us. Our sensorial-perception of space, which is necessarily *local*, has a Euclidian character. Perceptual space at large can be in principle elliptic or hyperbolic, but it must be *locally Euclidian* to be consistent with perception. *Scientific* representations of space, to the extent that perception is the highest court of science, must agree, within reasonable limits of variability, locally with the *perceptual* representation of space.

Helmholtz, among others, agreed that one *could* indeed perceive space as non-Euclidian if it were so, but Husserl is at odds with this view. Although he believed that physical space could be *scientifically* represented as non-Euclidian, he did not think that we could ever *perceive* it as non-Euclidian. Helmholtz fails to see, Husserl claims, the distinction between *physical* and *psychological* experiences of space. In the latter, the mind is at work making *sense* of (or bestowing sense on) sensorial experiences; for us, he thinks, the *perception* of physical space only makes *sense* in the Euclidian mold. For Helmholtz, on the other hand, as Husserl interprets him, perception is a *physical* experience involving more than the purely sensorial-perceptual, and thus capable of "correcting" the perceptual representation of space

[9] What sense one could attribute to the hypothesis that the world and everything in it change in such a way that, in principle, one cannot notice any change? As Weyl tells us (Weyl 2009b, p. 118), a metaphysically real difference that *as a matter of principle* one cannot detect is non-existent.

to accommodate what Husserl would consider as anomalous and thus dismissible perceptions.

Summing up, one *perceives* space, informed by our sensorial systems working cooperatively, as a continuous, non-homogeneous, simply connected, tri-dimensional, unbounded, approximately Euclidian manifold. This representation is strictly subjective (*my* space) and self-centered. One's sensorial-perceptual representation of space may not coincide with the representation of other space-sentient beings, men or beasts. Moreover, it may not correspond in some respect or another to transcendently real space. Husserl makes a strong claim to this respect; for him, congruence and *all* geometrical properties seem to be in the *nature* of our sensations, not in transcendent reality. Our space-constituting functions, like our tone and color-perception systems, may give us something that is not strictly speaking out there. Transcendent space, he says, may be only an *analogue* of perceptual space.[10] However, be as it may, and Husserl agrees with this, *we* are *perceptually justified* in representing space the way we do; our representation of space has a role to play in our lives even if it does not correspond to *transcendent* reality.

It would be interesting to pause for a moment to ask which features we consider essential to spaces *as such*, those that enjoy the status of necessary features of the *concept* of space. Essentially, space is the condition of possibility of a multiplicity of coexisting things (i.e. things existing simultaneously). Manifoldness is, then, an essential aspect of space; space is necessarily a manifold of parts (a part of space is the abstraction of a possible thing in the multiplicity of things in space).[11] The system of relations that things establish among themselves for merely coexisting in space is contingent; no one in particular is necessarily required. As a multiplicity of elements, space is structured (a structure is a system of relations), but no particular structure imposes itself necessarily upon space. Another conceptual feature of space is, arguably, dimensionality.[12] If the removal of a single point of space separates it in two parts (i.e. there is no way of moving continuously from one part to the other by remaining in space) one says that the space has dimension 1. If the separation of a space in two parts requires the removal of a subspace of dimension n-1, then the space has dimension n. If no finite dimensional subspace separates the space in two parts, the space is infinite dimensional. A manifold is finitely or infinitely dimensional, but not dimensionless. The concept of space does not necessarily require any other feature, or so it seems to me.

[10] It is tempting to read this analogy in terms of the notion of homeomorphism. Perceptual space may be only a homeomorphic copy of transcendent space. However, homeomorphism may still be too strong a requirement (homeomorphisms preserve dimensionality, and transcendent space may have more dimensions than perceptual space).

[11] *A manifold* is no more no less than a structured multiplicity of things; we would call it today a structured system.

[12] Provided one defines a topology in space, that is, a way of characterizing proximity, dimensionality proves to be a topological property of space. A topology allows one to talk of continuous deformations of space (continuous 1–1 maps of space onto itself), and a topological property of space is one that is preserved under continuous deformations of space, for example, dimensionality.

On purely conceptual grounds, any space is a manifold with a dimension. Continuity and the particular dimensional number three are not necessarily required on conceptual grounds, but are fundamental properties of *perceptual* space, prior and independently of any metric that it may eventually receive. In other words, *perceptual* space, considered as the *form* of our experience of the outer world, is fundamentally a metrically amorphous continuous tri-dimensional manifold. Kant's mistake was to go a step further and add a Euclidean metric that is not available prior to the perceiving subject getting involved with certain practices (namely, measuring lengths with rigid meters).

Sensorial-perceptual space has also features that strictly speaking are not extracted from experience but required on transcendental grounds, namely, the a priori transcendental determination of the field of all possible spatial perceptual experiences. Since the ego presupposes that the field of experiences available to it is an *open* domain extending over the horizon of its actual experiences, it *idealizes* space as unbounded or even infinite. Since it is at the center of its space, it presupposes that all parts of space, even those that are not actually accessible to it, relate to the center and to one another; i.e. that space is necessarily a simply connected manifold. These presuppositions are *not* perceptual hypotheses, but constitutive presuppositions. The indefinite divisibility of perceptual space seems also to follow necessarily from the indefinite extensibility of experience, being in this sense also transcendentally *a priori*.[13] However, infinite divisibility is not actual continuity. Although we experience space as a continuum, even if this may only be a perceptual "illusion", we do not experience it as infinitely divisible; the actual *continuity* of sensorial-perceptual space is a given but its indefinite *divisibility* is an a priori presupposition. However, indefinite divisibility is not the same as infinite divisibility, which is an idealization (therefore, by continuously moving in space one is *not* infinitely dividing it).

The Euclidean character of sensorial-perceptual space and its three dimensions depend on *contingent empirical facts* and are consequently *a posteriori*.[14] A note of caution, however; as already discussed, we are perceptually justified in attributing a Euclidean structure to space only *locally*. To extend it globally is not justified on conceptual, transcendental, or empirical grounds.

It is not necessarily true that any sentient creature that can develop a representation of space will represent it as we do. More dramatically, we should not expect extraterrestrial beings to share our geometry: fluid beings in a fluid world would probably have no concept of rigidity, similarity or congruence, and no Euclidian geometry either. Now, can perception induce us to change our perceptual

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[13] This, as already observed, is the transcendental synthetic a priori in Husserlian version.

[14] If we believe, with Husserl, that built-in perceptual system responsible for the spatial perception constrain sensorial data into a Euclidean mold, we might believe that the Euclidian character of space is a priori (giving Kant some reason). However, even in this case it would rest on a contingent fact of experience, namely, our biological history. However, some (Poincaré, Helmholtz) have argued that we can *imagine* situations in which, even built as we are, we would *perceive* space, not only theoretically conceive it, as non-Euclidean.

representation of space?[15] Husserl seems forced to accept that it can; after all, our perceptual systems are a product of nature and could have been different. For him, however, although the psychophysical functions responsible for the constitution of our spatial representation require, for performing their task, an input of sensorial data, the incoming data cannot, he thinks, alter how the system works, being as they are products of our adaptation to the environment. Given that they developed to function in a certain way, they will function that way faced with any perceptual experience (what does not mean that during million years radically new experiences could not give our descendants altogether different space-constituting functions, long after we, their no-longer-spatially-adapted ancestors, vanished from earth). According to Husserl, we do not see effects of perspective and like phenomena, for instance, as distortions of space, but as visual *illusions*. If *we* remain what we are our perceptual representation of space will not change, or so Husserl thinks.

This may count as an argument against Poincare's style of conventionalism (if taken to apply to sensorial-perceptual space), according to which we are to a large extent free to choose this or that spatial mold for our intuitions, for experience by itself is not compelling. Husserl on the contrary believed that our representation of perceptual space, the one we *effectively* have, is the only *we* can have, even if it is not the only one we can *conceive* (in fact, as *thinking*, not only *perceiving* subjects, considering factors other than perception only, we may even decide *against* perception in our scientific representation of space). We may be wrong as to how space *is* or the best way of representing it scientifically, but if we were, we are not free to be right if we derive our spatial representation from sensorial perception alone.

What about *radically new* experiences, light rays bending in some region of space without any physical reason for so doing or, more prosaically, otherwise rigid bodies behaving in the strangest ways; could they force or suggest a change in our spatial representation of the outer world? Again, Husserl answers in the negative. In fact, he explicitly says that if the field of vision were altered, we would say this was no longer the field of vision, but a new experience; no longer space, but something else. Our perceptual representation of space is not altered, but experiences of a different type or reason could suggest different *scientific* representations of space. Of course, we can also *always* hypothesize some physical reason to account for the strangeness of observed phenomena; we can even consider this behavior as *evidence* for the action of hitherto unknown forces.

However, as Riemann has suggested and Einstein has shown, this is not always the best (or methodologically sounder) solution. Things may be simpler if we change our conception of space. But here we are already talking of *mathematical-physical* space, not *sensorial-perceptual* space. The history of science has apparently

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[15] In answering this question, Husserl seems to be addressing Helmholtz, who argued *for* the sensorial-perceptual representability of a non-Euclidian space. For Helmholtz, one can show this by *imagining* spatial sense impressions captured by our sense organs according to the known laws but that would force nonetheless a non-Euclidian representation of space. Husserl, I believe, explicitly denies this possibility; for him, we would probably not – or maybe *should not* – interpret imaginary "non-Euclidean" sensations as *proper* sensations.

shown Husserl to be wrong on this particular aspect: we *have changed* our *scientific* representation of physical space, pressed not only by experience, but by *reason* as well. However, the representation of space Husserl is concerned with is *not* the scientific one, but the sensorial-perceptual representation; representing space sensorial-perceptually is prior to any scientific reasoning. On scientific grounds, however, Husserl has no reason not to agree with Helmholtz.

Sensorial-perceptual space is not geometrical; it does not contain extensionless points, flat surfaces or perfectly spherical bodies, but it is not devoid of structure either. Our Ur-space is a proto-geometrical manifold with morphological counterparts to most geometrical structures.[16] Sensorial perceptions of spatial character are comparable as to shape and size. One can establish among spatial percepts relations of similarity and dissimilarity with respect to shape, and larger, smaller and equal (congruent), with respect to size without resorting to measurements. These are *not* metric relations *stricto sensu*, but proto-metric. Sensorial-perceptual space only admits a metric in the proper sense, no matter if only approximately, by the selection of a rigid metric standard that in principle (a presupposition) one can take along to any position in space, along any direction, without deformation. *Exact* metric determinations involve idealization and naturally require the concept of real number, which as we have seen is also a product of idealization. In sensorial-perceptual space metric determinations are inexact and approximate vis-à-vis a mathematical metric proper.

Our perception system allows us a representation of space that contains more than what meets the senses. Sensorial-perceptual space is a construct, although not yet an intentional one, since its constitution is not fully conscious. It is a subjective, ego-centered space whose structure is perceptually determinable, but only *locally*.[17] Physical space, on the other hand, is *communal* sensorial-perceptual space, constituted intersubjectively in the life-word in a concerted effort to harmonize in a single objective (i.e. intersubjectively valid) spatial representation all subjective representations of space. Physical space is the objective medium where we live our daily lives, interacting with physical bodies and other space-perceiving egos. It is not yet the space of the mathematical sciences of nature, whose constitution requires higher-level intentional acts and a scientifically disposed ego.

Physical Space The constitution of physical space presupposes tacitly that all space-perceiving subjects are equivalent. Objective space is that which all perceivers agree upon based on their personal perceptions of space.[18] The most obviously

[16] See Husserl 1962 § 74 for the distinction between the morphological and the mathematical.

[17] Weyl takes the exclusive local determinability of the structure of an ego-centered space as a (phenomenologically inspired) methodological principle in the development of his infinitesimal geometry. See his opus magnum on the General Theory of Relativity, *Space, Time, Matter* (Weyl 1952) where he attempts, by a carefully step-by-step geometrization of space, to open room for the geometrization of electromagnetism, not only gravitation.

[18] "[...] the unique 'I' of pure consciousness, the source of meaning, appears under the viewpoint of objectivity as but a single subject among many of its kind [...] Thou art for thyself once more what I am for myself, conscious-existing carrier of the world of phenomena" (Weyl 2009b, p. 124)

subjective feature of sensorial-perception space is having a center. Objective physical space, on the other hand, has no center. In fact, objectivation yields the homogeneity and isotropy of space, all its points and all its directions are equivalent.[19] The truly objective features of physical space are those that are invariant by changes of perspective. Mathematically, objectivity is characterized as invariance under a group of transformations (in case of mathematical-physical space this is the Euclidean group of similarities).[20] Each transformation of the group is a change of perspective.[21] One usually classifies properties of space according to the group of transformations under which they are preserved. Isometries (or motions) preserve *metrical* properties; parallel displacements preserve *affine* properties, and so on. The groups of isometries and parallel displacements are invariant subgroups of the group of similarities of space (the Euclidean group); the group of parallel displacements is in fact a subgroup of the group of isometries.[22] Continuous transformations, which includes properly the Euclidean group of similarities, preserve topological properties.[23]

The constitution of objective physical space is a sort of non-verbal, mostly tacit compromise among cooperating egos implicit in common practices; it consists essentially in "deciding" either which space relations are objective or which changes of perspective (transformations) are irrelevant as to the objective structure of space. As Weyl said, "the immediate experience is subjective and absolute […] this objective world is of necessity relative" (Weyl 2009b, p. 116). The Euclidean structuring

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[19] "Our knowledge stands under the norm of objectivity […] all points in space are objectively alike and […] so are all directions" (Weyl 2009b, p. 71).

[20] A *similarity* is a 1–1 transformation of space onto itself that preserves the relations of collinearity, coplanarity, order and congruence (these are the four fundamental relations in Hilbert's axiomatization of Euclidean geometry).

[21] "A point relation is said to be objective if it is invariant with respect to every automorphism" (Weyl 2009b, p. 73). The notion of automorphism requires that certain spatial relations be selected as basic, for example, those mentioned in the previous note. One usually reverses the procedure by first choosing a particular group of transformations (continuous transformations, similarities, isometries, parallel displacements, etc.) and then *define* objectivity in terms of invariance under transformations of this group.

[22] A subgroup Δ of a group Γ of automorphisms is an *invariant* subgroup of Γ if and only if portions of space that are Δ-equivalent, i.e. equivalent under a transformation of Δ, continue to be Δ-equivalent under transformations of Γ. The object of investigation of the geometry determined by Δ are all the properties that the *normalizer* of Δ preserves. For example, the group of affine transformations is the normalizer of the group of parallel displacements and affine geometry is the study of properties preserved under affine transformations. This means that the fundamental notion of affine geometry is that of *parallelism*. The group of *motions*, translations and rotations, determines the notion of *congruence* in space: two portions of space are congruent if one can take one into another by a motion. Now, the normalizer of the group of motions is the group of similarities; therefore, the notion of congruence is the fundamental notion of Euclidean geometry. For a detailed analysis of the problem of objectivity with respect to space, see chapter III of Weyl 2009b.

[23] Affinities and projections also include similarities. An affine transformation is a composition of parallel projections and a projection is a composition of central projections. The group of similarities is a subgroup of the group of affinities and this is a subgroup of that of projections. The wider group of transformations is that of continuous transformations.

of physical space follows from the communal “agreement” that *congruence* is an objective relation among portions of space (or bodies, on the material level of space perception).[24]

Obviously, measurement is a practice of the life-world developed very early in human history. Practical necessities related to land surveying and agriculture have probably pressed the community to select objectively (i.e. intersubjectively) valid standards of measurement, rigid bodies that could be moved freely in space without any non-detectable, and thus compensable, changes of form or size. A piece of metal, for example, dilate with heat, but one can detect this dilatation and take it in consideration in using the piece of metal for measurements. One’s foot or hand is acceptable provided it is a preeminent foot or hand (for example, the king’s) and one has reliable (congruent) copies of it easy at hand.

The presupposition of free mobility is essential. Given any two bodies separated in space, it is a matter of fact whether they are congruous or not, but this is actually decided only by bringing the bodies into contact or, more practically, by bringing each of them in contact with the metric standard, presupposing of course that the relation of congruence is transitive.[25] The free mobility of rigid bodies is a fact of experience, and we are perceptually justified in believing it; the problem, of course, is that we cannot claim on purely experiential basis that free mobility is valid for the space *at large*. Although not a fact of *actual* experience, one can say that the presupposition of free mobility of rigid bodies is an *a priori anticipation of experience*, and then play a transcendental role in our representation of physical space.

Since Euclidean geometry has at its basis the notion of congruence, which is invariant under the group of motions, *if* the presupposition of free mobility were sufficient for characterizing the group of motions, one could say that our shared experience of space, together with some presuppositions concerning it (free mobility), were sufficient to determine the Euclidean character of physical space, locally and at large. However, free mobility does *not* characterize the group of motions. It *almost* does though; Helmholtz and Lie have shown that free mobility characterizes a projective space with a Cayley metric, which can be Euclidean but also non-Euclidean, hyperbolic or elliptical.[26] In short, our experience and presuppositions about the experience of space only allows us to say that space is *everywhere locally Euclidean*. Mathematics has shown that physical space could also be globally hyperbolic or elliptical. This knowledge, of course, is not perceptually available. In fact, the sort of pre-mathematical and pre-scientific involvement we have with space in the life-world only allows us to say that the space around us will always manifest itself, no matter where we are, as approximately Euclidean. The *scientific*

[24] Supposing that congruence is determined by means of freely movable rigid metric standards. But although necessary this condition is not sufficient for determining the Euclidean character of space.

[25] This is the first *common notion* of Euclid’s *Elements*.

[26] See von Helmholtz 2007. By specifying the Cayley metric on the projective space, one can obtain any of the three geometries, Euclidean, hyperbolic or elliptical.

representation of our spatial experience must consider this, even if it does not represent physical space globally with a Euclidean structure.

Mathematical-Physical Space The mathematical considerations of the previous section do not apply, strictly speaking, to physical space proper, only to its mathematical idealization, mathematical-physical space. They concern physical space only insofar as one takes mathematical-physical space to represent it. Whereas mathematical-physical space is a mathematical manifold, physical space is not, at least not to the same extent. At best, physical space is proto-mathematical and can only become properly mathematical by idealization, i.e. an intentional process of exactification. However, and this is an important remark, idealization is not a way of uncovering the "true" mathematical skeleton of physical space, which is *not* at its inner core mathematical. To claim otherwise is to take the ideal for the real, a categorial mistake. By idealization a mathematical manifold is constituted out of physical space by taking to their limits certain possibilities available in physical space. For example, one can conceive arbitrarily small portions of physical space; in the constitution of mathematical-physical space one intentionally closes these series of possibilities by conceiving extensionless points. Mathematical-physical space becomes, then, a continuous made of atoms tightly packed together, an idea that not only does not have a basis in our experience of space but also, in fact, contradicts it.[27]

Although the structure of physical space, where portions of space relate to portions of space in many ways (near or far, contiguous or apart, closer or farther, bigger or smaller, similar or dissimilar), is not strictly speaking a mathematical structure, it *suggests* one. Idealization takes the hints and, by "polishing" the rough structure of physical space, constitutes a proper mathematical domain out of it. However, mathematization also *adds* to mathematical-physical space features that do not belong to physical space but are consistent with it. As we have seen, our representation of physical space, founded on our sensorial perception of space, although attributing to it a local (approximately) Euclidean structure, does not say anything about its global structure. Mathematization not only turns the local metrical structure of space into a properly Euclidean structure, but also extends it globally to the whole space. By so doing, as we have seen, the scientific ego makes a choice, excluding the other available alternatives, namely, the non-Euclidean geometries. One can only see this as a scientific hypothesis that one must confront with experience, not the sensorial experience available in the pre-scientific world of our daily life, but the refined experience available to the men of science.

Weyl suggested that mathematical theories of intuitive concepts (such as his theory of the continuum) often fail to be completely faithful to intuition and are at best approximations whose epistemological value one must put to empirical

[27] Weyl says it explicitly (Weyl 1994, p. 91–2). One can construe Aristotle's rejection of Zeno's "paradoxes" on the distinction between reality and ideality. Movement occurs in *real* physical space, points only exist in *ideal* mathematical-physical space; therefore, bodies do *not* move by passing through points.

testing.[28] Science can force us to revise our mathematical-physical representation of space; physical geometry (but not pure mathematical geometry) may prove to be wrong or inadequate, like any scientific theory of nature, mathematical or not.

As Husserl claims in his "The Origins of Geometry", idealization has its roots in practices of the life-world. We polish surfaces, smoothening them, at least in principle, to arbitrary degrees of smoothness. By idealizing a plane as a perfectly smooth surface, one is idealizing a practice, i.e. carrying it out to its limit in *imagination*. The central notion that constructions occupy in Greek geometry suggest that practices of the life-world played a central role in the constitution of both the object and the idea of geometry. For the Greeks, for example, instead of being actually infinite, lines were considered as only arbitrarily extendable segments. To show that there is an equilateral triangle seating on any given segment of line, one had actually to construct it, and so on.

For the Egyptians, geometry was essentially a technology for measuring lengths, areas and volumes. With the Greeks, it became a science; later axiomatized by Euclid of Alexandria. Euclid saw his axioms as statements concerning *constructions* that one could carry out in ideal space with ideal straightedges and compasses, given the constructions one knows one can carry out in principle in physical space, with real straightedges and compasses. Euclid was obviously able to appreciate the difference between ideal and real spaces and ideal and real constructions, but he was nevertheless convinced that our intuitions about real space and real constructions are relevant to what we know about ideal space and ideal constructions.

All the five postulates of Euclid's tell us that one can do something. (1) one can draw a (straight) line connecting any two given points; (2) one can extend any given line; (3) one can draw a circle with center at any given point through any other given point; (4) one can, by moving them in space, if necessary, make any two given right angles coincide (i.e. all right angles are equal). And then the contentious postulate (5), one can extend two given coplanar lines until they meet, provided they make with a third line intercepting them, on the side one is prolonging, interior angles that make, together, less than two right angles.

The problem with the fifth postulate is obvious; whereas the remaining four involve finite tasks, the fifth does not. From the first to the fourth, the things that one does are clearly *finitely* delimited; the line of the first postulate begins and ends in points that are *given*, the extension mentioned in the second is limited by the case at hand – and so on. However, one does not know a priori for *how long* one must extend the lines mentioned in the fifth before they meet. If they did not meet, one would never know it by simply extending the lines indefinitely. But, if not so, how? No one before Gauss in the last years of the eighteenth century seems to have thought that the fifth postulate could actually be *false*, only that it lacked intuitive support and therefore one should prove it.

However, how to prove, on the sole basis of finite searches, that a possibly infinite search (for the point of intersection) ends? The task was of course doomed to failure; Euclid's fifth postulate was eventually shown to be independent of the

[28] Weyl 1994, p. 93.

others. The search for a proof of the problematic postulate, however, lead, first Gauss and then Lobachevsky and Bolyai, to the creation of the first non-Euclidian geometry, the now-called hyperbolic geometry, in which the *denial* of the fifth Euclidian postulate holds. That the given lines do *not* necessarily intercept implies that there is *more* than one parallel to a given line passing through a given point (the so-called Playfair axiom – there is *only one* straight line parallel to a given line passing through a given point – is logically equivalent to Euclid's fifth postulate). The remaining postulates in fact forbid that *no* such line exists (otherwise, there should be a line through the exterior point meeting the given line in *two* distinct points, which is impossible). However, with convenient alterations in the system Riemann conceived the now-called elliptic geometry, in which no parallel exists to a given line through a given point.

One important aspect of the mathematization of our representation of physical space is that the latter sub-determines the former. There is more in mathematical-physical space than one encounters in physical space. For example, as already mentioned, although physical space is only *locally* Euclidean, mathematics idealizes it as *globally* Euclidean. Of course, science can and has actually changed its conception of space, assisted by the mathematical science of nature, though, not direct perception. However, the conveniences of the mathematical sciences of nature are sometimes mathematical, not exclusively empirical.

Mathematical-physical space, the idealization of physical space, an abstract non-independent *moment* of the structured system of bodies in space, can be ideated as an empty mathematical form (an *eidos*) instantiable in structured systems that have nothing to do with physical space. The fact that the mathematical theory of idealized physical space can be abstracted of its content, ideated and applied to other structured systems shows that the mathematization of our experience of empirical reality only touches the *formal surface* of it.

In his masterwork *Raum – Zeit – Materie*, Weyl is quite explicit about this[29]:

> Geometry contains no trace of what makes the space of intuition what it is in virtue of its own entirely distinctive qualities which are not shared by "states of addition-machines" and "gas mixtures" and "systems of solutions of linear equations". […] We as mathematicians have reasons to be proud of the wonderful insight into the knowledge of space which we gain, but, at the same time, we must recognize with humility that our conceptual theories enable us to grasp only one aspect of the nature of space, that which, moreover, is most formal and superficial.

Weyl enumerates three structured systems that are formally equivalent to mathematical-physical space to stress the fact that the mathematical theory of physical space, even if it has a material content, is not in fact *about* this content, but the structure it instantiates that different systems also instantiate. Contentual mathematical theories are is this sense also *formal*.

Although, unlike physical space, mathematical-physical space is not perceptually accessible, there is a connection between forms of intuition typical to each domain. In physical space, intuition is sensorial perception, which has no role in

[29] Weyl 1952, p. 26.

mathematical-physical space, where the only form of donation is geometric intuition. Geometrical intuition, however, is the exactification of perceptual intuition, which explains the relevance of perceptual structures such as diagrams, drawings, etc. in geometric reasoning. These structures *represent* geometric structures by *analog*y, that is, by perceptually displaying perceptual facts that stand in a relation of quasi-isomorphism, i.e. structural similarity rather than structural identity, with respect to geometrical facts. Even though one cannot perceive, in the sensorial sense, geometrical truths in the diagrams one draws, one can geometrically perceive the geometrical facts they represent by idealization. Geometric intuitiveness rests on this relation of analogy.[30]

Therefore, ostensive constructions in physical space, although not geometrical in the proper sense, have some bearing on geometrical truth. The relation of analogy between perceptually displayed truths (in physical space) and non-perceptual geometrical truths can be expressed thus: that which is *seen* in the former as morphological facts is *thought* in the latter as geometrical facts. Therefore, to the extent that mathematical-physical space idealizes physical space, one can claim intuitive foundations for physical geometry, the mathematical theory of mathematical-physical space. Based on perceptual intuitions (in actual perception or imagination) one can posit by *idealization* what one does not perceive with the senses, that is, the geometrical fact not amenable to sensorial perception. Geometric points are ideal *limits* of sequences of vanishing spatial regions; geometric lines are limits of sequences of narrower and narrower perceptual lines; geometric surfaces are limits of sequences of thinner and thinner perceptual surfaces. One can reason about objects one does not see by reasoning about their perceptual representatives provided one does not allow "irrelevant" properties of the representatives, that is, properties that do not play any representational role, to interfere. In geometry, as Husserl claims, (geometric) intuition and thinking are intimately connected.

Not only the fundamental geometric elements – point, line and plane – are idealized from perceptual correlates, but the *morphological* notions of same size and same shape generate, by idealization, the geometric notions of similarity and congruence: two geometric structures are congruent (resp. similar) when they have *exactly* the same size (resp. shape). The notion of congruence captures that of constancy of size and shape independently of place (rigidity); that of similarity the notion of invariance under change of scale (similarity, not congruence, is the quintessentially Euclidian notion). The morphologic notion of order also gives origin, still by idealization, to the correspondent ideal mathematical (properly topological) notion of order: point A lies *exactly* in between points B and C. Having points, lines and planes as basic elements, and the relations of belonging, congruence and order (betweenness) as the fundamental notions, one can sketch an axiomatization of geometry as those of Husserl and Hilbert.[31]

[30] One can also think of diagrams as symbols standing to geometric figures as numerals to numbers. However, whereas numerals entertain with one another precisely *the same* ideal relations numbers do, perceptual diagrams display only approximately geometrical relations proper.

[31] See Husserl 1983 and Hilbert 1980. Although Husserl's approach to the axiomatic of geometry is remarkably like his colleague Hilbert's in his famous axiomatization of 1899, it was for the most

Generalized Mathematical-Physical Spaces As we have already discussed, the intuitive acceptability of the foundations of Euclidean geometry had since antiquity been put under suspicion (for Proclus, the fifth century commentator of Euclid, the non-existence of asymptotic straight lines is not intuitively obvious). The fifth postulate of the system seemed, for the reasons already mentioned, to be lacking in intuitiveness. It took, however, more than 2000 years for one to realize that Euclidian geometry is not the only possible consistent theory of space or the only to have a rightful claim to being the *true* theory of space.

Which, then, is the true geometry of physical space? Throughout the nineteenth and into the twentieth century, after the discovery of non-Euclidean geometries, this remained a much-debated scientific and philosophical question. Free from Euclidean constraints, mathematicians did not take long to come up with formal conceptions of space more general than non-Euclidean ones, Riemannian *n*-dimensional spaces in particular (the word "Riemannian" here does not refer to elliptic or spherical spaces, which are, with Euclidean space, particularizations of the generic Riemannian space). Riemannian spaces are continuous *n*-dimensional spaces (arbitrary *n*) in which points have coordinates and a quadratic differential form defines a local metric such that the lengths of any two line-segments are commensurable. But even the Riemannian conception of space can be generalized. Mathematical spaces can be finite or infinite, discrete or continuous, there are non-Riemannian spaces with non-Riemannian metrics and spaces with no metric at all.[32] One could no longer avoid the question: which geometry has the best claim for being a better, more adequate or maybe truer idealization of physical space and on which basis should one decide this? Mathematicians are free to consider many different *mathematical* spaces, but a real space, perceptual, physical or transcendent, can only instantiate one of them.[33]

What is the true structure of space? This was the "problem of space", which engaged mathematicians, physicists and philosophers from the early nineteenth into the twentieth century, particularly after Einstein's general theory of relativity. There is some ambiguity in the statement of the problem, for it not clear if one means sensorial-perceptual, physical, mathematical-physical or transcendent space. My

part developed before they became colleagues in Göttingen, in 1901. This is easily explainable: both were buds of the same Paschian branch. In Hilbert's system, the basic elements are point, line and plane, and the fundamental relations those of incidence (point lies on line or plane, line lies on plane), order (betweenness) and congruence. The notion of continuity (which in Hilbert's system is given by the axiom of completeness and the Archimedean axiom) does not appear in Husserl's sketchy system explicitly.

[32] As Riemann observed (Riemann 2007), discrete (finite or infinite) manifolds (nets) have a natural notion of distance: we can define the distance between two points as the smallest number of points one must go through to reach one from the other. Non-discrete manifolds, continuous ones in particular, on the other hand, have no "natural" notion of distance and can accommodate various.

[33] "Under the influence of modern mathematical axiomatic investigations one has come to distinguish the 'mathematical space', whose laws are logical consequences of arbitrarily assumed axioms, from the 'physical space', the ordering scheme of the real things, which enter as an integral component into the theoretical construction of the world" (Weyl 2009b, p. 134).

impression is that those involved with the problem those early days did not make any differentiation. They believed that transcendent space coincided with physical space, which had an intrinsic mathematical structure. They also thought that perception was a poor means of access to the structure of space, being as it is "imperfect" vis-à-vis the "mathematical perfection" of spatial structure. Husserl was one of the few who saw the necessity of drawing distinctions and that there are many variants of the problem, psychological, physical, and metaphysical.

The problem of space occupied mathematicians such as Riemann, Helmholtz, Poincaré, and Weyl, physicists such as Einstein, and philosophers such as Husserl and Cassirer. Riemann and Helmholtz thought that the question as to the geometrical structure of physical space was ultimately an empirical question. It is an observable fact, they say, that we live in a world in which *rigid* bodies can move freely in space without changing size or shape (without some *physical* reason for so doing, that is). The *simplest* space in which rigid bodies can so move is the Euclidian space. There are other spaces that satisfy the constraint of free mobility, but they are less simple than Euclidean space (if we presuppose that space is infinite, not only *unbounded* – a difference our perception cannot appreciate –, both Euclidian and hyperbolic geometries satisfy the principle of free mobility. If we drop infiniteness, all three rival geometries qualify, for free mobility requires only that space be of constant curvature, which Euclidean, hyperbolic and elliptic spaces are – respectively with zero, negative and positive values). If physical space were effectively non-Euclidian, this would eventually manifest in our experience, not necessarily only immediate sensorial perception, but also scientifically informed experience.

It is in principle possible that bodies apparently rigid deform in movement but no measurement can detect the deformation because everything else deforms in the same way. How can we tell whether rigid bodies in fact *exist*; that there are bodies that can move freely in space without deforming themselves? Helmholtz claims, in a typical transcendental argument, that one can tell this on *a priori* grounds. Since, he thinks, the *notion* of measurement requires the rigidity of measuring rods, the *possibility* of metric geometry requires the constancy of space curvature. Therefore, space has constant curvature out of necessity; its value, positive, negative or zero, however, must, he thinks, be empirically decided, by astronomical observations and measurements, for example, or hypothesized, if no conclusive verification is possible. Poincaré held similar views on the a priori character of the constancy of space curvature, but he believed that no experiment would be conclusively in favor of any value for the curvature of space, for the interpretation of experiments depends on physical theories which we have no reason, logical, methodological, epistemological, or metaphysical, to prefer to the detriment of any geometry. For the sake of simplicity, he thinks, we can always endow space with zero curvature (and hence, Euclidian structure) and change physics accordingly. One usually label this view *conventionalism*. Notice, however, that the representation of space to which Poincaré is alluding relies on means of verifying spatial curvature that goes beyond the purely perceptual.

Provided one is referring to *physical* space, Husserl thought that both constancy of curvature and its value, positive, negative or zero, are *empirical* facts, only experience can – and *will* – tell us; not scientifically informed experiences though, but relevant sensorial perception pure and simple (and therefore one always reaches a decision, by default if needed). Husserl seems to reason in the following way. Suppose the curvature of space is not, *as a matter of transcendent fact*, constant, but that we remain ignorant of it for all dimensions change in displacement in such a lawful manner that an *appearance* of constancy remains. Bodies not actually rigid will be, *ex hypothesi*, experienced as rigid. Would not one then be *intuitively justified* in believing space had a constant curvature even if superhuman beings observing us from outside *our* world would tell a different story (because they can see the deformations we do not)? Suppose that, like the beings inside Poincare's ball, who mistake a gradient of temperature that changes lengths in some lawful manner for a deformation of space, and adopt consequently a non-Euclidean geometry, physical factors one is unaware of influence one's perception of space. In this case, we may be wrong as to the transcendent reality of our perceptual representation of space, but, nonetheless, we are intuitively *justified* in believing space to be how we represent it to be. For Husserl, the *perceptual* representation of physical space need not to be faithful to transcendent space nor depend on scientific contributions. The *scientific* representation of physical space, of course, is an altogether different matter. Husserl believed one *naturally* and *irrecusably* represents physical space intuitively, within reasonable limits of approximation, as Euclidian. It is worth noticing in this argument that Husserl neatly separates the perceptual-physical space where we live our lives from the scientific picture of physical space and its mathematization, i.e. mathematical-physical space, where non-perceptual criteria of adequateness also play a role.

When only Euclidean geometry was available, there was little doubt, if any, that it correctly describes the spatial structure of physical reality, i.e. physical space. After the non-Euclidean geometries were discovered, one had the right to ask which conception of space the entire system of science suggests as a more adequate picture of physical space. Husserl, as mentioned before, conceded that science is not compelled to accept the Euclidean picture of physical space if criteria of validation other than perception are allowed. Nonetheless, Husserl insisted, no matter which representation science chooses to embrace, when it comes to sensorial perception it must take the Euclidean form. In other words, the perception-based Euclidean representation of physical space serves, for Husserl, as a contour condition for more elaborate scientific pictures of space. Hilbert was of the same opinion. He believed that there was an intuitive, perceptual, anthropomorphic picture of space, which fitted the Euclidean mold, and a scientific representation, namely, that provided by relativity theory until a better one emerges.[34]

[34] "Hitherto, the objectification of our view of the processes in nature took place by emancipation from the subjectivity of human sensations. But a more far reaching objectification is necessary, to be obtained by emancipating ourselves from the subjective moments of human intuition with respect to space and time. This emancipation, which is at the same time the high-point of scientific objectification, is achieved in Einstein's theory, it means a radical elimination of anthropomorphic slag, and leads us to that kind of description of nature which is independent of our senses and intuitions and is directed purely to the goals of objectivity and systematic unity" (Hilbert 1995, p. 158).

These views raise many questions. In what sense is the scientific picture of physical space a “better” picture than that provided by the senses? Is the fact that science transcends the perceptual experience of space an indication that it may be closer to the true structure of physical, or maybe even transcendent space? Is a “better” representation of physical space necessarily ontologically truer? If so, to the extent that physical space as scientifically represented differs from physical space as perceptually perceived, what does this imply for our conception of physical reality and its relation to perception? Do mathematically more sophisticated scientific theories necessarily provide better access to the intimate structure of reality? Is physical space essentially mathematical and our unmathematical perception of it a sort of distortion?

Husserl’s *Crisis* addresses these questions directly, but from a more general perspective, the relation between empirical reality and mathematical empirical science. For Husserl, empirical reality is fundamentally that which we perceive with the senses. The task of science is to organize our perceptions of empirical reality into an organic system that is both explicative and predictive. But, Husserl conceded, science can *idealize* perceptions to better investigate the abstract aspects of perceptual reality by mathematical means, which could be freely used, even when mathematics extrapolated the limits of representability of perceptual reality. For Husserl, mathematization is a scientific *methodology* (which influenced strongly the development of modern philosophy) devised by Galileo and other scientific revolutionaries of the seventeenth century to deal with *perceptual* reality, *not a means of disclosing the inner structure of either physical or transcendent reality*. By abstracting the formal structure of the world as perceived and idealizing it as a mathematical manifold proper (to be delivered to the care of mathematics), the “Galilean” mathematical science of empirical reality turns empirical reality into a mathematical object on equal terms with other mathematical objects. However, Husserl insists, one must not forget that this is a *method* of representation, not the uncovering of the *true* structure of empirical reality, supposedly more willing to reveal itself in mathematical garb.

This method has a predictive, but also a heuristic dimension. Husserl could appreciate the predictive efficacy of mathematization, but with a proviso. It could not deliver more than what perception alone, maybe less effectively, could disclose. The heuristic dimension of mathematization in science, on the other hand, seems to have escaped Husserl’s attention altogether, although it makes with predictability the two sides of a single coin. Let us dwell a bit on this. One represents reality mathematically by substituting perceptions, actual or possible, by mathematical representatives of the appropriate sort. Mathematization starts with idealization from perception but usually go well beyond this; often, mathematical “avatars” do not represent perceptions in any sense of the word.

Quantification is usually the first step into full mathematization; it consists in representing the states of a continuous magnitude numerically in terms of a standard state of this magnitude taken as unity. This is how one "measures" temperature, pressure and volume of gases, for example. One can then express lawful relations among correlated magnitudes in terms of functions correlating series of numbers. In our example, assuming ideal conditions, the laws of Boyle and Gay-Lussac, for example. All this indicates that we are *not* operating within perceptual reality itself, but an *idealized* and *mathematizable substitute of it*. General mathematical correlations, formulas, equations and laws are usually established by *induction* on the basis of conveniently idealized actual experiences, but usually extrapolating them. This renders them amenable to correction in face of further experience. Formulas, equations and laws can then be used to *predict* further experiences, but – and this is important – provided there is a system of representation available for *decoding* numbers and other mathematical avatars in terms of perceptions.

It may happen that numbers representing possible "values" of physical magnitudes when fed as inputs into, say, a formula, generate as output a number that *can* correspond to a possible "value" of a correlate magnitude. In this case, the formula is said to have *predicted* that value. Whether this is a good prediction or not remains to be seen when it is submitted to empirical (i.e. perceptual) verification. But it may also happen that the output value does *not* correspond to any possible value of the relevant magnitude. There are two possible ways of dealing with this situation; one can either dismiss the absurd "prediction" as non-sense, thus restricting the range of validity of the formula, or change the standard system for decoding numbers into physical terms to give the "prediction" a physical sense, thus enlarging the domain of validity of the formula. The second alternative can sometimes pay off; when it does we say that the formula played a *heuristic* role. We may then be tempted to think that our formula "offered more than what was put into it", showing that mathematics has the mystical power of uncovering hidden layers of reality.

This is a mistake. First, the *material* content of perception cannot be mathematized, only its abstract formal content can; the mathematical manifolds with which science covers perceptual domains only capture their abstract form. Material content can only be reintroduced via a fixed semantics. This opens some possibilities; either the semantics that gives material meaning to predictions is the same on which the mathematical representation depends or is another that somehow consistently enlarges it. Such an enlargement amounts to a broadening of the field of applicability of the mathematical schema, which one may be tempted to see as induced *by mathematics*. But this is never the case, since mathematical symbolisms do not predetermine their semantics. Attribution of material meaning is a task for the ego who manipulates mathematical systems of formal representation. Therefore, to attribute heuristic powers to mathematics is, to say the least, misleading.

Mathematical systems of representation in science offer possibilities that largely extrapolate the domain of the effectively perceivable. Husserl understood this, but saw in it only a pragmatic strategy that could not substitute perception proper. For example, one can introduce in our scientific representation of thermic phenomena the notion of thermodynamic entropy, whose definition requires in an essential manner sophisticated mathematical machinery. Its perceptual correspondent is not as evident as, for example, temperature. One may reach a situation in which theoretical notions, sometimes essentially mathematical, have no correspondent in perceptual experience. Husserl recognized that these constructions can help us organize our experiences more effectively, but refrained from believing that they give us a privileged access to reality *itself*. Empirical reality, he thought, is primarily that which we *perceive*, and our privileged means of accessing it is perception. Mathematization, in short, provides a method, and a most effective one, but it does not have metaphysical relevance. It operates with an idealization of reality, not reality proper.

For Husserl, therefore, more elaborate, scientifically motivated representations of physical space do not have a claim of being "truer" representations of physical space. In fact, for him, physical space is, strictly speaking, not even Euclidean; Euclidean geometry is only a more convenient idealization of its abstract structure. It is, however, that which best approaches our actual spatial perception; no other representation of physical space better serves perception. More scientifically elaborate representations of space are, in a way, mere convenient *formal fictions* devised for methodological purposes.

Husserl's views seem very conservative, unjustifiably privileging direct sensorial perception. For this reason, Weyl, for whom Husserl was a major influence, decided to turn his back to phenomenology. For him, science must overcome the subjective via symbolization, i.e. mathematization, by which means one can express the formal aspects of experience, those only that are objective, he claims. Adequately mathematized, science can freely erect its system of concepts, which must, however, be confronted with experience and stand or fall as wholes (Weyl's scientific holism). Nonetheless, even if the mathematical science of nature has the right to impose a representation of physical space that overcomes the perceptual one, as Weyl thought, against Husserl, it remains an *idealization* that touches only the formal surface of reality. It is *not* the whole of reality and it is not properly speaking reality; it is only an idealization of formal aspects of reality.

Therefore, to suppose that mathematical representation of physical space, whatever they may be, give us access to the *true* structure of space is a categorial mistake that takes the ideal for the real. The situation is more delicate with scientific representations that depend essentially on elaborate mathematical constructs. If they could tell how physical space *really* is, one had to accept that mathematical instruments created without particular attention to reality and often to completely different purposes could be "unreasonably" effective in telling how reality actually is at its inner core. The most natural way out of this embarrassment, the only that does not require a sort of pre-established harmony between man and nature, is provided by Husserl: mathematical idealizations are *instruments* of formal investigation that play, to certain extent, a representational role, albeit a purely formal one, but that often include elements that, despite helping the internal articulation of the system, do not stand for anything, like complex numbers in the mathematical investigation of the formal properties of the real numbers.

The Euclidean representation of physical space, despite its intuitive foundations, is no less an ideal construct. It falsifies to non-negligible extent perceptual features of physical space and often attributes to it features that are not perceptually discernable. Moreover, *perceptual* space itself is to some extent already a *construct*. We are, already at the level of perceptions, justified in asking whether space as perceived corresponds, and to what extent, to space as it *really* is, that is, transcendent space.

Physical space, which is a communitarian consensus based on spatial perception, is not yet, as already stressed, a mathematical manifold. At best, it is a proto-mathematical domain structured in terms of morphological rather than proper mathematical concepts. One can well inquiry how much of it is due to our perceptual systems rather than to transcendent space. It is conceivable that the proto-mathematical categories *themselves*, which structure physical space, is a contribution, although a non-intentional one, of the *ego* itself. All this considered, one sees how unjustified it is to take the mathematical representations of space, the "naïve" perceptual representation mathematically exactified or more elaborate mathematical representations, as faithful descriptions of transcendent, physical or even perceptual space, and conclude that empirical reality is at its core *essentially mathematical*.

Analytical Geometry I would like to consider now an important point related to mathematical methodology.

Points in space are indistinguishable and there are non-denumerably many of them. Since non-congruent continuous portions of space have the same numbers of points (Cantor' theorem), one cannot express metric notions in terms of numbers of points. To express, for example, the length of a line segment in terms of its end-points, one must take into consideration their *positions* in space, not only their separation in terms of the quantity of points between them. This requires that points in space be objectively distinguishable and identifiable. This can be done by labeling them, thus allowing the separation between two points to be expressed in terms of the labels identifying them. If the space is Euclidean, the labeling must somehow give information about Euclidean distances. This can be done by means of a system of three mutually perpendicular lines (the coordinate lines) that meet at a single point, the origin of the system. One then labels points on each line by real numbers such that the label of P is $+r$ if P is at a distance r from the origin on the one side of the semi-line and –r if it is at the same distance on the other side (it does not matter which side is positive, which is negative). The distance 1 corresponds to the standard metric unit. One now labels each point Q in space by three real numbers, providing information about the distances from Q to each of the coordinate lines. Now, in Euclidean spaces, one can easily express the distance between two points in space in terms of their labels by using Pythagoras' theorem.

One can also introduce systems of coordination in general Riemannian spaces that do not contain any information about distances.[35] Their sole purpose is to identify points by labels in a way to respect relations of proximity and order, i.e. topological relations among points in space. In his manner, one can analytically express different metrics of space in terms of point-labels. Riemann himself gave a general expression for all *local* metrics satisfying certain "reasonable" hypotheses ("the

[35] See Riemann 2007.

hypotheses that lie at the foundations of geometry"). The value of these metrics may vary from point to point (non-constant metrics).

Greek geometry (which was Euclidean) involved in a substantial manner constructions with rulers and compasses, which, as already explained, offered a basis for geometric intuition. The only way one had of determining a point in space was by the intersection of two lines, two circles or a line and a circle. Now, once points in Euclidean space have been labelled in the manner described above, algebraic relations involving the coordinates of the points of geometrical configurations can represent these configurations algebraically (their equations). This means that Euclidean constructions are substitutable by algebraic operations. One can "construct" a point algebraically from given points. Suppose, for instance, that given two points on a plane one must find the third point that form with the given points an equilateral triangle on the same plane (*Elements* 1.1). It suffices to find the equations of two circles, centered each on one of the given points and passing through the other, and find the solutions of the resulting system of two equations. These are the coordinates of the point searched. One will find two such solutions, corresponding to the two possible solutions of the problem.[36]

One has now the following situation, geometric relations among points in space correspond to numerical relations among their coordinates and geometric constructions in space correspond to algebraic manipulations of algebraic equations. On the one side, synthetic geometry, on the other, *analytic* geometry. This, however, is not only a correlation, but an *isomorphic* correlation, insofar as algebraic manipulations remain within the field of real numbers.[37] Geometry, as any mathematical theory, with a privileged interpretation or not, is essentially formal, that is, it is concerned only with relations among "points" independently of whether points are idealizations of positions in space and space is physical space. Therefore, one does not lose anything by taking "points" to be triplets of numbers provided there is an isomorphic correlation between triples of numbers and relations among them and points and relations among them. One can do geometry analytically; mathematically, *nothing* is lost.

There are both a methodological and a philosophical lesson here. Methodological, one can investigate any mathematical manifold by moving to one that is isomorphic

[36] The fact that "constructible" points are solutions of systems of equations of first and second degrees, respectively equations of lines and circles, plays an essential role in showing that the classical problems of geometry, the squaring of the circle, the doubling of the cube and the trisection of the angle, are unsolvable. Their solutions, if they existed, would require the construction of points that are not constructible. The proof of this fact by exclusively algebraic means constitutes an eloquent testimony of the efficacy of formal-symbolic algebraic methods in geometry.

[37] There is no reason why it should. One can conceive "imaginary" points in space corresponding to imaginary numbers, thus enriching the real plane with imaginary entities. This can give the enlarged plane a richer structure that may provide information about the structure of the real plane. Projective geometry, for example, which focus exclusively on the projective properties of the Euclidean plane, works with an enlarged plane containing imaginary points and a line "at infinity". Analytic projective geometry finds great utility in the concept of imaginary points corresponding to imaginary solutions of equations.

to it. Philosophical, from a strictly mathematical perspective, objects do not matter. It is not so much that mathematical objects do not exist, they do, but mathematics is not about them, or not exclusively about them, but whatever behaves *formally* like them. In mathematics, isomorphic copies, which are formally identical, are indistinguishable. Unicity theorem in mathematics are valid only modulo isomorphisms. The philosophy of mathematics should take this lesson seriously. The obsession with ontology and mathematical objects shows that most philosophers of mathematics have not learned it. In fact, one can *invent* (or intentionally posit) mathematical entities with the sole purpose of positing mathematical forms (or structures), which can eventually be either instantiated in existing (or already posited) domains or relate formally to forms (structures) of our interest *even if they have no relevant instantiations*. Of course, this fact plays a central role in accounting for the applicability of purely formal mathematics in mathematics itself and science.

The philosophical task I impose myself, and will carry out in later chapters is, first, to examine from a phenomenological perspective the epistemological and ontological questions related to mathematical forms as intentional constructs. Second, to understand, at least in its general lines, how the knowledge of some forms can be useful in understanding other forms. Third, to derive from these considerations an explanation for the applicability of mathematics in natural science – or better, the applicability of mathematical forms to forms one abstracts from our empirical experience and idealizes into mathematical forms proper.

Concluding Remarks This chapter had one main concern, to measure the distance that separates transcendent and physical spaces from mathematical conceptions of space, and the extent to which the latter can be said to represent the former. I take this as a paradigmatic case of the relation between mathematics and empirical reality. There is a sequence leading from the given (transcendent and sensorial spaces) to proto-intentional and intentional constructs from the given (perceptual space, physical space and mathematical spaces): transcendent space → subjective sensorial space(s) → subjective perceptual space → objective physical space → mathematical-physical space → mathematical spaces.

The only thing we can say of transcendent space is that it is a space, and thus must fall under the concept of space. It necessarily is a manifold and, probably, has a definite dimension. In the most general sense, a space is the abstract aspect of a multiplicity of coexisting elements insofar as they coexist. If the transcendent world is a multiplicity of coexisting things, it must exist "in" some sort of space. The transcendent world supposedly "causes" sensations, which coexist in sensorial space. If we consider sensorial systems separately, visual, tactile, kinesthetic, etc. we can conceive different sensorial spaces, but also of a unified sensorial space insofar as sensorial systems work together cooperatively. Sensations, which are necessarily subjective, may have as much to do with how sensorial systems operate as with the transcendent reality that causes them. It is not then obvious that sensorial space is a faithful copy of transcendent space, whatever sense we may attribute to this.

Sensations, however, are not yet perceptions, which require the action of intentionally motivated built-in psychophysical systems that articulate sensorial impressions into a multiplicity of stable physical bodies in a unified ego-centered spatial mold. The essential difference between sensations and perceptions is that the former are affections of the body and the latter are represented as external to the body. Perceptions are constructs whose matter are sensations. Therefore, perceptual reality, and consequently perceptual space, may have as much to do with sensorial inputs and their spatial mold as with how perceptual systems operate. Perceptual space can only be objectified when other perceivers are taken into consideration; physical space is the spatial frame of a physical world objectively the same for all perceivers.

Considered abstractly, all these spaces have some sort of structure, and who says structure says mathematics, for mathematics is the science of ideal abstract structures. From the perspective of physics, physical space is space par excellence, and it makes sense to ask which geometry it, or rather its mathematical idealization, has. The answer will depend on whether we ask perception or mathematics. Perceptual properties of physical space, insofar as they are structural properties, are expressible in mathematical terminology, but allowing for some degree of imprecision. Mathematical description proper is only possible by idealizing physical space into a mathematical-physical space, which is physical space turned into a mathematical manifold *stricto sensu*. Physical space is not a manifold of idealized "points" that can be said to admit a topological or a metric structure in the strictest sense; only mathematical-physical space is.

The mathematical science of physical reality throws a mathematical net, so to speak, over physical space to capture and express in mathematical terms its formal properties. Mathematical-physical space is only a formal mold, in the sense that it can be given a material content that has nothing to do with the manifold of possibly coexisting physical bodies. As any mathematical manifold, mathematical-physical space can be structurally enriched or extended for methodological purposes, in our case the investigation of the formal properties of physical space (or its mathematical idealization). Such extensions, however, whose elements and structure may not all be said to represent by idealization physical space, cannot be taken to represent the non-perceptual inner core of physical space, insofar as physical space is based on perceptual space. Empirical science has, of course, the right to entertain a more elaborate conception of physical space and impose on it any spatial mold that it deems adequate considering its overall interests, but it does not have the right to impose over physical space as *perceived* a structure that physical space is not *perceived* as accommodating, which is the Euclidean structure (with some degree of imprecision allowed), considering the prominence of rigid bodies in our experience of the physical world. Therefore, to the extent that (communal) perception remains the privileged form of accessing empirical reality, and certain communal practices such as measuring distances with rigid bodies, together with the presuppositions that go with it, are respected, mathematical representations of physical space, no matter how removed from perception, must necessarily approximate locally the Euclidean structure.

Mathematical representations of physical space in general, be they founded on perception proper or more freely elaborated, do not represent physical space directly, only a *mathematical surrogate* of it. Although itself an intentional construct requiring the contribution of *both* transcendent reality *and* our sensorial and perceptual systems to come into being, physical space is not a mathematical construct. Further intentional action is required for it to become one. Therefore, mathematics represents physical space proper only indirectly, by falsifying it to some extent, and exclusively for strictly methodological purposes. Mathematical-physical space is *not* the perceptually hidden core of physical space. The relations of mathematical representations of space with *transcendent* space are even less direct. We see, then, how off the mark is the claim that mathematical theories of space expose the mathematical core of an independently existing reality out there.

Having established that there are many layers of intentional and proto-intentional action leading from sensorial reality to its mathematical representation, I will now develop in more details a conception of mathematics as a science of ideal structures that better serves, I claim, the task of accounting for the pragmatic problem in the philosophy of mathematics, that is, the problem raised by the many uses of mathematics in science, particularly the representational and the heuristic.

Chapter 7
Structures

Structuralism is usually presented in philosophy of mathematics as a nominalist strategy for avoiding commitment to mathematical objects. As relational systems made of empty "points", "vacancies", or "places" in relation, structures can do without the objects that would fill these vacancies or occupy these places. Of course, this does not avoid ontological problems altogether, for one can always raise questions about the ontological status of "vacancies" or structures themselves. However, if one is not particularly concerned in advancing the nominalist agenda, but simply identifying the true objects of mathematics, one has the option of embracing ontological Platonism and grant structures ontological independence. I will present here a version of structuralism that does not side with nominalism, eliminative structuralism, or Platonism.

My approach, however, is not motivated by ontological, but by pragmatic questions, where by "pragmatics" I mean that aspect of the philosophy of mathematics that deals with the applicability of mathematics. The supposedly "mysterious" applicability of mathematics to the empirical sciences has attracted a lot of attention lately, but few philosophers have paid the same amount of attention to a related, no less "mysterious" problem, the applicability of mathematics to mathematics itself. This phenomenon is widely disseminated and the utility of mathematical theories in areas of mathematics other than those for which they were conceived is often a measure of the success of these theories. Algebraic and analytic techniques are extremely useful in geometry, as are geometry and complex analysis in the theory of numbers or topology in mathematical logic, to mention a few cases. Mathematicians are always alert to the possibility of importing mathematical concepts and techniques from other areas of mathematics, for this has often proven to be conceptually enlightening and technically useful. However, this would be difficult to explain if the particular nature of mathematical objects were *essential* for their theories. If geometry were *only* the theory of space, how could we explain that geometry can be abstracted of its material content and reinterpreted salva veritate as the theory of an altogether different domain? If a theory can be formally abstracted, divested of its original material meaning, and reinterpreted as the theory of *another* domain, i.e.

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given another material meaning, then both domains must have *something* in common and this something must be that which the theory is *really* about. But if domains completely differ from one another as to the nature of their objects, i.e. materially, then materially determined objects cannot be what mathematical theories are really about. But if not that, what? The answer that imposes itself is that mathematics is, in fact, concerned only with how objects *relate* or, better, *can possibly relate* to one another *independently of what they are*, i.e. formal patterns or structures considered *idealiter*. The fact that an *interpreted* theory can be formally abstracted and reinterpreted in materially different domains salva veritate tells us that the nature of the objects the theory is supposedly "about" is not essential for its truth; the theory is not "about" the objects of its privileged interpretation, but any objects that have the same formal properties, regardless of their nature. This is, I believe, a strong argument for mathematical structuralism, the view that the true objects of mathematical theories, be they interpreted or purely formal, are *abstract ideal relational systems* or, for short, *structures*.

No matter how well designed the mathematical theory of a *given* materially determined realm of objects – mathematical or otherwise – is, no matter how adequate for the task the language of the theory is, it will always have non-intended interpretations, even if the theory is categorical and all its interpretations are isomorphic. The conclusion forced upon us is that theories, mathematical or otherwise, can only capture that which is most superficial in their domains, namely, their abstract form (or structure), which they share with other domains, in special with those that are isomorphic to them. In short, theories, mathematics or otherwise, are in this sense *formal*.[1] In fact, not only mathematics, *any* science is formal, even if it cannot for reasons we will examine later take profit of formal means of investigation (for example, if the theory is not completely axiomatized or, in case it is, if it does not have any other relevant interpretation).

The non-logical symbols of an *interpreted* theory surely denote *something*, but the theory cannot capture *what* this thing is, no matter how well designed for the task the theory is. Fixing the reference of the terms of an interpreted theory is not a task for the theory itself. The distinction between intentional directness and intentional meaning is useful here. Grasping a particular object, a particular something, is the job of that which I called *intentional directness*, the *immediate* relation intentionality establishes between the intentional subject and the intentional object; intentional meaning, which is expressible in the form of a theory of the object, is the meaning *attached* to *this* object (the *determinable X* in Husserl's parlance) to which intentional consciousness is directed. Meaning may change without the intentional object changing, for different meanings can be identified as meanings of the *same* object (this is the intentional act of *identification*). One can communicate intentional

[1] "La Mathématique estl' art de donner le même nom à de choses différentes. Il faut s'entendre. Il convient que ces choses, différentes par la matière, soient semblables par la forme, qu'elles puissent, pour ainsi dire, se couler dans le même moule [...]. Dans un group la matière nous intéresse peu, [...] c'est la forme seule qui importe, [...] quand on connaît bien un group, on connaît par cela même tous les groups isomorphes." (Poincaré 1908).

directness by pointing or other *non-verbal* forms of communication. Linguistically, one can do this by a definition or a description, but only if the terms of the language involved in either the definition or the description have already fixed references. By following the regress back to its necessary origins, one will necessarily reach *non-linguistic* forms of establishing intentional directness. Fixing a particular interpretation of a theory can be seen as a form of intentional directness to an object whose intentional meaning the theory expresses.

Theories of material sciences such as zoology or physics are interpreted theories with privileged interpretations, the realm of animals and the empirical world, respectively. Even though these theories admit in principle non-standard interpretations, this would be a change of subject matter with no methodological relevance. It is different in mathematics; substituting intended domains for convenient isomorphic copies may offer immense methodological advantages. Analytic geometry, where the geometrical domain proper is substituted by an isomorphic arithmetical copy is a classic example.

Although this methodology is in principle available for any theory, for some, the material sciences in general, it offers no advantage. For example, one does not know any interesting domain of objects isomorphic to the realm of animals. For this reason, zoologists cannot give up animals as sources of insights. If any such domain were available, one could do zoology by investigating domains others than that of animals. Or still, if one had a *complete* axiomatization of a theory of animals, one could do zoology only by deriving the consequences of this theory. But, in general, one does not even know what the relevant concepts are so a theory of animals can be developed by intuitively inquiring those concepts. In mathematics, however, we usually know precisely what the basic relations of the domain under consideration are and there are plenty of available isomorphic copies of mathematical domains of interest. Better still, we can *create*, or better, intentionally posit mathematical structured domains of objects with the sole purpose of making useful isomorphic copies available. In fact, the situation is a little more general; instead of isomorphic copies, we often create mathematical domains where domains of our interest are *interpretable*. For example, one can interpret the domain of real numbers in the domain of complex numbers and use the theory of complex numbers as a tool to investigate the real domain. By investigating the *formal* properties of *complex* numbers, we may disclose properties that are valid for or transferrable to the *real* numbers. All this suggests that formal structures, patterns, or whatever one wants to call them, instantiated in particular domains of objects, mathematical or otherwise, or intended as correlates of formal mathematical theories, are the real objects of mathematics. For these reasons, a philosophy of mathematics attentive to mathematics *as practiced* can only be a structuralist philosophy (one, however, that is *not eliminative* and accepts the existence of all the usual objects of mathematics).[2]

No theory is material in the sense of being the theory of a single domain of objects, its subject matter, but not all sciences can extract enough formal structure from their domains of interest to be able to dissociate themselves from them, thus

[2] Ontological parsimony is not in itself desirable if it is misplaced or plainly wrong.

becoming self-consciously formal, like mathematics and, to a lesser degree, theoretical physics. The empirical world can be experienced from different perspectives, described in many different manners, no empirical theory is final. Moreover, as a transcendent being, empirical reality is ever willing to disclose new aspects. Hence, no empirical theory can be sure of having completely captured the formal structure of the corner of empirical reality under its responsibility. Therefore, empirical sciences cannot afford to give up intuitive access to their privileged domains, even though they cannot expect to grasp nothing beyond their formal aspects. In a formula, material sciences are formal sciences of *particular* domains; mathematics is the formal science of *all possible* domains.

Form and Matter Let me first characterize in a more systematic manner the notions of *formal* and *material* as I understand them here. I say a domain of objects is a *material* domain or that it is *materially determined* if its objects are materially determined, i.e. if they belong to some particular ontological type (numbers, animals, stars, etc.) The objects of a material domain have characteristic traits *other* than merely being objects. A *formal domain* of objects is a domain of formal objects, i.e. objects considered merely as such, abstracted from their other attributes. Formal domains can be obtained from material domains by abstracting the particular material content of their objects. But also, as will be explained below, by being posited as objective correlates of *formal* theories. A *formal object* is an object considered merely as such, with no material attributes, an object-form or mold that can in principle fit any object. Formal objects can, however, have formal properties, expressible in non-interpreted (formal) assertions.

A language *L refers* to a material domain if its non-logical symbols are interpreted as denoting objects and relations of *that* domain, no other. By *formally abstracting L*. i.e. by divesting it of its interpretation, *L* becomes a *formal language*. A *formal language* is, then, a *non-interpreted language*. Now, a theory is *material* if it is the theory of a materially determined domain of objects; i.e. the language of the theory refers to this domain and the theory is *true* in it (in general, material theories are devised by inquiring the *concepts* that rule over their domain). By formally abstracting the language of an interpreted theory, the theory becomes a *formal*, i.e. *non-interpreted theory*; the properties of the original domain expressed in this theory become, by formal abstraction, formal properties of the corresponding formal domain. Formally abstracting a domain is simply considering the objects of this domain merely as objects in the most general sense and the relations that may be defined therein as relation-forms determined only as to their logical types (i.e. n-ary relations of determinate orders for fixed n's). A formal domain determined by a formal theory is a materially indeterminate domain of objects considered simply as such having all the properties expressed in the theory or following logically from it. These are the *formal properties* of this domain. In general, it is presupposed that the domain can have properties that are not derivable in its positing theory (when, for example, the domain is posited as *objectively complete* although its theory is *not logically complete*). If a formal theory has non-isomorphic materially determined interpretations, the formal domain of the theory is *not* identifiable with the formal abstract of any of its interpretations. However, if the formal theory is *categorical*, i.e. if it is interpretable and all its interpretations are isomorphic, then the formal abstracts of these domains are all *equal* and the formal domain determined by the theory is the *ideal* domain instantiated in all equal domains. The reason is that isomorphic materially determined domains satisfy the same properties expressible in their language and cannot be distinguished in it.

Structures A scientific domain is essentially a manifold of entities and relations among them.[3] The anthropologist may be interested in manifolds of people and kinship relations; the geometrician in manifolds of geometrical configurations and relations among them or their parts; the physicist in physical bodies and their mutual relations, considered either statically or dynamically, i.e. physical systems, their states and their temporal evolution. In short, the objects of science are structured systems of objects, i.e. objects in relation. Now, given any structured system, one may consider its elements *only* as instances of their logical types, i.e. objects of a certain logical category as nothing but objects of that category and n-ary relations of determinate degrees as only n-ary relations of those degrees, with no further *material* determination. By doing so (i.e. by formal abstraction) one reduces the system to its abstract form, one ignores the material meaning originally attached to it. The intentional theory of the original system, i.e. the logically closed system of logical consequences of the intentional meaning originally attached to the system, once formally abstracted, becomes a non-interpreted theory whose formal domain is *instantiated* as the abstract form of the original (materially filled) domain.

Suppose, for example, that our original domain is that of the natural numbers and the successor relation. As already explained, although the objects are not themselves, all of them, given individually to consciousness, the generative process is, and by describing how it acts one obtains a description of the ordering of the system of natural numbers by the successor relation. One has the collection of numbers and a single relation, the successor relation. Although the non-logical vocabulary of the intentional theory is determined – it has only a unary function symbol for the successor – the logical context is not. One may, abiding to empiricist prejudices, choose to restrain ourselves to a first-order language. The resulting intentional theory is first-order arithmetic (*PA*). Once *PA* is formally abstracted, it becomes a formal theory, *FPA*, where non-logical constants are devoid of any interpretation. Since *FPA* has many non-isomorphic interpretations, the formal domain of *FPA* is no longer a well-determined domain of formal objects; it is not even numerically determined.

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[3] One can define functions as particular relations: if f is a n-ary function defined in a domain D, one can define a $(n+1)$-ary relation R such that $R(x_1, \ldots, x_n, y)$ iff $y = f(x_1, \ldots, x_n)$. I say that a subdomain B of D is *closed* under f iff for every sequence $x_{1\ldots}x_n$ of elements of B, $y = f(x_1, \ldots, x_n)$ is also an element of B. In general, when considering subdomains where functions are defined one supposes they are closed under these functions. To *restrict* a n-ary relation S defined in a domain D to a subdomain B of D is to consider the n-relation S' defined in B thus: for all sequences $x_1, \ldots, x_n$ of elements of B, $S'(x_1, \ldots, x_n)$ iff $S(x_1, \ldots, x_n)$. If R is defined as above, its restriction R' in B is such that for all sequences $x_1 \ldots x_n$, y of elements of B, $R'(x_1, \ldots x_n, y)$ iff $y = f(x_1, \ldots, x_n)$. There may exist elements $x_1 \ldots x_n$ of B but no y in B such that $R'(x_1, \ldots x_n, y)$ unless B is closed under f.

One could construe the notion of formal domain of a formal theory in general as the *family* of all the abstract forms of its interpretations (isomorphic interpretations have the same abstract form). In this sense, I say that the abstract form of the original domain of numbers *instantiates* the formal domain of *FPA*. The formal domain posited by a theory is, of course, given by this theory, but it is not exclusively accessible through it, as we will see later when discussing interpretations of structures into one another. Ontologically, a formal structure can depend on a positing theory, but not necessarily epistemologically.

If, however, we decide to use a second-order language to describe the structured system of numbers, then all the material interpretations of the resulting theory are isomorphic (second-order arithmetic is *categorical*). In this case, the formal domain of the associated formal theory, understood as a family of forms, has only one element, *the* ω-structure. As I understand the notion, a *structure* (*formal structure* for emphasis and to distinguish it from materially filled structured systems) is either the idealized abstract form of a (materially filled) structured system or the uniquely determined formal domain of a *categorical* formal theory. Non-categorical theories determine only *families* of structures, the ideal abstract forms of all their interpretations.[4]

There are, then, only two types of structures, abstract forms of structured systems considered *idealiter*, i.e. structures *in re*, and formal domains of categorical theories, i.e. structures *de dicto*. Given a structure instantiated in a domain *D*, and chosen a convenient language *L* for describing it, the sentences of *L* that are true in *D* are, when formalized (formally abstracted), true of the structure instantiated in *D*. The set of all such sentences is a complete structural description of this structure (in *L*). In case of structures posited by theories, although the theories are also structural descriptions of the structures they posit, they may fail to be complete descriptions (second-order arithmetic is a standard example).[5] Different theories describing a *given* structured system count, when formalized, as different structural descriptions of the *same* structure in different logic contexts. *Any* theory is a *structural description*, even interpreted theories, which are structural descriptions of the abstract structures of their intended interpretations. Descriptions (i.e. theories) can only express what is formal, since they can, as already emphasized, if materially determined, be formally abstracted and reinterpreted salva veritate. Interpreted and formal theories only differ as to their intentional focus, the former being intentionally directed to privileged domains by a "focusing" that is not expressible descriptively in their theories.

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[4] Again, there is a phenomenological difference between the structure as *abstract* (i.e. non-independent) *aspect* of a particular domain, which is only an aspect of *this* domain, and the structure as an *ideal* entity, which is *not* an aspect of anything, but an object indifferently instantiable *as* the abstract aspect of materially distinct, but formally equal, domains. The ideal structure is the *type*, all the equal abstract structures the *tokens*.

[5] If a categorical structure positing theory is incomplete, and some sentence undecidable in the theory is proven in a consistent extension of it, then the categoricity of the original theory implies that this sentence is true in the structure it posits. This is a trivial example of how one can investigate the structure posited by a theory by means other than the positing theory itself.

Once one has structured a domain of objects by imposing on it a system of relations, one has also singled out a structured system and, in part, a language for describing it, which must have a non-logical symbol for any relation of the system, and only for those (the logical context of the language being left undetermined). One can express in this language different structural descriptions. In this case, the structure precedes its theory. In fact, there is no single theory of a structure *in re*. Although the non-logical elements of the language of any such theory are determined (for the system of relations is determined), its logical elements are a matter of choice (first-order, second-order, infinitary languages, particular rules of derivation, etc.) One is free to choose the most adequate logical ambiance where to carry out the investigation of a given structure *in re*. Different descriptions may capture different aspects of the structure. Structures *de dicto*, on the contrary, are posited by their theories, and more often than not they come in families; groups, for example, which have many non-isomorphic interpretations. However, consistent extensions, *in the same language*, of structure-positing theories that are logically incomplete are still descriptions of the structures the narrower theories posit.

Structuralism As I understand it, *structuralism* is the view that the privileged objects of mathematics are formal structures, but that mathematics is sometimes interested in structures that are instantiated in particular domains of objects, usually mathematical objects. Unlike other perspectives that go by the same name, I do not claim that mathematical objects do not exist; they do, sometimes as intuitable objects, sometimes as purely intentional objects. However, I claim, since mathematical theories are capable of grasping only the formal structure of whatever structured systems of objects they purport to describe, mathematics is essentially a science of either instantiated or non-instantiated ideal abstract forms or structures. Even when mathematics considers a *specific* type of objects, for example, numbers, it finds it unnecessary to maintain the focus of interest on objects of this type once it has grasped the formal structure of its domain. It often finds it advantageous to redirect intentional focus to isomorphic copies of the original domain, where the same structure is instantiated.

According to the version of structuralism that I am advancing: (*a*) there are no free-floating, i.e. independently existing structures; they are either attached to structured systems as their form or posited as correlates of theories of a certain type (categorical theories). (*b*) The objects of structured system can be anything, aspects of empirical reality or mathematical objects proper; mathematical domains may be intentionally meant with the sole purpose of carrying a structure. (c) Structured systems are *given* either extensionally in (adequate or partially adequate) intuitions or intensionally, as extensions of either intuitable or emptily meant concepts.

Some examples are in order. Let us consider first the structure of the domain of real numbers. There can be no adequate intuition of this domain, for it is infinite, nor of its generative process, for there is none. But we have the concept of real number – a real number is a quantitative binary relation among continua of the same species –, which we can inquire. By so doing, it becomes evident that one can operate with real numbers, for example, adding and multiplying them. Upon reflection, one sees that these operations have certain formal properties that make the domain of real numbers into what one calls a *field*. The following are also intuitive truths about real numbers. (1) Given a real number, either it or its negation has a square root; (2) the equation $x^2 + 1 = 0$ has no real solution; and (3) if a, b are real numbers, the equation $x^2 = a^2 + b^2$ has a real solution. One can easily convince ourselves of these things geometrically, i.e. by taking non-negative real numbers in concreto as lengths of line segments and operating with them geometrically.[6] These three facts taken together with the fact that the domain of real numbers is a field constitutes a logically *complete* theory of real numbers, *RA*.[7] This is an interesting property, for although the resulting theory is not a categorical characterization of the operational domain of the real numbers, it has the property that any assertion of its (first-order) language *true* in the domain in question is logically derivable in the theory (so, this theory does not fall prey to Gödel's first incompleteness theorem).

A few things are worth remarking. From a strictly mathematical perspective, real numbers are of interest only as things upon which one operates, i.e. mathematically, the domain of real numbers is nothing beyond an operational domain. If *RA* is formally abstracted, it can be interpreted (materially filled) in many different ways, by things that may have nothing to do with real numbers. However, as a theory of real numbers proper, *RA* tells all one wants to know about them as things upon which one operates, although it is silent about their other aspects. All the other conceptual features of the concept fall out of the picture. This justifies the identification of natural and rational numbers, which are *also* things upon which one operates, *as* real numbers, given that operations with them, as real numbers and as what they originally were conceived to be, *formally* coincide. In principle, one can then prove *in RA* facts about *natural* or *rational* numbers merely *as things upon which one operates* (in some cases one must, if one can, isolate them from other real numbers by means of predicates). As objects with which one can operate, rational and natural numbers can be seen as real numbers. As said above, the non-categorical theory we have obtained does not single out the structure of the real domain. This is in general the case with theories expressed in first-order languages, i.e. first-order theories, the sole exceptions being theories of finite domains. One can always return to the domain of real numbers for intuitive insights and extend the theory in many convenient ways.

In general, the same domain of objects can instantiate different structures, if differently structured with different sets of relations. A given structured system can also have different structural descriptions. However, as far as one considers the real numbers only as an operational domain describable in the language of the theory, all one needs for settling questions about them is this theory, given its logical completeness. There is, however, no *decision procedure* that would decide, for any assertion in the language of the theory, whether it is true in the domain or not. When one has to answer *specific* questions about the domain of real numbers, the theory is in general of little assistance; one must find other, usually specific techniques for dealing

[6] See, for instance, Descartes' *Geometry* for how to operate with numbers geometrically.

[7] See Shoenfield 1967, chap. 5, for details.

with specific problems. This often involves *enriching* the original structure of the domain in convenient ways.

The axiomatization given above is not that which Hilbert presented in 1900, where he considered also the relation of order among real numbers, with which he tried to capture the Archimedean property (there is a multiple of a given real number that is bigger than any given real number).[8] He also added an axiom (axiom of completeness) to guarantee that the domain of real numbers could not be further enlarged and still obey all the other axioms. Nowadays, one prefers to state the completeness of the real domain with a different axiom: each bounded set has a least upper bound (this, of course, requires a second-order language). Both these axiomatizations try to get a better picture of our original conceptual intuition of the real domain; the completeness axiom, in particular, expresses the intuition that real numbers form *themselves* a continuum.

Structures: Ontology Numbers, geometrical forms and sets are classical examples of mathematical objects. They are not real objects immersed in the flux of time, but ideal objects existing a-temporally. Nor are they ontologically independent concrete objects, but abstract objects ontologically dependent on other objects. Numbers, geometrical forms and sets are nothing but abstract *forms*, existing as either *actual*, *effectively actualizable* or *in principle actualizable* forms of collections of things and physical bodies, in case of geometrical forms. The intentional generation of mathematical objects often requires idealization; for instance, geometrical shapes idealized from spatial shapes given in spatial perception. Abstract aspects, idealized or not, however, only attain full mathematical status by ideation, the action that turns idealized abstract forms into ideas or eidos proper. Despite all the intentional action that goes into the constitution of mathematical objects, they exist *objectively*, not mental or platonic entities living either in the mind or in an independent realm of their own.

For centuries, mathematics was just the study of these objects, particularly numbers (arithmetic) and space (geometry), either for their own sake or for practical purposes. In pure geometry one can, for instance, study the geometrical properties of the chiliagon, a form never instantiated in geometrical intuition or perception, but which is a *conceivable*, constructible form, and therefore a legitimate object of geometrical concern. Pure arithmetic also extends well beyond the effectively intuitable and our practical concerns. For example, it admits the existence of infinitely many numbers, even though no one will ever intuit or have any use for most of them. Geometrical constructability and numerical productivity are theoretical idealizations, despite their origins in mundane practices. Nonetheless, geometry and arithmetic, at least in classical times, never lost their ties with empirical reality and mundane practices. There is arithmetic proper, the pure science, but there is also logistics, the practical science of reckoning. And no one ever doubted that despite its ideal character geometry could be applied, that real constructions carried out with ideal compasses and rulers could also, abstracting from inessential restrictions,

[8] Hilbert 1900.

be carried out to a reasonable degree of approximation with real compasses and rulers. The space of physical geometry, i.e. mathematical-physical space, is physical space considered abstractly, i.e. independently of the bodies in it, and idealized; in other, more precise words, it is a mathematical surrogate of perceptual space. Physical geometry is the *mathematical* theory of mathematical-physical space and can be applied to our perceptual representation of space (physical space) by reverting the constitution process.

The applicability of mathematics, at least of this type of mathematics, to reality is not a mystery, for this type of mathematics is, in a sense, "extracted" (i.e. abstracted and idealized) from reality, even if it often adventures beyond the bounds of the strictly perceivable. The limitation Greek geometers imposed on geometrical constructions reflected their understanding of what the fundamental forms of reality are. By allowing only straight lines and circles as basic figures and thus admitting only rulers and compasses in constructions, the Greeks expressed their understanding of which lines are physically "natural", i.e. which trajectories are allowed for bodies in free, unforced movement. These were, according to them, only movements along straight lines (up and down) and circles (around the center of the finite universe).

But there are other types of mathematical objects, *second-order forms* that came into mathematical consideration much later, when the instruments for their investigation became available. I believe that the inaugural act of the new mathematics, concerned with abstract formal structures rather than individual objects, was the invention of complex numbers in the sixteenth century. Italian algebraists of the Renaissance realized that dealing with *meaningless symbols*, "imaginary numbers" such as square roots of negative numbers, which one could operate with by extending to them the rules of operations with numbers proper despite the absurdness of the procedure, was not only safe but also immensely useful. It did not take too long for the mathematical community to realize a few facts, namely: there are things that are not numbers but behave operationally like numbers, one can add these things to numbers proper into a formal-operational domain devoid of material content and use this enlarged context to deal in a most fruitful manner with numbers proper. Of course, this would be impossible if mathematical knowledge of numbers proper did somehow involve the material meaning of numbers, not only their formal (operational) meaning, i.e. how they behave operationally. Only considered as a formal-operational domain the domain of numbers proper can be enriched with "imaginary" numbers. Solving algebraic equations by radicals is only a matter of finding clever formulas for *calculating* the desired solutions from the coefficients of the equation, known also its degree. Such calculations depended solely on the formal-operational properties of numbers, not on *what* they are, and could be performed in any formal-operational *equivalent* of the numerical domains. Adding the "imaginary" numbers added room, so to speak, for convenient formal manipulations.[9]

[9] The question of whether operating with imaginary entities produces always true results about real entities depend on how both operational domain, the narrower (real) and the larger (imaginary), relate to one another. See below.

Another important step in the direction of a formal-structuralist conception of mathematics was the creation of analytic geometry by Descartes and Fermat. In the analytic treatment of geometry, one can substitute algebraic constructions for geometrical constructions, and, again, this is possible only because both domains of construction are formally identical. With the development of the method of representing points in space by triads of numbers and geometrical forms (collections of points) by equations, a formal domain was found that instantiated the *same* structure of the geometrical domain of points, collections of points and relations among them. Geometrical constructions are operations on geometrical forms and can be substituted by formally identical symbolic constructions (i.e. algebraic manipulations). From a formal point of view, *what* one operates upon is immaterial. The fundamental insight imposes itself that, from a strictly mathematical perspective, all formally identical domains are essentially the same. In other words, in mathematics, form only matters.

The creation of abstract algebra in the early years of the nineteenth century was the final step on the way to making abstract structures the privileged objects of mathematical inquiry. Two major accomplishments showed beyond doubt that one can investigate objects of one type by investigating objects of another type if both types of objects were formally analogous. One was the proof that the old problems of Greek geometry, squaring the circle, doubling the cube and trisecting the angle were unsolvable with ruler and compass.[10] Finding intersection points of straight lines and circles corresponds, on the algebraic side, to solving systems of equations of the first and second degrees. Therefore, "constructible" points – solutions of those systems of equations – belong either to the ground field, the rational numbers, or to field extensions of degree equal to two. Thus, a purely algebraic characterization of constructible points was found. Since the solution of those three problems by the required methods required the construction of demonstrably non-constructible points, they are unsolvable problems.

Another momentous breakthrough in the way to the recognition of the formal-structural character of mathematics was the proof that algebraic equations of degree larger than four were not, in general, solvable by radicals.[11] This involved a more elaborate algebraic representation. A tower of fields, each a "Galois extension" of its antecessor, is constructed leading from the field of the coefficients of the equation to the field of its solutions. This tower represents the "actions" involved in solving an equation by radicals. This tower of numerical fields is then put in correspondence (the Galois correspondence) with another, a tower of groups, each associated with a field extension. This last tower enjoys a special property; it is a composition series in which each factor is a cyclic group. The existence of the tower

[10] Pierre-Louis Wantzel, in 1837

[11] Niels Abel, in 1824, showed that there are equations of the fifth degree that were not solvable by radicals. However, it was Évariste Galois (died in 1932 aged 20) who developed the general theory known today as Galois Theory, where the quintic is shown to be in general unsolvable by radicals. Galois was the first to use the word "group"; today we consider him one of the founders of abstract algebra.

of fields representing the solvability of the equation is thus represented in terms of the "solvability" of a particular group associated with the original equation. Not only the operations of solving an equation by radicals are represented in terms of abstract algebraic notions – field extensions – but a series of groups is found that corresponds, from a convenient formal perspective, to the series of field extensions. In the end, to verify whether a given equation is solvable by radicals, it suffices to find its Galois group and verify whether it has the algebraic property of being "solvable".[12]

In all these cases, the situation is essentially the same. A problem involving objects of a certain domain is given; the problem is associated with another, involving objects of another type and one can transform the solution of the latter problem into a solution of the former. The relevant methodological and epistemological questions are *how* and *why* this works. A fact is immediately clear; the method works because the *particular nature* of the objects involved does not play any essential role in the problem, one can change subject-matter provided enough formal-structural properties are preserved, or still, the formal-structure of both domains of objects overlap in some significant way. Isomorphism is the strictest such condition; two isomorphic domains are the same domain as far as formal-structure is considered. But there are other possibilities.

One can easily explain how imaginary numbers can be used to solve real equations. First, notice that the domain of "imaginary" numbers contains a subdomain formal-operationally identical with the domain of real numbers. Now, given an equation with *real* coefficients, it suffices to take it as an equation whose coefficients are the "*imaginary*" correspondents of the real coefficients and operate freely in the "imaginary" domain. If the "imaginary" equation has a solution in the subdomain that corresponds to the domain of real numbers, the real number that corresponds to this solution is a real solution of the original equation. In this case, the original domain of objects is not isomorphic with the second domain, but isomorphically embeddable into it. If finding the solution of the equation in the field of real numbers took into consideration the peculiar nature of numbers, not only their operational properties as things upon which one operates, such a method would not work.

All this suggests that despite the fact that some mathematical domains are materially determined, i.e. their elements are individuals of some particular type, for example, quantitative or geometrical forms, mathematics is interested only in the abstract structures these domains instantiate. By closing the focus on formal rather than material content, mathematics was able to develop the most efficient methods for investigating one domain by investigating another, whose form happens to be conveniently similar to that of the original domain. Other similar methodological strategies are available, of course; for instance, we can structurally enrich the original domain to obtain a formal context for which stronger mathematical tools of investigation are available.

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[12] For a detailed analysis of the rise of mathematical structures, see Corry 1996.

It is not my goal here to conduct a systematic logical-epistemological investigation of the strategies of formal-structural investigations in mathematics. Enough was said to make my point that mathematics has, throughout its history, gradually realized that it is essentially a formal science, i.e. a science of contentless forms, and that its true objects are abstract structures, even though they may require mathematical objects to be instantiated. Abstract structures can sometimes present themselves *in concreto* (structures *de re*) as instantiated structures, but they can also be intentionally posited *in abstracto* (structures *de dicto*) as intentional correlates of formal theories often designed especially for this purpose. In fact, if all that matters are structures, one could simply *invent* them. *Invented* structures are no less efficient, and could very well be a convenient context of representation or interpretation of other formal structures, invented or not. These insights resume my conception of what sort of objects mathematical structures are and my understanding of the reasons why mathematics has such a wide range of applicability in all domains, scientific or otherwise.

The brand of structuralism that I endorse does not deny the existence of the usual mathematical objects; it is not a form of nominalism. Philosophers who find the existence of mathematical objects, Platonically construed, inconvenient because of the ontological and epistemological problems they raise are not better off by substituting objects by structures, for these entities invite the same embarrassing questions. Do structures exist? In what sense? How can we access structures? As I have already discussed, we can give a uniform answer to all questions concerning mathematical existence. Higher-order objects such as structures or lower-level ones such as numbers exist essentially in the same way as intentional correlates of positing acts.

Let us go into details, beginning with explicit definitions. A *relational system* is a (materially determinate) collection (*not necessarily a set*) D of objects (i.e. bearers of attributes, or, from a linguistic perspective, referents of nominal terms of some arbitrary language),[13] called the *domain*, where some relations R_i are defined. They can be first-order relations among objects or higher-order relations among higher-order entities; I denote a relational system by (D, R_i). To specify the orders and arities of these relations is to specify the *signature* of the system.

A relational system is a *structured* system, the relations R_i being its structuring relations. The (materially determinate) domain D can be *given* either extensively or conceptually (intensionally). A *structural description* of (D, R_i) is a collection of sentences of a convenient language L, called the *structural language*, that are true in (D, R_i). Any sentence of L true in (D, R_i) expresses a *structural feature* or *property* of the structured system from the perspective of L. The *complete* structural description of (D, R_i) is the set of all sentences of L true in (D, R_i). The only non-logical symbols of L are the symbols "R_i" denoting the relations R_i. The logical

[13] "Being (*Seiend*) in the broadest sense, in that of theory of science and formal ontology, is each and every thing that can figure as the subject of a statement, each and every thing about which we can in truth speak, each and every thing that can in truth be referred to as being (*seiend*)" (Husserl 2008).

strength of L will depend on what it is required for expressing what one *sees* as the characterizing properties of the structuring relations, first-order, second-order, infinitary, or any other. (D, R_i) satisfies, of course, any structural description of it, but that which makes (D, R_i) the particular structured system it is, i.e. the particular nature of its objects and relations, is *not* expressible in any language; they are *material* properties of the domain that cannot be captured descriptively since structural descriptions of (D, R_i) always admit materially different interpretations.

By accessing the given system (D, R_i) somehow, one may be able to single out salient structural properties of the system, a collection A of L-sentences, from which all the truths of the system can be logically derived; in this case I say that A is an axiomatization of the *complete* structural description of (D, R_i). In case a structural description is not logically complete, there may be structural properties of the system that are not derivable in this description. Nonetheless, an L-sentence φ can be shown to be true in the system either by directly accessing (D, R_i) (in extension or conceptually) or indirectly, by ways I will briefly analyze soon. (For example, φ can be shown to be true in a system that is isomorphic to, but more easily accessible than (D, R_i).)

The *abstract structure* of (D, R_i) is simply the structured system *itself* abstractly considered, i.e. its form is maintained whereas its material content is "abstracted out". In details, we take each element of D merely as a "something", not an object of a particular ontological type, but an object of the largest ontological type, i.e. an object merely as such (an *etwas überhaupt* in Husserl's way of speaking). I call D_{ab} the *formal abstract* of D. Of course, D and D_{ab} are in a 1-1 correspondence (in particular, they have the same number of elements); two elements in D_{ab} are different if they correspond to different elements in D, etc. D_{ab} just *is* D, but without taking into account *what* its elements are, only that they are the elements. Now, for each structuring relation R of D, its abstract R_{ab} is a relation of the same type and arity with no further *material* properties or determinations. By formal abstraction, any structural description of (D, R_i) originates a structural description of the *abstract* structure (D_{ab}, R_{iab}). Given a L-sentence φ, φ is *true* in (D_{ab}, R_{iab}) iff φ is true in (D, R_i). One can investigate the abstract structure (D_{ab}, R_{iab}) by investigating (D, R_i) or any structured system that happens to have the same abstract structure, i.e. any system *isomorphic* to (D, R_i), for their abstract structures are *equal*.

It is important to notice that relations S_j may be defined (or definable) in the domain D that are *not* structuring relations of (D, R_i) *if they are not defined in L*. Sentences in a language that includes symbols for these new relations may be true in (D, R_i, S_j) but they are *not* structural properties of (D, R_i). For example, the assertion $\varnothing \in \{\varnothing\}$ is true in the system of the von Neumann finite ordinals $Ord = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\} \ldots\}$ with the structuring relations defined in terms of the function $S(x) = x \cup \{x\}$ and the distinguished element $\varnothing$, but it is not a property of the abstract structure of the system (Ord, $\varnothing$, S) for it is not expressible in its language.

The abstract structure of a structured system is an abstract aspect of *it*. An *ideal structure*, on the contrary, is by definition realizable as the abstract structure of *any* structured system of a class of *equally* structured systems. We then need a characterization of what it means for two structured systems to have equal (but not identical) abstract structures. Consider the systems (D, R_i) and (F, S_i); the first requirement for their structures to be equal is that $D_{ab} = F_{ab}$ or, equivalently, there must be a 1-1 correspondence between D and F. Obviously, the signatures of $\{R_i\}$ and $\{S_i\}$ must be the same so both admit the same structural language L. However, the most important requirement is that both structured systems be L-equivalent; therefore, a sentence φ is true in (D, R_i) iff it is true in (F, S_i). The strongest condition that meets all these requirements is (D, R_i) to be *isomorphic* to (F, S_i).There is also a weaker condition: (D, R_i) and (F, S_i) instantiate equal abstract structures iff there is a 1-1 correspondence between D and F and (D, R_i) is L-equivalent to (F, S_i). However, the latter condition depends on the language L; change it and two systems will no longer instantiate equal structures. Therefore, it seems better to adopt the stronger condition: two structured systems instantiate *equal* structures iff they are isomorphic; *ideal* structures are the *same* (identical) iff they are instantiated in isomorphic systems. *Abstract* structures are *different* if they are instantiated in different systems even if these systems are isomorphic; *ideal* structures, on the contrary, are ideations capable of realizations or instantiations as different abstract structures provided they are equal (i.e. they are the abstract structures of isomorphic systems). One can access an *ideal* structure by accessing any structured system of a class of isomorphic systems. In the end, it all boils down to accessing some structured system and describing it. Moving from it to its abstract structure, or the ideal structure this aspect of the system instantiates, is only a matter of intentional refocusing.

The question now is how one can single out an ideal structure purely *descriptively*. Given a language L and a non-interpreted theory T in L, which conditions must T satisfy to single out an ideal structure *some* of whose structural properties T expresses (maybe all if T is logically complete)? Although T posits a formal domain (in Husserl's sense) that is materially filled in any of the interpretations of the theory, it is not always the case that any two such interpretations instantiate equal abstract structures. For this to be the case all instantiations of T must be isomorphic, i.e. T must be a *categorical* theory. Of course, one supposes that T is logically consistent (although maybe not actually instantiable). Now, if T has an interpretation, it has, up to isomorphism, only one interpretation; T posits *the* ideal structure instantiable in all its interpretations. One can investigate the structure T posits theoretically, by deriving the logical consequences of T or intuitively, by investigating L-structural properties of any of its interpretations. The second strategy is necessary in case T is not logically complete, but may be useful, if in principle dispensable, even when T is complete. If T does not have any interpretation, the structural investigation of the ideal structure it posits must be exclusively theoretical, although not necessarily restricted to the positing theory T, as we will see.

There are then two ways an ideal structure can be given, realized as the abstract structure of a given structured system or intentionally posited by a categorical theory. In the latter case, the ideal structure the theory posits is realized as the abstract structure of its essentially unique instantiation, if the theory is actually instantiable. I say an ideal structure is given *in concreto* (de re) in the first case and *in abstracto* (de dicto) in the second. In most cases, structures are instantiated; if they are not, they exist only as intentional correlates of logically consistent theories – *that* which the theory purports to describe. By studying structures, instantiated or not, mathematics is doing formal ontology in the sense Husserl understood it, i.e. the a priori science of the possible ways (or forms) that objects considered simply as such (no matter whether real, ideal, concrete or abstract) and objective domains in general can *in principle* present themselves to consciousness.[14]

The fact that mathematical theories are not in general categorical, particularly all first-order theories interpretable in infinite domains as most mathematical theories are, is not a serious drawback. One may take non-categorical theories as structural positing of families of structures, each instantiated in one of the interpretations of the theory. There is, however, a limitation one must reckon with; structural descriptions (structure-positing theories in particular) are not in general logically complete. This means that there are a priori possible structural properties that the structural description is incapable of deciding. Structures may transcend the logical powers of the language chosen to describe them; they are, in this sense, *transcendent* entities. However, as already discussed, it is presupposed, as with mathematical entities in general, that any meaningful structural property is *in itself* decided; in other words, (ideal) structures are *objectively complete* (remember that this presupposition has a transcendental-phenomenological character and is required for classical logic to apply).

Structures in concreto are instantiated structures whose material supports are always available as sources of information about the structures they instantiate. One can always go back to structured domains in order to improve on their structural descriptions, no matter whether they are given extensionally or conceptually. The same is true for structures posited by *conceptual* theories; their ruling concepts being always available as sources of intuitive insights (structures can also be posited by freely created formal theories not originally concerned with any concept; in this case, the improvement of structural descriptions must rely on other strategies, as we will see below). Gödel, for instance, believed that to decide questions that are logically independent of our set theories we should inquire our *concept* of set. By going back, *intuitively*, to particular structured domains or concepts one can improve structural descriptions. The fact, however, that one cannot expect to surmount logical incompleteness of structural descriptions in general gives structures the character of transcendent entities, always capable of presenting new aspects to consciousness.

Some Examples Consider the sequence **N** of natural numbers 0, 1, 2, etc., structured by the successor function, i.e. $1 = S0$, $2 = S1$, etc. and having one distinguished element, the constant (0-ary relation) 0. The structural language of **N** has only two non-logical symbols, one binary relation symbol for the binary relation R such that $R(x, y)$ iff $y = Sx$ and a constant symbol denoting 0. A convenient description of this structure is second-order Dedekind-Peano arithmetic, whose formal abstraction is a

[14] For this reason, Husserl placed *formal* mathematics, insofar as their objective correlates are concerned, in the realm of formal ontology, the ontological side of the third realm of the three-stored edifice of formal logic (the other side, the apophantic, concerns itself with theories and logical relations among them).

categorical theory whose interpretations are all isomorphic to (**N**, *R*, 0). Second-order arithmetic, however, is not a complete theory, so one cannot expect to disclose all the structural properties of this structure by simply deriving the consequences of its theory.

Now, let **N'** be von Neumann finite ordinals ∅, {∅}, {∅, {∅}}, … with the relation *R'* such that *R'*(*x*, *y*) iff $y = x \cup \{x\}$ and having a distinguished element ∅. It is not difficult to see that (**N**, *R*, *0*) and (**N'**, *R'*,∅) are isomorphic, and therefore instantiate equal structures; either one is a realization of the ideal ω-sequence.[15]

Another categorical theory is second-order analysis, which fully characterizes the structure instantiated in the domain **R** of real numbers where two (ternary) relations, associated respectively with the usual operations of addition and multiplication, and two (0-ary) relations given by the constants 0 and 1 are defined. One can also define a (binary) order-relation in this domain with which to express, together with the other relations, in a second-order language, that **R** is a complete ordered field (completeness captures the intuitive continuity of **R**, it essentially says that **R** contains all the elements that it could contain, that there are no "gaps" in **R**). As second-order Dedekind-Peano arithmetic, second-order analysis is not logically complete either.

Mathematics is a formal science. This means that the objects of mathematics are empty forms considered in themselves, for their own sake, independently of the material content that actually fills or can potentially fill them. Examples of empty forms are numbers, geometrical forms, sets and structures. Contrarily, material sciences such as zoology, for example, are concerned with the abstract structure of *particular* materially determined domains of objects; for example, the animal realm in the case of zoology. This distinction, however, is much more superficial than usually acknowledged. Despite focusing on particular domains of materially determined entities, material theories, to the extent that they are articulated systems of *assertions*, can only express formal-structural properties of their domains; material sciences, to the extent that they are linguistically expressible, can only touch the formal surface of their privileged domains. Both mathematics and zoology, or any science for that matter, are *essentially* structural sciences, which differ only with respect to the *method* they most often rely on for carrying out structural investigations. Compared to material sciences, mathematics is less dependent on *particular* material instantiations. Material sciences, on the other hand, seldom, more often never have access to domains that are formally equivalent or similar to the domains where the structures they investigate are instantiated.

Suppose, for the sake of argumentation, that zoologists had developed a satisfactory description of the realm of lions to the point that they are confident that everything derivable in the theory is true of lions (the theory is adequate, and conversely, if they are lucky, everything true of lions is derivable in the theory; i.e. the theory is complete). The objects of lion theory are, supposedly, things like lions, lion organs, lion parts, lion habits, in short, anything that has a place in a theory of lions, whose

[15] Again, since the elements of **N'** are sets, it is *true* (in set theory) that $\varnothing \in \{\varnothing\}$, but this does *not* express a structural property of **N'**; it has no sense as a property of the ω-sequence.

goal is to disclose relevant properties of these things and relations among them. Suppose now that zoologists discover a domain of animals that are not lions, let us call them quasi-lions, where lion theory is, nonetheless, *reinterpretable salva veritate*. The domain of quasi-lions is simply another interpretation of lion theory. No matter how the interpretations differed materially, formally they would be the same theory. The zoologists would probably be puzzled with this discovery and feel that they missed something important in their description of lions, which would differentiate them from quasi-lions. Suppose, however, that in whatever way they further specify lion theory, quasi-lions still satisfy it. They would know *intuitively* that lions and quasi-lions are different types of animals, but the language they use could not capture this difference; it would not be capable of expressing any differentiating property. Suppose, however, that for some reason or other quasi-lions were much more approachable and easier to study than lions; zoologists would then be *methodologically* justified in turning their attention to *quasi-lions*, study their properties, and reinterpret whatever they found back as reasonable hypothesis or conjectures about *lions*. (*The heuristic use of mathematics in empirical science follows essentially along the same lines*.) Suppose now that zoologists, after much trying to find a formal, linguistically expressible distinction between lions and quasi-lions (in lion-language), were convinced that there is none and there could be none; that both domains, although differing materially, were formally the same domain, *insofar as the means of expression are concerned*. Then, they would be *logically* justified in carrying out their investigations about lions by investigating quasi-lions and vice-versa.

In short, any science is essentially formal-structural, the difference between material and formal sciences lies in that the former have privileged interpretations and the latter in general do not. Even when a mathematical science has a privileged domain (like arithmetic or physical geometry, for example), it is not particularly troubled by the substitution of this domain by formally identical ones (for example, space by a domain of triads of numbers). In mathematics, one can solve geometrical problems by solving algebraic equations; in physics, one can investigate empirical reality through mathematical models; in material sciences, these techniques are in general unavailable for the lack of conveniently formally similar contexts. Material scientists cannot in general turn their backs to the domains they privilege, by considering other domains, to investigate their common structures as mathematicians and physicists often do simply because domains formally similar to the domains of material sciences do not in general exist. Material scientists are bound to the concrete. In mathematics, however, as Hilbert said, we still do geometry when “points”, “lines” and “planes” are read as tables, chairs and beer mugs.

The notion of a structure *in abstracto* is a particularization of the Husserlian notion of formal domain. Let us recapitulate. For Husserl, a formal domain is, by definition, *the* domain of formal objects theoretically mastered by a formal theory.[16]

[16] In mathematics, formal theories are typically introduced thus: “let there be a domain of objects, no matter what they are, where certain unspecified relations are defined, no matter how, such that …”, the blank being filled with the formal axioms of the theory. If this theory is consistent and

Formal domains do not preexist and are dependent on their theories, which simultaneously posit and describe them (better, posit by describing). A formal domain is such that *there is no fact of the matter* about the *nature* of its elements and its structuring relations. Therefore, we cannot know, even *in principle*, *what* they are. All that one can know of a formal domain are the formal truths derivable in its formal theory. So, the formal properties of a formal domain *expressible in a determinate language* can only be exhaustively determined by a syntactically complete theory written in that language. If a formal theory is not syntactically complete, there are situations in the domain that are not objectively determinate. For Husserl, I assume, any theory extending an incomplete formal theory consistently would pose *another* formal domain, a *specification* of the domain of the narrower theory (for instance, *the* formal commutative group as a specification of *the* formal group). As is obvious, a formal domain is only the other face of its formal theory, that is, its noematic counterpart; and conversely, the formal theory is the noetic counterpart of its formal domain. One cannot live without the other. Consequently, one can see the pair formal theory/ formal domain as only different aspects of the same thing, one the noetic, another the noematic.

One can, however, separate the formal domain posited by a theory and the theory that posits it by presupposing[17] that the domain is objectively complete; i.e. that any assertion expressible in the language of the theory is either true or false in its formal domain, even if it is not decidable in the theory, and that an untrue assertion is necessarily false. This requires that the theory be such that no assertion is true in some interpretations of the theory and false in others; i.e. all interpretations must have the same formal properties. Categorical theories satisfy this requirement; since all its interpretations are isomorphic, they are a fortiori L-equivalent, L the language of the theory (i.e. if an assertion of the language of the theory is true in one interpretation, it is true in all interpretations). Therefore, one can consistently presuppose that the formal domain of a categorical theory is objectively complete. I just gave this domain another name, namely, the formal structure the (categorical) theory posits. *The objective completeness of structures posited in abstracto opens the possibility that their positing theories are not the only way of epistemically accessing the structures they posit.*

The epistemic independence of structures in abstracto from their positing theories is consistent with their ontological dependence. Structures, no matter if in concreto or in abstracto, are ontologically dependent entities; the existence of structures in concreto depends, on the noematic side, on the existence of materially determined relational domains and, on the noetic, on intentional acts such as abstraction and ideation. The existence of structures in abstracto, on their turn, depends on judging as an object-positing act (who judges, judges something *about* something;

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categorical, it posits a structure (i.e. it brings it to intentional existence) whose structuring relations, albeit materially indeterminate, must obey the formal stipulations established in the axioms.

[17] Keeping in mind that this presupposing is part of the *intentional constituting act*, not a hypothesis.

theories are systems of judgments). Analogously to numbers, structures are *ideal abstract* entities[18] depending ontologically on structured systems of entities given to us somehow, intuitively or conceptually,[19] or formal theories that intentionally posit them as their formal domains. Although structures in abstracto as correlates of purely formal consistent categorical theories[20]express themselves in particular languages, the languages of their positing theories, there are no privileged languages for structural descriptions, and certainly not first-order languages. Quine's predilection for elementary theories is essentially an empiricist prejudice. He believed that reality, empirical or mathematical, is constituted of objects only; properties, structures and like higher-order entities belonging to the *discourse about* reality, not reality itself. Needless to stress that phenomenology does not share these prejudices; it considers mathematics *as practiced* independently of metaphysical *idées reçues*, and certainly no mathematician would seriously consider restricting mathematics to first-order languages.

Identity As was seen, structures come in two varieties: (a) instantiated in intuitively or conceptually given structured systems; (b) intentionally posited by formal categorical theories. (The way they are meant, however, does not affect their ontological status; structures exist objectively as objectively complete entities.) One can easily derive from this characterization a criterion of identity for structures. Given two structures, one of the following is necessarily the case: (a) both are instantiated in

[18] "Abstract" simply means "non-independent". Husserl's explanation of the concepts of dependent existence and abstract objects can be found in the *Logical Investigations*, 2nd Investigation for abstraction, and 3th Investigation for the notion of ontological dependence.

[19] When a mathematician says, for instance, "let us consider (or imagine) the system of all motions in space with composition as the structuring operation", he is positing a structured system of entities. What in this way is intended is not each of the infinitely many motions individually, but the operational domain as a whole, with the intuitive properties associated with its ruling concept (for instance, motions are continuous rigid point transformations that can be reversed to cancel themselves). The intuition of a mathematical domain (and imagining is intuiting) does not require the intuition of all of its elements individually, but, in most cases, the intuition of a concept (in our case, rigid motions in space) under which these materially determined elements fall. To think of mathematical intuition as the intuition of objects and to construe the intuition of a mathematical manifold as the summation of the intuition of its elements individually betrays the wrong conception that mathematics is a science of objects. The *adequate* intuition of the objects of a mathematical domain individually is not a necessary condition for the intuition of the structure underlying this domain. We can furthermore *abstract* the structure of the system of motions in space (a particular group) by ignoring the nature of its entities or the nature of the operation structuring it and concentrating only on its *formal properties*. By considering generically domains of unspecified entities structured by unspecified binary operations having the properties of associativity, existence of compositionally neutral elements and inverses, we move to a higher level of theoretical interest, a general theory of abstract structures of a type (group theory in this case).

[20] Of course, I do not claim that any consistent theory has a *set-theoretical* model in the sense of model theory; I am not particularly interested in this type of models. From my phenomenological perspective, any consistent formal theory posits a *formal* domain (although there are – second-order – consistent theories that do not have set-theoretical models), whose existence the consistency of the theory is sufficient to grant, and that is accessible *only* through the theory. The fact that domains such as these do not have material instantiations does not mean that they do not exist.

concreto; (b) both are posited in abstracto; and (c) one is given in concreto, another posited in abstracto. Structures are then identical if (a) they are instantiated in isomorphic domains; (b) their positing theories are logically equivalent (both intend the same unique ideal structure) or (c) the system where one is instantiated is an interpretation of the theory positing the other.

As we have seen, structures, or formal domains in general, can be taken as structured systems of *formal* objects. The question, then, imposes itself: what are the identity conditions for formal objects (or *object-forms*) when they belong (*a*) to the same structure and (*b*) to different structures? Before approaching this question let us recall what formal objects are. They differ both ontologically and epistemologically from ordinary, materially determinate objects. Ontologically, for formal objects are abstract, and then dependent objects whose existence depends on the existence of other objects. Formal objects only exist in a system together with other formal objects, and systems of formal objects depend ontologically on either materially filled domains (to whose abstract structure they belong) or theories positing them as objective correlates. Epistemologically, for systems of formal objects are not directly accessible, they can only be accessed by accessing materially determinate structured systems or convenient formal theories, in particular the theories whose correlates they are. Since formal objects exist *only in a system*, their identity is determined contextually by how they relate with other objects of the system. If the system of relations in a structured system of formal objects changes, or new objects are added to the system, or both, so that the objects of the system establish new ways of relating with one another, *all* the objects of the system become different objects. For example, the number 2 in the structure of *real* numbers is *not* the same number 2 in the structure of *natural* numbers (although one can *interpret* the system of natural numbers in the system of real numbers so the real number 2 can play the role of the natural number 2). The real number 2 can under certain circumstances stand for the natural number 2, but it is not identical with it since it has formal properties that the natural 2 does not have (for example, the real 2 has a square root, the natural 2 does not). In short, *formal objects of different structures are different objects*.

Now, let L be the structural language of the structure S and t, t' are terms of L. If S is instantiated in D, then, of course, the identity $t = t'$ is true in S if, and only if, it is true in D. The objects in D for which t and t' stand are, in fact, the same object, regardless of how they are denoted. This definition is independent of the particular instantiation of the structure for truths in D are also true in any domain isomorphic to D. Suppose now that S is posited in abstracto by T in L. If T has an interpretation D, then S is instantiated in D, and $t = t'$ is true in S iff $t = t'$ is true in D (and then all interpretations of T), *even if T does not prove $t = t'$*. Now, if T has no interpretation, despite being logically consistent, the truth or falsity of $t = t'$ in S can be established if T proves either $t = t'$ or $t \neq t'$. However, if T is logically incomplete and does not prove either $t = t'$ or $t \neq t'$, I still admit (by the objective completeness of the structure posited by T) that one, and only one of these two possibilities is the case. The determination of which is actually the case may depend on extensions of T or indirect means (such as interpretations of the structure posited by T in other structures – see below). In short, *identities are always objectively determined in any* *structure*. For example, SS0 and SSSS0 denote different formal objects in the ω-structure posited by second-order arithmetic because this is true in all interpretations of this theory (in this particular case, arithmetic proves that SS0 ≠ SSS0 and, consequently, SS0 $=_{df}$ 2 ≠ 3 $=_{df}$ SSS0 in all ω-structures).

Interpretations It follows from previous considerations that formal objects from different structures are always different, no matter how the structures are posited. However, one can *interpret* a structure *S* in another structure *S′* in many different ways. This is a particularly efficient methodological strategy, for besides allowing structural descriptions to be improved it may open new logical possibilities for structural investigations. Before considering the methodological advantages of interpreting structures into other, usually richer structures, let me define the notion case by case:

(a) *S* and *S'* are both instantiated. Let *S* be instantiated in (D, R_i) and *S'* in (D', R'_j), whose structural language is *L'*. Suppose that there is a subset *F* of *D'* where we can define, in *L'*, relations S_i such that (F, S_i) is *isomorphic* to (D, R_i). (If functions are involved, we suppose that they are closed in *F*.) In this case, I say that (D, R_i) is *interpretable* in (D', R'_j) by (F, S_i).

(b) *S* and *S'* are both posited in abstracto. Let *T* (in the language *L*) and *T'* (in the language *L'*) be the positing theories of respectively *S* and *S'*. An *interpretation* of *L* in *T'* is given by a unary predicate *U* of *L'*, which *T'* proves to be non-empty, together with a correspondence that associates to each *n*-ary predicate symbol R_i of *L* an *n*-ary predicate symbol R_i' *definable* in *L'*. (If *L* has *n*-ary function symbols f_i, there should correspond to each of them an *n*-ary function symbol f_i' definable in *L'* that *T'* proves to be closed in *U*.) Now, given a formula φ of *L*, one defines a formula φ' of *L'* in the following way: replace each non-logical symbol *u* of φ by its correspondent symbol *u'* and restrict all variables, bound and free, of φ' to *U*. I say *S* is *interpretable* in *S′* if *T* is *interpretable* in *T'*, i.e. if *T'* proves φ' whenever φ is an *axiom* of *T*. This property is extendable to the *theorems* of *T*.[21]

(c) *S* is instantiated and *S'* is posited in abstracto. Suppose that *S* is instantiated in (D, R_i) and *S'* is posited by *T'* (in the language *L'*). I say that *S* is *interpretable* (or *definable*) in *S'* if the theory *T'* is as in (*b*) above and for each sentence φ of the language of (D, R_i) there is a sentence φ' as in (*b*) such that *T'* proves φ' for each φ that is *true* in (D, R_i). The predicate symbol *U* plays the role of *D* and to every sentence true in *D* there corresponds a sentence in *L'* "referring to *U*" provable in *T'*. This definition is equivalent to saying that the *complete theory* of (D, R_i), i.e. the set of all sentences of the structural language of (D, R_i) true in (D, R_i) is interpretable in *T'* according to (*b*) above.

(d) *S* is posited in abstracto by the theory *T* in the language *L* with relations symbols R_i and *S'* is instantiated in (D', R_j'), whose structural language is *L'*. *S* is *interpretable* in *S'* if there is *U* contained in *D'* and one can define in *L'* relations S_i in *U* (functions should be closed in *U*) such that *T* is satisfied in (U, S_i).

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[21] See Shoenfield 1967, pp. 61–65. This definition could be generalized by dropping the requirement that *T'* is categorical, i.e. that it is itself a structure-positing theory.

Epistemology Let us turn now to the question of how structures can be epistemically accessed. The "epistemic access" that plagues Platonist accounts of mathematics has a straightforward solution here: structures in concreto are primarily accessed by accessing the structured systems where they are instantiated and structures in abstracto via the theories positing them. Structures posited in abstracto are given with a privileged means of access, their positing theories; descriptions (theories) of structures in concreto, on their turn, are obtainable by directly accessing the structured systems where they are instantiated. But these are not the only ways structures can be accessed.

There are essentially two ways of obtaining structural descriptions of structures in concreto. In a few cases, when their domains are finite and reasonably small, structured systems can be intuitively accessed extensionally, since their elements and structuring relations are displayed before our eyes. One just look and see. Of course, what one sees depends largely on the perspective one takes. The same domain, with the same signature, may support different structural descriptions, depending on which properties one perceives the structuring relations to have. Structural descriptions are ways of seeing structured systems. As before, I suppose that the only non-logical symbols of structural languages (i.e. the languages chosen to carry out structural descriptions) are those denoting the structuring relations. In cases structured systems can be intuitively accessed, either extensionally or, as we will see below, conceptually, structural descriptions are intuitively founded. In cases of theories posited in abstracto, there are two possibilities, either the positing theory, which is also a structural description, is a free creation, or it strives to describe a concept (in whose extension the structure it posits is instantiated). The positing theory, in this case, describes the structure it posits through the concept that rules over the system where this structure is instantiated. Interpreted structural descriptions can always be formally abstracted into purely formal descriptions of abstract structures.

If structures in concreto are instantiated in large finite and infinite domains they are no longer *extensionally* accessible. In these cases, we must turn to the *concepts* characterizing them, that is, we must consider them *intensionally*. It may happen, as with the natural numbers, that domains of objects come with a generation-and-limitation principle that allows one to characterize them. For example, one can "*see*" a domain of entities *as* a ω-sequence (as opposed to, for instance, a very large finite sequence or a $(\omega + \omega)$-sequence) from *actually* seeing a few of these elements ordered in a finite linear sequence with an initial element by *interpreting* this intuitively given segment as an *initial* segment of a sequence generated by two principles. One, a principle of *generation*: *any* object has an *immediate* successor and *only one* object is not a successor, another, a principle of *limitation*: the *only* principle of generation is the production of successors (one cannot take "limits", for example). Directly "perceiving" large or infinitary structured systems based on the actual perception of substructures of them depends on preconceptions (*conceptual* predeterminations) about *what* one is perceiving. The intuition of the *concept* that presides over the relational system tells us what we are looking at, thus providing a structural description of the domain. In this case, seeing requires thinking, and thinking is expressed in saying.

Although structural descriptions provide knowledge about the structures they characterize and describe (characterize by describing), they may not be able to tell everything that there is to know about them. We must cope with the fact that structural descriptions may be logically incomplete. *Despite* this, one presupposes (idealizing presuppositions, remember, *always* have a *transcendental* character) that any meaningful assertion expressible in the structural language is *in itself* decided, even if it is not *actually* decidable in the context of a particular structural description. The categoricity of structure-positing theories does not imply their logical completeness. Categoricity and completeness are, in general, relatively independent concepts; there are categorical incomplete theories, such as second-order arithmetic, and complete non-categorical theories, such as any complete first-order theory of an *infinite* mathematical domain. However, categoricity (and even categoricity in a power, for powers at least as big as the cardinality of the language of the theory) implies completeness for first-order theories that have no finite interpretation. In any case, categoricity always implies *semantic* completeness (a theory is *semantically complete* when all its interpretations satisfy the same sentences of the language of the theory; i.e. no two interpretations of the theory can be distinguished within the expressive power of its language), but may not imply *syntactic* completeness (any meaningful assertion expressible in the language is decidable in the theory).

Extracting the logical consequences of structural descriptions (structural theories), be they structure-positing theories or theories obtained by intuitively accessing structured systems, is the obvious way of carrying out structural investigations. Suppose that φ is a particular assertion of a structural language L and that T is a structural description in L. The most obvious way of showing that φ is true in the structure that T describes is to prove φ in T. But remember, classical logic can be used, since by hypothesis for any sentence φ of L, either φ or not-φ is a structural property of the structure T describes. So, T proves φ if T proves not-(not-φ). The presupposition that structures are objectively complete entities logically *justifies* the use of the principle of double negation.

Suppose now that there is, still in L, a theory *T' extending T*; i.e. theorems of T are also theorems of T', but *not necessarily the converse* (if T' is not a *conservative* extension). Since all possible interpretations of T' are also interpretations of T, T' is also categorical and posits the same structure as T. Therefore, if T' proves φ, this sentence is true in the structure posited by T, even if it is *not* proven in T. So, logically incomplete structure-positing theories can be *arbitrarily* extended consistently. Suppose, then, that neither φ nor not-φ are proven in T; one can extend T in two possible ways $T \cup \{\varphi\}$ and $T \cup \{\text{not-}\varphi\}$; if both these theories were consistent, we would have an inconsistency, because both φ and not-φ would both be true in the essentially unique interpretation of T. Therefore, only one of them is a consistent extension of T. *If* we had an independent way of deciding which, we would have a way independent of T of showing that certain assertions *independent of T* are true in the structure posited by T. In any case, one can derive from this a *heuristic*, although not strictly logically justified *methodological strategy* of structural investigation. Suppose that we are *pragmatically confident* of the consistency of some extension *T'* of *T*; therefore, we are *pragmatically justified* in taking theorems of *T'* as true in the structure *T* posits, or at least take these theorems as *conjectures* to be considered seriously.[22]

Now, from our definition of structural identity, in order to decide whether φ is true or false in the structure instantiated in (D, R_i), for example, it suffices to verify whether this is the case in *any* domain (D', R_i') isomorphic to (D, R_i). This may be done by means of structural descriptions of (D', R_i'). An isomorphic copy may offer a *more convenient* context of justification. One example is so obvious as to go almost unnoticed: suppose, for instance, one wants to verify whether 245,612 + 4,6 57,312 = 4,902,924. One might try to operate directly with the numbers these signs denote, i.e. collections of units, but one would soon realize the difficulty of the enterprise. Fortunately, one can perform this operation indirectly by operating *symbolically* with numerals by means of the usual algorithms. The reason why one can be sure that the result obtained is *correct* if the algorithms were *correctly applied* is that the domain of numbers proper with intuitive numerical operations and that of numerals with symbolic operations are *isomorphic*. Both domains instantiate the same structure and since the identity under exam is purely formal (it does not matter *what* numbers are, only how they relate to each other operationally), one can verify if it is true in one domain by verifying whether it is true in a formally equivalent domain. One finds innumerably many analogous examples in mathematics of answering questions about a domain by investigating formally equivalent domains.

Structures in abstracto allow even more flexibility. One may access them either through their theories, theories logically equivalent to them or, as just seen, consistent extensions of their positing theories. But also by directly accessing any of its interpretations. Any assertion of the language of the theory true in one interpretation of a structure-positing theory is true in all of them (for they are all isomorphic to each other). If the theory is expressible in a first-order language, this implies, as already observed, that this assertion is derivable in the theory. Even if the structure-positing theory is not logically complete, it is capable of singling out the structure it posits, and we can directly access this structure independently of the positing theory to decide what the theory cannot to. For example, Dedekind-Peano second-order arithmetic characterizes the ω-structure, but it is not logically complete; given any assertion φ of the language, one cannot expect always to be able to decide whether φ expresses a structural property of the structure or not. However, one presupposes that either φ or not-φ is true in it. One may then verify this directly in some (any) particular instantiation of the ω-structure. If one verifies that, for instance, φ is true in $(\mathbf{N}, 0, S)$, then φ is a structural property of the ω-structure even if it is not decidable in Dedekind-Peano arithmetic. If the theory were not categorical, this would

[22] This strategy is widely used in mathematics and mathematical physics. In this last case, it sometimes pays off to take logical consequences in purely mathematical extensions of mathematical models of empirical reality as *maybe* expressing facts about reality (even though mathematical theories of reality may not be categorical). Here, logic gives place to heuristics.

not be possible, for there might exist interpretations of the theory where φ were true and interpretations where it were false.

One can also access structures via interpretations. Let us see how:

(*a*) By interpreting structures in concreto in structures in concreto. Interpreting *S*, instantiated in (D, R_i), into *S'*, instantiated in (D', R'_j), opens new possibilities for accessing *S* by means of the system (F, S_i) isomorphic to (D, R_i). This can be advantageous for the system (D', R'_j) may be more easily accessible than (D, R_i) (if, for instance, the concept ruling over the second system is intuitively more accessible than that ruling over the first). Another possibility is that the structural description *T'* of (D', R'_j) is a richer theory than that of (D, R_i). Suppose, for example, that *T'* proves a sentence φ and that the restriction of this sentence to (F, S_i) is true (see the different notions of interpretation above). Given the isomorphism between (D, R_i) and (F, S_i), one has managed to derive a truth about *S* by investigating *S′*. For example, one can interpret the ω-structure in the structure **R** of real numbers by taking *F* as the closure of {0} by *S* and defining $S(x) = x + 1$ (there are many other possibilities). If, for example, φ is a sentence of the language of the theory of real numbers whose only non-logical symbols are 0, 1 and + and whose variables are restricted to *F* (a copy of the natural numbers in the real numbers) that is true in the structure of real numbers is true in the ω-structure as well, *even if the truth of* φ *depends on properties of the structure of the real numbers as a whole* (it may, for instance, have been proved by appealing to the axiom of completeness of the system of real numbers).

Let me give another simple illustrative example. Consider the domain (**N**, 0, *S*), the set of natural numbers with zero and the usual successor operation, instantiating the ω-sequence, and the domain (**Z**, 0, *S*, *A*), the set of integers with zero, the successor and the antecessor functions, instantiating the Z-sequence. Let second-order Dedekind-Peano arithmetic (*T*) describe the ω-structure and second-order arithmetic of the integers (*T'*) describe the Z-structure. In details: let *L*, the language of *T*, have the constant symbol 0 and the unary function symbol *S*. The language *L'* of *T'* have both these symbols and the unary function symbol *A*. *T'* states that *S* and *A* are bijective, that $SAx = ASx = x$, and that any subset of the universe that contains 0 and is closed under *S* and *A* is the whole universe. One can define addition in (**N**, 0, *S*) recursively as usual and prove in *T*, by induction, the cancellation law: $a + c = b + c \rightarrow a = b$ (this requires that one has already proven inductively the associative law $(a + b) + c = a + (b + c)$). One can define an associative operation of addition in (**Z**, 0, *S*, *A*) extending the addition in **N** (i.e. the restriction of the addition of integers to the natural numbers is the addition of natural numbers). Now, one can prove in *T'* the cancellation law thus: $a + c = b + c \rightarrow (a + c) + (-c) = (b + c) + (-c) \rightarrow a + (c + (-c)) = b + (c + (-c)) \rightarrow a + 0 = b + 0 \rightarrow a = b$. One has then proved the sentence $a + c = b + c \rightarrow a = b$ for all *a*, *b*, and *c* in **Z**. In particular, $a + c = b + c \rightarrow a = b$ is true for all *a*, *b*, and *c* in **N**. The richer theory *T'* has provided a different proof of the cancelation law in **N**. This strategy is particularly interesting if the larger theory is capable of providing proofs that the narrower theory *cannot*, or *simpler* proofs of those that it can.

Strategies can be iterated. Let **R** be the field of real numbers. First, let us devoid **R** of its material meaning; that is, let us consider real numbers as mere object-forms (formal objects) with which one can operate formally, operations having their usual formal properties. One can now extend this symbolic operational domain by adding to it an extra formal object, denoted by *i*, and define formal operations in the extended domain as if they were performed on real numbers but such that $i^2 = -1$.[23] The symbol *i* does *not* have any material, only formal meaning. Let us denote the extended formal domain by **R**(*i*), the field of complex "numbers". Let us consider now the domain **R**x**R** whose elements are pairs of real numbers with operations of addition and multiplication defined as usual. It is a trivial matter to show that **R**(*i*) is isomorphic to **R**x**R**. That is, **R**(*i*) and **R**x**R** instantiate the same (ideal) structure. Now, both **R** x {0} and {0} x **R** are subdomains of **R**x**R** and both are isomorphic to **R**; therefore, both instantiate the same structure as **R**. Suppose now that we can, in some theory of **R**(*i*) – for example, complex analysis –, prove some φ in the language of **R**(*i*). This counts as a proof of the truth in **R**x**R** of some φ' in the language of **R**x**R** that corresponds to φ under the isomorphism. The truth of φ' in **R**x**R** can yield the truth in **R** x {0} of some φ_1 in the same language (or of φ_1 in **R** x {0} *and* φ_2 in {0} x **R**). Hence, the translation of φ_1 (or φ_1*and* φ_2) back into the language of **R**, is (are) true in **R**. In short, we managed to prove something (or more than one thing) about **R**, and thus the structure it instantiates, by interpreting **R** in a larger structure and then carrying our investigation in an isomorphic copy of this latter structure.[24] The mathematician carries these transpositions without even taking notice of what he is doing, changing constantly the *material* meaning of the terms and concepts he uses. The philosopher cannot be so nonchalant; he must explain why the method works, justifying it logically and epistemologically.

(*b*) By interpreting structures in abstracto in structures in abstracto. Let *T* be the theory, in the language *L* with relational symbols R_i, of *S* and *T'*, in a language *L'* containing relational symbols R_j' and a unary predicate *U*, the theory of *S′*. Suppose that *T* is interpretable in *T'* is as in (*b*) above. First, observe that if *T'* had an interpretation (D', R'_j), *T* would have one too. Indeed, consider the interpretation of *U* in *D'*; by assumption, all sentences φ' corresponding to axioms of *T* would be true in (U, R_i') (R_i' as in (*b*)), which would then be an interpretation of *T*.

Suppose now that *T'* proves a sentence φ' of *L'*, the interpretation of φ. Is φ provable in *T*? Suppose that it is not (this would in particular be the case if φ were *false* in *S*). If *T* were *complete*, it would then prove not-φ, but since all theorems of *T* are interpretable as theorems of *T'*, (not-φ)' = not-φ' would be provable in *T'*, a contradiction. Therefore, φ must be provable in *T*, and then true of *S*. If *T'* were a richer or more resourceful theory, it might offer a better context for the structural investiga-

[23] In fact, the new formal object is originally only a symbol; it acquires "formal objecthood" only when the way it relates operationally with the other formal objects of the domain where it is introduced is determined. With the introduction of this new object the entire domain, and the objects in it, change, they become other objects and another domain.

[24] "The true value of such numbers [*complex numbers*] lies in the fact that they enable us to form connections between entirely different parts of mathematics." (Waismann 2003).

tion of S than the original positing theory T. This is, of course, a methodological advantage. The completeness of T implies that T' is also complete, *but only with respect to the set of interpretations in L' of sentences of* L, not with respect to *all* sentences of L'. This is precisely Husserl's notion of relative definiteness.[25]

Now, what if T is *not* complete? In this case one could not infer the *provability* of φ in T. But we would still have that not-φ is *not*, short of contradiction, provable in T. In general, if T' proves φ', then either T proves φ or φ is undecidable in T. Now, what about the *truth* of φ in S? In any possible interpretation of T', φ' would be *true* in a subdomain of this interpretation that is an interpretation of T; therefore, φ, which is the equivalent to φ' in S would be true in S. Suppose, for instance, that T' has a realization (or interpretation) (D', R'_j); as noticed, T has a realization (U, R_i'), U in D'. If T' proves φ', φ' is true in (U, R_i'). But since the structural property of S that φ expresses is identical to the property of (U, R_i') expressed by φ', the fact that of φ' is true in (U, R_i') implies that φ is true in S. In short, if T' is realizable, proving facts in T' "referring to its subdomain U" is tantamount to proving facts about S, independently of its positing theory T. T' functions in this case as an improvement of T. We can, then, in general, take φ to be *true* in S whenever φ' is *provable* in *T'(notice that the categoricity of T' plays no role here).*

(*c*) By interpreting structures in concreto in structures in abstracto. Suppose that S is instantiated in (D, R_i) and S' is posited by T'. In this case, the complete theory T of (D, R_i) is interpretable in T' and we are back to the previous case when T is complete.

Finally, (*d*) by interpreting structures in abstracto into structures in concreto. Let S posited by T in the language L whose only non-logical symbols are relation symbols R_i and S' be instantiated in (D', R_j'); suppose also that S is interpretable in S'. This, as defined, means that there are relations S_i definable in the structural language of (D', R_j') and a subset U of D' such that (U, S_i) is an interpretation of T. If φ is a sentence of L true in (U, S_i) and T is complete, then φ is provable in T. However, even if φ is *not provable* in T, it is *true* in S, since it is true in a realization of its positing theory. The methodological advantage of this strategy is that the direct (intuitive) inspection of (D', R_j') or its structural theory may offer better chances for showing that φ is true in (U, S_i) – and then in S – than trying to show that it is derivable in T.

Interpreting structures into one another is a powerful instrument of accessing and investigating structures. A complete taxonomy of such strategies would certainly be a great contribution to the study of mathematical methodology. Here, I only considered a few simple pure cases; there are certainly other ways structures can be interpreted into other structures and, of course, interpretations can be iterated without limit. We could also consider the case of non-categorical theories, positing families of structures, each one instantiated in one of the many formally different, actual or possible interpretations of the theory. The theory is in this case incapable of singling out a structure and can be considered as a partial positing. I leave these investigations for a future work.

[25] da Silva 2016a.

Other Structuralisms Michael Resnik characterizes structuralism thus[26]:

> In mathematics, I claim, we do not have objects with an "internal" composition arranged in structures, we have only structures. The objects of mathematics, that is, the entities that our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or feature outside of a structure.

I find almost every line in this claim objectionable. Resnik downplays the role of mathematical objects and interpreted theories in mathematics, as if they were mere historic accidents, pre-mathematical at best. Mathematics proper, he believes, is a strictly formal science of empty abstract forms (or "patterns"). More often than not, however, "patterns" (structures) are instantiated in structured systems of "things". Structures are abstract, ontologically dependent entities, which only exist if other things exist. If not mathematical objects, then structure-positing formal theories. Although, as is clear, mathematics is only concerned with abstract structures, these structures require a support, which is usually provided by the usual mathematical objects. These objects exist, if only as bearers of structures. Moreover, mathematical objects or, more often, the concepts under which these objects fall, also serve as means of accessing the structures they support.

But Resnik admits only systems of free-floating *formal* objects as bearers of structures. Since these objects have indeed no material determination (and have never had one), we are deprived of a relevant way of accessing the structures they instantiate. This is to me an unjustified and crippling prejudice. As I see the matter, structures exist only if actually instantiated or in principle potentially instantiable in structured systems of materially determined objects of no matter which ontological category, even physical objects, but usually mathematical objects proper, such as numbers, spatial forms, or sets. Empty formal objects (object-forms) are *themselves* dependent and exist only as either forms of materially determined things or referents of terms of (formal) languages. When structures are posited by purely formal theories, its objects are indeed devoid of "internal" structure (or, in my terms, a material content) and "have no identity or feature outside of a structure". But not when structures are instantiated or posited by formally abstracted material theories; the objects of such structures are formal only upon abstraction and their material content can be relevant for accessing their formal properties.

Contrary to Resnik, I take as a self-evident truth that there are mathematical objects with an "internal" structure. They may be *irrelevant* from the perspective of the structured system where they belong, but not non-existent. However, non-structural properties of a system, that is, properties of the objects of the system *that are not expressible in the structural language* do in general play a relevant role as means of *access* to the structure of the system and its structural properties, those precisely that are expressible in the structural language. One example suffices. To obtain a convenient structural description of the system of natural numbers, one must know *what* a natural number is, namely, ideal abstract forms of *a certain type*, which one can operate upon in certain ways, etc., even if this knowledge is not

[26] Resnik 1981, p. 530.

structural. Of course, there is a purely formal theory, i.e. a theory of internally unstructured formal objects that I called “numbers”, positing the ω-sequence, the structure of the domain of numbers proper, and it is true that anything true about numbers is also true about “numbers” and vice-versa. But to claim that “numbers”, not numbers, is all that mathematics is about is historically false and more a source of problems than solutions. Why are we interested in “numbers”? Why do “numbers” have the formal properties they have, not others? The fact is that “numbers” and the formal theory of “numbers” are only formal abstractions of numbers proper and their theory, whose axioms express intuitive properties of the *concept* of number. This is a fact of the history and the methodology of mathematics, which we must cope with, not something one could pull under the rug of philosophical analyses only because one does not like the notions of abstract, in particular mathematical object or conceptual intuition.[27] The philosophy of mathematics must take mathematics at face value, not reinterpret it to satisfy philosophical parti-pris.

Resnik’s claim that mathematical objects do not have “intrinsic” features has an undesirable consequence, even if it is indeed true that an interpreted mathematical theory can be formally abstracted and all that we can know about the domain it describes, *within the theoretical context in question*, can be obtained by investigating by formal means its formal abstract. However, since this is not only true of mathematical objects, but of *any objects whatsoever*, Resnik should, out of consistency, take all objects, even empirical objects, from a strictly scientific perspective, as empty things that only exist in relation with other objects in a system. The fact that formal abstracts of empirical theories are not in general very useful in practice is a logical-methodological accident, not a relevant ontological fact.

A few other questions can be raised concerning Resnik’s structures. Are they independently existing entities or do they only exist as correlates of their theories? If structures exist in themselves, i.e. independently, is there some sort of formal intuition that gives us access to them? But what is this if not formal abstraction from instantiated structures? If, however, structures are exclusively posited theoretically by purely formal theories, what logical properties must these theories have? What if a structure-positing theory is logically incomplete, would this mean that the structure it posits is also incomplete? If yes, are incomplete structures capable of being completed, how? Are sentences undecidable in a structure-positing theory meaningless from the perspective of the structure it posits? What would count as a meaningful, epistemically (not only pragmatically) justified extension of an incomplete structure-positing theory? If one extends an incomplete structure-positing theory by adding an extra independent axiom to it, is the new theory still the theory of the old structure?

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[27] Commenting on Resnik’s characterization of structuralism, C. Parsons (1990) says that structuralism “is most persuasive [...] in the case of pure mathematical objects such as sets and numbers […]. In these cases, we look in vain for anything else to identify them beyond basic relations of the structure to which they belong […]”. What about the *concept* of number as quantitative form? Numbers proper are *something*; they fall under a well-determined (and intuitable) concept; telling *how* numbers relate to one another is *not* telling *what* numbers are.

Structuralism along Resnik's lines has known many different formulations – if-thenist structuralism, plural-reference structuralism, modal structuralism – and variants. And, of course, there are other brands of structuralism, some even admitting the existence of at least some mathematical objects (usually sets). But they all share in a way or another the same naturalist or empiricist parti-pris that I have already pointed out. There is no reason to fear abstract objects, the usual objects of mathematical theories, or formal abstract structures themselves. There is no need for "eliminative" strategies in the structural approach to mathematics. Resnik's and other variants of structuralism raise more problems than they solve, if adjusting mathematics to preconceived ontological and epistemological doctrines is solving a problem, besides being unfaithful to mathematical practice and history.

My approach to structuralism has an altogether different motivation, which has nothing to do with qualms about the existence of mathematical objects. Once empiricist misconceptions about existence are put aside and the true nature of existence in mathematics recognized (as essentially intentional), one can easily overcome the empiricist discomfort with the existence of abstract objects. Structuralism, as I see it, imposes itself as the *correct* philosophy of mathematics for different reasons – not ontological, but logical-methodological. Mathematical theories, as in fact all linguistically expressible theories, are essentially formal; that is, they are by themselves incapable of fixing privileged interpretations, they cannot grasp material, only formal meaning. And who says form says structure. However, unlike material sciences, which are exclusively concerned (mostly for practical and methodological reasons) with structures of particular materially filled systems, mathematics is in principle concerned with *all* structures (thus fulfilling its role as a priori formal ontology). Mathematics is the science of idealized abstract structures in general. Now, even a superficial analysis of mathematical methodology shows that mathematics often investigates particular objective domains by "reinterpreting" them as domains of a different material nature. If theories could express the material, not only the formal meaning attached to their domains, this would be an utterly unjustified method of investigation: why can one investigate a domain of objects by examining others that have nothing to do materially with it? So, the *logical* fact that theories are interpretable in different domains salva veritate and the *methodological* fact that mathematical methodology is only compatible with a form of structuralism imposes this view as the correct philosophy of mathematics. As I see it, structuralism imposes itself *from within*, not from without, as a way of accommodating preconceived ideas about existence with mathematical practice.

Structuralism is also, I believe, the correct perspective for explaining the applicability of mathematics, the subject of our next chapter.

Chapter 8
Science

The traditional "schools" in the philosophy of mathematics have paid very little attention to a problem to which they should have been more attentive, the applicability of mathematics. Mathematics is extremely useful, sometimes indispensable, in daily-life, the empirical science and, not less significantly, mathematics itself. The possibility of reinterpreting mathematical theories of particular domains as theories of *other* domains, which alone accounts for the applicability of mathematics in empirical science, also accounts for its self-applicability. The fact that mathematics has anything to say about nature is no more, or less, surprising than the fact that number theory, concerned as it is with numbers, has anything to say about the structure of space. Recently, the problems of the applicability of mathematics in empirical sciences has attracted a lot of attention from philosophers of science and mathematicians, but, surprisingly, not as much attention has been paid to the correlate problems of the applicability of mathematics to itself. However, both have the same explanation.

The utility of mathematics in daily life activities such as counting and measuring is less puzzling, for mathematics originated in such practices. Frege had a straightforward explanation why number theory, for example, can be applied. Since, for him, numbers are attached to concepts, empirical or scientific, number theory is applied by the intermediation of concepts. Two apples on the table plus 3 oranges on the table makes 5 fruits on the table because if 2 is the number of a concept *A* ("apples on the table") and 3 is the number of a concept *B* ("oranges on the table"), then 5 is the number of the concept *A* (disjoint) or *B* ("apples or oranges on the table" = "fruits on the table, if no other fruit is on the table"), for 2 + 3 = 5 is a logical truth. As we have seen, Husserl has essentially the same explanation. Since $(x)(y)(z)\ ((2(x) \wedge 3(y) \wedge z = x \cup y) \rightarrow 5(z))$, i.e. for all collections *x*, *y* and *z*, if *x* has the quantitative form 2, *y* the form 3 and *z* is the disjoint union of *x* and *y*, then *z* has the quantitative form 5, the fact that 2 apples plus 3 oranges makes 5 fruits follows by instantiation, for 2 + 3 = 5 is a conceptual truth of numerical forms. The difference is that where Frege has concepts (with sharp boundaries, as he says) Husserl has quantitative forms. Whereas, for Frege, the extension of a concept is quantitatively

J.J. da Silva (ed.), *Mathematics and Its Applications*, Synthese Library 385,
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determined by the number attached to it; for Husserl, a quantitatively determined collection is numerically determined by its quantitative form, an abstract aspect of the collection itself. The fact that, for Husserl, numbers are *forms*, allows this explanation to be extended to *all* forms, spatial forms, sets, and also structures. Essentially, mathematics is so widely applicable because mathematical forms (numbers, sets, structures …) are in principle capable of being materially filled by any content whatsoever.

Although the interplay of structures in mathematics poses no serious problems, it seems that the applicability of mathematics in the empirical sciences requires that nature be *mathematically* structured, and this is puzzling. Why should it? Moreover, how can it be that the mathematical structures of nature so often coincide with mathematical structures that have not been invented (or "discovered", if you prefer) with any particular regard for nature? Is nature "tailor-cut" for mathematics? Why should it? Are nature and human mathematical imagination in pre-established harmony? How can *that* be explained? And more puzzling still, how can mathematics have, as it seems to be the case, a *heuristic* role in science, i.e. how can mathematics be used for *discovering* how nature works? Some mystical "explanations" have been suggested – if not explicitly advocated – that are utterly unacceptable from a scientific perspective. The applicability of mathematics to science *must* have a straightforward explanation. I believe that such an explanation has eluded the best philosophical (and scientific) minds because, again, of the inadequate empiricist perspective from where they consider the questions and advance the answers. The puzzlement itself indicates that the perspective is wrong. The empiricist presupposition that the empirical world is simply "out there", mathematically structured in itself, only waiting to be scientifically unveiled is simply false. "Empirical nature" is an *intentional construct* and it is mathematical because we have, in a sense, *made* it so. Once this is understood, the "miracle" of the applicability – even heuristic applicability – of mathematics in science vanishes. There is no better place to begin my account of the matter than with Husserl himself.

Husserl's "Crisis" Husserl wrote a few introductions to phenomenology, where by "introduction" I do not mean a presentation of phenomenology, its concepts, ideas, methods and goals, to the non-initiated, but a defense of phenomenology as a first (and exact) philosophy. Whereas *Ideas I* and other expository books have this goal, others like *Crisis* and *Formal and Transcendental Logic* (*FTL*) argue for the view that a transcendental phenomenological perspective is *necessary* in philosophy. In *FTL*, Husserl takes us from formal logic, whose scope and task he delimitates and clarifies, to transcendental logic, presented as a necessary philosophical complement to formal logic. In Chap. 3 of this book I spelled out ideas put forward in *FTL* but not fully developed therein, showing how logical principles can be carried back to transcendental consciousness, from where meaning emanates, and thus justified if not proven, which would be impossible since principles cannot be reduced to anything more fundamental than themselves.

In *Crisis*, the point of depart is science, and the task presented to phenomenology is that of overcoming what Husserl believed to be a "crisis" in modern culture, science in particular. To put it in the simplest possible terms, the "crisis" Husserl detected had to do with loss of meaning and personal responsibility. Or still, the rupture of the ties between man's actions and thoughts and the sources where they acquire meaning. The result, he claims, is a form of "alienation", of "going" through the moves" without a clear understanding of what the moves mean. By looking backwards to the history of European culture, philosophy in particular, Husserl detects in science and mathematics the origins of this process of "alienation", more precisely in the change geometry underwent at the beginning of modern times and the concomitant creation of the modern mathematical science of empirical nature, which Husserl calls "Galilean" science. In analytic geometry, geometry detaches itself from the human perception of space, becoming a "game with symbols"; on being mathematized, science substitutes experienceable empirical reality by mathematical surrogates of it. In both cases, that which Husserl believes to be the only source of meaning for both mathematics and science, namely, human experience, drops off the picture altogether or is dramatically downgraded.

Husserl does not criticize the deflection of mathematics and science into the non-experienceable, the non-intuitive, the purely symbolic, and the formal per se; he agrees this has proven to be a very efficient and successful *methodology*. The crisis Husserl detected was not one *in* science, but *of* science, or rather culture in general. He believes that science and mathematics are *right* in adopting symbolic methods of reasoning without a foothold in the experienceable, but he thinks this should be done *consciously*, with complete awareness of what is being done, the range of validity of purely symbolic methods of reasoning in science and mathematics, and how they can be justified. And here is where transcendental phenomenology comes in. By uncovering the many layers of intentional constitution of the *object* of science, phenomenology can clarify, delimit, and ultimately justify the symbolic, intuitively empty methods of modern "Galilean" science, but also of modern mathematics.

The only word in the last sentence above with which I do not agree is "delimit". Husserl believed that there were justified and *unjustified* symbolic methods in science and mathematics. Symbolic methods of reasoning, he thinks, are justified only if they are *essentially unnecessary*; symbolization can only have a meaning as a *tool*. Resorting to symbols as surrogating devices is fine, but not as non-eliminable substitutes of intuitions. Scientists such as Hermann Weyl, who were influenced by Husserl as they themselves admitted, have also felt uncomfortable with the consequences of such a view for the methodology of mathematics and science. The fact is that science cannot afford to give up essentially symbolic means of reasoning. This is why Weyl substituted Husserl's intuition-founded epistemology by a more symbolic-friendly holist approach he called *cognition*.[1]

The Applicability of Mathematics in the Empirical Science But this is not relevant here. My interest lies elsewhere. In discussing the mathematization of empirical nature in the dawn of modern science, Husserl provides a key to understanding the

[1] See da Silva n.d.

applicability of mathematics in the empirical science. In few words, mathematics is applicable in science *because* the object of empirical science is not nature as given directly in sensation or perception, but a mathematical substitute of it. The problem of the applicability of mathematics in science was brought to the attention of scientists and philosophers emphatically by the much-quoted paper by the physicist Eugene Wigner "The unreasonable effectiveness of mathematics in the natural sciences".[2] Wigner forcibly points out what, for him, remains a mystery, the use of mathematics as a heuristic devise in science. How can mathematics, which is often created with no concern for empirical reality, have anything to say about reality to the point of being capable of unveiling hitherto unknown aspects of it? Wigner believes this should be accepted as a gift (from whom, it is left for us to wonder) that we do not deserve (he does not explain why); it is a mystery, like, maybe, transubstantiation. The philosopher Mark Steiner, in a few papers and a book, pushed the issue a bit further.[3] He argues that the "unreasonable" effectiveness of mathematics in physics poses a problem for what he calls "naturalism", the quite reasonable view that man has no privileged place in the natural schema of things. Steiner obviously believes that man *does* have such a privileged place; that man is, supposedly, a special being whose mind, where mathematics is created, is somehow fine-tuned to the innermost structure of nature. How can this be explained? A consequence of natural evolution? Pre-established harmony? Natural evolution is not of much help here, for since we have obviously been selected to cope with nature, this "match" is practical, not theoretical. Moreover, it does not go beyond the macroscopic level and only long enough to keep us alive until we procreate. It does not explain, for instance, why mathematics has come to play such an important heuristic role in microphysics, which has hardly a direct role in human survival. Pre-established harmony opens a door to the mystical, which is definitely not the right way of doing philosophy.

I believe that, as is often the case, the problem appears so difficult only because it is posed against a background of unquestioned and unnoticed preconceived ideas, and once these preconceptions are examined the problem loses its aura of mystery, being easily dealt with and solved. The most questionable presupposition is that mathematics is applicable in science because nature, the object of empirical science, supposedly an ontologically independent realm of being, is *itself* mathematical. Some are happy to believe that nature is *nothing but* a mathematical manifold, existing in itself "out there". Here, I argue, with Husserl, that although the object of the mathematical science of nature is indeed a mathematical manifold, this manifold is not simply given but, instead, *intentionally* constituted as a mathematical *surrogate* of real nature. Mathematical reality is an elaborate intentional construct, resting ultimately on perception – itself a proto-intentional construct –, some of whose formal-abstract (or structural) aspects are selected to be represented mathematically in idealized form. The structure of mathematical reality, then, like any mathematical structure, can be *interpreted* in whatever mathematical structure we find convenient

[2] Wigner 1960.

[3] Steiner 1989, 1995, 1998.

as a means of accessing it epistemically. It is not nature *itself*, out there, that is mathematical, only *our* intentional reconstruction of *our* experience of nature, and for strictly scientific methodological purposes. Mathematization is only a method. I have already dealt with this problem when discussing the intentional genesis of physical space and its mathematical representation; this chapter is dedicated essentially to flexing out these ideas in a more general context. First, let us see what Husserl himself has to say about the mathematization of nature.

Husserl and the Mathematization of Nature Despite the intensification of the process of mathematization in science from the end of the Middle Ages on, this was hardly a novelty, given the extensive utilization of geometry in astronomy. About this Husserl says[4]:

> For Platonism, reality had a relation of 'méthexis' (participation) more or less complete in the ideal. This opened for ancient geometry possibilities of application – primitive application – to reality. But in the *Galilean mathematization of Nature* it is Nature *itself* that, under the direction of the new mathematics, is idealized: it becomes, to employ a modern expression, a mathematical manifold.

The applicability of mathematics in science in antiquity and in modernity, as pointed out by Husserl, display radically different patterns. Whereas in the old days mathematics, on the one hand, and reality, on the other, were two separate realms, in the modern world, the gap between them is completely eliminated. For Plato, mathematical forms are not to be found in reality; forms and reality are only in a relation of participation (*méthexis*). Reality only displayed *imperfect* copies of mathematical forms. However, the heavens, made of perfect matter, seemed to invite naturally a mathematical approach. Sub-lunar reality, on the other hand, was an altogether different matter; mathematics was too good for it.

For Galileo and the creators of modern science, differently, there was no such a neat division; the trajectories of projectiles, they thought, could be mathematically represented just like those of stars. But this is not, of course, the whole story. Reality was not simply taken as *participating* in mathematics while still preserving its non-mathematical "imperfections". On the contrary, Galilean reality is *itself* mathematical, it is itself a *mathematical manifold*. Reality is no longer deemed imperfect, only our *perception* of reality is. Not any longer that which we perceive with our senses but that about which we can *think* in terms of mathematical forms, reality becomes itself mathematical, removed to a realm of being untouchable by perception if not only imperfectly. Perception is downgraded as a means of accessing nature, reason – in particular mathematical reason – takes its place. Here the philosophical problem of the applicability of mathematics appears already with full force: is the mathematics we produce somehow inspired or induced by the mathematics of reality (in which case the applicability of mathematics to reality would pose no problem)? If it is not, how can we account for the fact that mathematics created independently of reality has a place and a role in our theories of reality, in scientific

[4] *Crisis* §9.

methodology and heuristics? How can this be so short of some mysterious (maybe mystical) connection between reality and the human mind?

Husserl says[5]:

> Mathematics and the mathematical science of Nature', or still the *dressing with symbols* of symbolic-mathematical theories, contains all that that, for the expert and the cultivated men, replaces (as the objectively real and true Nature) the life-world, substituting it. It is this covering of ideas that makes us take for the true Being what is only a method – a method that is there to correct, in an infinite progression, by "scientific" anticipations, the "rough" anticipations that are originally the only that are possible in the realm of the effectively (really and possibly) experienced in the life-world. It is this covering of ideas that renders the authentic sense of the method, formulas and theories incomprehensible, and that, in the naiveté of the method at its birth, was never understood.

Let me spell this out. For Husserl, instead of a method for inquiring nature – to be explained soon –, mathematization was, in the hands of Galileo and the founders of modern science, a sort of epiphany of the true nature of empirical nature. With them, mathematics is not only a methodological tool, but constitutes the very *essence* of reality. Nature is mathematical and can only be approached via mathematics. Galileo's famous words about the book of nature being written in geometrical characters has always been taken, and was probably meant as the clearest expression of this pathos. If, however, we reaffirm, with Husserl, the purely methodological role of mathematics in science, we must face the task of explaining how and why it works so well. If we prefer to side with Galileo, however, we have a much more serious metaphysical, not simply methodological, question to answer, namely, how can we account for the "mysterious" match between *man-independent* mathematical nature and *nature-independent* mathematical man?

For Husserl, as the quote shows, *real* nature is that which we perceive, it is the nature of the life-world, which we deal with on a regular basis. Real nature is approachable, felt, perceivable, even if we have to resort to instruments that enhance our sensorial abilities (microscopes, telescopes, things of that sort); it can also be measured, but not with mathematical precision. Real nature admits only morphological concepts. However, Husserl says, in the mathematical theories of nature real nature loses its *ontological priority*, giving place to mathematical nature. Mathematical nature *becomes* true nature and perceptual nature is degraded to an inferior status, only a necessarily unsuccessful, imperfect attempt at grasping the true essence of reality by inadequate means, namely, perception. Here, as Husserl notices, a problem emerges: how, for the purposes of applying and testing them, can mathematical theories of nature be confronted with perceptual nature when they are not theories of perceptual nature, but of a mathematical surrogate of it? I will deal with this question soon.

Another important point that Husserl's quote brings to our attention is the *symbolic* character of the mathematical theories of nature. By transubstantiating real nature into a mathematical manifold, the mathematical science of nature becomes, of course, essentially symbolic. Mathematics is often carried out by manipulating

[5] *Crisis*, §9 h.

symbols according to rules. The questions then impose themselves: do the symbols of the mathematical theories of nature correspond in some sense to entities of perceptual (real) nature? In what sense do symbols *represent* what they symbolize? How can we know anything about real nature by manipulating mathematical symbols?[6]

This problem divides in two. One when it is possible to associate real entities to mathematical symbols. In this case, symbols correspond to entities of the real world, capable *in principle* (but maybe not in practice due to factual limitations) of being perceived. Another when symbols do not, even in principle, correspond to *anything* in perceptual reality. Husserl considered a particular case of the first problem when, in *PA*, he justified symbolic arithmetic. The approach of *PA* can be generalized. When symbols have material content, symbolic relational systems and symbolic theories represent the context where they are materially interpreted thus: relational systems represent provided they instantiate the *same* or *approximately the same* structure of the system of entities for which they stand; symbolic theories represent by being *true* (within certain limits) in the system of real entities.[7] However, it is not the real that "imperfectly" approximates the mathematical, but the mathematical that somehow unfaithfully captures the real, touching only its *formal-abstract aspects* and only after *idealizing* them.

Another problem altogether is how to account logically-epistemologically for the role empty symbols devoid of any possible material content of perception play in our theories of nature. Are they simply elements of internal articulation of properly representing manifolds and their theories? For Husserl, meaningless "playing with empty symbols" would open an unsurmountable gap between empirical sciences and (real) perceptual reality if not logically-epistemologically justified. However, his approach to the problem was somewhat conservative (I have dealt with it in details elsewhere[8]). In few words, he believed that the only possible justification for the use of materially meaningless symbols in science was strictly pragmatic; it should be possible to eliminate them without cognitive loss. In case empty symbols could not be eliminated, the ties of theorizing with perception would be cut and the doors to "alienation" opened; in short, loss of meaning. This particular aspect of the treatment Husserl gave to the problem of symbolic knowledge in science was particularly displeasing to Weyl, who thought that Husserl misrepresented the role of intuition, perception in particular, in science. For Weyl, only as a *whole*, not in each of its statements, as Husserl seems to believe, a theory can face the test

[6] This last question brings in with full force the problem of symbolic knowledge. See da Silva 2010, 2012a.

[7] Husserl refers explicitly to a correspondence between the system of perceptions (*objectified sensations*) and its mathematical representative: "[…] the specifically sensible qualities […] that we experience on bodies given to intuition *are intimately connected according to a rule*, in a particular manner, to the *forms* that belong to them according to their essence" (*Crisis* §9c). Still, according to him, the perceptual world has "*its double in the realm of forms*, in a way that *any change in the manifold of contents* [*the perceptual world, my note*] *has a causally induced copy in the sphere of forms*." *Ibid*.

[8] da Silva 2010.

of experience. I too believe that Husserl's approach to this question is inadequate from the point of view of scientific methodology and that there is a better way of dealing with this problem via the notion of *structural interpretation*. I will be back to this.

Now, in what sense is mathematization, for Husserl, a *method* of science? The outputs of the mathematical theories of nature, he says, are essentially mathematical correlations among mathematical entities, often equations. As we have seen, these entities have sometimes a surrogating role, i.e. they correspond to something in experienceable reality, but sometimes they only have a role *interna corporis* as elements of articulation of the theory. In any case, the theory offers *more* than what was put into it, i.e. its experiential content. From what has *actually* been experienced, which constitutes its empirical basis, the theory often generalizes, thus offering *anticipations of experience* or *predictions*. For Husserl, in this essentially consists the methodological relevance of mathematization. Anticipations of experience, however, anticipates only the *formal* aspects of future experience; its material content being only disclosable by going back to what the symbols of the theory *mean*. For this reason, they must mean something, and this is why materially meaningless cannot occur in predictions. Moreover, or so Husserl thinks, they cannot, for epistemological reasons, occur in any essential way in the derivation of predictions either. In short, for Husserl, formal predictability covers the full extent of the applicability of mathematics in empirical sciences. Mathematics is for him essentially a context of formal representation in which we can carry out formal investigations of the formal structure of reality that allows us to make predictions. The material content of reality remaining at the background as a permanent context of interpretation. However, by ignoring that the standard semantics, the perceptual context where empirical theories are anchored can be *modified* to make sense of *senseless* symbols and *meaningless* symbolic manipulations, Husserl bypasses one important role symbolical-mathematical reasoning plays in science, the heuristic.

The possibility, of course, is always open that anticipated experiences are not confirmed, i.e. they are disconfirmed by being in contradiction with what is actually experienced. In this case, the theory is contradicted by experience, failing in its anticipatory role. Theories are also, of course, explicative, not only predictive, but a theory that fails in being predictive fails also in being explicative. Reality cannot be as the theory says it is if the theory is unable to predict how reality behaves, now and in the future. Although, for Husserl, mathematization can indeed offer theoretically more refined means of anticipation of experience than perception, we must refrain from inferring from this fact that mathematization reaches a deeper level of reality *in principle* inaccessible to perception, a supposedly purely mathematical inner core of empirical reality.[9] Mathematics, Husserl thinks, serves science only as a tool; nature, *real* nature, is not mathematical, only our representation of nature, i.e. idealized nature, devised for *methodological* purposes, is.

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[9] Of course, scientific theories can refer to levels of empirical reality that are, *in practice*, inaccessible to experience, but that can be experienced indirectly via chains of physical causation. It remains always open the possibility *in principle* of directly accessing them.

This brings us back to the issue of perceptual experience as the ultimate judge of theorizing. Since mathematization is, in a sense, a falsification of experienceable reality, would not mathematical theories of reality be disconfirmed every time they are confronted with perceptual experience? But, as Husserl notices, scientific theories are not confronted with *raw* perceptual experience, but with *scientifically interpreted experience*. In other words, the intentional elaboration of empirical reality that serves the mathematical science of nature affects not only nature as theorized, but also *nature as experienced*. For example, one will never *actually* read the value π on a scale in the lab, one will maybe see the pointer somewhere between 3, 14 and 3, 15, but will, considering the "imperfections" of perception, take this value as a confirmation of the value π that the theory anticipates. It is presupposed from the start that discrepancies are, within a certain range, tolerable; empirical verification is not in general a yes-or-no event. The theorized world and the world where theories are tested are, Husserl thinks, the *same* world. Therefore, the intentional action that goes into constituting this world is never put to test, it is not a *scientific hypothesis*, but a genuine transcendental-constitutive trait of reality as conceived in science.

Husserl believes it is a task for transcendental phenomenology to investigate how such a world comes to be[10]:

> The sense of being of the world given in advance in life is a *subjective formation*; it is the work of life in its experiencing, of pre-scientific life. It is in this life that the sense and the validity of being of the world is built, that is, at any time of *this* world that is at any time effectively valid for the subject of experience. With respect to the "objectively true" world, that of science, it is a *formation of a superior degree*, whose foundations lie in the pre-scientific thinking and experiencing and their operations of validation. Only a radical regressive inquiry on subjectivity, that is, the subjectivity that renders ultimately possible all validation of the world with its content, in all its scientific and pre-scientific modalities, an inquiry that considers the what and how of rational performances, can render comprehensible the objective truth and attain the *ultimate sense of being* of the world. Then, it is not the being of the world in its unquestioned evidence that is in itself what exists primarily, and it does not suffice to pose simply the question of what belongs to it objectively; *on the contrary, what is primarily in itself is subjectivity*, and it is *as such* that it pre-gives naively the being of the world, and then rationalizes, or, what amounts to the same, objectifies it.

The world of science has, then, its roots in the life-world and is constituted from it in a series of intentional acts. Let us see, of course in a sketchy form, what acts these may be. Perceptions are, of course, private. My perception is not necessarily your perception. The first step into the constituting of an *objective* reality is the identification of *objective* features of the world that could account for subjective experiences. It must be possible for everyone to identify his personal experiences as *signs* of objective events in the world. Another layer of meaning comes with the mathematization of the objective, communal world. As a consequence of this, private sensations and perceptions are degraded to *imperfect* signs of a mathematical objective reality. Sensations of warmth or coldness, for instance, become

[10] *Crisis*, §14.

manifestations of the objective *temperature* of bodies. Strictly speaking, one does not *feel* temperature; this is only a theoretical construct.

The constitution of objective time is another example. Originally, time is a *subjective* experience, objectified as an immaterial continuous flux that accompanies all events of the world. Newton conceived it as a single *universal* flux, Einstein, as a tourbillon of fluxes related to one another in ways his theory establishes. Subjectively, time is only an abstract aspect of the flux of *our* experiences, which either *succeed* one another or happen *simultaneously* with one another. Objective time is the objectification of this abstract aspect of experience, idealized beyond the limits of actual experience, and eventually mathematized. Our subjective perception of time becomes, then, a rough and imperfect approximation to supposedly more real mathematical, clock-measurable time.

It is important to remember, as already observed, that the structure of the perceptual, and then, the objective world depends *to some extent* on the perceptual systems themselves. Perceptions are *structured* systems of sensations by the action of psychophysical systems that *make sense* of them and often contain elements that are not rigorously speaking sensed. Moreover, not all subjective perceptions are objectifiable, only those that are invariant by a change of subjective perspective. In short, objective reality is a *communal* construct that need not contain everything that someone or everyone perceives whose structure derives from the structure of perception, which already has the mark of intentional action of some sort. By being a structured system, objective reality is already proto-mathematical. By being mathematically represented, it becomes fully mathematical. Mathematics applies *indirectly* to the objective world by applying *directly* to its mathematical representative. However, as already stressed, mathematics can only touch the formal-structural aspects of reality. I will be back to this soon.

Let us dwell a little longer with time. After being projected into reality as a background tic-tac, as a formal aspect of the system of our experiences, time is ready to be mathematized.[11] The way of doing this is by means of a *clock*, i.e. any periodical process. But one must first presuppose that the process is periodically uniform. Of course this is not something that can be verified for it would require that the accuracy of a clock be verified by this same clock. Unless one uses a different clock that we believe to be more accurate. Whereas any clock can be checked as to its accuracy, there is no way that all clocks can be so checked. A clock being selected (if many clocks are used, they must be synchronized, i.e. in phase), one can now take each period of the process as a unit of time. The uniformity of the process allows one to subdivide the unit in as many equal subunits of time as one wants. Now, to associate a point in time to *any* event *E*, one must choose an arbitrary event *O* as the initial point in time and count how many periods of the clock separate *E* from *O*. This clock *measures* the tic-tac of background time flowing independently and

[11] Husserl conducted an extensive investigation of the constitution of internal, subjective time in his essay *On the Phenomenology of the Consciousness of Internal Time* (Husserl 1990). The important thing to be noticed is that subjective time, as any object of consciousness, is *constituted*, not simply *given*.

equally for all events (at least in any given frame of reference; according to the special theory of relativity, when one considers different frames one must make corrections).

The interesting thing, for Husserl, is that subjective time, the original experience of time, is now disqualified as an imperfect, emotionally charged misrepresentation of supposedly real objective time. But, still, it is possible to make any *subjective* experience to correspond to a point in objective time in such a way that time order is preserved – if experience *A* is subjectively labelled as prior to experience *B*, then the objective time of *A* must be anterior to the objective time of *B*. It must also be true that *subjective* time-durations in the continuous flow of conscious experiences correspond to objective time-durations, and vice-versa, at least potentially. One can then access the structure of internal, subjective time-flow by means of external, measurable objective time-flow. But, for Husserl, this is only a method of structural investigation; external time has nothing to do with the particular experiential *content* of subjective time-flow; all it gives us access to is the topological or maybe proto-metrical structures of the flux of experience, and even though in idealized form.

Another example may be helpful. Let us consider the mathematization of that particular realm of empirical reality that has to do with thermal phenomena. It all begins with sensations. One can feel, mostly through our skins, that some bodies are what we call warm and others cold, and that bodies have different degrees of coldness and warmth. These are sensations, they have a subjective character. One can also verify that a warmer and a colder body when put in contact will, under certain conditions, when they are not "isolated" from one another, change with respect to the sensation of warmth or coldness they produce until a point of equilibrium is reached when both bodies cause, under appropriate conditions, the same sensation of warmth (or coldness). We call this the point of *thermal equilibrium*. Bodies in thermal equilibrium, if isolated from other bodies, will from that point on produce the same thermal sensation. All these things belong to the field of sensations.

Objectivation starts by supposing that the bodies we touch have a *property*, which does not depend on their size or the matter they are made of, one for which we still do not have a name, which *causes* the sensation of warmth or coldness when we touch them. And also that there is something that "flows" from one body to the other, from the warm to the cold, until they both reach an intermediate state where both bodies are at the same state with respect to that still unnamed property. Let us call this property *temperature* and that thing that "flows" between bodies at different temperatures until they are in thermal equilibrium, *heat*. Note that we have no clues as to what these things are *objectively*, or whether and how they can be measured. All we are presupposing is that they are *objectively real*.

Now, bodies *must* be in thermal equilibrium with themselves, for otherwise temperature would not be a well-defined property. Moreover, we know that if *A* is in thermal equilibrium with *B* then, of course, *B* is in thermal equilibrium with *A*. But we still do not know, nor could have known, whether *any* two bodies *B* and *C* that are, both, in thermal equilibrium with a body *A* are in thermal equilibrium with one another. By *presupposing* this fact – which is usually called the *0*th law of thermodynamics – we can classify *all bodies* in equivalence classes; bodies in one class are in thermal equilibrium with one another, bodies in different classes are not. We can now define the relation *sameness of temperature* thus: bodies have the *same temperature* if they are in thermal equilibrium with one another. This is not yet a definition of temperature, which requires an act of ideation[12]: a *given* temperature is that which all bodies of same temperature have in common (note, there is *still* no numbers associated to temperatures). Temperature, then, is conceived as a *universal* that has instantiations; all bodies in thermal equilibrium have the *same* temperature. Now, we *presuppose* that the domain of thermal sensations and that of temperatures are both *continuous* domains and that the first stands in a 1–1 correspondence with the second. Of course, as Husserl observes, both presuppositions are idealizations, but they are essential if one wants to develop an *objective* science of thermal phenomena that can, moreover, be conveniently *mathematized*.[13]

The next step is where mathematization properly speaking comes in. We want to quantify temperature or, which is the same, *measure* it. But before a few relevant remarks. It is *not* the domain of sensations that is being directly mathematized, but the domain that corresponds to it objectively. Therefore, *before* mathematization can properly take place, a relevant amount of intentional action must be performed. Now, one can choose any body, or better, any physical system with adequate properties, and associate somehow numbers to its thermal states or degrees of temperature. Given the *0th* law of thermodynamics, we can use this system, which we call a *thermometer*, to associate a number to the temperature of any body. We just put the body in contact with the thermometer until the thermometer, by being more sensitive, changes thermal state until it reaches the point of thermal equilibrium with the body. Then, it is just a matter of verifying which number corresponds to this state of the thermometer; this number is the temperature of the body (and any that is in thermal equilibrium with it). It is a practical problem how to associate numbers to all possible thermal states of the thermometer, but there is a way out of this difficulty. Suppose that some property of the thermometer, to which one can associate numbers in an easier way, change with temperature *according to a rule* (which, as a "phenomenological law", can be inferred from perception; to express this law mathematically one must, of course, perform the required idealizations). For example, we see that the volume of a gas under constant pressure or the height of a tiny straight cylinder of mercury of constant section change lawfully with temperature. In this way, temperatures can be measured indirectly by measuring other magnitudes associated with it by a rule. Volumes or lengths are easily measured and, then, systems whose volume or length vary in a lawful manner with temperature make more convenient thermometers.

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[12] "But logical concepts are not concepts extracted from the simplicity of the intuitive; they grow by an activity proper to reason: the formation of ideas, the exact formation of concepts; for example, by this idealization that, in opposition to vague empirical lines and curves, produces the geometrical line, the geometrical circle". Ibid.

[13] "If it is necessary that all experienced qualities have their right to objectivity, this is only possible to the extent that they indicate something mathematical." Annex I to §9 of *Crisis*.

Now, having, on the one hand, an objective universal property of physical systems, temperature, and on the other the continuum of the real numbers with which to measure it, one can take the latter for the former. Sensations of warm, warmer, or less warm and cold, colder, or less cold dropped out of the scientific picture of reality altogether as physically irrelevant subjective sensations that have no place in objective science. They are, at best, necessarily imperfect subjective impressions caused by something entirely objective. The problem, for Husserl, is not the method itself, but an erroneous interpretation of the method. According to him, objectivation and mathematization only serve as indirect ways of dealing with our sensations from a strictly formal and idealized objective perspective. But instead, he claims, intentional constituting acts are obliterated and intentional meanings fossilized in hidden layers of sedimented deposits and "forgotten". The mathematized objective reality being consequently taken for the only real reality.[14] A reversal of ontological priorities takes place.[15]

This example purports to display the many presuppositions and the amount of intentional action that lies between sensorial perception and mathematized empirical reality. These actions are moments of the intentional constitution of empirical reality as understood in the mathematical science of nature. Now, for theoretical and methodological reasons, other mathematical objects and concepts can be introduced in mathematized reality, corresponding in a much less obvious way, if at all, to sensations and perceptions. Thus, empirical reality is mathematically enriched, a common strategy of mathematical investigation. Correlations discerned by the senses give place to objective correlations, which translate into mathematical correlations. If semantics remains unaltered, provided mathematical manipulations do not extrapolate the bounds of material meaningfulness, one can use the latter to anticipate the former. Husserl believed that the mathematization of perceptual reality essentially serves this purpose. It is a method. Since science can only touch the formal surface of perceptual reality, one can conduct our scientific investigations in a formally equivalent mathematical context, even if some degree of idealization is required, and return to experiential reality when convenient.

According to Husserl, the ideal of the mathematical science of nature is that mathematical substitutes of perceptual reality be *definite*. This means that a family of concepts can be selected so that anything that can be meaningfully said about mathematical representatives of perceptual reality (and indirectly about perceptual reality itself) can be expressed in terms of these basic concepts and decided, i.e. either *proved or disproved*, in their theories.[16]

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[14] "[A] meticulous intentional analysis, strictly free of prejudices and in absolute evidence […] does not deprive in the least of its sense the natural conception of the world, that of daily life and also of the exact science of Nature, but proceeds to the lecture of what is effectively and properly contained in this sense." Appendix I to §9 of *Crisis*.

[15] "[…] the general hypothesis according to which empirical Nature is experienced as an approximation of the mathematically ideal Nature." Appendix IV to §12 of *Crisis*.

[16] "The general mathematical legality is definite in the sense that it has the form of a finite number of fundamental mathematical laws (the axioms) in which all laws are included, in a purely deductive manner, as consequences". Ibid.

In the words of Husserl[17]:

> It follows that physics, forgetting its theoretical situation, which governs its idealizations and hypotheses, would fall in an error analogous to that of mechanic atomistic physics, if it believed, with its original formations, to be able to deduce mathematically all the formations of the concrete world. As if, to speak the language of the ideal, one could form the project of a mathematics that would dominate in this way those ultimate formations in their interrelations, from which it would be possible to extract deductively all the formations of a possible concrete world.

Husserl is conscious that the method, by leaving behind the original domain of perceptions and their material content (for example, sensations of warmth and coldness) and substituting it with another domain only formally similar to it, where only formal-structural properties are represented, science – as Weyl also realized – touches only the formal surface of the only effective world, the perceptual world.[18]

> Mathematical physics is an extraordinary instrument of knowledge of the world in which we live effectively, of the Nature that maintains always, in all its changes, a concrete and empirical unity in identity. It makes it practically possible a physical technique. But it has its limits, not in the fact that we do not, empirically, leave the level of approximation, but in the fact that it is only a narrow layer of the concrete world that is in this way effectively grasped.

I find Husserl's analyses of the first steps of the process of mathematization of modern science exemplar and I believe that they clarify at least two important aspects of the applicability of mathematics in science, namely, the representational and the instrumental. However, yet another essential aspect, the heuristic, is not touched. It is time now that we take a closer look at the applicability of mathematics in science inspired by Husserl's perspective on the issue but not embracing it in all its aspects, in particular not in its limitations. The structuralist approach that I have developed earlier will play a central role in my account.

Mathematics in Science There are essentially three roles that mathematics plays in science, the *representational*, the *instrumental* and the *heuristic*. Of course, these pure types often occur in conjunction.

As we have seen, from the perspective of the mathematical empirical science, empirical reality is, in fact, a mathematical surrogate of perceptual reality.[19] In physical-mathematical reality, there are things as temperature, but not thermal sensations, a physical-mathematical space but not visual, kinesthetic or tactile spatial sensations. In short, nothing subjective exists in empirical reality. The mathematization of perceptual reality proceeds in steps; at the most basic level, that closest to perceptions, mathematical reality is such that everything that exists therein has a correspondent in perceptual reality, regardless of whether it is effectively accessible to direct perception. However, this first mathematical "draft" of perceptual reality is

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[17] Ibid.

[18] Ibid.

[19] "The science of nature does not deal with nature itself, but with nature as man considers and describes it", in W. Heisenberg, *Discussione sulla fisica moderna*. Turin: Enaudi, 1959.

usually still mathematically too limited and is often mathematically enriched in multiple ways for methodological purposes. Of course, there is no rigid delimitation of which entities of physical-mathematical reality have and which do not have correspondents in perceptual reality. Take, for example, the concept of *instant velocity*, which can only be properly defined mathematically. Of course, it does not have nor can have a correspondent in *perception* proper; extensionless instants of time are not objects of perception, but mathematical idealizations. However, there is some perceptual content in the idea of the average velocity of a body as it moves from A to $B \neq A$; we idealize by taking the limit as A gets closer and closer to B. The concept of instant velocity lies at the limit of a series of concepts with perceptual content. There are also empirical concepts with an even more tenuous connection with experienceable reality. The notion of *entropy*, for example, which cannot be formulated without mathematics, does not have a *direct* correspondent in perception, even if we allow the use of instruments to enhance our perceptual powers. Entropy as a measure of disorganization of a system has a sense only in microphysics, a realm hardly accessible to direct perception. Less perceptual still is the concept of *field*. We can observe that an electric particle moves in a certain way in the proximity of another electric particle; our theory of electricity tells us that these particles are being affected by the fields they create, but we cannot observe these fields directly. In fact, a field is a purely mathematical construct, an entity of mathematical reality without any correspondent in perceptual reality. As I said, there are no sharp boundaries in physical-mathematical reality between what corresponds somehow to perceptions and "imaginary" mathematical elements. But one thing they have in common, they are all mathematical.

Let us assume that the concept of empirical reality is reasonably well-defined, and let us denote it by R. R is a mathematical manifold that can be described by some convenient language L. I suppose initially that all terms of L denote entities that correspond, no matter how indirectly, to perceptual entities. Mathematics is primarily a provider of concepts with which we "concoct" a mathematical surrogate of perceptual reality and a language to describe it. This is *the representational role of mathematics*. In this role, mathematics offers a context of representation of objectified perceptual reality; not of all its aspects, of course, only those that admit mathematical representation. At the most basic level, mathematics represents by providing mathematical surrogates of perceptual reality (rather, aspects of it) that share with it roughly the same structure. The most important thing to keep in mind is that the mathematical representation of perceptual reality is *not* perceptual reality and strictly speaking, *not* real reality either, only an intentional construct playing the role of real perceptual reality by somehow representing it. This is the aspect of the application of mathematics in science explained in *Crisis*. Phenomenologically clarified, it does not seem to pose any problem.

In the terminology of last chapter, R is a structured domain and L its structural language. It is in principle possible to access R intuitively (perceptually) to describe it in L, but the description will necessarily be a mathematically rectified version of "imperfect" perceptual experience (embodied in *phenomenological* theories). Any such description is a theory T of R in L. But, as we have seen before, we do not have to confine ourselves to *T* in order to know *R*. One can extend *T*, in *L* or extensions of *L*, in many ways. Here are some. One can enlarge *R* by introducing new purely mathematical terms that have *no* counterpart in perception, not even as ideal possibilities (these new terms are the equivalent of imaginary numbers). This amounts to defining a larger system R_1 containing in some sense *R*, and a theory T_1 containing *T* and some new axioms referring to the new terms. I suppose that T_1 is expressed in a language L_1 containing symbols for the new terms. Let us suppose this is done and that there is a sentence φ of the language *L* provable in T_1. The question is whether φ is or not true in *R*, the "lower level" physical-mathematical reality. There are situations in which we can definitely say that φ is indeed true in *R*, as I have shown in the previous chapter.

Let us suppose first that there are *logical* reasons for φ to be true in *R*, some of them analyzed and clarified previously. In this case, the introduction of the new theoretical terms, and the mathematics for dealing with them, the theory T_1, by extending the logical powers of *T*, plays a purely *instrumental role* in the theoretical investigation of *R*. I suppose, remember, that the new terms do not have *any* perceptual content. If theorems of T_1 in *L* are justifiably true in *R*, we may feel tempted to take T_1 as a more convenient theory of *R* than *T*. In this case, we are justified in *believing* that the new theoretical terms also have an empirical content, even though there is no *logical* guarantee that they indeed have.

One possible way of enriching *T* is by adding "principles" that are supposed to be valid in *R*; we may be convinced, no matter the amount of empirical evidence, that a certain principle Φ is true of *R*, for instance, conservation of energy.[20] Now, if $T_1 = T \cup \{\Phi\}$ proves φ we are entitled to accept the truth of φ, "moduleΦ", that is. Now, if φ can be proven by means not involving Φ, one may take this as an *empirical evidence* for Φ. I will give an example soon.

Let us now suppose that although there are no logical reasons for φ, proven in the extension T_1, to be true in *R*, we *conjecture* that it may be. No matter for which reason, maybe by analogy. Suppose now that φ is effectively verified to be true in *R*, maybe by accessing *R* intuitively. In this case, the theory T_1, incorporating perceptual "imaginaries", has played a *heuristic role* vis-à-vis *R*. Note however that heuristics is *not* logic, the derivability of φ in T_1 is not in general a guarantee that φ is indeed true in *R*. Although the manipulation of extensions of empirical theories can serve the heuristics of science, there are other, much more interesting mathematical scientific heuristic strategies.

Of course, my idealization of a single empirical reality *R*, a single language to refer to it and a single theory of it is as removed as possible from what actually goes on in science. In fact, empirical reality consists of a family of domains, which often overlap, different languages and theories that can, sometimes, be incompatible with one another, as is the case, so far, of general relativity and quantum theory. This

[20] Conservation of energy does have empirical support. It is nonetheless conceivable that a principle is adopted that has no or almost no evidential support. The principle of least action or conservation of parity come to mind.

allows for interesting logical and heuristic strategies of scientific investigation. Let us consider one in particular, called *modelling*.

Suppose R_0 is a subdomain of R, for instance, the domain of electromagnetism, mechanics, or any other. Let R_1 be another subdomain of R and suppose that there is a correspondence between elements of R_0 and those of R_1. In this case, I say that R_1 is a *model* of R_0. If this correspondence is an *isomorphism*, then, as already discussed, we can carry out our investigation of R_0 by concentrating our attention on R_1 the same way we can do geometry analytically. Both domains just happen to have the same structure and science cannot tell them apart. But interesting things happen even when the correspondence falls short of being a full isomorphism. A more realistic situation is when R_0 or the theory R_0 can be *interpreted* in R_1 or the theory of R_1, as defined in the previous chapter. But there are still more general situations. Suppose that there is a 1-1- function from a *subdomain X* of R_0 into but *not onto* R_1, and that some situations in X correspond, maybe not univocally, to situations in R_1, but not necessarily conversely. In other words, suppose that to all *known* relation symbols U definable in the language of R_0 there is a symbol V definable in the language of R_1 such that $U(x_1, \ldots, x_n) \rightarrow V(y_1, \ldots, y_n)$, for all $x_1, \ldots, x_n$ in X and where the y's are the images of the x's in R_1. Known situations (configurations, processes, etc.) in X expressed by $U(x_1, \ldots, x_n)$ are modelled by situations expressed by $V(y_1, \ldots, y_n)$ in R_1. What good can come out of this?

Suppose first that there is a relation V defined in R_1 such that $V(y_1, \ldots, y_n)$, where all y's represent x's in X. Does $V(y_1, \ldots, y_n)$ model a situation in R_0? Not necessarily. For example, take a standard subway map; it represents topologically the subway system; all situations concerning relative position and connections in reality are faithfully represented in the map, but distances in the map in general do not correspond, even in scale, to real distances. In scientific contexts of modelling, however, it may not be so clear that no situation in X is represented in R_1 by the situation $V(y_1, \ldots, y_n)$. The fact that $V(y_1, \ldots, y_n)$ is true in R_1 may, however, induce us to look for situations U in X such that $U(x_1, \ldots, x_n) \rightarrow V(y_1, \ldots, y_n)$, *even if we have to enrich X with new structuring relations*. If we find some, the model has helped us to uncover situations in X or structurally richer extensions of X that we had previously overlooked. There may even be *more than one* situation in X that can be modelled by $V(y_1, \ldots, y_n)$. Suppose now that $V(y_1, \ldots, y_n)$ is true in R_1 but *not* all y's represent x's in X. This could induce us to look for *extensions* of X and situations in this extended domain that would be represented by $V(y_1, \ldots, y_n)$.

In all these cases, if represented situations are effectively found, maybe by direct inspection of R_0, and happen to have a relevant *physical* meaning, for I am supposing we are investigating empirical domains, the model and the *mathematics* involved in disclosing situations in the model have played an important *heuristic* role in the investigation of R_0. If we are convinced that R_1 is a faithful model of R_0, *we can take truths in* R_1 *as indications of things that may be true in* R_0, that is, the model can be used as a source of scientific conjectures and hypotheses. The model can be a useful *heuristic* devise. *But for it to work*, *we must relinquish full isomorphism*. Isomorphic copies can be useful for representational reasons, but they are heuristically barren. On the other hand, finding non-isomorphic models of domains of theoretical interest, and there are many different ways of doing this, although not logically justifiable, can be a heuristically rewarding strategy.

This strategy has a long history of services rendered to science, it goes thus: find a model of the system you are interested in; it may fall short of being formally equivalent to your system but, provided that you can represent true situations in your system by true situations in the model, you can investigate the model searching for situations that may correspond to true situations in the original system. With the success of Newtonian mechanics as the science of movement, mechanicism became the official doctrine of science; all physical phenomena had to be explained in terms of matter in motion. Mechanicism knew a reasonable success, even in fields where phenomena were not mechanical in nature, electromagnetism, for example. This must mean something, how non-mechanical processes can be so well explained mechanically? The answer, of course, is that mechanics provides efficient *models* for electromagnetism. I will give an example shortly.

This is how Sir James Jeans describes a standard explanatory and heuristic strategy in theoretical physics, at least in the heydays of mechanicism[21]:

> The first step [is] to discover the mathematical laws governing certain groups of phenomena; the second [is] to devise hypothetical models or pictures to interpret these laws in terms of motion or mechanics; the third [is] to examine in what way these models would behave in other respects, and this would lead to the prediction of other phenomena – predictions which might or might not be confirmed when put to the test of experience.

The fact that modeling works as a heuristic strategy *because* of purely formal similarities was stressed by Maxwell[22]:

> Whenever [men] see a relation between two things they know well, and think they see there must be a similar relation between things less known, they reason from one to the other. This supposes that, although pairs of things may differ widely from each other, the *relation* in the one pair may be the same as that in the other. Now, as in scientific point of view the *relation* is the most important thing to know, a knowledge of the one thing leads us a long way towards knowledge of the other.

Consider also the following quote[23]:

> We shall take the view that the physical world is an abstract creation of the mind of man, modeling for him the pattern of his sense perceptions and so assisting him to understand and predict the course taken by this stream of events; he is therefore free to build into this model any features which render the model effective for its purpose, requiring only that the resulting structure shall be internally consistent and that those of its elements which possess an interpretation in terms of sense perceptions shall be in accord with experience. It is certainly not necessary that every element should possess a correlate in the flux of sense perception. Some of the elements will be introduced with the sole intention of simplifying the logical structure of the model and need not be directly observable; such are the vectors associated with the states of a physical system, since it is only the squares of the moduli of their scalar products with one another that are observable

[21] Jeans 1981, p. 155.

[22] "Analogies in Nature" (1856), *apud* Longair 2003, p. 88.

[23] Lawden 1995, p. 27.

A few claims in this quote deserve attention. The first, very Husserlian in spirit, is that the objective physical world is a construct that corresponds *formally* to the subjective perceptual world (modelling for man the *pattern* of his perceptions), devised to help man (emphasis on methodology) "understand and predict" the course of events in perception. The second is that the mathematical manifolds to which empirical reality has been reduced to can be arbitrarily enlarged by the introduction of purely theoretical constructs whose task is only to help organize the theory internally. This is what I call the *instrumental* role of mathematics in science. Now, says Lawden, the whole system is logically acceptable only insofar assertions with empirical content that are derived in the extended, mathematically more convenient context, are indeed empirically true. In other words, the enlarged theoretical construct must play a *predictive* role to full satisfaction. Obviously, if the theory makes predictions that are provably false, then it must be revised. That he does not consider the *explicative* role of scientific theories has to do with the fact that he is dealing with quantum mechanics, which according to standard views is essentially an instrumental theory only very indirectly representational.

There are, however, further possibilities that Lawden does not consider: what if the theory predicts empirical facts that are neither confirmed nor disconfirmed or even *purely formal* possibilities that not only have no actual correspondent in perceptual reality, but *cannot have any* insofar as perceptual reality remains unaltered? How should we react? The second possibility deserves special attention. Mathematical theories of nature are, of course, expressed in symbolic-mathematical languages. Symbols can have both material and formal meaning; the language of a given scientific theory has a semantic and a syntax. However, the possibility exists that syntactic manipulations are allowed that have no semantic content in the standard semantics, i.e. perceptual reality as it is supposed to be at that moment. When one "plays" with the symbols of a theory in conformity with syntactic rules but with no concern for the semantic rules of *meaningful* symbolic manipulation, one may stumble on *formally possible facts* that are sufficiently interesting for us to ask: how should the world be for these formal possibilities to be *materially* true, i.e. real facts of the world? We can, then, as an exercise of semantic speculation, *conjecture* empirical possibilities that would make the formal possibilities that the theory produced into empirical truths. Of course, the new semantics may not correspond *at all* to reality, in which case our exercise is in vain. But, in at least a few cases, we may actually discover something. In this case, the *syntax* of mathematical theories plays a *heuristic* role. Sometimes speculation may pay off, as the discovery of anti-matter suggests. I will consider this example later.

Case Stories Let us now analyze some high profile cases of heuristic use of mathematics in science and see whether and how they fit in my schema.

(1) *Maxwell and displacement current*. I begin with Maxwell's discovery of displacement current. First, Steiner's version of the story.[24] According to him, the discovery of Maxwell's equations was entirely based on second-order mathematical

[24] Steiner 1989, p. 458.

analogies; it was, that is, an instance of a heuristic strategy in which "physicists attempting to discover some physical description restrict their search to descriptions with the same mathematical properties of known, successful descriptions." (p. 452). He calls "such analogies *second-order mathematical analogies*, because they are based on properties of descriptions, rather than on the descriptions themselves." (ibid).

It is worthwhile quoting Steiner in full here[25]:

> Maxwell's procedure in writing down his immortal equations provides another example of this strategy. Once the phenomenological laws of Faraday, Coulomb, and Ampere had been given differential form, Maxwell noted that they contradict the conservation of electrical charge, though the phenomenological laws were strictly in accord with the evidence then available.[26] Yet, by tinkering with Ampere's law, adding to it the 'displacement current', Maxwell succeeded in getting the laws actually to imply charge conservation. With no other empirical warrant (Ampere's law stood up well experimentally; on the other hand, there was 'very little experimental evidence'[27] for the physical existence of a 'displacement current'), Maxwell changed Ampere's law to read that (the 'curl' of) the magnetic field is given by the sum of the 'real' current and the 'displacement current'. Ignoring the empirical basis for Ampere's law (magnetism is caused by an electric current), but by formal mathematical analogy, Maxwell now asserted the law even for a zero 'real' current! Thus did Maxwell predict electromagnetic radiation, produced later by Hertz.

Steiner's account of the course of events is a distortion of historical facts, but let us for now accept it. This is what he apparently wants us to believe[28]: from the original Ampere's law, curl $\mathbf{H} = 4\pi/c\ \mathbf{i}$ (where $\mathbf{i}$ is the density of electric flow, i.e. the amount of electric charge crossing per unit time a unit surface perpendicular to the flow), it follows that div $\mathbf{i} = 0$ (since div curl $\mathbf{H} = 0$ by definition of the divergence and curl operators). But this contradicts the equation of continuity, expressing the conservation of electric charge, namely, div $\mathbf{i} + \partial\rho/\partial t = 0$ (ρ = volume charge density). So, *in order for electric charge to be conserved*, Maxwell thought, there must be another term $\mathbf{X}$ in Ampere's law, which should then read curl $\mathbf{H} = 4\pi/c\ (\mathbf{i} + \mathbf{X})$; it follows from this and the equation of continuity that div $\mathbf{X} = \partial\rho/\partial t$. But, from Poisson's equation, div $\mathbf{E} = 4\pi\rho$, we have that div $\partial\mathbf{E}/\partial t = 4\pi\partial\rho/\partial t$; so, we can write $\mathbf{X} = 1/4\pi\partial\mathbf{E}/\partial t$. Therefore, if in Ampere's law $\mathbf{i}$ is replaced by $\mathbf{i} + 1/4\pi\ \partial\mathbf{E}/\partial t$, electric charge is conserved. Ampere's law now takes the form: curl $\mathbf{H} = 4\pi/c\ \mathbf{i} + 1/c\ \partial\mathbf{E}/\partial t$; $1/c(\partial\mathbf{E}/\partial t)$ being the *displacement current*. This additional term, moreover, "has the physical consequence that electromagnetic waves can exist."[29]

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[25] Id. Ibid. p. 458.

[26] In fact, Ampere's law, as Maxwell clearly recognized, was valid only for closed circuits [*my note*].

[27] These words are Maxwell's own; he however is not referring to the existence of the displacement current itself, but to the fact that, like "real", conduction currents, displacement currents can also produce magnetic effects [*my note*].

[28] See Pauli 2000, pp. 109–10, where the justification of the concept of displacement current is based in analogous manipulations. But, notice, Pauli is not committed to, and does not claim historical accuracy.

[29] Id. Ibid. p. 114.

Put this way, there is no analogy with anything whatsoever, only, apparently, outright "tinkering" with mathematical formulae. However, even if this were a historically faithful account (which it is not, as we will see below) Steiner's claim that Maxwell's formal manipulations had no *physical* basis, would not be warranted. There is one, of course, namely, *the principle of conservation of electric charge*. Mathematical manipulations would have been used only in trying to find out the *mathematical form* of the flow of "missing" charge (and a flow of charge is an electric current). This flow appears, as the formalism would have told him, *in the form* of the time derivative of the electric field, which could then be seen as *formally equivalent* to an electric current, the displacement current, and then also a source of magnetic action, *even in the absence of real* (*conduction*) *currents*. This is the case I discussed above, when the addition of an extra principle Ψ allows T to show φ, which must be true if T and Ψ are.

But this is *not* how things *actually* happened. The displacement current was in fact *naturally required* by Maxwell's *mechanical* model of electromagnetic phenomena. He was not reasoning by formal analogies purely and simply, without any reason for so doing, but using instead, for descriptive and heuristic purposes, a mechanical model of electromagnetism *that had already proven to be reliable*. Let us see how Maxwell has actually arrived at the notion of displacement current and the final form of the equations of electromagnetism that eventually took his name.[30]

Maxwell's mechanical model of electromagnetic phenomena pictured magnetic fluxes in both conductor and insulators (*including the vacuum*) on the model of rotating vortex tubes of fluid.[31] Magnetic flux was supposed to flow along these tubes (the analogy being based on the fact that lines of magnetic field tend to spread exactly like fluid vortex tubes when rotational centrifuge forces are unbalanced).

So, for him, space (including *empty* space, it is important to emphasize) was filled with rotating tubes of magnetic flux. In order to eliminate friction he inserted "idle wheels" between these tubes, whose movement he identified with electric current, and so could not move freely in insulators. This mechanical model was able to account surprisingly for all electromagnetic phenomena known at the time.

Now, Maxwell used this model to account also for the storage of electrical energy in insulators. Of course, this had to be done in some mechanical form. He supposed that the electric particles (the wheels between tubes of magnetic flux) in insulators (where they could not move freely), when placed under the action of an electric field, would be *displaced* from their equilibrium position, so storing potential mechanical energy.

Variable electric fields would then give origin to small displacements appearing as small electric currents that, due to the elastic character of displacements, would propagate as an electric current through the medium. Displacement current was then discovered. Since the displacement $\mathbf{r}$ is proportional to the electric field $\mathbf{E}$, displacement current, i.e. the time derivative of the displacement $\mathbf{r}$, is proportional to the time derivative of $\mathbf{E}$. The density of displacement current $\mathbf{i_d}$ can then be written as a

[30] I follow Longair 2003, pp. 88–98.

[31] See his "On Physical Lines of Force" of 1861–2.

multiple of $\partial\mathbf{E}/\partial t$ and allowed to join the density of *conduction* current **i** in Ampere's law.

In short, for Maxwell, a variable electric field, *even in the vacuum*, *even in the absence of conduction currents, simply because his mechanical model so required*, would originate a variable displacement current. From the new version of the equations of electromagnetism, where Ampere's law includes density of conduction *and* displacement currents, the existence of electromagnetic waves could be deduced. A variable electric field (which is then seen as formally equivalent to an electric current) would generate a variable magnetic field, which would generate a variable electric field, *and so on*; this disturbance traveling with the speed of light.

However, despite its descriptive and heuristic utility, the elaborate mechanical machinery Maxwell put in action is a *fiction*; these things do not exist. How, then, can a physical model of reality *that does not correspond to anything physically real* work? The answer is straightforward: because model and reality are *formally* similar, electromagnetic and mechanical processes are, to an important extent, *formally* the same process. Unlike Steiner's claims, it is not a matter of exploring *second-order mathematical analogies*, but *first-order structural similarities*.

Let us speculate how Maxwell may have come to his model. Electromagnetism has mechanical effects; a *natural thing* to do would be to ask what sort of *mechanism* (in the literal sense of the word) would have these *same* effects.[32] It was not difficult for Maxwell, who seems to have had a talent for mechanical engineering, to devise one. But, as he tells us explicitly, he did not believe this model was a picture of actual reality[33]:

> The conception of a particle having its motion connected with that of a vortex by perfect rolling contact may appear somewhat awkward. I do not bring it forward as a mode of connection existing in Nature […]. It is however a mode of connection which is mechanically conceivable and it serves to bring out the actual mechanical connection between known electromagnetic phenomena.

So, Maxwell was aware that his mechanical model was not a *materially faithful* depiction of actual facts; it only behaved, in regard to "the actual mechanical connection between electromagnetic phenomena" *as if* electromagnetic action occurred as the model indicated. But how can anything behave "as if" it were something else if model and reality, although possibly very different with respect to *material* content, did not have something in common? Since it is not matter, it can only be *form*, i.e. how the elements in each domain relate to one another independently of what they are and the particular nature of the relations involved. And here we reach the crux of the matter: models model because they are indistinguishable from reality with respect to *form* (or underlying structure; Maxwell's "relations"), at least as far

[32] Remember, in those days the idea that electromagnetic action was due to the presence of mathematical fields was still in the future. However, far-fetched as it is, Maxwell's mechanism at least offers a *picture* of physical reality that fields do not. The introduction of the concept of field in science (to which Faraday and Maxwell contributed) reinforced the Platonist trend in science initiated with Galileo and other scientists of the seventieth century.

[33] *Scientific Papers*, vol. 1, p. 486, *apud* Longair 2003, p. 102.

as the model "works". Now, if a model shares with reality a core of common formal properties, isn't it reasonable to explore the model for hints of further formal properties of reality? Why can't the model behave correctly beyond the limits where it has already been proved (formally) correct? Maxwell's (very successful, remember) model told him that variable electric fields generate displacement currents; if these currents had the same magnetic effects of conduction currents, electric charge would be conserved. Since electric charges *must* be conserved (they cannot just disappear!), the incorporation of displacement current in Ampere's law is *physically* justified (which does *not* mean that further *direct* empirical evidence would no longer be required). How I see it, this has nothing to do with "tinkering" with mathematical equations with little (or no) regard for empirical evidence.

We can summarize Maxwell's strategy thus: realms of scientific interest can be investigated through others, materially different from but *formally* identical in whole or in part with them because formal properties can be *identical* even when the things displaying these properties are materially different, and if phenomena we know well display *some* formal properties that are identical with formal properties of phenomena we know less well (for example, mechanical behavior of fluid vortex tubes vis-à-vis mechanical behavior of electromagnetic flux) it is a good idea to explore this formal identity for heuristic purposes. It may pay off or it may not; if the identity extends *further* than already observed (as we are justified to expect) it probably will.

It is now obvious how this strategy can be extended so as to allow *mathematics* to play a relevant heuristic role in science. Since the utility of mechanical models is solely due to formal similarities, material content being irrelevant, we can use *mathematical* models instead (a propos, this is a two-way road, we can also in principle discover mathematical facts by empirical investigation, a heuristic strategy not unheard of in the history of mathematics). It is then *possible*, but only *possible*, that *formal* properties of empirical domains reveal themselves in the mathematical formalism before they show up empirically. The formalism, however, *cannot* by itself determine either whether formal possibilities are materialized in the empirical domain or, in case they are, in which *particular* states-of-things. The next example will make this clear.

(2) Dirac and antimatter. Here is another case, also a favorite of Steiner's, the discovery of antimatter. By imposing on the generic form of the wave equation certain formal restrictions so as to guarantee invariance under Lorentz transformations – a relativistic must – and taking into consideration Klein-Gordon's earlier quantum relativistic equation, Paul Dirac managed to derive (1928) a new equation combining quantum theory and (special) relativity. By solving this equation for the electron he got two possible *positive* values for the energy, corresponding to the two possible states of energy (spin +1/2 and spin −1/2) and two *negative* values of energy that did not correspond to anything known. States of negative energy are characteristic of relativistic theories, since the relativistic energy E of a particle of mass m and momentum p is given by $E^2 = p^2c^2 + m^2c^4$, where c is the velocity of light in the vacuum. Dirac had two alternatives, either to dismiss electrons with negative energy as merely *formal* possibilities with no material content or take them as real

entities. However, not to give negative-energy states physical reality, although empirically justified, would be heuristically barren, so Dirac preferred to consider the alternative seriously. However, even though Dirac's equation opened *formal* possibilities that Dirac could conjecture to be *materially* realized, the theory could not tell him *what* this matter was. Dirac's equation, or the theory where it was derived, cannot *by itself predict* the existence of anything. Even if we could say that the equation predicts the *possible* existence of *something*, it did not predict the *actual* existence of *anything*, nor what this thing would be, if it existed.

Dirac was then faced with the following question: *why don't we see electrons transitioning to states of negative energy*, *as they could*? His answer to this question was ingenuous; based on Pauli's exclusion principle (valid for fermions like electrons) he conjectured that all the states of negative energy were already occupied by an undetectable "sea" of electrons. This would, however, had the consequence that if a negative-energy electron moved to a positive–energy state (as it *could*), it would leave behind an unoccupied negative-energy position that would behave like a positive electron. Thus a sort of anti-matter was conjectured to exist.

Crediting this conjecture to the heuristic powers of mathematical manipulations is like blaming the bullet for killing someone, not the man who fired it. The mathematical formalism only rendered explicit what was implicit in the physical presuppositions on which relativity theory and quantum mechanics rest. Mathematics served only as a means of expressing these presuppositions and deriving their consequences. Dirac's decision of taking *formal possibilities* for *material realities* can be credited to other, different factors. One, certainly, was his faith in the predictive power of the theory, but also maybe a certain Leibnizian picture of reality: nature actualizes the maximum of possibilities so as to make this if not the best, at least a more interesting world. A richer reality, Dirac could have thought, is preferably to a poorer one. Any physicist would *prefer* that mathematical possibilities were actualized if this rendered the world more exciting from the perspective of the theoretical and practical scientist.

The reality of positively charged counterparts to electrons, which were to be called positrons, was later experimentally confirmed (1932),[34] but the story has a few twists. First, there were suggestions that negative-energy "holes" could be identified with protons, an unviable alternative since the mass of the proton would have to be equal to that of the electron, which is not the case; protons are much more massive than electrons. Second, to postulate the existence of positrons is not the only way of giving negative-energy solutions a physical interpretation; we could *also* presuppose (even more fantastically) the existence of *electrons moving backwards in time*. Third, Carl Anderson, the man who actually discovered positrons experimentally said he knew Dirac's conjecture only superficially and that it did not play any relevant role in the discovery.

[34] One cannot avoid thinking of electrons and positrons as positive and negative integers, and consider the similarities between the strategies that led to the discovery of, respectively, positrons and negative numbers.

In short, there is much more between formal sub-products of mathematical manipulations and reality than is dreamt of in our philosophy. Mathematics can play a heuristic role in science by disclosing formal possibilities, but always in the context of a *theory*. These empty formal possibilities can either indicate the *limits of meaningfulness* of the formalism or suggest *extensions of the standard semantics* of the formal language. *Which* alternative to choose and *which* semantics are things the formalism is completely silent about. The manipulation of mathematical formalism by itself cannot be credited with the discovery of anything in the world, for if we could say that meaningless mathematical manipulations *showed* that positrons exist, we could also say, *on precisely the same grounds*, that it *shows* that there are electrons moving backwards in time. Or, for that matter, that it "predicts" whatever yet unknown interpretation of the mathematical formalism we may eventually stumble on. If these last claims are hard to swallow, so is the first.

The following case bears similarities with Dirac's "discovery" of anti-matter.[35] Johann Balmer, a Swiss mathematics teacher with an interest in numerology, found, for the fun of it, an arithmetical relation for the wavelengths of the then known four lines of the hydrogen spectrum (named alpha, beta, gamma and delta): $\lambda = b(m^2/(m^2 - n^2))$, where λ is the wavelength, b a constant (364.56 *nm*) and m and n (in the formula) integers. If we give n the value 2, and m the values 3, 4, 5 and 6 we obtain, respectively, an almost perfect match for the wavelengths of the four lines of the hydrogen spectrum (differing from them by less than one part in 40,000); interesting, but not very impressive – yet. Balmer, however, conjectured that $n = 2$ and $m = 7$ would give another line, unaware that such line had already been discovered and measured by Angström. More, setting $n = 1, 3, 4, 5$ and letting m run through a sequence of integers, new lines of the hydrogen spectrum in the infrared and ultraviolet range were predicted. All were effectively discovered. In few words, by extending, without any empirical reason for so doing, an arithmetical equation beyond its limits of validity, Balmer hit on a deep truth of atomic physics.[36] How can we explain this apparent instance of the "unreasonable" heuristic effectiveness of mathematics? Certainly not by supposing that nature is Balmer-friendly, or that it is particularly willing to disclosure its secrets to blind arithmetical manipulations, for there are innumerably many other equations that would fit the four original wavelengths perfectly but would fail for the others. Balmer's formula is not even the simplest "good" formula, for $\lambda = b(m^2/(m^2 - 4))$ is simpler, it gives the wavelengths of lines from alpha to delta, but has more limited heuristic virtues. The fact is that the formal possibilities opened up by Balmer's formula just happened, *as a matter of fact*, to correspond to facts of the empirical world. Balmer was just lucky in choosing this equation instead of others that he could have chosen.

One of Dirac's points of departure was the relativistic expression for the energy of a particle, *any* particle, not only the electron, and there is hardly a more *fundamental* and *general* theory of nature than special relativity. So, since his final

[35] See Kumar 2010, pp. 101–02.

[36] On seeing Balmer's formula, M. Kumar tells (op. cit. p. 102), Bohr immediately saw what caused spectral lines, it was electrons "jumping" between different allowed orbits.

relativistic equation predicted *correctly* the possible states of energy of the electron – and thus proved its worth –, besides states of negative energy that could not be accounted for but could not be ignored from a theoretical perspective, it was possible, and made good sense to suspect that his equation described *more* than the behavior of positive-energy electrons. The equation, which Dirac was *justified* in believing had a wider scope, was *hinting* at something, what it was the equation could not, even *in principle* say. So, if anything deserves credit for suggesting the existence of positrons, it is firstly the fundamental ideas that went into the makeup of Dirac's theory, quantum and relativistic ideas of *sufficiently large* scope, and, secondly, the adequacy of the formalism expressing these ideas and their interplay, which provided enough room for unexpected formal relations among basic ideas to emerge at the syntactic level. It is, after all, to be expected that the reality hidden under the surface of observable physical facts may hint at its existence, if only as meaningless *formal* correlations, in the context of theories correctly describing the observable level of reality, provided the manifest and the hidden are physically correlated and the theories stand on sufficiently general physical principles.

There are, as we have seen, many examples of this phenomenon in *mathematics*. Recall, for instance, that negative and imaginary numbers, particularly the latter, made their début in mathematics as pseudo-roots of algebraic equations, i.e. meaningless sub-products of algebraic manipulations performed in the search for the "real" roots. Cardano, among others, had the good idea of taking "imaginary" solutions as suggesting new numerical concepts intimately related to those then known, realizing that an enlarged numerical domain was probably the *right* formal context where to look for algebraic techniques for solving equations. It is important to notice that these mathematicians did not bother to tell *what* these new numbers were, i.e. to which concept they were subordinated (certainly not that of quantity); all they did was to postulate their existence satisfying certain *formal* requirements: there is a new "number", they admitted, which can be adjoined to the usual numerical domain so that we can extend formally to this new enlarged domain the usual numerical operations, under the sole requirement that the square of this new number is equal to -1. Mathematically, this was all that mattered. Questions of consistency remained open which later were answered by interpreting these new numbers into geometrical contexts. Imaginary numbers hinted at their existence, but only *formally*, and were initially accepted only as formal entities possessing only the formal properties they presented themselves as having. Anti-electrons also manifested themselves at first only formally, and mathematics was utterly unable to tell which objects in the world, if any, had such fingerprints.

Purely symbolic manipulations in the context of an *interpreted* theory, even if not allowed by the meaning attributed to the symbols of its language, can possibly suggest fruitful ways of extending or reformulating semantically the theory. Theories may not fully actualize their potential for formal expressivity, that is, their capacity of expressing formal properties of some given domain if confined to *particular* semantics. This is why it is sound scientific methodology to investigate *formal* possibilities contained in successful mathematizations of physical phenomena, even if at first sight they seem absurd or impossible given the standard interpretation of the formalism. Semantically meaningless, purely formal byproducts of symbolic manipulations can sometimes be how the formal structure of a different, more convenient semantic domain projects on an inadequate one. The strategy of not dismissing apparent absurdities before a closer look will not work every time, but it is *bounded* to work sometimes, and when it does we are to credit the success to the capacity of the symbolic context in question to express relevant formal relations by the power and scope of the basic ideas it was designed to express and the productivity of its syntax rather than to a mystical predisposition of domains of our scientific interest for disclosing their secrets to man-made artifacts.

To represent empirical objects and relations by mathematical entities has the immediate effect of providing empirical theories with a symbolic language and mathematical instruments of theoretical inquiry. Both can be put to at least three uses: to make empirical previsions by *deriving*, within the relevant mathematical context, consequences of established empirical facts; to frame empirical hypotheses and conjectures that are not logical consequences of anything previously established but are in conformity with the standard interpretation of the formalism, and, finally, to frame somehow "unreasonably", by purely formal means, stressing the standard semantics of the symbolic apparatus, hypotheses that although expressible symbolically do not correspond to anything known in the standard interpretation of the symbolic system.

(3) *Pauli and the neutrino*. In his book *La matière dérobée* (*the hidden matter*),[37] particularly chapter IX, entitled "Modèle mathématique et réalité physique", French philosopher of science Michel Paty points to Galileo (as Husserl before him, although Husserl is not mentioned) as a turning point in the historical development of the relations between mathematics and physics. Among the ancients, Paty says, the real was in a relation of *analogy* with the ideal; ideal forms were simply imposed on facts of observation (as we have seen, Husserl thinks of this relation in terms of Platonic *méthexis*).[38] With Galileo, this becomes a relation of *implication*; mathematics becoming essential for the construction of the theoretical forms of physics. Before Galileo, mathematics provided at best a *language* into which to *translate* empirical reality. After him, mathematics becomes explicitly involved in the *elaboration* of physical concepts. As we have seen, all this agrees perfectly with what Husserl says in *Crisis*.

The "mystery" that mathematical concepts are applicable in physics even though they were not created for this end dissolves when one recognizes, with Paty, that the concepts of modern physics are *from their very conception* mathematical (an essential aspect of the mathematization of empirical reality). Given that mathematical concepts *must* be involved in the clarification of mathematical concepts (for how else could they be clarified? which other concepts could be used for this end?), it is to be *expected* that mathematical concepts *must* be applicable in physics, no matter what they were invented for.

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[37] Paty 1988.

[38] I suppose we can understand this relation of analogy as *formal* analogy.

The fact that mathematics is *necessary* to physics, however, never ceases to cause wonder, as clearly expressed by A. Lautman, as quoted by Paty (all translations are mine): "There is a physical reality and the miracle to be explained is that it requires the most developed mathematical theories to be interpreted".[39] More sober, but no less emphatic is Brunschvicg, also quoted by Paty: "[Physics] sees the precision of its results subordinated to the degree of perfection reached by the mathematical instrument",[40] and D'Espagnat: "the only entities stable enough for physics to regard them as fundamental are numbers, functions, and other even more abstract mathematical entities".[41]

"In its use in physics", says Paty, "mathematics can be conceived as a tool that constructs or delivers structures – not a language that translates".[42] This is relevant. For Galileo, mathematics offered a *language* for science; reality is *expressible* in mathematical terms. But if this is all that mathematics did, we could well puzzle at why reality *preferred* to express itself mathematically. However, by using mathematics as something *more* than a language, as an instrument for structuring reality, science *induces* reality to have this penchant for mathematics.

Mathematics, for Paty, has different levels of involvement with physics. From the most elementary, where "phenomenological" models of experience are constructed ("phenomenological models" are one step higher in the hierarchy of mathematization than the purely perceptual), to the highest, the axiomatic level. In his words[43]:

> Mathematical modeling in physics happens in different ways, from the so-called phenomenological model, at the most rudimentary stage, nearer to the empirical data, to attempts at complete axiomatization. By using phenomenological models the physicist proposes to account for a distribution observed with the help of a calculus founded on an anterior theory, hypothetically accepted, by means of adjustments of one or many variables let free. The model acquires a physical dimension, beyond the stage of simple parametric representation, if these variables are referred to elements of a more profound, more 'explicative' representation. For example, the parametric representation of a process of diffusion of particles with relevant transfer of impulsion expressed in function of cinematic variables, which constitutes a first model, allows information of a dynamic nature to be obtained if considered with reference to a model about the structure of the particles in iteration – for example, the quark model. The integration of these two models, situated at different epistemological levels, with the success of their confrontation with experimental data, revealed itself capable of making what was nothing but a 'mathematical representation' acquire the status of a truly physical constitution.

A more abstract and active use of mathematics is what Paty calls "the theoretical production of physical reality" ("production théorique de [sic] réel physique"): "the theoretical process […] develops out of its own inertia, following the logic of relations, the chain of deductions […], reaching the point of formulation of a new

[39] Lautman 1977, p. 281.

[40] Brunschvicg 1972, p. 569.

[41] D'Espagnat 1979, p. 12.

[42] Patty 1988, p. 323.

[43] Ibid. pp. 324–5.

property",[44] when mathematics and theoretical elaboration becomes *predictive*, not only *descriptive*. Examples of this are Planck's ad hoc and purely mathematical hypothesis of the quantum of action and the prediction of the positron, the neutrino and the meson "as more or less formal consequences of equations expressing laws of motion (Dirac's equation for the positron), laws of conservation (of energy in the case of the neutrino) and the law of a field of force (the nuclear liaison for the meson)".[45]

A case-story Paty analyzes in more details is the discovery of the neutrino. Before being anything real, he says, the neutrino was "only" a mathematical hypothesis, invented by Pauli as a "desperate remedy" to save the laws of conservation of energy, angular moment and the "statistics". Paty explains: "He supposes that neutral particles of spin ½, obeying the principle of exclusion, of very small mass and very penetrating, are emitted simultaneously with the electron [*in β-emissions - my remark*] so that the sum of the energies of this particle and the electron is constant. These particles are the 'neutrinos'".[46] At first, Paty observes, they were called "neutrons", for the neutron we know had not yet been discovered. As Paty tells us, Perrin proposed to give the neutrino the denomination of "ergon", which, he notes, betrays its conceptual origin. At its origins the neutrino was "a little grain of energy and spin", a theoretical construction to fill a "gap". The hypothesis became physically more robust when Fermi introduced it in his theory of the β-decay. In Paty's words: "[I]t was no doubt its inscription in the theory of weak interactions, by making it indispensable, that the neutrino turned into something other than a ghost; one of the points of articulation of a system of concepts and theoretical relations with great predictive power".[47] The existence of the neutrino could be granted only because Fermi's theory gave it the power of interaction; its capacity of "absorption as well as emission that, together with its theoretical characterization […] is all that we have the right to demand of a particle to be assured of its existence", more in fact than its "visibility".[48]

Although, as I have already noticed, Paty does not mention Husserl, he comes very close to repeating what the latter said in *Crisis* about the "mathematical substruction of reality". According to Paty, echoing Husserl: "Physical theory, in fact, is not only founded on the mathematization of its concepts, but in general on a *substitution* [*emphasis added*]: it substitutes the complex and yet unknown determinations of the real offered in experimental observation by a set of principles on the basis of which the theory develops. They serve as the formal frame and

[44] Ibid. pp. 327–8.

[45] Ibid. p. 328.

[46] Ibid. p. 331.

[47] Ibid. p. 332.

[48] Ibid. p. 333. For Paty, however, not all theoretical previsions follow the same pattern. There is the case of the neutrino, originally a mathematical hypothesis (if only in disguise), but there is also the case of de Broglie's wave-particle duality hypothesis, from the beginning a physical hypothesis, even though mediated by mathematical considerations (the theory of relativity in de Broglie's case).

methodological protocol of the theory".[49] Paty provides examples. Newton's three laws, the principle of least action, the laws of conservation (energy, momentum, electric charge), the principle of relativity (the laws of nature are covariant with respect to changes of system of reference), the axioms of quantum theory.

And he continues: "[I]t is then on a *substitution* [*emphasis added*] of the real by an abstract rational construction – which is not only that of principles […] but also that of mathematically expressed concepts, models and a methodological protocol – that mathematical reasoning operates. *And operating thus it does not create, after the initial creative step, anything concerning physical content; it only translates, all consequences being already potentially in the premises* [*emphasis added*]". "So, the neutrino was at the beginning only in appearance a simple 'mathematical hypothesis'. It was already, from the first formulation, a disguised and non-admitted physical hypothesis".[50] This is a clear statement of the point I want to make here: mathematics does not apply to perceptual reality directly, only to a *mathematical reconstruction* of it, the facts of experience being subsumed under concepts and general principles that are *from their very inception mathematical*. Reality is not *in itself* mathematical; only our representation of reality, as we happen to perceive it, is.

Let me sum up. For Paty, there are different levels of involvement of mathematics with physics. From a purely external one in which mathematics is a tool devoid of physical meaning, as in calculations, to higher levels, in which mathematics is involved in concept formation and the formulation of laws with the status of principles. There is also a level in which mathematics intervenes, but only as a tool. For example, procedures of renormalization in quantum field theory to eliminate undesirable and physically non-interpretable "deviations" of the formalism, infinite quantities in this case (as problems of the formalism, these "deviations" are corrected by adjustments of the formalism). The important thing, at least for the views I want to advance here, is that Paty's analyses of cases and the general view he derives from them, points to mathematics as actively involved in the constitution of the very field of science, *reality itself* (or better, reality *as devised for scientific purposes*, reality as an *intentional construct*) and the methodological strategies of scientific investigation. From this perspective, the applicability of mathematics in science, descriptive, predictive and heuristic, is not the mystery challenging "naturalism" that Steiner's empiricist perspective, in which reality is simply *given*, revealing itself "unreasonably" apt to being mathematically treated, forces on him and all those who cannot accept the "idealist" perspective that the history of science *itself* suggests.

Steiner According to Steiner, "the strategy physicists pursued […], to guess at the laws of nature, was a Pythagorean strategy", and that to use "the relations between the structures and even the notation of mathematics to frame analogies and guess according to these analogies" (pp. 4–5) poses a challenge to the view that nature is

[49] Ibid. p. 335.

[50] Ibid. p. 339.

indifferent to man (since mathematics is quintessentially a product of man's culture). As I believe my previous considerations made clear, the appeal to formal analogies is a *logically justified* scientific strategy. Since science, any science, is essentially formal, it is irrelevant in which particular material context scientific investigations are carried out provided they instantiate the required form. Moreover, the fact that mathematics has a role in science *at all* follows, as I have argued, from the fact that the object of science is not, at least not directly, raw perceptual reality, but a mathematical substitute of it. Mathematics can be heuristically relevant in the empirical sciences because empirical reality is *already* mathematical (better, a mathematical *construct*) and mathematics is heuristically relevant in mathematics.

As sufficiently stressed, reality as directly experienced is, for scientific purposes, "purified", that is, mathematically idealized, before scientific theorizing takes place. In the end only formal mathematical representatives suitable for mathematical investigation remain. To rely on formal mathematical analogies as a heuristic strategy in science is only a matter of good sense, and the success of "Pythagorean" strategies in science has absolutely no relevance for the place of man in the world and poses no threat to "naturalism".

For Steiner, "in formulating conjectures, the working physicist is gripped by the conviction (implicit or explicit) that the ultimate language of the universe is that of mathematics." (p. 5). Indeed, but only because his universe is already mathematically shaped. This, however, has nothing to do with what Steiner calls "metaphysical Pythagoreanism", the view that the things of the world "out there" just *are* mathematical entities, or "conceptual Pythagoreanism" as he construes it, i.e. "the view that the ultimate properties or 'real essences' of things are none other than the mathematical structures and their relations" (p. 5).

The main argument of his book, Steiner says, "is that, given the nature of contemporary mathematics, a Pythagorean strategy cannot avoid being an *anthropocentric* strategy" (p. 5). What he has in mind is that, besides being species-specific (not essentially different from music or art), mathematics, particularly modern mathematics, is not always inspired by natural science, and so could not play such a major role in the description and investigation of nature if nature were not willing to accommodate itself into the all-too-human structures of mathematics. Again, the main problem here is the assumption that science describes nature *itself*, not only our all-too-human *perception*, *categorization*, *idealization* and *mathematization* of nature. Steiner marvels at the fact that we have managed to *invent* abstract formal structures that just *happened* to coincide with the formal structure of empirical domains without seemingly realizing how much he is taking for granted. His solution of the puzzle – that nature has a weak spot for us – testifies to his unwillingness to see the extent to which empirical reality of scientific enquiry is a *human* construct. The real mystery would be if the idealized structures *we* impose on abstract aspects of *our* perception of nature were completely strange to mathematics, *our* science of structures in general.

For Steiner, anthropocentrism is more blatant when Pythagorean strategies rely on mathematical notation, not on what the notation expresses. But, as already suggested, competently designed notational systems express *syntactically*, that is, by means of their rules of symbolic manipulation, relevant formal relations among the entities the symbols denote. Ideally, symbolic notations aim at *displaying* at the symbolic level the formal relations they are designed to express. By exploring notational analogies we are again only reasoning by formal analogy.

The true reason why some thinkers believe that there is a philosophical problem and a methodological puzzle with the applicability of mathematics in science is their Platonist presuppositions. If you believe that empirical reality is a *given* independent of the ego's intentional action, particularly willing to unveil its secrets to man-made mathematics, you will naturally believe also that there is a "mysterious" link connecting man and nature. Transcendental idealism is a more scientific perspective. According to it, reality is what we perceive or can possibly perceive, directly or indirectly, with our senses, but perceiving is not a passive experience, we are actively involved in organizing the raw material of perception. Not in the way we want, of course, for we are not free to perceive the way we like, but in the only way at our disposition for making sense of the mass of sensations. Ultimate reality is a regulative idea, the complete maximally consistent system of all possible perceptions. Reality is in principle perceivable. The reality of the mathematical science of nature, however, is not perceptual reality, but a mathematical surrogate of it devised for purely methodological purposes. Mathematical reality is not perceivable; it is a creature of reason. There is no gap between man and the world, because the *real* world is the perceptual world, which, however, is *already* a construct: perceptual world = raw sensorial matter (hyle) + form (determined by intentionally loaded psychophysical perceptual systems). Perceptual reality has *already* the marks of human action, but mathematical reality, being essentially pure form, is still more markedly human. Mathematical reality is not real reality, only a substitute of it, that only indirectly and formally represent perceptual nature. Mathematics has all the applications it has in science for the object of science, or rather, the mathematical science of nature, is mathematical, not directly perceptual reality. Transcendental idealism, I believe, is a more consistent and less problematic view than traditional realism.

It is curious that Steiner does not consider this alternative, since the official, the so-called Copenhagen interpretation of quantum mechanics, is definitely not realist. Nature, at least at the quantum level, according to this interpretation, is not independent of us, being determined only at the moment of observation; there is no sharp separation, so the view goes, between observer and physical reality.[51] The relation of causality, in particular, does not appear as a relation among phenomena, but as a functional dependence among *descriptions* of phenomena.[52]

[51] "Then the very idea of facts prevailing independently of observation becomes dubious." (Weyl 1963, p. 258).

[52] "The principle of causality holds for the temporal change of the wave state, but must be dropped as far as the relation between wave and quantum states is concerned" (Weyl 1963, p. 263). From this perspective, the formation of ideas in quantum mechanics must obviously rely less on physical and more on formal or mathematical models and analogies. See "L'influence des images méthaphysiques du monde sur le développement des idées fondamentales dans la physique, particulièrement chez Louis de Broglie", in de Broglie 1994, pp. 103–114.

Steiner's conclusions, being so dependent on his realist, Platonist presuppositions, cannot survive in an idealist environment, and Steiner's arguments, for those who find their conclusions abhorrent, rather than a successful challenge to naturalism, stand instead as a *reductio ad absurdum* of their metaphysical presuppositions.

Chapter 9
Final Considerations

I would like to close with brief critical assessments of some arguments put forward for and against the existence of mathematical objects that I find objectionable from the perspective adopted here. In the sequence, I will state, also briefly, my conclusions before ending by confronting my transcendental-idealist approach to structuralism with more traditional views.

Benacerraf's dilemma It is generally believed, at least in analytical circles, that as a philosophy of mathematics structuralism is a reaction to Benacerraf's argument against numbers being objects, sets in particular.[1] Although this is true for more recent theories, structuralism is not new in the philosophy of mathematics, as the work of Dedekind[2] and the development of abstract algebra, whose central notion is that of an *abstract* algebraic structure, clearly show. Despite its historical role and impact, I believe that Benacerraf's argument is faulty and not a good foundation for structuralism. In my approach, structuralism is compatible with the existence of mathematical objects and imposes itself for logical-methodological rather than ontological reasons.

Benacerraf's argument can be summarized thus: from the obvious fact that the ω-structure can be instantiated – or materialized – in whatever category of objects and in infinitely many different ways, one should conclude that numbers cannot be any *particular* type of objects. The conclusion does not follow, the fact that an abstract ideal structure can comport different materializations does not imply that it has no privileged materialization. As the abstract entity it is, the ω-structure depends on either a material support or a positing theory. Historically, it is quite clear that formal number theory was not an *original* theory, only formally abstracted from one. The original was an interpreted theory, which presupposed an originally given, conceptually framed domain. The original instantiation of the ω-structure is the

[1] P. Benacerraf, "What Numbers Could Not Be" (1965), in Benacerraf & Putnam (eds.) *Philosophy of Mathematics* (2nd ed.) Cambridge: Cambridge University Press, 1983, pp. 272–294.

[2] R. Dedekind, *Was sind und was sollen die Zahlen?* 1. Auflage, Vieweg, Braunschweig 1888.

J.J. da Silva, *Mathematics and Its Applications*, Synthese Library 385,
DOI 10.1007/978-3-319-63073-1_9

domain of numbers proper. However, the formal identity (isomorphism) between the material domain of numbers and the purely formal domains of "numbers" justifies taking, from a strictly mathematical perspective, one for the other. Mathematically, we can do without numbers, what does not mean that numbers do not exist or that only "numbers" do. Other objects – in fact, any objects – can, by instantiating the ω-sequence, *behave* like numbers, but they are not numbers. Numbers, as I have argued here, are objects of a particular type, ideal quantitative forms, and since they belong to a *proper s*ubclass of the class of all objects, numbers have a material content of their own.

Benacerraf can, of course, appreciate the fact that numbers have properties other than structural properties, the properties that they have as materially empty elements of the generic ω-structure. For instance, "10 is the number of commandments". If God had decided otherwise, as He could (and maybe should) this statement would be false. Hence, this is not an *essential* property of the number 10 capable of characterizing it. It is an accidental property capable of singularizing this number in *this* world, but not in *all* possible words. According to Benacerraf, the only essential properties numbers have are structural properties; therefore, numbers are nothing beyond places in the ω-structure.

Being the number of commandments is certainly not an essential property of the number 10 as the object it is, but this number, as all numbers *do* have *essential non-structural* properties, as I show next. Let us define the second-order predicate $I(X)$ thus: for any class A, $I(A)$ iff $\exists x(x \in A \ \& \ \forall y(y \in A \rightarrow y = x))$. A class A of objects has the *property* of being a singleton iff $I(A)$; being a singleton is an *abstract aspect* of A such that $I(A)$. Now, by ideation, one can posit a new ideal *object*, the *number* 1, as that which all singletons, and only singletons, have in common: $1(A)$ iff $l(A)$. The quantitative form denoted by 1 is *instantiated* in a collection A iff A is a singleton. The assertion "1 is the ideal quantitative form instantiable in all singletons and only in the singletons" is an *essential*, but *non-structural* property of the number 1, capable of characterizing, not only singularizing it. The same is true for all numbers. This is enough to block the derivation of "numbers are not objects" from "any category of objects can instantiate, in infinitely many different ways, the ω-structure". It is also a proof that numbers have essential non-structural characterizing properties. Mathematics may be *uninterested* in properties other than structural properties, but this does not imply that mathematical objects do not *have* essential non-structural properties.[3]

The Causal Inertness of Mathematical Objects Benacerraf has also famously argued against Platonism along the following lines. Suppose that mathematical objects exist *objectively*. Therefore, mathematical knowledge is *objective knowledge* of objective entities. Question: how can we obtain knowledge about mathematical objects or even refer to them? This is the famous *access problem*. There must be a way in which mathematicians "grab" mathematical objects to investigate their properties. How? In case of the empirical objects of the empirical world, there

[3] Of course, Benacerraf may have difficulties in accepting empty forms as bona fide objects.

is a simple answer: because they act upon us along causal chains. We can just *look* at them, or anything to the same effect. This, argues Benacerraf, cannot work for mathematical objects because they are *causally inert*. Two possibilities, either we give up a uniform semantics, i.e. a uniform *causal* theory of reference and truth for both empirical and mathematical objects or give up our Platonist belief in mathematical objects.

The truth, as I explain below, is that there is no "access problem"; referring to and "grabbing" mathematical objects pose no special difficulties. The only reason why someone might think there is one, as I have already emphasized, is a wrong conception of the nature of mathematical objects. Since they are *objective* entities, so the view goes, they must *also* exist *independently*. Moreover, since mathematical objects are "just like" empirical objects, existing independently "out there", a uniform semantics seems the natural way to go. After all, since mathematical and empirical objects enjoy the same ontological status, there is no reason why the same *causal* semantics should not work for both. However, it does not and cannot work; therefore, mathematical objects do not exist objectively. Hence, if mathematics is an objective science, it cannot be a science of objects.

Of course, the only conclusion this line of reasoning allows is that mathematical objects are *not* "just like" empirical objects. Indeed, albeit being objective, mathematical objects do *not* exist independently – objectivity and independent existence are neither identical nor equivalent notions. Precisely by requiring intentional action to come into being, a road lies open for accessing mathematical objects, namely, by inquiring intentional positing and intentional meaning. In a formula, to constitute is *also* to access.

It is the identification of objectiveness with ontological independence that originates the problem of access. Making mathematical objects things about which we have no responsibility also makes them inaccessible. I would like to make a parallel with semantics. If objective meanings exist independently out there and speaking meaningfully consists in "grabbing" meanings with words, how can we do this? A coherent answer must, I think, begin by questioning the independent existence of meanings. They are, of course, objective, but not ipso facto independent. As soon as we realize that meanings are infused into words by intentional action, that *objective* meanings are constituted in *subjective* experiences, we can see how meanings are "grabbed'. However, and this is important, the ego is, in this case, the entire linguistic community, and consequently it befalls on the linguistic community at large the responsibility of securing linguistic correctness. To use words meaningfully is to use them in accordance with their intended meaning, which requires, objectively, abidance to shared criteria of meaningfulness (the correct use of words in accordance with their meaning) and, subjectively, the intention to mean, a sort of implicit participation in the meaning-positing act. The words of a parrot, for example, lack meaning for no meaning-intention accompanies them; they are meaningless, even if the same words could, in different circumstances, say, a regular conversation among people, be uttered meaningfully. There are *subjective* meaning-intention and *objective* criteria of successful meaning-intention. Of course, no one is individually responsible for meaning constitution; by learning a language and joining a linguistic community, we join other competent users of that language as the communal meaning-given ego. No individual ego can take upon itself the task of changing the meaning of words, for words are communal possessions and can only acquire meaning in community. Community surveillance establishes a right and a wrong as to the meaning of words. Any individual is free to give words the meaning he likes, but by so doing, by giving up objective parameters of right and wrong, he may be subtracting himself from a particular linguistic community. Meaning is not simply use, but that which words acquire in meaning intending acts, manifesting itself in use under the surveillance of the community of language users, the intentional ego in this case.

The Platonist picture of independent mathematical objects on the one side and the knower on the other is bound to create a problem of access. The solution is, again, to abandon the Platonist view of things, to preserve the objectiveness of mathematical objects whereas denying them independent existence. Mathematical objects are intentional constructs and the ego can access them in the constituting acts themselves. No object exists that is not an object-for-the-ego; no ego exists that is not an object-intending ego. The intentional ego and the object it intends are at the opposite poles of a relation of codependence. An object is "grasped" when it is intended, as it appears in the act, with the intentional meaning attached to it in that act. The privileged way of accessing intentional objects is by unveiling their intentional meaning or extending it by direct analyses, if the objects are given intuitively (keeping in mind that *intuition* is a generalized form of *perception*). Mathematical objects are "out there" indeed, objectively, but non-independently. Accessing them goes along with meaning them. The gap between the object and the knower that originates the access problem is no longer there.

The Indispensability Argument One of the most popular arguments for Platonism in mathematics is the so-called indispensability argument. It purports to establish the existence of mathematical objects from the indispensability of mathematics in empirical science. I agree with both the premise and the conclusion, mathematics is indeed indispensable in science and mathematical objects do indeed exist, but the former has no consequence for the latter. Of course, existence, as I mean it, is intentional existence, not Platonically construed existence. Let us consider the argument from closer up. The general form of an indispensability argument is as follows: one must believe something for so believing is indispensable for certain purposes.[4] Colyvan presents some variants of indispensability arguments; here they are[5]:

1. (Scientific Indispensability Argument) If apparent reference to some entity (or class of entities) ξ is indispensable to our best scientific theories, then we ought to believe in the existence of ξ.
2. (Quine/Putnam Indispensability Argument):
   (a) We ought to have ontological commitment to all and only those entities that are indispensable to our best scientific theories.

[4] See H. Field, *Realism, Mathematics and Modality*, Oxford: Blackwell, 1989, p. 14.

[5] M Colyvan, *The Indispensability of Mathematics*, Oxford: Oxford University Press, 2001.

(b) Mathematical entities are indispensable to our best scientific theories. Therefore,
(c) We ought to have ontological commitment to mathematical entities.

3. (Semantic Indispensability Argument) If apparent reference to some entity (or class of entities) ξ is indispensable to our best semantic theories of natural (and scientific) languages, then we ought to believe in the existence of ξ. Abstracta are indispensable to our best semantic theory of natural (and scientific) languages. Thus, we ought to believe in such abstracta.

As far as the existence of mathematical objects is concerned, we can amalgamate these arguments thus: we must believe in the existence of mathematical entities because our best scientific theories and the semantic of scientific theories are committed to them.

But I prefer still another version: mathematical theories are indispensable in science; nothing could be indispensable in science that were not true (why should *falsities* be *indispensable* in science?) Therefore, mathematical theories are true. But, if a theory is true, that to which the theory refers, that which it is *about*, must exist (for, otherwise, how could it be true?) Therefore, mathematical objects, that which true mathematical theories are about, must exist.

The presuppositions of the argument are the following:

(a) Mathematics is indispensable in science.
(b) If a theory is indispensable in science, it is true.
(c) Theories refer to their objects and if true, they are true *of t*hese objects.
(d) If a theory is true, the objects to which the theory refers exist.

From (*a*) and (*b*), it follows (C_1): Mathematical theories are true. From (*c*), (*d*) and (C_1) it follows (*C*): Mathematical objects exist.

Now, I have no qualms about (*a*). (*b*), however, is unacceptable. To start, it has a hidden presupposition, namely, that scientific theories are, to some extent at least, true of empirical reality and anything that is *indispensable* to formulate, express and articulate scientific truths must be true as well. If one believes in such a thing, one also believes, I guess, that a novel that contains psychological truths must be about real people. Of course, implicit also in the argument is a theory, the *correspondence theory of truth*, and a *realist presupposition*, namely, that the domain of a theory is independent of the theory. This comes out in (*c*) and (*d*); there are mathematical objects independently existing out there, on the one side, and theories that describe them, on the other. Once this is accepted, it follows that these objects must exist if the theory is true.

The problems with this argument are premise (*b*), which misconstrues the ways mathematics works in science, and premise (*d*), which incorporates realist presuppositions. There are problems with (*c*) too, which seems to presuppose that the objects of a mathematical theory are the objects to which the individual terms of the theory refer, not, as is the case, the structure they instantiate. With the correct reading, however, namely, that the object of a mathematical theory is an ideal structure, (*c*) would be acceptable, provided one does not give the structure the theory describes independent existence instead of, as is always the case, intentional existence.

Premise (*a*) is the only that is immune to criticism because it expresses an obvious fact. Mathematics is indeed indispensable to science, but from this, one cannot infer neither that nature is *in itself*, intrinsically, mathematical nor that mathematical objects are *themselves* part of nature, or rather, a "Platonic" extension of nature. The indispensability of mathematics in empirical science follows from the fact that empirical reality, as explained before, is a mathematical intentional construct devised, for methodological reasons, to organize and articulate our perceptual experiences, explain them, help to predict future experiences, and sometimes disclose formal possibilities that may or may not be given a material content in experience. As any mathematical manifold, empirical reality and its mathematical theory benefit from our knowledge of other mathematical manifolds and theories.

Mathematical theories are either structural descriptions of ideated abstract structures, embodied in particular structured systems, or descriptions of formal domains the theory itself posits. A formal domain, on its turn, is the common formal core of a family of ideal structures in principle instantiable (in case the theory is categorical, the formal domain of its interpretations coincide in the structure the theory posits). *All mathematical theories are structural descriptions*; mathematical objects may serve as privileged supports of mathematical structures, but mathematical theories are *not* primarily concerned with them, only with their formal-structural properties or the structures they instantiate.

A mathematical theory benefits from another mathematical theory insofar as one can be translated into another, maybe a more adequate or methodologically richer theory. We have seen some examples of such "translations" in terms of interpretations of structures into structures. Mathematics is *indispensable* in science *to the exact extent* that empirical reality is mathematical, and *useful* to the exact extent that (mathematical) theories of empirical reality are, for methodological purposes, translatable into richer mathematical theories.

In details:

(a) Empirical reality is a mathematical construct devised for methodological purposes.
(b) Scientific theories are theories of empirical reality, and as any mathematical theories, they are structural descriptions.
(c) Scientific theories that merely *describe* conveniently idealized abstract aspects of *perceptual* reality are the so-called *phenomenological theories*.
(d) As any mathematical structure or any mathematical theory, mathematized perceptual reality and phenomenological theories are interpretable in other structures and theories. This extends the range of mathematics in science, refining the *predictive* and *explicative* roles of scientific theories and opening a *heuristic* dimension in science.
(e) Interpretations can be iterated.

Although the indispensability argument does not establish the existence of mathematical objects, mathematical objects exist, only not as Platonists believe. However, objects, mathematical or not, are *not* the primary focus of interest in mathematics; they serve mostly as supports of mathematical structure, which are the real objects of mathematics. Mathematical objects also have non-structural properties, their material properties, in particular, which, however, are not mathematically relevant. Being thoroughly formal is a *choice* that characterizes mathematics, which explains its total disregard for matter. Mathematics also *chooses* to investigate mathematical objects *only* with respect to certain of their mutual relations; any other properties they may have that are not expressible in terms of these relations fall off the picture – they are the mathematically irrelevant non-structural properties, *even if they could be structural properties under different structuring of the system of objects*. Objects of mathematics, either the individuals where structures are instantiated or ideal abstract structures themselves, exist, but not on their own right; mathematical objects, instead, *come into existence* by intentional action. Intentional existence is granted to all objects whose positing is consistent, and *nothing but consistency* is required for a mathematical theory to play a role in science. Utility and indispensability of mathematics in science have no ontologically "robust" (realist) consequences.

The "Unreasonable Effectiveness" of Mathematics I offer the following arguments against the so-called "unreasonable effectiveness" of mathematics in science and the mystical consequences some have drawn from this supposedly unreasonableness.[6] "Unreasonable effectiveness" arguments have their presuppositions, which, unsurprisingly, are realist presuppositions passing off as established wisdom. One of them is that science describes nature *itself*; that there is no intermediation between nature out there and science over here. If this were so, it would be indeed mysterious how mathematics, a human creation, could have any role in science. Unless, of course, nature were intrinsically mathematical and mathematics created by observing nature. The indispensability and applicability of mathematics in science would be, in this case, unproblematic. But this is not how mathematics is in general created, with some notable exceptions, and a single application of free-invented mathematics in science would indeed indicate, as Steiner suggests, that nature is perhaps not only intrinsically mathematical but that there is some inexplicable link between man's mathematical creativity and "natural" mathematics.

To explain the scientific indispensability of mathematics, however, one does not have to presuppose that nature is at its inner core mathematical, it suffices to establish that our *perception* of reality is at least proto-mathematical. And this is *obvious*, since our perceptual experiences necessarily form a structured system. The system of perceptions, however, is not a mirror reflection of the system of things, but a proto-intentional production of our perceptual systems based on raw sensorial impressions. The ideally complete system of objectively valid *perceptions* is *nature for us*. But nature for us is not yet the empirical reality of the mathematical science of nature. Empirical reality is a mathematical "exactification" of the system of perceptions, which is then given over to the care of mathematics. Instead of the

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[6] I mean, of course, mainly, E. Wigner (Wigner 1960) and M. Steiner (Steiner 1989, 1995, 1998)

"unreasonable effectiveness" immediate relation: nature out there → mathematical science of nature, we have instead: metaphysically real nature → sensations → perceptions → empirical reality → phenomenological theories → full-blown mathematical science of nature.

The first step, nature → sensations is beyond the reach of science, for would require a non-sensorial form of access to nature, and should, maybe, be handed over to metaphysics. The step sensations → perceptions is, of course, object of theories of perception, which have shown beyond reasonable doubt that perceptions are intentionally loaded constructs. The step perceptions → empirical reality is largely a matter of mathematical modelling, involving intentional actions such as abstraction, idealization, and ideation, besides probably others. The requirement is that empirical reality somehow represents to a reasonable degree of effectiveness, and objectively, the system of perceptions. After this step, mathematics comes in with full force. Phenomenological theories are a first mathematical "draft" of empirical reality; they come into being concomitantly with the first mathematization of perception. At this point mathematics plays essentially a *representational* role, it is a language and a conceptual system with which one tries to capture and "improve" the proto-mathematical aspects of the system of perceptions. In the last step phenomenological theories → full-blown mathematical science of nature, mathematics is allowed to play other roles, for example, as a context of internal theoretical articulation (the *instrumental* use of mathematics) and as a *heuristic* instrument.

"Unreasonable effectiveness" philosophers are particularly impressed by the heuristic role of mathematics in science. But as I have shown, as soon as one becomes conscious of the mundane and trivial fact that mathematical structures and structural descriptions can profit, for methodological reasons, from immersion in other mathematical structures and descriptions the mystery vanishes. With one important observation, mathematical "predictions" concern only *form*, not material content. Mathematics can point out interesting formal facts, which however correspond to facts of the system of possible perceptions only if they can be given a material content therein. Otherwise, when the established semantics determined by the system of possible perceptions has no way of filling in a consistent manner the formal predictions of the mathematical-physical theory (e.g. positive charged electrons), talk of "discovery" is misplaced; if discovery there is, it must be credited to the scientist's genius who comes up, by extending the system of possible perceptions, with a new semantics that gives apparently materially meaningless formal "predictions" a material content. And even though only after the facts predicted have actually been verified, since they may very well not be.

"Platonism" Without Platonic Heaven The quotes in "Platonism" are not scare quotes; they have a technical function. As "the battle of Waterloo" does not refer to a battle, but to the meaning of the name of a battle, "Platonism" does not refer to a metaphysical view, but to the meaning attached to it. The question is this: can we endorse the meaning without embracing the view, i.e. "Platonism" without Platonic heaven? The answer is, yes, we can.

As explained before, once we have contemplated a few small numbers directly by abstraction and ideation, and the way they relate operationally to one another, we have no difficulties in imagining the whole arrays of numbers, of course in a purely intentional manner. A domain of a particular type of objects *opens up* to us, a domain of forms that can receive different material instantiations. Some numbers present themselves as numbers of collections we actually perceive, as the number of the fingers in my hand. But most of them are idealizations only potentially instantiable. I have never seen 10^{10} objects together, but I can *conceive* the possibility, no matter how ideally, of contemplating such a collection. A domain is posited, and it makes sense to inquire the sense with which it is posited. How do we *conceive* this domain to be? The answer, of course, is that we think it to be ontologically complete, in the sense that any possible situation in the domain is objectively determined. In other words, anything that *makes sense* to say about numbers i.e. anything that we can phrase in the *language* we use to refer to numbers, either describe or does not describe how things stand in the numerical domain. We also conceive the domain to be such that if a possible situation is not a fact, the complementary situation must be. Alternatively, if a sentence of the language of number-theory is false its negation must be true, and conversely. In short, we conceive the numerical domain to be objectively complete, and we can reason about such a domain using classical logic.

This is not something for which we have a proof or any type of evidence in the usual sense. This is not a hypothesis that future investigation could confirm or disconfirm, but the way we *conceive* the domain of numbers to be. Although it has been *proved* that our number-theory cannot, under reasonable constraints, decide any meaningful numerical assertion, number theoreticians are *convinced* that they all have a definite truth-value. This is often taken as a clear sign of allegiance of mathematicians to metaphysical Platonism, at least as working, not philosophizing mathematicians. But the conclusion does not follow. The fact that mathematicians are convinced that the domain of numbers is objectively complete only shows that this is how mathematicians *conceive* the numerical domain to be, which has no bearing on how this domain is *independently* of being so conceived, *if* it indeed were so. Some philosophers think that this way of conceiving must be *justified*. But how, they wonder, if not by a metaphysical presupposition, namely, that this is how the domain *actually is*? The conception is made to rest on a metaphysical parti-pris. It takes phenomenological insight for one to realize that there is a gap between intentional conceptions and metaphysical theses. Bracketing a sense of being, as phenomenologist say, is withdrawing ontological commitment to the being that has this sense. But the sense remains as intentional sense, a way of thinking that goes along with conceiving such a being. The sense of being is *justified* insofar as it is an essential element of the conception. If this is a presupposition, it is a transcendental presupposition, not a provisory hypothesis. In scientific contexts, intentional conceptions are not *in the mind* of the individual subjects who conceive it; on the contrary, the conception is a *communal objective* possession shared by all the individual subjects that together constitute the intentional subject. The numerical domain, objectively given as the same to all, is an objectively complete domain for this is how the constituting community *conceives* it to be. No metaphysical commitment should result from this.

The possibility of existence of objective domains that are *not* ontologically independent is something that we learn from phenomenology, but that seems ignored in analytic philosophical circles. The reason is clear. Empiricism together with anti-psychologism blocks any possibility of conceiving existence and being that is not as ontologically independent existence and being, in the manner of the real empirical object of the external world. Platonism is the consequence. Phenomenology opens new ways of conceiving existence that are more appropriate for the mathematical being. One does not have to give up the “Platonic” way of thinking of practicing mathematicians, but we are allowed to let the Platonic heaven go.

Structures Let us recall. A structured system is a collection of objects – where by object I mean anything that is a bearer of attributes – where a system of relations is defined. Our perceptions, for example, constitutes a structured system (and this is fundamentally the reason why mathematics has a place in science). When describing a structured system, we first select a language convenient for the task – I called this the structural language. The objects of the system may have properties and stand in relations that are not of interest to us; consequently, the structural language will not have symbols for them and the structural description will pass them by in silence. By selecting the relevant structuring relations, we determine the structure of the system and with it the structural language. Now, an important fact: descriptions are not true only of the particular system described, but of all systems that are isomorph to it. Structural descriptions are completely indifferent to the material content of the systems described. Hence, *that* which the descriptions *actually* describe are, obviously, something else that is identically the same in all structured systems satisfying, under material reinterpretation, the description. The ideal structure instantiated in a structured system is that which the abstract structures of all the systems isomorphic to it instantiate. Ideal structures are instantiated as abstract structures of particular structured systems. Mathematics is essentially the science of these idealities. Mathematics can access instantiated structures intuitively by accessing *any* system that instantiates the structure. The mathematician must, of course, remain within the limits of the structure in question (i.e. he must ignore non-structural properties of the system), but also perform the necessary acts by which the ideal structure comes to consciousness. Ideally, a structural description should leave no question that can be meaningfully raised, i.e. any question that can be phased in the structural language, without an answer. If the structural description can answer all possible questions about the system, we say the description is complete. Complete descriptions are rare, but incomplete descriptions can be improved, either directly by a more thoroughly intuitive examination of the structure in question, or indirectly by other means, examples of which were given before.

Although a number of structured systems offer themselves naturally to mathematics, such as the systems of numbers and idealized spatial forms, or more abstract ones such as the system of symmetries of a solid, mathematics also *creates* (posits) systems of objects, such as, for example, imaginary numbers and quaternions, with the sole purpose of bringing ideal structures into existence. Mathematics is not only concerned with structures it finds instantiated in nature, but also actively involved in creating structures that *could* in principle be instantiated no matter where. The mathematical investigation of structures are carried out completely a priori, even when they are extracted by abstraction from nature (and idealized). One can access a structured system, and the structure it instantiates, by accessing the system, either directly or indirectly, via the concept that delimits the domain. If the structure is posited by a formal theory, it can be accessed through this theory, which at the same time posits and describes it. Structure-positing descriptions are required to be categorical and even if they are not logically complete, it is presupposed (transcendental presupposition) that any meaningful question that can be raised about the structure is *in itself* decided, and can be actually decided by means other than deriving the logical consequences of its positing theory. Although materially indeterminate, the formal objects of a structure in abstracto may have non-structural properties too, if relations are definable in the system of objects that are not structuring relations.

A theories that is neither complete nor categorical, as many mathematical theories, is an incomplete structural descriptions not of one, but a family of structures. Structure-positing theories must be consistent, or at least not manifestly inconsistent. If, however, an inconsistency should manifest, the positing is cancelled. I do not suppose that structure-positing theories have models in the usual sense of the term. Even if no set-theoretical model is available, the description, if consistent, *intends* to characterize a structure that it also describes. Structures or structured systems do not have to be *sets*, pure or impure, but collections, multiplicities, universes, with all the vagueness these terms comport. The universe of mathematical sets themselves, for example, is a structured system that is not a set. This system is particularly interesting for other mathematical structures are *interpretable* in it. This fact, however, has no ontological consequences; one cannot infer from the willingness of the system of sets to accommodate other structures via interpretations that sets are somehow the fundamental objects of mathematics; they are not.

Abstract structures, and the ideal structures they "incarnate", can be considered as structured systems of formal objects (or forms of objects), i.e. objects considered merely as such, devoid of any attributes or properties other than those they have for standing in relation to other equally materially empty object-forms in a system. Objects of different structures are different objects, for they stand in different systems of relations. The object we call 2 in the ω-structure is not the same object as that which we call by the same name in the R-structure. For example, the real 2 has the property of having an inverse additive that the natural number 2 does not have. However, we can interpret the ω-structure in the R-structure and have the real 2 play the role of the natural number 2. But this requires that we restrict the system of relations into which the real 2 participates in the R-structure. Taken as the natural number 2 the real 2 does not have an inverse additive. Formal objects are dependent objects that exist only as nodes of abstract structures. Under the presupposition of objective completeness, a structured system, of materially filled or empty objects, must satisfy one, and only one, of any pair of contradictory assertions expressible in the structural language, even if the positing theory, for being logically incomplete, cannot decide which. There are, as we have seen before, other means of decision. Even if a particular assertion is effectively undecided, we presuppose nonetheless that it is in itself decided; a method for deciding it may eventually become available. This is not a hypothesis, but a transcendental presupposition of standard mathematical positing.

One of the most relevant fact about structures is that one can investigate structures and improve structural descriptions by means of other structures and descriptions via interpretations. *The applicability of mathematics, in empirical science in particular, rests entirely on this fact.*

Structuralism Structuralism is the view that mathematics is essentially the study of ideal abstract structures; that the real (maybe only) objects of mathematics are structures, *regardless of the objects on which they stand*, sometimes *mathematical objects posited expressly for this end*. As I said before, modern versions of structuralism are attempts at answering Benacerraf's dilemma by eliminating the usual objects of mathematics leaving only structures behind. Unfortunately, contemporary structuralism shares with the traditional philosophies of mathematics an inadequate notion of existence, originating a series of perfectly avoidable ontological and epistemological problems. Consequently, structuralism faces the same problems it was supposed to overcome. There are a mathematical, a Platonic, an Aristotelian, and a modal, essentially nominalist, version of structuralism. Essentially, they are ways of dealing with the age-old ontological issues. As an ontology-free perspective, structuralism is a failure.

(a) Set-theoretical Structuralism This is the mathematical version; it identifies structures to structured sets. In this perspective, all structures are what I have called structures in concreto instantiated in the domain of pure sets. From the perspective of set-theoretical structuralism, abstract structures do not exist independently; they are always instantiated. Of course, the empiricist would prefer them to be instantiated in the empirical world, but this cannot be, for the empirical world is not mathematically rich enough, no matter how idealized; therefore, the set-theoretical structuralist must find a substitute. The natural choice is the universe V of sets, since all mathematical structures are interpretable in V. There are two obvious problems with this alternative. First, sets must exist and so one has not gotten rid of mathematical objects, as one might have wanted. Second, mathematical structures are not uniquely instantiable in V; so, of the many isomorphic instantiations of a given structure, which is the real one? Benacerraf's dilemma is back in great style.

There is a common misconception among philosophers of mathematics that sets are more concrete entities than, say, structures, supposedly more abstract. This is blatant in set-theoretical structuralism. The truth, however, is that both are equally ideal abstract forms. Pure sets are constructible out of the empty set and the "materiality" of the former depends on that of the latter. Set-realists such as set-theoretical structuralists usually take the empty set as a bona fide object, on a par maybe with tables, chairs and beer mugs. Denoting the empty set by $\varnothing$ helps to give this illusion, but giving this set a proper name does not make it into an object with form *and* content. In truth, the empty set is a pure contentless form; for this reason, I prefer to denote it thus { }. This notation shows clearly that there is nothing there but the form of a frustrated set-formation intention. The empty set is the objective correlate of an intention that *cannot be fulfilled*. It is an object because it stands at the objective pole of an object-positing act: set-unification, but it is an *empty* object because there is nothing there to be collected. Sets have sometimes form and content (they are called impure sets), such as, for example, the singleton of a chair, whose material content is the chair and whose form is the collecting an unifying that go into collecting the chair and taking the resulting collection as an object. As a collecting intention that collects nothing, the empty set is an objectified pure form. It is a more abstract entity than, say, the singleton whose sole element is a chair, and as formal as, say, the number 1. *V* is a structured system of empty forms. Denoting sets by symbols such as, for example, {{ }, {{ }}}, renders explicit the formal structure of the series of set-collecting acts. The innermost vacancies can be filled in principle by anything whatsoever, in particular real objects like, say, table, chairs and beer mugs. The resulting object is a *materially filled* intentional construct with many layers of superposing collecting and unifying intentions. Unfilled as they are, *pure sets are nothing but pure forms*. They display, nonetheless, an internal structure that reveals the constituting intentionality that goes into their making. The more complicate the internal structuring is, the higher the set lies in the set-theoretical hierarchy. In short, *pure sets are forms*, and there is no reason why supposing their *independent* existence is ontologically more satisfying than supposing the independent existence of structures, which are intentional objects of the same ontological type, namely, forms.

(b) Ante rem Structuralism This is the Platonic version.[7] According to this perspective, structures are independently existing entities. The main problem with this view is the usual one, how to access structures that exist independently of us. One possibility is via their instantiations; but this only makes the problem worse. Although finite structures can have physical instantiations, the vast majority of mathematical structures cannot. The group of symmetries of a cube, for example, is physically instantiable, but the ω-structure is not. It is not clear to me how one can access the supposedly independent ω-structure from finite sequences of objects that are immediately accessible to perception. How can we "see" finite sequences of whatever objects structured *as* initial segments of an ω-structure independently of a meaning-giving act determining what is it that we are seeing? Seeing the possibility in principle of indefinitely extending the originally given finite sequence of objects *in imagination*, conceiving "arbitrarily and indefinitely" as meaning "provided it remains finite but without an upper bound", is a way of *meaning* it *as* an initial segment of a ω-sequence, and *seeing* it as such requires this subjacent meaning-intention. *Meaning* a given structured system as *part* of a larger system instantiating a certain structure is *not* simply *abstracting* this structure from the given system. For one to *abstract* a structure from a system, this system must *actually* instantiate the structure. Platonic structures, if they exist, cannot be instantiated in anything

[7] This view has been advanced by, among others, S. Shapiro (see, for instance, Shapiro 1997).

that is *merely* given. It is not clear to me, then, how they can be given to us. If intentional acts are required so the knowing subject can access supposedly independently existing structures, there is not much content to their supposed independence. Independent entities, I think, must be capable of simply *manifesting* themselves.

However, if we accept that numbers exist and the ω-structure is primarily instantiated in the domain of numbers we can access the ω-structure by accessing numbers. Recall that given *any* number to consciousness (via abstraction and ideation) one immediately "sees" (intuits) that one can obtain further numbers by withdrawing or adding units to the given number in imagination. There is a limit for withdrawing but none for adding; in other words, there is a first but no last number. If numbers are conceived as finite collections of units, one also "sees" that all numbers can be obtained by either adding or withdrawing units from the given number. By such *acts,* one constitutes the succession of all numbers and is then able to access the ω-structure that underlies this succession. It is not a matter of "seeing" the ω-structure standing "out there", but "seeing" this structure being *constituted*. By generating numbers from a given number in accordance with a *conception* of number (finite collections of abstract units), one constitutes a sequence of numbers and its underlying structure, the ω-structure. However, since Platonic structuralists believe in the independent existence of the ω-structure, but not in that of numbers, it is not clear to me where this structure is supposed to be instantiated in order to be accessed.

For the Platonist structuralist, structures are, from a strictly mathematical perspective, structured systems of places taken as objects. Excluding the metaphysics, I have no problem with structures being taken as structured systems of places as objects. I prefer, however, to think of "places" as formal objects, or object-forms that can materialize as no matter which objects proper. Some have argued that this way of seeing raises some difficulties generically known as "identity problems". As was already discussed, formal objects are a particular type of objects, with peculiar properties. They, for instance, have no internal structure and no properties other than structural properties, i.e. properties they have in virtue of standing with other formal objects in the system of relations that characterize the structure to which they belong. Some things follow by way of consequence. One, already discussed, is that one cannot identify formal objects in different structures. This is indeed the case, since a formal object does not exist if not as an element in a web of relations that gives it its "personality". Change the system of relations the object changes. Nonetheless, since structures are interpretable into one another, objects of one structure are interpretable as objects of another structure. This, however, is an *interpretation*, not an *identification*.

Another "identity problem" relates to the fact that formal objects are only identifiable up to isomorphism.[8] Indeed, if i is an automorphism of a structured system S of formal objects onto itself, S and $i(S)$ are exactly the *same* system. If one looks at S and then at $i(S)$ one would be incapable of detecting any change, *except by somehow identifying elements of S by non-structural means*, such as naming.

[8] See Keränen 2001.

Suppose that we give *names* to formal object of *S* and the objects named *a* and *b* are such that $i(a) = b$. In this case, *a* and *b* are names of the *same* objects. The automorphism may change *names*, but names can change with objects remaining the same: what we had previously *called a*, we now *call b*. Naming is tantamount to giving the things named non-essential, although characterizing properties: the object named *a* is the only object that has this name; being named *a* characterizes *a*, but non-essentially and non-structurally.

I do not see any of these so-called "identity problems" as a problem. The fact that formal objects cannot be identified as materially filled objects only shows that there are ways of identifying material objects, namely, by their *material* properties, that are not available to formal objects, as it is reasonable to expect,. Taking this as a "problem" is to confuse formal objects with material objects, a category confusion.

(c) In re Structuralism This is the Aristotelian variety; according to this view, all structures are structures in concreto, i.e. instantiated; they are abstract aspects of structured systems. The first, obvious, question this view raises is where, precisely, structures are supposed to be instantiated. It cannot be in empirical reality, for the empirical world is not large enough for it; it does not have enough objects. A natural candidate is, as always, the universe of sets; but then we would be back to set-theoretical structuralism. There is a way out, however; when asserting a mathematical truth, so the view goes, we are in fact asserting a universal truth: *for all* structured systems of such-and-such a type, so-and-so is true. The problem now, as many have pointed out, is that if there is no system of the required type anything that we feel like saying about it turns out to be vacuously true.

One can avoid this problem with the introduction of modal notions.[9] *If* so-and-so a structured system *actually existed*, *then* so-and-so *would* be necessarily true of it. One does not claim that structures exist, only that they *may* exist, whatever this means. Structured systems are *possibilia*, whose structural properties mathematics investigates a priori. Mathematics is thus a sort of pre-occupation. Such a view comes close to capturing an important aspect of mathematical activity, but misses the target for its incapability of conceiving a way of existing that is not *real* existence, i.e. existence in the mode of the real, temporal object. To begin, what is it supposed to mean, that a structured system *may* exist? In what sense of existence and which sense of possibility? Of course, it does not make much sense to conceive *all* structured systems as *real* possibilities, for this would require a substantially enlarged conception of reality. And if not real possibility, what? Logical possibility is an alternative; but what does it mean if not that a certain *conception* is internally and externally consistent, i.e. consistent with itself and the system of conceptions to which it belongs? If this is all that logical possibility means, why cannot it apply to *abstract* structures directly? Why is the difference between a concrete and an abstract logical possibility? In fact, I do not see any substantial ontological difference between structured systems conceived as logical possibilities and abstract

[9] See Hellman 1989.

idealized structures similarly conceived. Unable to eliminate abstract structures in favor of supposedly more "real" structured systems, but incapable of finding all the structures mathematics needs actually instantiated or effectively instantiable, *in re* structuralism takes a nominalist turn. It is enough that structured systems exist only as mere logical possibilities, the notion of logical possibility being taken as "primitive" (taking a notion as primitive usually means that one does not know what it is). Why cannot we take *abstract* structures as logical possibilities in the same sense?

The fact however remains that mathematicians refer to mathematical objects, including structures, as actual objects, not mere possibilities, whose existence is granted provided their theories are consistent. Moreover, they are not real and concrete, but abstract and ideal objects. The task of the philosopher of mathematics is to take mathematical discourse and practices at face value; then, clarify and justify them. In particular, explain what sort of objects mathematical objects are and in what sense they exist. However, most philosophers of mathematics, out of empiricist prejudices, prefer to dismiss mathematical discourse as a "manner of speaking", get rid of mathematical objects in one way or another and reinterpret mathematical assertions accordingly, or attribute to the objects of mathematics an absurd way of existing. I, on the contrary, prefer to dismiss empiricist prejudices and take the existence of abstract objects seriously as a philosophical problem: what are they, in what sense they exist? The phenomenological notions of intentional object and intentional existence offer, I claim, a possibility of conceiving abstract and ideal objects as correlates of intentional actions, existing as such with the sense they are given in positing experiences. By conceiving structures as *cogitata*, as intentional constructs existing as correlates of particular positing acts, worlds intentionally meant that could, *even when not actually instantiated or effectively instantiable*, help us understand structures that *are* or *can actually be* instantiated, we can construe the notion of *possibility* of the modal variant of structuralism as *intentional actuality*.

It is tempting to read my approach to structuralism as a sort of fictionalism. This is not so for, as I see the matter, mathematical objects are *not* posited *as* fictions. *Not* existing as posited is not part of the intentional meaning attached to the positing of ideal abstract structures, rather the opposite. Structures are actualities, although not real actualities. Besides, there are structures that are quite *real*; namely, the abstract structures of structured systems of *real* objects. Although abstract structures only acquire mathematical status when made into ideal entities by ideation, some ideal mathematical structures are either actually instantiated or effectively instantiable in real systems. They are definitely not fictions. Structures in abstracto come close to being fictions, but not quite so. Fictions are intentionally posited with a sense of being that they fail to have. For example, when I say that Sherlock Holmes is a fictional man, I say that there is no real, living, breathing man who is Sherlock Holmes. *As a character* of a series of books, however, he exists with all the rights of (fictional) existence; he exists *in the story*. A fiction is an intentional object that fails to exist according to the sense of existence of the category of being to which it is supposed to belong. An intentionally posited structure, on the other hand, if the positing does not vanish due to internal or external inconsistencies, exists with the sense of existence proper to mathematical entities, which is *not* real existence. A structure would only be a fiction if it were posited as an object of an ontological type, for example, as a real object, that happens not to exist as an object of *this* type. In short, structures in abstracto are not fictions, but perfectly (intentionally) existing entities, at least as far as the positing remains consistent.

From the pragmatic perspective, however, as instruments of research, structures can behave just like fictions, but fictions are very important things. Charles Swann and Odette de Crécy are not real people; they are fictional characters (although inspired on real people). But by reading their story in Proust's *Recherche* ("Un amour de Swann") we learn a lot about the mechanisms of *real* love and the torture of *real* jealousy. How can this be so? Simple, the genius of Proust in inventing these characters – who come out alive, as we say – was such that we can *project* ourselves on them and our experiences on theirs to learn *about us*, our real experiences of love and jealousy, from their fictional behavior and feelings. All great art creates worlds where we can live. Lower quality art, on the other hand, by creating poor, fake, low-quality worlds, on which projections are either difficult or useless, is incapable of teaching us anything of value about us. Mathematical structures, even if not actually instantiated or instantiable in the real world, can teach us a lot about it by means that bear similarity to the ways art helps us to understand reality. As we can project reality in art, we can *interpret* empirical reality, which as I have sufficiently emphasized is a mathematical construct, into mathematical structures and derive in larger, richer contexts truths about empirical reality that are not accessible more directly.[10] This contains the essence of the applicability of mathematics in science and the life-world.

Matter and Form My views on the nature of mathematics and its applicability depend heavily on the distinction between matter and form. Mathematics, I claim, is a science of forms and its wide applicability rests entirely on it being a formal science. All objects of mathematics are, in some sense or other, forms. Numbers, spatial configurations, sets, abstract algebraic structures are all forms.

It is not so easy to characterize precisely what forms are. They are of the same nature of concepts, but differ from them in important aspects. Both are universals, in the sense that different objects can have the *same* (not only similar) forms and fall under the *same* concept. Matter, on the other hand, is an individualizer; different objects can be different for being materially different, although formally identical. Objects present themselves to us with many different *aspects*, which are ontologically dependent parts of them, like their color, if they are visual objects, or their shape, if they are spatial objects. Aspects *belong* to the objects whose aspects they

[10] There are, of course, many important differences between fictional literature and mathematics. After the original positing, unless they are somehow re-posited, mathematical "fictions" no longer admit changes, whereas true fictions can be arbitrarily reshaped. After the original positing of numbers, for example, they cannot be given, unlike fictional characters, further arbitrary properties. Moreover, mathematics is constrained by logic whereas fictional literature is not. An "illogical" mathematical structure cannot be "invented", whereas a piece of fiction can take more liberties concerning logical consistency. However, a work of fiction that is consistently inconsistent with human experience may fail to fulfil its goals.

are. Objects can have *equal* but not *identical* aspects; physical bodies can have equal colors or shapes, for example. Each body has its own color and shape as constitutive aspects of it, but *we* can take, by ideation, aspects as manifestations of a universal, a certain ideal color or shape.

The act of ideation – turning aspects into ideas - rests on a relation of equality. In the case of color and shape the equality has a sensorial quality, a certain visual impression, which obviously has a physiological basis that does not interest us here. In the case of shape, equality is spatial coincidence. We can then define: for spatial bodies A and B, the shape of A = the shape of B iff one can bring, in principle if not actually, A and B together and perfectly superpose one onto the other. The same is true for collections of objects considered quantitatively. Two collections A and B are equinumerous iff A and B are in a bijective correspondence. Equinumerous collections share the same *cardinal number*. All these things, shape, color, number are ideal forms. Of course, for theoretical (mathematical) reasons, we usually idealize shapes[11]; idealization (which, remember, is different from ideation) is a process of "exactification", of "taking the limit".

Concepts are not, in general, reified aspects of things but reified properties of things. Aspects are to properties as forms are to concepts; the essential difference, as I see it, is that unlike aspects, properties are *not* parts of objects. We could say, in a formula, that *properties are ways of seeing and aspects are ways of being*. An object, for example, can be a book, i.e. have the property of being a book, but being a book is not an aspect of the book. The same book can present itself as, say, a pile of paper sheets instead of a book without transmuting itself into another object, but it cannot be *that* object of perception with a color that is not, in fact, the color that it effectively has. However, I admit, these distinctions are not so neatly drawn. One can, for example, make aspects into concepts. A green object can be seen as instantiating the concept of green object. I think the essential difference is that aspects appear at a lower, perceptual level, whereas concepts only at a higher-level, that of judging. I *see* the redness of the book (I see *it*), but I judge *that* the book is red (I see *that* it is so).

Mathematics is a formal science; i.e. it is essentially a science of forms irrespectively of their materializations. It is interested in ideal quantitative forms (i.e. numbers) irrespectively of *what* they determine quantitatively. Although, as I showed before, any science is in the end concerned only with forms (because theories can only fix their domains up to isomorphism), the material sciences cannot completely forget where the forms they are interested on materialize. Unlike mathematics, material supports are *practically* (although not in principle) *indispensable* for material sciences to access abstract structures. However, not all forms are mathematical objects. Color, for example, as a visual form, is not. However, if conveniently objectified and mathematized in terms of wavelength of luminous radiation, it finds its way into mathematics. More precisely, colors become mathematical by being formally represented in terms of mathematical objects and concepts. We can in fact

[11] Although in Hilbert and Cohn-Vossen's famous book (Hilbert and Cohn-Vossen 1996) the authors develop a geometry of non-idealized geometrical forms and shapes, which they call "intuitive".

"define" mathematics as the science of objective forms. The major difference between mathematics and the empirical sciences, however, is that mathematics can create its forms freely, provided it does so consistently.

Although the objects of mathematics are forms, like any other science mathematics utilizes concepts as instruments for knowing. Concepts bring things together and allow them to be seen from their perspective. For example, one can assemble all numbers under the concept of number and by inquiring intuitively the concept of number we can bring properties of numbers, *as numbers*, to consciousness, in particular, how they relate operationally to one another. From the mathematical perspective, it does not matter so much what numbers are (matter does not matter), only how they relate to one another. But one cannot bring the operational relations among numbers clearly to consciousness without intuitively grasping the *concept* of number. Once this is done, however, numbers fall off the picture and the mathematical interest centers on the *form* of the numerical domain, which can be instantiated by other number-like entities. In this case, the concept of number provided a way of accessing the structure of the numerical domain. By formal abstraction, this structure becomes an object on its own right, which, once ideated, acquires the status of a mathematical object.

Structures are a kind of form, and since the creation of abstract algebra, they became the quintessential objects of mathematics. However, doing arithmetic is, in many ways, like doing zoology, with the difference that numbers are a completely different type of "animals", accessible in completely different ways. Zoology is an ontological science, whose objects are directly accessible intuitively, whereas arithmetic is a conceptual science, whose objects are indirectly accessible conceptually. Although arithmetic has a clearly defined concept of number, zoology does not have a well-defined concept of animal. Without mentioning, of course, that the intentional constitution of numbers involves the subject more substantially than the intentional delimitation of the realm of animals. The major difference between both sciences, however, is that mathematics can shift its interest from numbers to the *structure* of the numerical domain, which is instantiable in domains materially different, but formally identical to that of numbers proper. By so doing, mathematics has access to much more powerful instruments of investigation of the numerical structure and, derivatively, numbers proper.

However, the greatest difference between mathematics and empirical science is that mathematics is free to posit its own structures, or objects with the sole purpose of instantiating structures, whereas empirical science is stuck to empirical objects and forms it finds instantiated in empirical reality. Only derivatively, and exclusively for *methodological* purposes, empirical science can contemplate extending empirical reality into richer domains. For this purpose, empirical science usually turns to mathematics, which provides science with a variety of structures where empirical reality can be immersed, but always for *strictly methodological reasons* and not before perceptual reality is first somehow mathematized.

Refreshing our "definition", mathematics is the science of forms, structures in particular, either actually instantiated or only potentially instantiable, for their own sake. These forms come in many varieties, natural and positive real numbers and geometric patterns are the oldest; ideal abstract structures are late comers. Notice that structures can be filled by other forms, such as, for example, the ω-structure materialized in the numerical domain; in this case, the distinction matter-form is only relative.

Mathematics as Formal Ontology One of the most serious errors of some current philosophies of mathematics, which follows almost by necessity from the empiricist prejudices they subscribe, is to view mathematics on the model of the empirical sciences. The supposedly desirable "uniform semantics" for science and mathematics points in this direction. From the obliteration of the specificity of mathematics, it follows pseudo-problems and "solutions" to problems that should not be raised in the first place. Mathematical knowledge is a peculiar form of knowledge; it has many similarities with empirical science, but many *essential* dissimilarities as well. Like science, it has a domain of investigation, it has its own methodology and it generates technologies. However, on the other hand, unlike the natural sciences, mathematics does not find its objects ready-made (in fact, not even the empirical sciences do, but at least the objects of science relate somehow to perceptions, which relate to sensations, which are somehow "caused" by a world out there). Mathematical objects are constructs, they *exist*, but not in nature or in a *topos uranos* that is not nature but looks a lot like nature; mathematical objects are not simply "given" to us, they are constituted, brought into existence. Not, however, as private possessions, confined to individual minds; on the contrary, mathematical objects are objectively given as common possessions, objectively the same for anyone who approaches them and is in principle capable of reenacting the adequate constituting acts. To hold such a view requires that one turns one's back to the empiricist dichotomy that has objective but independent existence on one side and dependent but subjective existence on the other. There is an alternative, which is to exist dependently of constituting acts of a communal ego but objectively, i.e. intersubjectively in the public space.

Mathematical objects are empty forms, sometimes abstracted from aspects of the world given to us, idealized (exactified) and ideated (turned into universals), and sometimes freely invented as forms of *possible* experience (and the only condition for them to be possible forms of intuitable contents is the consistency of their positing). This puts mathematics somewhere in between empirical science and art. It is art in the sense that it involves free creations (not because it seeks beauty); it is science insofar as it is a form of knowledge, *a priori knowledge of actual or in principle possible forms of experience*. In short, mathematics is part of formal ontology.

It also creates tools and generates technology. The tools are the mathematical forms, which are useful in two different ways, by giving form to or representing in idealized manner the form of actual experience or as contexts of "immersion" where other forms are interpretable. Mathematization as a method of investigation in empirical sciences constitutes the most important technology mathematics generates; all sciences are essentially formal, and by providing a plethora of forms with which one can understand other forms, mathematics serves science (including mathematics itself). This, by the way, is its major raison d'être, regardless of mathematicians thinking of themselves as the purest of the pure scientists.

Why Phenomenology? Psychologism has since long been completely discredited as a legitimate approach to the ontology and epistemology of mathematics. Frege argued against it, thus inoculating anti-psychologism directly into the veins of analytic philosophy; Husserl argued against it, even more vehemently than Frege, but was accused of slipping back into it (which is probably why analytic philosophers mistrust phenomenology). After having dedicated many pages to fighting the errors of psychologism, Husserl goes on to create phenomenology, a philosophy markedly ego-centered – Husserl once defined phenomenology as "egology". An avalanche of criticism followed, as if Husserl had forgotten all he had said before about the impossibility of grounding objective knowledge in subjectivity. What critics failed to see, however, despite the many explanations Husserl offered, was that he was not thinking of subjectivity in naturalistic terms; *Husserl was an enemy of naturalism, psychologism in particular*. The ego and subjectivity are, for him, abstract concepts not concrete instances, although they can sometimes be concretely instantiated in individuals and minds. The fact is that Husserl, more acutely than Frege, saw the gap between objectivity and subjectivity as a problem and sought throughout his philosophical life to build a philosophically solid bridge connecting both sides.

This opened possibilities for thinking about objective being and objective knowledge, in particular *mathematical* being and knowledge, in connection with the intentional life of a meaning-giving ego that, however, is *not* the empirical ego. This new perspective offers new answers to the usual ontological, epistemological, logical and pragmatic questions about mathematics. Do mathematical objects exist? Yes, they do, objectively but not independently. Can we access mathematical objects, how? Yes, not via some form of intellectual intuition that takes us to an independent realm of being but, instead, through the constituting acts by which mathematical objects are posited with the meaning they are intended to have. Is there a mathematical form of intuition? Yes, intuitions are intentional acts of presentification, and anything in principle can be brought to consciousness (to awareness) as being present not merely represented. Can we apply classical forms of reasoning in mathematics without presupposing the independent existence of mathematical objects? Yes, we can, provided mathematical domains are intentionally posited as objectively complete (which is not a hypothesis but a constituted sense of being). Does the "unreasonable effectiveness" of mathematics in science pose a threat to the view that man does not occupy a privileged position in the natural scheme of things? No, it does not, for mathematics is applicable in science only because empirical reality, the domain of natural sciences, is already a mathematical intentional *construct*, tailor-cut to be mathematically represented and investigated by mathematical means. Is nature itself a mathematical manifold? As intentionally constituted out of perceptions as an instrument for organizing, making sense and predicting perceptions, it is indeed, but not intrinsically, independently of intentional action – empirical nature is a mathematical manifold because it was so constituted to serve as a methodological instrument of science.

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