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Canonical Extensions and Discrete Dualities for Finitely Generated Varieties of Lattice-based Algebras

Abstract. The paper investigates completions in the context of finitely generated lattice-based varieties of algebras. In particular the structure of canonical extensions in such a variety $\mathcal{A}$ is explored, and the role of the natural extension in providing a realisation of the canonical extension is discussed. The completions considered are Boolean topological algebras with respect to the interval topology, and consequences of this feature for their structure are revealed. In addition, we call on recent results from duality theory to show that topological and discrete dualities for $\mathcal{A}$ exist in partnership.

Keywords: canonical extension, natural extension, topological algebra, interval topology, natural duality

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1. Introduction

This paper investigates the structure of particular completions of algebras in any finitely generated variety of lattice-based algebras, and dual representations of such algebras. A *lattice-based algebra* (also known as a *lattice expansion*) is an algebraic structure which is a lattice equipped with a (possibly empty) set of additional operations. Completions of algebras of this kind have received recent attention also from Gehrke and Vosmaer [23, Section 5] and Vosmaer [35, Section 3.4]. Our approach is quite different from theirs, though there are some results in common. We present a direct route to the principal conclusions: this is tailored to the finitely generated case and thereby has merit in its own right. Critically, our treatment, unlike that in [23, 35], is independent of the machinery required for the general lattice-based setting. In particular we make no use of the extension to this setting of the δ-topology methodology as developed by Vosmaer (see, specifically, [35, Section 2.1.4]); the δ-topology (formerly known as the σ-topology) was introduced by Gehrke and Jónsson [21] to study distributive lattice expansions. The second author of this paper acknowledges with gratitude fruitful

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discussions with Mai Gehrke during 2007 on canonical extensions in the finitely generated case, concerning in particular Theorem 3.4 below.

Let us fix until further notice a finitely generated variety $\mathcal{A}$ of lattice-based algebras. Underpinning our approach to completions are some well-known facts from universal algebra, to be found for example in [4]. The variety $\mathcal{A}$ is necessarily congruence distributive and so, by Jónsson's Lemma and Birkhoff's Subdirect Product Theorem, can be expressed as $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M} = \{M_1, \dots, M_\ell\}$ is a finite set of finite algebras, each having a lattice reduct. Here $\mathbb{ISP}(\mathcal{M})$ denotes the class of isomorphic copies of subalgebras of products of algebras drawn from $\mathcal{M}$.

We equip each member of $\mathcal{M}$ with the discrete topology. Each $A \in \mathbb{ISP}(\mathcal{M})$ can then be regarded as carrying the induced product topology. Thereby we can form the closure $\overline{A}$ and view it as a member of a category $\mathcal{A}_{\mathcal{T}}$ of Boolean topological algebras; this means that the underlying space is Boolean (that is, compact and totally disconnected) and that the algebraic operations are continuous. In addition $\overline{A}$ is complete as a lattice. So $\overline{A}$ is an obvious candidate for a completion of A. This observation, in conjunction with ideas underlying the theory of natural dualities, led Davey, Gouveia, Haviar and Priestley [9] to introduce the concept of the *natural extension* $n_{\mathcal{A}}(A)$ for any algebra A in a class $\mathcal{A}$, as specified above (and more generally in any class $\mathbb{ISP}(\mathcal{M}')$, where $\mathcal{M}'$ is a set (not necessarily finite) of finite algebras of common type (not necessarily lattice-based)). In [14, Theorem 2.4] we proved, for $A \in \mathcal{A}$, that the lattice reduct of $n_{\mathcal{A}}(A)$ is a canonical extension of L_A, the lattice reduct of A. We stress that the proof relies solely on the *definition* of canonical extension, without the intervention of other possible means of construction. In particular no reference is made to profinite completions. These, and profinite lattice-based algebras more generally, are a recurring theme in the thesis of Vosmaer [35]. In Section 4 we outline how profiniteness fits into the overall picture.

We now provide further contextual background for the order-theoretic and topological aspects of our study. The theory of canonical extensions is very well established. It originated with Jónsson and Tarski's work [29] on Boolean algebras with operators (BAOs) but its scope has now widened to encompass lattice-based (and even poset-based) algebras with arbitrary non-lattice operations [20, 19, 21, 22, 23, 35]. Jónsson and Tarski sought to devise an algebraic means of analysing operations, and the equations they satisfy, by lifting these operations from algebras to their canonical extensions. This aim has remained a central plank of canonical extension theory ever since. The methodology is of most value for varieties which are *canonical*, that is, closed under the passage to canonical extensions.

The strategy, in general, for building a canonical extension of a lattice-based algebra A is to form the canonical extension L_A^δ of the underlying lattice L_A and then to lift any non-lattice operations from L_A to L_A^δ to form an algebra of the same algebraic type as A. The properties of density and compactness which characterise canonical extensions of (bounded) lattices (we recall the details in Section 2) ensure that any operation f has two natural liftings, denoted f^σ and f^π (the definitions can be found in Section 3); f is said to be *smooth* if $f^\sigma = f^\pi$. In lattice-based algebras in general, the σ- and π-extensions may not coincide and may exhibit a rich diversity of behaviours, which can be fully captured only through a subtle analysis involving, alongside order-theoretic arguments, conditions expressed in terms of a proliferation of topologies and order-theoretic properties; see [21, 35].

For almost as long as canonical extensions have been studied there has been evidence that they must behave especially well in the case of finitely generated varieties. However the foundational literature contains no analysis focusing on this case and only recently have these varieties attracted the attention they deserve. To specialise to a lattice-based variety $\mathcal{A}$ generated by a finite algebra K, Gehrke and Vosmaer [23, 35] exploit the fact that, by Jónsson's Lemma, $\mathcal{A}$ can be expressed as $\mathbb{HSP}_B(K)$, where $\mathbb{P}_B$ denotes Boolean product; the key point is that ultraproducts are absent (see [21, Section 3]). By applying successively to K the operators $\mathbb{P}_B$, $\mathbb{S}$ and $\mathbb{H}$, the properties of canonical extensions of members of $\mathcal{A}$ are then developed, with heavy reliance on ideas employed to study canonical extensions in general.

By contrast, we invoke our realisation of the canonical extension in terms of the natural extension. We are able immediately to say that $\mathcal{A}$ is *hyper-canonical*: each $A \in \mathcal{A}$ embeds in a Boolean topological algebra which, viewed as an algebra, belongs to $\mathcal{A}$ and, viewed as a lattice, serves as L_A^δ; moreover, each algebraic operation f is smooth, with $f^\sigma = f^\pi = f$. Hyper-canonicity was established in [14] by a very direct method, keeping the set $\mathcal{M}$ of generating algebras for $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ explicitly in play. This approach revealed, transparently, how the topological structure and the algebraic operations relate to notions of order-convergence. However this tactic does have limitations. An important feature of the canonical extensions methodology in general is that much can be achieved abstractly, that is, without reference to how the canonical extension is constructed. Therefore it is appropriate to recast our representation in a freestanding way and to capture topological structure on canonical extensions directly from the order structure, and so to expose the internal structure of the extensions. Our natural extensions, and hence canonical extensions too, are linked bi-algebraic lattices equipped with the interval topology, and the non-lattice operations are interval-continuous. The term 'linked bi-algebraic' comes from the theory of continuous lattices as presented for example in [24]. Although we draw on this theory to a limited extent, we keep our treatment as self-contained and elementary as possible.

Finally, in Section 5, we bring in recent results from the theory of natural dualities. We show how to set up linked dual equivalences between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$ and between $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{X}$, where $\mathcal{X}$ and $\mathcal{X}_{\mathcal{T}}$ are, respectively, a category of structures and a category of topological structures, the relational structure being the same for both. This puts on a more precise footing than hitherto the relationship, in the finitely generated setting, between topological dualities and discrete dualities.

The paper of Gehrke and Vosmaer [23] provides a valuable summary of the way in which canonical extensions are employed in the semantic modelling, in the style of Kripke semantics, of a range of deductive systems. Many such logics, in particular modal logics, are modelled algebraically by algebras based on the variety $\mathcal{B}$ of Boolean algebras or on the variety $\mathcal{D}$ of bounded distributive lattices. The associated relational semantics can be seen as being provided by a 'semantic platform' built using Stone duality (for $\mathcal{B}$) or Priestley duality (for $\mathcal{D}$), with non-lattice operations and their relational counterparts overlaid. The reason that non-lattice operations often cannot be incorporated into the basic platform is precisely because the varieties under consideration are rarely finitely generated, so that it is necessary first to consider reducts in a variety which is, such as $\mathcal{B}$ or $\mathcal{D}$. Our duality framework promises to provide, likewise, a uniform way to derive relational semantics for certain many-valued logics modelled by algebras having reducts in a finitely generated lattice-based variety. These ideas will be taken forward in a subsequent paper.

2. Complete sublattices of products of finite lattices

This section summarises, and adds to, material from [14]. We begin by recalling some basic definitions. Let L be a sublattice of a complete lattice C. Then C is called a *completion* of L. (More generally, if $e\colon L \to C$ is an embedding of the lattice L into the complete lattice C, then the pair (e, C) is also called a *completion* of L.) Write $T \Subset S$ to mean that T is a finite subset of S. The completion C of L is *dense* if every element of C can be expressed both as a join of meets and as a meet of joins of elements of L. In addition, C is called a *compact* completion of L if, for all non-empty subsets A and B of L, we have that $\bigwedge A \leqslant \bigvee B$ implies $\bigwedge A_0 \leqslant \bigvee B_0$, for some $A_0 \Subset A$ and $B_0 \Subset B$ or, equivalently, if for every filter F of L and every ideal I of L, we have that $\bigwedge F \leqslant \bigvee I$ implies $F \cap I \neq \varnothing$. A further equivalent condition is obtained if A and B are replaced by (non-empty) sets which are respectively up-directed and down-directed. A *canonical extension* of a lattice L is a completion C of L that is both dense and compact. Gehrke and Harding [19] proved that every bounded lattice L has a canonical extension and that any two canonical extensions of L are isomorphic via an isomorphism that fixes the elements of L.

Now assume $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ is the quasivariety generated by $\mathcal{M}$, where $\mathcal{M}$ is a finite set of finite lattice-based algebras. We shall identify suitable subalgebras of products of algebras in $\mathcal{M}$ as candidates for the canonical extensions of algebras in $\mathcal{A}$. We recall that a non-empty subset L of a complete lattice K is called a *complete sublattice* of K if it is closed under joins and meets (taken in K) of arbitrary non-empty subsets. Our first result, Proposition 2.1 below, generalises [10, Lemma 2.2] and amplifies [14, Proposition 2.1]. In preparation, we note the following well-known description of the closure in topological products. Let $\{M_s\}_{s\in S}$ be a family of topological spaces indexed by a non-empty set S. Let A be a subset of $\prod_{s\in S} M_s$. An element x of $\prod_{s\in S} M_s$ is *locally in* A if, for every $T \Subset S$, there exists $a \in A$ with $x{\restriction}_T = a{\restriction}_T$. The set of all elements of $\prod_{s\in S} M_s$ that are locally in A will be denoted by $\operatorname{loc}(A)$. If each M_s is finite and endowed with the discrete topology, then $\operatorname{loc}(A)$ is the topological closure of A in $\prod_{s\in S} M_s$.

PROPOSITION 2.1. *Let S be a non-empty set, let M_s be a complete lattice, for all $s \in S$, and let L be a sublattice of $\prod_{s\in S} M_s$.*

(i) *Let $x \in \prod_{s\in S} M_s$ and assume that x is locally in L. Then, with the joins and meets calculated pointwise in the product, the following hold:*

  (a) *$x = \bigvee\{\, \bigwedge A_i \mid i \in I\,\}$, for some non-empty set I and non-empty subsets A_i of L, and*

  (b) *$x = \bigwedge\{\, \bigvee A_i \mid i \in I\,\}$, for some non-empty set I and non-empty subsets A_i of L.*

(ii) *Assume that M_s is a finite lattice, for each $s \in S$. Then $\operatorname{loc}(L)$ forms the complete sublattice of $\prod_{s\in S} M_s$ generated by L.*

(iii) *Assume that M_s is a finite lattice, for each $s \in S$. Then following are equivalent for all $x \in \prod_{s\in S} M_s$:*

  (1) *$x \in \operatorname{loc}(L)$;*

  (2) *$x = \bigvee\{\, \bigwedge A_i \mid i \in I\,\}$, for some non-empty set I and non-empty subsets A_i of L;*

(3) $x = \bigwedge\{ \bigvee A_i \mid i \in I \}$, *for some non-empty set* I *and non-empty subsets* A_i *of* L.

(iv) *Assume that each* M_s *is a finite lattice endowed with the discrete topology. Then* L *is a topologically closed sublattice of* $\prod_{s\in S} M_s$ *if and only if* L *is a complete sublattice of* $\prod_{s\in S} M_s$.

PROOF. Assume that x is locally in L. For each $T \Subset S$, define

$$k_x^T := \bigwedge\{ a \in L \mid x{\restriction}_T = a{\restriction}_T \}.$$

As x is locally in L, we have $k_x^T{\restriction}_T = x{\restriction}_T$. Clearly, $\bigvee\{ k_x^T \mid T \Subset S \} \geqslant x$, so to prove equality it remains to show that $k_x^T \leqslant x$, for all $T \Subset S$. Let $T \Subset S$ and $s \in S$. Then there exists $a \in L$ with $x{\restriction}_{T\cup\{s\}} = a{\restriction}_{T\cup\{s\}}$. It follows that $k_x^T \leqslant a$, whence $k_x^T(s) \leqslant a(s) = x(s)$. Thus, $k_x^T \leqslant x$, as required. This proves (i)(a), and (i)(b) follows by duality.

Now assume that each M_s is a finite lattice. That $\mathrm{loc}(L)$ forms a complete sublattice of $\prod_{s\in S} M_s$ will follow easily once we prove that (with the joins and meets calculated pointwise in the product):

$$\varnothing \neq A \subseteq L \implies \bigvee A \in \mathrm{loc}(L) \ \ \& \ \ \bigwedge A \in \mathrm{loc}(L). \qquad (*)$$

Let A be a non-empty subset of L and let $x := \bigvee A$. Let $T \Subset S$ and let $t \in T$. Then, since M_t is finite, $x(t) = \bigvee_{a\in A} a(t) = a_1^t(t) \vee \cdots \vee a_{j_t}^t(t)$, for some $j_t \in \mathbb{N}$ and $a_1^t, \dots, a_{j_t}^t \in A$. Define $a := \bigvee\{ a_1^t \vee \cdots \vee a_{j_t}^t \mid t \in T \}$. Then $a \in L$ and $a \leqslant \bigvee A = x$. We have $a(t) \geqslant a_1^t(t) \vee \cdots \vee a_{j_t}^t(t) = x(t)$ for $t \in T$. Thus, $x{\restriction}_T = a{\restriction}_T$. So $\bigvee A \in \mathrm{loc}(L)$, and $\bigwedge A \in \mathrm{loc}(L)$ by duality.

By replacing L by $\mathrm{loc}(L)$ in $(*)$, we conclude at once that $\mathrm{loc}(L)$ is a complete sublattice of $\prod_{s\in S} M_s$. The remainder of (ii) follows from (i). Now another application of $(*)$ shows that (iii) follows from (i). Finally, (iv) is an immediate corollary of (ii). ■

It is noteworthy that Proposition 2.1(iii) asserts that the two order-theoretic conditions specifying density are equivalent. This would not be expected to occur for dense lattice completions in general. While the lattices M_s in parts (ii) and (iii) of the proposition are assumed to be finite, it is quite easy to show that the conclusions are valid under the weaker assumption that each M_s has no infinite chains: use the fact that lattices of this kind are complete and have the property that every join, or meet, reduces to the join, or meet, of a finite subset (see [13, Theorem 2.41]).

We now want to identify dense completions which are also compact. To this end we specialise to lattices which are represented as sublattices of powers of finite lattices in such a way that each exponent in the representation is a Boolean power of the base. That is, the exponents are algebras of the form $\mathcal{C}(Z, M)$, namely the continuous functions from Z to M with pointwise-defined operations, where Z is a Boolean space.

THEOREM 2.2. [14, Theorem 2.4(ii)] *Let $Z_1, \ldots, Z_\ell$ be compact spaces, let $M_1, \ldots, M_\ell$ be discretely topologised finite lattices and let L be a sublattice of $\prod_{1 \leqslant i \leqslant \ell} C(Z_1, M_1)$. Then the topological closure of L in $\prod_{1 \leqslant i \leqslant \ell} M_i^{Z_i}$ is a compact and dense completion, and so a canonical extension, of L.*

For Theorem 2.2 to be of value, we must demonstrate how its assumptions can be met. Given the quasivariety $\mathcal{A} = \mathbb{ISP}(\{M_1, \ldots, M_\ell\})$ and $A \in \mathcal{A}$, we want to choose compact spaces $Z_1, \ldots, Z_\ell$ in such a way that L_A embeds into $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ in the manner demanded. As in [9, 14], we do this by introducing the natural extension construction. We take $Z_i := \mathcal{A}(A, M_i)$. Here the underlying set of Z_i is the set of homomorphisms from A into M_i and Z_i is endowed with the subspace topology derived from the power M_i^A, where M_i is equipped with the discrete topology. Since Z_i is a closed subspace of the product, it is compact; indeed it is a Boolean space.

We embed A into $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ by means of the map

$$e_A \colon A \to \prod_{1 \leqslant i \leqslant \ell} M_i^{\mathcal{A}(A, M_i)}$$

given by $e_A(a)(i)(x) = x(a)$, for $i \in \{1, \ldots, \ell\}$ and $x \in \mathcal{A}(A, M_i)$; we call the map $e_A(a)$, for $a \in A$, a *multisorted evaluation map*. The map e_A is a homomorphism and, because $A \in \mathbb{ISP}(\mathcal{M})$, it is also an embedding. We then define the *natural extension* $n_{\mathcal{A}}(A)$ of A (relative to $\mathcal{M} = \{M_1, \ldots, M_\ell\}$) to be the topological closure of $e_A(A)$ in $\prod_{1 \leqslant i \leqslant \ell} M_i^{\mathcal{A}(A, M_i)}$, where each M_i carries the discrete topology. Since each evaluation map $e_A(a)$ is continuous we can restrict the codomain of e_A and write $e_A \colon A \to \prod_{1 \leqslant i \leqslant \ell} \mathrm{C}(Z_i, M_i)$. The map e_A embeds A as a topologically dense subalgebra of its natural extension in just the way we require in order to be able to apply Theorem 2.2.

THEOREM 2.3. [14, Theorem 2.4] *Let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ where $\mathcal{M}$ is a finite set of finite lattice-based algebras (of the same type). Then, for each $A \in \mathcal{A}$, the lattice reduct of the natural extension $n_{\mathcal{A}}(A)$ is a dense and compact completion of the lattice reduct L_A, and so a canonical extension.*

We shall wish in Section 5 to view the natural extension categorically, as a functor $n_{\mathcal{A}}$ from $\mathcal{A}$ into a category $\mathcal{A}_{\mathcal{T}}$ of topological algebras. We denote by $\mathcal{M}_{\mathcal{T}}$ the set of members of $\mathcal{M}$, each endowed with the discrete topology.

We then define $\mathcal{A}_{\mathcal{T}} := \mathbb{I}\mathbb{S}_{\mathrm{c}}\mathbb{P}(\mathcal{M}_{\mathcal{T}})$, the *topological prevariety generated by* $\mathcal{M}_{\mathcal{T}}$, that is, the class of isomorphic copies of topologically closed non-empty substructures of products of members of $\mathcal{M}_{\mathcal{T}}$. We make $\mathcal{A}_{\mathcal{T}}$ into a category by taking as morphisms the continuous $\mathcal{A}$-homomorphisms. See [9, Section 2] for the definition of the action of $n_{\mathcal{A}}$ on morphisms, and also for the natural extension construction in maximum generality and with all its bells and whistles.

We stress that $n_{\mathcal{A}}(A)$ belongs to $\mathcal{A}_{\mathcal{T}}$, and so is a topological algebra with algebraic reduct in $\mathcal{A}$. The realisation of the canonical extension of $A \in \mathcal{A}$ as $n_{\mathcal{A}}(A)$ also leads to very simple proofs of some well-known properties of operations and homomorphisms (cf. Gehrke and Harding [19], p. 360).

Proposition 2.4. *Let $\mathcal{A}$ be a finitely generated lattice-based variety.*

(i) *Let f be an n-ary basic operation. Let $A \in \mathcal{A}$ and $C := n_{\mathcal{A}}(A)$. Then, if the interpretation f^A of f on A preserves $\vee$ respectively $\wedge$, in any coordinate, the interpretation f^C of f on C preserves arbitrary non-empty joins, respectively meets, in that coordinate. In particular, if f^A is an operator on A (that is, it preserves $\vee$ in each coordinate), then f^C is a complete operator (preserves non-empty joins in each coordinate).*

(ii) *Any homomorphism between algebras in $\mathcal{A}$ lifts to a complete homomorphism between their natural extensions.*

Proof. Because C is a complete sublattice of a product of finite lattices, joins and meets in C are computed pointwise and in each component any join (meet) reduces to a finite join (meet). Also, any non-lattice operations are given pointwise on C. Hence (i) and (ii) hold. ■

3. The structure of canonical extensions and hypercanonicity

In this section we investigate more closely the topological and order-theoretic structure of a topologically closed sublattice of a non-empty product of finite lattices, where these lattices carry the discrete topology. As noted in Section 1 we make our exposition as self-contained as we can. However, without proof, we do set our results in context by stating a portmanteau result, Theorem 3.5. The continuous lattices methodology which is required to prove this theorem also provides alternative derivations of those of our prior, elementary, claims which the theorem subsumes.

We denote the completely join-irreducible elements of a complete lattice C by $J^{\infty}(C)$ and the completely meet-irreducible elements by $M^{\infty}(C)$. A lattice is *bi-algebraic* if it and its order dual are algebraic.

LEMMA 3.1. *Let C be a complete sublattice of a product of finite lattices. Then C is bi-algebraic. Consequently, C is meet-generated by $M^\infty(C)$ and join-generated by $J^\infty(C)$.*

PROOF. Any finite lattice is algebraic, a product of algebraic lattices is algebraic (see for example [24, I.4.14]) and it is elementary that a complete sublattice of an algebraic lattice is algebraic ([13, Exercise 7.7]). For the final claim, see for example [13, Proposition 10.27]. ■

Let P be an ordered set. The interval topology on P, denoted ι_P, is defined by taking a sub-basis for its closed sets consisting of all sets $\uparrow x$ and $\downarrow x$, for $x \in P$. Recall that an ordered topological space X is a *Priestley space* if it is compact and totally order-disconnected, that is, given $x \nleqslant y$ in X, there exists a clopen up-set containing x but not y.

PROPOSITION 3.2. *Assume that C is a topologically closed sublattice of Y, where $Y = \prod_{s\in S} M_s$, with each M_s a finite lattice with the discrete topology. Then the induced product topology $\mathcal{T}$ on C coincides with the interval topology ι_C, and with respect to this topology C is a Priestley space.*

PROOF. By Proposition 2.1(iv), C is a complete sublattice of Y. For any $a \in C$, the set $\uparrow_C a := \{\, y \in C \mid y \geqslant a \,\}$ is the intersection of C with the complete sublattice $\{\, y \in Y \mid y \geqslant a \,\}$ of Y and hence itself a complete sublattice of Y. By Lemma 2.1(iv) again, $\uparrow_C a$ is $\mathcal{T}$-closed, and likewise for $\downarrow_C a$. Therefore $\iota_C \subseteq \mathcal{T}$.

Since the topology on each M_s is discrete, $\mathcal{T}$ has a sub-basis for its closed sets consisting of the sets $\{\, a \in C \mid a(s) = m_s \,\}$, where s varies over S and m_s varies over M_s. It follows immediately that the family of sets of the form

$$U_{s,m_s} := \{\, a \in C \mid a(s) \leqslant m_s \,\} \text{ and } V_{s,m_s} := \{\, a \in C \mid a(s) \geqslant m_s \,\}$$

together form a sub-basis for the $\mathcal{T}$-closed sets. Since, by Proposition 2.1(iv), C is closed under joins, calculated pointwise, $b := \bigvee U_{s,m_s}$ exists in C and belongs to U_{s,m_s}. Because U_{s,m_s} is a down-set in C we conclude that $U_{s,m_s} = \downarrow_C b$, and hence that U_{s,m_s} is ι_C-closed. Likewise, the set V_{s,m_s} is ι_C-closed. Therefore $\mathcal{T} \subseteq \iota_C$. Since each M_s is trivially a Priestley space, the final assertion holds since the class of Priestley spaces is closed under the formation of products and closed subspaces. ■

It is clear that on each M_s the discrete topology coincides with the interval topology, and, on a product of complete lattices, the product topology derived from the interval topologies on the factors is the interval topology on the product (see [15, Theorem 2.6]). So on the full product $\prod_{s\in S} M_s$ the product topology is the interval topology. Proposition 3.2 says more; in general the interval topology on a closed subspace will not coincide with the subspace topology derived from the interval topology.

We must now venture into the foothills of the theory of continuous lattices. The next result is not new (cf. [24, 32, 34]), but for completeness we outline a direct proof.

PROPOSITION 3.3. *Let C be a bi-algebraic lattice. Then conditions* (1)–(4) *are equivalent:*

(1) *C is a Priestley space with respect to ι_C;*

(2) *the topology ι_C is Hausdorff;*

(3) *the topology ι_C coincides with the Lawson topology, with the dual Lawson topology and with the bi-Scott topology;*

(4) *for each compact element k of C there exists a finite subset F of C such that $C\backslash\uparrow k = \downarrow F$, and the order-dual assertion also holds.*

PROOF. We note the following very basic facts from [24]. On any complete lattice C the interval topology ι_C and the Lawson topology $\lambda(C)$ coincide if ι_C Hausdorff (because $\lambda(C)$ is necessarily compact and contains ι_C) and the dual assertion also holds.

It is elementary that the Lawson-open up-sets are exactly the Scott-open sets (that is, those whose complements are closed under directed joins) [24, III-1.6], and dually. Hence (2) implies (3). Assume (3). Then a Lawson open up-set is a union of principal up-sets [24, III.1.6 and III.1.9]. Consider a non-empty clopen up-set U in C. Every element of U lies above a minimal element of U, and each minimal element is compact. Since U is a closed subset of C it is compact. Consequently U has only finitely many minimal elements. Therefore non-empty clopen up-sets are of the form $\uparrow G$, with G a finite set and, order dually, clopen down-sets are of the form $\downarrow F$ with F finite. Hence (4) holds. Certainly either condition in (4) implies (1) because the compact elements in an algebraic lattice are join-dense, by definition. ■

Theorem 3.4 focuses on completely join- and meet-irreducible elements. By Proposition 3.3 and Theorem 2.2 this theorem will apply to the lattice reduct of the canonical extension of an algebra in a finitely generated lattice-based variety. As it applies to canonical extensions, essentially the same result is also given by Gehrke and Vosmaer [23, Section 5]. Our proof of Theorem 3.4 takes as its starting point property (4) in Proposition 3.3.

THEOREM 3.4. *Let C be a bi-algebraic lattice which is a Priestley space in its interval topology ι_C. Let $x \nleqslant y$ in C. Then there exists $j \in J^\infty(C)$, with $j \leqslant x$ and $j \nleqslant y$, and a finite set $\mathcal{M}_j \subseteq M^\infty(C)$ such that $C \setminus \uparrow j = \downarrow \mathcal{M}_j$. The order dual statement also holds.*

PROOF. Let $x \nleqslant y$ in C. Because C is dually algebraic, it is join-generated by $J^\infty(C)$, so there exists $j \in J^\infty(C)$ with $j \leqslant x$ and $j \nleqslant y$. Since C is algebraic, j is the directed join of the compact elements in $\downarrow j$ and because $j \in J^\infty(C)$, we deduce that j is compact. Now we can apply (4) in Proposition 3.3 to write $C \setminus \uparrow j$ as $\downarrow \mathcal{M}_j$, where $\mathcal{M}_j$ is a finite set and each member of $\mathcal{M}_j$ can be assumed to be maximal in $C \setminus \uparrow j$ since each member of the clopen down-set $\downarrow \mathcal{M}_j$ lies below a maximal element belonging to this set. Let $m \in \mathcal{M}_j$. If m were not completely meet-irreducible, then we could write m as the meet of a set S of elements of C with $s > m$, for all $s \in S$. But then $s \geqslant j$ for all $s \in S$, so that $\bigwedge S \geqslant j$, which is a contradiction. ■

In the non-distributive case, we cannot strengthen the above result to assert that $\mathcal{M}_j$ can be taken to be a singleton for every $j \in J^\infty(C)$. If we had $\mathcal{M}_j = \{m\}$, then (j, m) would be a splitting pair, and j necessarily completely join-prime. But a complete lattice that is join-generated by its completely join-prime elements is necessarily completely distributive (see for example [10], in particular Theorem 2.5, for this classic result, and full references). A completion of a non-distributive lattice cannot, of course, be completely distributive. We do however, have what may be seen as a weak form of complete join-primeness: the finite set $\mathcal{M}_j$ above has the property that, for any down-directed set F with $\bigwedge F \in \downarrow \mathcal{M}_j$, we have $F \cap \downarrow \mathcal{M}_j \neq \varnothing$ (cf. [34, Proposition 1.7 and Theorem 2.2]).

The interval topology on a complete lattice being compact and Hausdorff, as occurs in Proposition 3.3, already signals a weakened form of complete distributivity. In [16, 17], Erné discusses topological equivalents of various distributive laws on complete lattices. In particular, for a complete lattice C, the topology ι_C is compact and Hausdorff if and only if L is *ultrafilter distributive* in the sense that

$$\bigwedge \{ \bigvee Y \mid Y \in \mathcal{Y} \} = \bigvee \{ \bigwedge Z \mid Z \in \mathcal{Y}^\# \},$$

where $\mathcal{Y}$ is the family of ultrafilters on C and

$$\mathcal{Y}^\# = \{ Z \subseteq \bigcup \mathcal{Y} \mid (\forall Y \in \mathcal{Y})\, Y \cap Z \neq \varnothing \};$$

see [16, Theorem 5] and also [17], where it is revealed how ultrafilter distributivity relates to, and combines with, additional properties imposed on C.

We now set our elementary treatment above in a wider context. We recall that a lattice is *linked bi-algebraic* if it is bi-algebraic and has a Hausdorff interval topology. Combining the conditions of Lemma 3.1 and Proposition 3.3 we see that a complete sublattice of a product of finite lattices (each equipped with the discrete topology) is linked bi-algebraic. We record in Theorem 3.5 equivalent characterisations of linked bi-algebraic lattices. For the proof, we refer to [24, VII.2.6].

THEOREM 3.5. *For a complete lattice C, conditions* (1)–(3) *are equivalent:*

(1) *C is bi-algebraic and satisfies one, and hence all, of the equivalent conditions given in Proposition* 3.3*;*
(2) *C is a Boolean topological lattice with respect to ι_C;*
(3) *C is a Boolean topological lattice with respect to some topology.*

In the remainder of this section we consider a variety (or more generally quasivariety) $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite lattice-based algebras of common type. Let us fix $A \in \mathcal{A}$. We shall make direct use of the preceding results to reconcile the operations of $C := n_{\mathcal{A}}(A)$ with the σ- and π-extensions of the operations on A. We identify A with its image under e_A in C. The sets $K(C)$ of *filter elements* and $O(C)$ of *ideal elements* of the canonical extension C are defined as follows: $p \in K(C)$ if and only if p is a meet of elements from A and $q \in O(C)$ if and only if q is a join of elements from A. (Filter and ideal elements are known in the older literature as closed and open elements, respectively.) As in [19, 21] when considering extensions of maps we can restrict attention without loss of generality to the unary case: formation of canonical extensions and of filter and ideal elements commutes with the formation of finite products (see [19, Section 5] or [21, Section 2]).

Let $f\colon L_A \to L_A$ be any map. We recall that the maps f^σ and f^π on C are defined as follows:

$$\begin{aligned} f^\sigma(x) &:= \bigvee\big\{ \bigwedge\{ f(a) \mid a \in A \text{ and } p \leqslant a \leqslant q \} \mid \\ &\qquad\qquad p \in K(C), q \in O(C) \text{ and } p \leqslant x \leqslant q \big\}, \\ f^\pi(x) &:= \bigwedge\big\{ \bigvee\{ f(a) \mid a \in A \text{ and } p \leqslant a \leqslant q \} \mid \\ &\qquad\qquad p \in K(C), q \in O(C) \text{ and } p \leqslant x \leqslant q \big\}. \end{aligned}$$

It is elementary that f^σ and f^π extend f (note that every element of L_A is in $K(C) \cap O(C)$). Throughout this paper (and also in [14]) we have been at pains to stress that the full machinery for analysing canonical extensions in general can be circumvented in the finitely generated case. We shall however assume the basic result that $f^\sigma \leqslant f^\pi$. This was proved by Gehrke and Harding [19, Lemma 4.2(ii)], taking advantage of a restricted form of distributivity valid in the canonical extension of any bounded lattice [19, Lemma 2.3]. In [14, Lemma 3.4] we showed how this result could be sidestepped in the finitely generated case.

In [14, Theorem 3.5] we established, for any finitely generated variety $\mathcal{A}$, that, for each $A \in \mathcal{A}$ and for each operation f in the algebraic type, (the interpretation of) f on $n_{\mathcal{A}}(A)$ coincides with both the σ- and π-extensions of (the interpretation of) f on A. We now present yet another proof of the smoothness of the basic operations. Theorem 3.6 implies in addition that $f^\sigma = f^\pi$ is an interval-continuous map (this was proved in a quite different way by Gehrke and Vosmaer [23]). In fact Theorem 3.6(ii) proves a little more, since it applies to any map with an interval-continuous lifting.

THEOREM 3.6. *Let $A \in \mathcal{A}$, where $\mathcal{A}$ is a finitely generated lattice-based variety.*

(i) *The operations of the natural extension $n_{\mathcal{A}}(A)$, viewed as a topological algebra, are continuous with respect to the interval topology.*

(ii) *Let C be a dense and compact completion of L_A, viewed as a topological lattice. Assume that a map $f\colon A^n \to A$ (for some $n \geqslant 1$) has an ι_C-continuous extension g to C^n. Then $f^\sigma = f^\pi = g$. In particular $f^\sigma = f^\pi$ whenever f is an operation in the algebraic type.*

PROOF. The first assertion is immediate from the remarks at the end of Section 2 and Proposition 3.2.

We now let C be any dense and compact completion of A. By the uniqueness of the canonical extension, C can be identified with $L_{n_{\mathcal{A}}(A)}$. Hence C is algebraic, dually algebraic, is a Priestley space with respect to its interval topology and has the additional properties listed in Theorem 3.4. In addition we recall that the interval topology on a product of complete lattices is the product of the interval topologies ([15, Theorem 2.6]). Therefore our comment above that we may restrict f to be unary when considering f^σ and f^π applies equally to consideration of g. So assume $f\colon L_A \to L_A$.

As noted already, $f^\sigma \leqslant f^\pi$. Therefore, making use of order duality, it will suffice to show that $g(x) \leqslant f^\sigma(x)$, for all $x \in C$. Suppose for a contradiction that this fails for some x. Then we can find a compact element $k \in C$ such that $g(x) \in {\uparrow} k$ and $f^\sigma(x) \notin {\uparrow} k$. By ι_C-continuity of g, the set $W := g^{-1}({\uparrow} k)$ is ι_C-clopen. The set W is a finite union of sets each of which is the intersection of a clopen up-set and a clopen down-set. There are various ways to verify this claim (for example, we may exploit the fact that we are dealing with a Priestley space; see [13, Lemma 11.22]). Hence there exist clopen sets U and V, respectively an up-set and a down-set, such that $x \in U \cap V \subseteq W$. Then there exist p minimal in U and q maximal in V such that $x \in [p, q] \subseteq g^{-1}(\uparrow k)$. Here p is a compact element and so is a finite join of elements from $J^\infty(C)$ (because $J^\infty(C)$ is join-dense in C) and so is a filter element, and dually, q is an ideal element. For every $a \in [p, q] \cap A$ we have $f(a) = g(a) \geqslant k$ and so $\bigwedge f([p, q] \cap A) \geqslant k$. The definition of $f^\sigma(x)$ implies that $f^\sigma(x) \geqslant \bigwedge f([p, q] \cap A) \geqslant k$, and we have the required contradiction. The final assertion is now immediate. ■

4. The role of profiniteness

In Section 2 we deliberately identified the natural extension as a canonical extension without mentioning profinite completion. We now indicate how profiniteness fits into the picture. For background we refer to Banaschewski [1] and to Clark, Davey, Jackson and Pitkethly [7].

The general setting for the theory of profinite completions of algebras is that of classes $\mathcal{K} := \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a set of finite algebras of the same type. In [9] such a class $\mathcal{K}$ is called an *internally residually finite prevariety* (IRF-prevariety). Given an IRF-prevariety $\mathcal{K}$, any algebra in $\mathcal{K}$ embeds in its $\mathcal{K}$-profinite completion (see [9, Section 2] for details). In general an IRF-prevariety need not be a variety; when $\mathcal{K}$ is a variety, it is residually finite in the sense that this term is traditionally used in algebra.

It is very well known that a profinite completion of an algebra in any IRF-prevariety is in fact a Boolean topological algebra (see for example [1, 6] and also [10] for the case of distributive lattices). Theorem 3.7 of [9] shows that, for any IRF-prevariety $\mathcal{K}$, the $\mathcal{K}$-profinite completion and the natural extension of any member of $\mathcal{K}$ are isomorphic, both algebraically and topologically. IRF-prevarieties encompass many classes of algebras which are not lattice-based, and so the scope of [9], even for varieties, is very much wider than that of the present paper; see [9, Section 5]. A variety $\mathcal{A}$ of lattice-based algebras of finite type (that is, with a finite number of operations in the signature), is known to be residually finite if and only if it is finitely generated (Kearnes and Willard [30]), so that for such varieties the results of [9] apply only when $\mathcal{A}$ is finitely generated.

We now focus on a finitely generated variety $\mathcal{A}$ of lattice-based algebras. The work of Harding [26] and Gouveia [25] directly relates the canonical extension of an algebra A in $\mathcal{A}$ to its profinite completion $\widehat{A}$. It was Harding's work, and also that of Bezhanishvili, Gehrke, Mines and Morandi [3], which led the authors of [9] to develop the theory of natural extensions for IRF-prevarieties. In [26] Harding proved that $\widehat{A}$ serves, at the lattice level, as the canonical extension of L_A and that, for each additional operation f which is monotone (in each coordinate), the corresponding operation of $\widehat{A}$ agrees with each of f^σ and f^π. By refining Harding's arguments, Gouveia [25] removed the restriction of monotonicity. Independently, and using quite different methods, Vosmaer [35, Theorem 3.4.12] established that $\widehat{A}$ and A^δ coincide as algebras. Vosmaer's treatment relies on substantial technical machinery concerning the δ-topology and liftings of maps, much of which is applicable without the assumption of finite generation.

Therefore, for $A \in \mathcal{A}$, the identification of

- A^δ, the canonical extension, as constructed and characterised in [19, 22];
- $\widehat{A}$, the profinite completion of A;
- $n_{\mathcal{A}}(A)$, the natural extension of A, as presented in [9],

and hence the full panoply of properties of these algebras, can be arrived at by a variety of routes. Knowing that two pairs from A^δ, $\widehat{A}$ and $n_{\mathcal{A}}(A)$ can be identified, the third pair can be identified too. Which of the equivalent representations, or simply an abstract characterisation, is most illuminating or most convenient will depend on the context and the user's preference.

An algebra in a variety $\mathcal{V}$ is *profinite* if it is (isomorphic to) the inverse limit of finite algebras in $\mathcal{V}$, so that, for $A \in \mathcal{V}$, the algebra $\widehat{A}$ is certainly profinite. In this paper, we do not study profinite algebras in general, nor profinite completions outwith the finitely generated setting. We do note, however, that without a residual finiteness assumption, we cannot expect A to embed into $\widehat{A}$; see for example [9, Lemma 2.1], and [3] for a striking example: an infinite Heyting algebra H for which $|\widehat{H}| = 2$. Profinite algebras are often best handled via the universal mapping property which characterises them. This is the philosophy adopted by Vosmaer in [35, Section 3.4.3], where the relationship between $\widehat{A}$ and A^δ is explored for arbitrary lattice-based algebras A.

5. Exploiting duality theory

In this section we indicate briefly what our approach owes to, and gains from, duality theory. Thus far we have taken advantage of the theory of natural dualities only insofar as we have employed its formalism to arrive at the definition of the natural extension and hence to arrive at Theorem 2.3.

As before, we focus on a class $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M} = \{M_1, \ldots, M_\ell\}$ is a finite set of finite lattice-based algebras. In the special cases that $\mathcal{A}$ is the variety $\mathcal{B}$ of Boolean algebras or the variety $\mathcal{D}$ of bounded distributive lattices, we may take $\mathcal{M} = \{M\}$, where M is a single 2-element algebra. In general we shall need to work with the multisorted set-up, either because the class $\mathcal{A}$ we wish to consider is not generated as a quasivariety by a single finite algebra, or because $\mathcal{A} = \mathbb{HSP}(M)$ for a single algebra M, but $\mathcal{A} \neq \mathbb{ISP}(M)$; in the latter case we may invoke Jónsson's Lemma to express $\mathcal{A}$ in the form $\mathbb{ISP}(\{M_1, \dots, M_\ell\})$, with $\ell > 1$.

Traditional duality theory is lopsided: it seeks to set up a dual adjunction or, better, a dual equivalence, between a category $\mathcal{A}$ of algebras (without topology) on the one hand and a category $\mathcal{X}_\mathcal{T}$ of structures (with topology) on the other. We have already seen that topological algebras arise naturally in our investigation of natural, *alias* canonical, extensions. We therefore wish to look at a related dual adjunction between a category $\mathcal{A}_\mathcal{T}$ and a category $\mathcal{X}$ of structures, obtained by removing topology from $\mathcal{X}_\mathcal{T}$ and adding it to $\mathcal{A}$. The key to this topology-swapping is the TopSwap Theorem of Davey, Haviar and Priestley [11]. Here our structures may include, besides operations and relations, also partial operations. To convey the flavour while keeping the notation simple, we state the result, and the necessary preliminaries, only for the single-sorted case.

Let $N_1 = \langle N; G_1, H_1, R_1\rangle$ and $N_2 = \langle N; G_2, H_2, R_2\rangle$ be finite structures on the same underlying set. Here the G_i, H_i and R_i are respectively sets of finitary operations, partial operations and relations on N. Assume that N_2 is *compatible with* N_1, that is, each (n-ary) operation $g \in G_2$ is a homomorphism from N_1^n to N_1, for each (n-ary) partial operation $h \in H_2$, the domain of h forms a substructure $\mathrm{dom}(h)$ of N_1^n and h is a homomorphism from $\mathrm{dom}(h)$ to N_1, and each (n-ary) relation $r \in R_2$ forms a substructure of N_1^n. The structure N_1 is said to be *total* if $H_1 = \varnothing$. The topological structure $(N_2)_\mathcal{T}$ obtained by adding the discrete topology to N_2 is denoted by $\underset{\sim}{N}_2$ and is referred to as an *alter ego* of N_1. For further details see Davey [8].

We then have four categories: two categories of structures,

$$\mathcal{A} := \mathbb{ISP}(N_1) \text{ and } \mathcal{X} := \mathbb{IS}^0\mathbb{P}^+(N_2),$$

and two categories of Boolean topological structures,

$$\mathcal{A}_\mathcal{T} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{N}_1) \text{ and } \mathcal{X}_\mathcal{T} := \mathbb{IS}_c^0\mathbb{P}^+(\underset{\sim}{N}_2).$$

(A technical note: here the class operator $\mathbb{P}$ allows empty indexed products and so yields the total one-element structure while $\mathbb{P}^+$ does not; the operator $\mathbb{S}$ excludes the empty structure while $\mathbb{S}^0$ includes the empty structure when the type does not include nullary operations.)

There are naturally defined hom-functors $\mathrm{D}\colon \mathcal{A} \to \mathcal{X}_{\mathcal{T}}$ and $\mathrm{E}\colon \mathcal{X}_{\mathcal{T}} \to \mathcal{A}$, given on objects by

$$\mathrm{D}(A) := \mathcal{A}(A, N_1) \leqslant \underset{\sim}{N}_2^A \text{ and } \mathrm{E}(X) := \mathcal{X}_{\mathcal{T}}(X, \underset{\sim}{N}_2) \leqslant N_1^A,$$

for all $A \in \mathcal{A}$ and all $X \in \mathcal{X}_{\mathcal{T}}$. The evaluation maps

$$e_A\colon A \to \mathrm{ED}(A) \text{ and } \varepsilon_X\colon X \to \mathrm{DE}(X)$$

are embeddings and $\langle \mathrm{D}, \mathrm{E}, e, \varepsilon\rangle$ is a dual adjunction between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$. Likewise, we have hom-functors $\mathrm{F}\colon \mathcal{A}_{\mathcal{T}} \to \mathcal{X}$ and $\mathrm{G}\colon \mathcal{X} \to \mathcal{A}_{\mathcal{T}}$, given on objects by

$$\mathrm{F}(A) := \mathcal{A}_{\mathcal{T}}(A, \underset{\sim}{N}_1) \leqslant N_2^A \text{ and } \mathrm{G}(X) := \mathcal{X}(X, N_2) \leqslant \underset{\sim}{N}_1^X,$$

and evaluation maps $e_A\colon A \to \mathrm{GF}(A)$ and $\varepsilon_X\colon X \to \mathrm{FG}(X)$, for all $A \in \mathcal{A}_{\mathcal{T}}$ and all $X \in \mathcal{X}$. These functors give rise to a new dual adjunction $\langle \mathrm{F}, \mathrm{G}, e, \varepsilon\rangle$ between $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{X}$. The definition of the natural extension functor given in Section 2 extends without change to the more general setting now under consideration and the functor $n_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}_{\mathcal{T}}$ factors as $n_{\mathcal{A}} = \mathrm{F} \circ {}^{\flat} \circ \mathrm{D}$, where ${}^{\flat}$ denotes the functor which forgets the topology.. The set-up is now as indicated in Fig. 1.

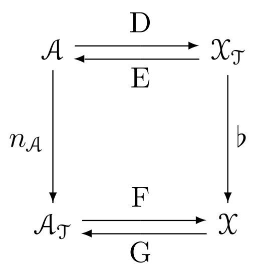

Figure 1. The functorial set-up for the TopSwap Theorem

We say that $\underset{\sim}{N}_2$ *yields a duality* on $\mathcal{A}$ if e_A is an isomorphism, for all $A \in \mathcal{A}$, that is, the only continuous homomorphisms $\alpha\colon \mathcal{A}(A, N_1) \to \underset{\sim}{N}_2$ are the evaluations maps $e_A(a)$, for $a \in A$. If, in addition, ε_X is an isomorphism, for all $X \in \mathcal{X}_{\mathcal{T}}$, we say that $\underset{\sim}{N}_2$ *yields a full duality* on $\mathcal{A}$. In case that e_A an isomorphism for all *finite* $A \in \mathcal{A}$ we say $\underset{\sim}{N}_2$ yields a duality at the finite level, and a full duality at the finite level if in addition each ε_X is an isomorphism for all *finite* $X \in \mathcal{X}$.

The following result is the main theorem obtained by Davey, Haviar and Priestley in [11]. We note that the principal novelty here lies in (i); the techniques of Hofmann [27] and Davey [8] then yield (ii). The proof of the theorem makes essential use of the finiteness of the signature of $\mathcal{A}_{\mathfrak{T}}$: this assumption is required in order that the Duality Compactness Theorem (see [27, Theorem 2.3] and [8, Theorem 4.8], generalising [5, 2.2.11]) can be applied.

TopSwap Theorem 5.1. *Let N_1 be a finite total structure of finite type, let N_2 be a structure compatible with N_1 and define the categories $\mathcal{A}$, $\mathcal{A}_{\mathfrak{T}}$, $\mathfrak{X}$ and $\mathfrak{X}_{\mathfrak{T}}$ as above.*

(i) *If $\underset{\sim}{N}_2$ yields a duality at the finite-level between $\mathcal{A}$ and $\mathfrak{X}_{\mathfrak{T}}$, then N_2 yields a duality between $\mathcal{A}_{\mathfrak{T}}$ and $\mathfrak{X}$.*

(ii) *If $\underset{\sim}{N}_2$ yields a full duality at the finite-level between $\mathcal{A}$ and $\mathfrak{X}_{\mathfrak{T}}$, then a dual equivalence between the categories $\mathcal{A}_{\mathfrak{T}}$ and $\mathfrak{X}$ is set up by the adjunction $\langle \mathrm{F}, \mathrm{G}, e, \varepsilon \rangle$.*

It may be helpful at this point to emphasise how topology-swapping operates when $\mathcal{A} = \mathcal{D}$. We may take N_1 to be the 2-element lattice and N_2 the two-element chain, regarded as an ordered set. The categories $\mathfrak{X}_{\mathfrak{T}}$ and $\mathfrak{X}$ are, respectively, the categories of Priestley spaces and of ordered sets. The natural extension of $A \in \mathcal{D}$ is then simply the canonical extension of $\mathcal{A}$, as this was first defined by Gehrke and Jónsson in [20]. The category $\mathcal{A}_{\mathfrak{T}}$ is the category of Boolean topological distributive lattices with continuous lattice homomorphisms (which is isomorphic to the category $\mathcal{D}^+$ of completely distributive bi-algebraic lattices with complete lattice homomorphisms). The duality set up by F and G is due to Banaschewski [2]. For a full account see Davey, Haviar and Priestley [10].

All the results presented so far in this section extend, *mutatis mutandis*, to the multisorted case. By way of illustration, we draw attention to the application of this theory to discriminator varieties. These varieties have attracted attention in connection with substructural logics; see Galatos *et al.* [18]. Further background can be found in Burris and Sankapannavar [4] and also in [9] and the references cited there. In [9, Section 5], a multisorted natural duality is derived for any finitely generated discriminator variety; there is no assumption that the variety is lattice-based, though in many applications it would be. In addition, the natural extension is described explicitly as an algebra of structure-preserving maps and shown to be a full direct product of quasiprimal algebras.

Very many finitely generated varieties of algebras having affinities with logic are lattice-based. This feature is a major asset in that it automatically guarantees the existence of a full duality. Because the median term, *viz.*, $m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$, in a lattice is a 3-ary near unanimity term, we can apply the NU Strong Duality Theorem from natural duality theory; see [5, Section 3.3] and in particular Theorem 3.8, or, for the multisorted version, [5, Theorem 7.1.2]. (We do not need to discuss the concept of strong duality here; we note only that it provides a sufficient, but not necessary, condition for a duality to be full.) For simplicity we state Theorem 5.2 only for the single-sorted case.

THEOREM 5.2. *Let $\mathcal{A} = \mathbb{ISP}(N_1)$, where N_1 is a finite lattice-based algebra.*

(i) *Let $R_2 = \mathbb{S}(N_1^2)$ and $G_2 = H_2 = \varnothing$, Then $\underset{\sim}{N}_2 = \langle N_1; R_2, \mathfrak{T}\rangle$ dualises N_1.*

(ii) *If the duality in* (i) *is not already full, then a full duality can be obtained by taking, as before, $R_2 = \mathbb{S}(N_1^2)$ and adding to the structure N_2 all partial homomorphisms from N_1^k to N_1 for $k = 0, 1, \ldots, n$, where the bound n can be explicitly computed from N_1.*

Combining the TopSwap Theorem 5.1 with Theorem 5.2 we see that it is possible in the lattice-based case to ensure that both adjunctions in Fig. 1 are dual equivalences (and this extends to the multisorted setting). Summing up what we have achieved so far, we may assert that, given any finitely generated lattice-based variety $\mathcal{A}$ of finite type, we can define $\mathcal{X}$ and $\mathcal{X}_{\mathcal{T}}$ in such a way that we have

- a dual equivalence between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$, and
- a dual equivalence between $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{X}$.

We shall refer to these as, respectively, a *topological duality* and a *discrete duality*. (We shall also use the term 'discrete duality' when we replace $\mathcal{A}_{\mathcal{T}}$ by an isomorphic category; this aligns our usage with the customary one.)

We now consider once more $\mathcal{A} := \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras, and investigate $\mathcal{A}_{\mathcal{T}}$ more closely,. We denote by $\mathcal{A}_{\mathrm{Bt}}$ the set of Boolean topological models of the axioms of $\mathcal{A}$. Since $\mathcal{A}_{\mathcal{T}} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathcal{M}_{\mathcal{T}})$, each member of $\mathcal{A}_{\mathcal{T}}$ belongs to $\mathcal{A}_{\mathrm{Bt}}$. The question then naturally arises as to whether $\mathcal{A}$ is *standard*, that is, whether $\mathcal{A}_{\mathcal{T}}$ coincides with $\mathcal{A}_{\mathrm{Bt}}$. We have already noted in passing that this is so when $\mathcal{A} = \mathcal{D}$. To address this question more generally we call on recent investigations at the interface of natural duality theory and universal algebra. The following theorem, due to

Clark, Davey, Jackson and Pitkethly, specialises [7, Theorem 2.13] and the discussion following it.

Theorem 5.3. *Assume that* $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, *where* $\mathcal{M}$ *is a finite set of finite lattice-based algebras, and assume that* $\mathcal{A}$ *is a variety* (*or equivalently, assume that every homomorphic image of every subalgebra of each algebra in* $\mathcal{M}$ *is in* $\mathcal{A}$). *Then* $\mathcal{A}$ *is standard.*

We remark that congruence distributivity of the variety $\mathcal{A}$ (a consequence of $\mathcal{A}$ being lattice-based) is crucial in Theorem 5.3: it serves to guarantee that $\mathcal{A}$ has the property FDSC (finitely determined syntactic congruences) that suffices to ensure standardness. In fact, FDSC also implies that every Boolean topological algebra whose underlying algebra belongs to $\mathcal{A}$ is necessarily profinite. This result is due to Clark, Davey, Freese and Jackson [6, Theorem 8.1]. (We allude below to profiniteness issues as regards members of $\mathcal{X}_{\mathcal{T}}$; there we shall consider structures, rather than just algebras.)

We may argue that Theorem 5.3 gives $\mathcal{A}_{\mathcal{T}}$ a right to be regarded as the natural home for the natural extensions of the members of $\mathcal{A}$. Note, too, that profinite algebras ought always to be seen as topological algebras. Thus the most appropriate way to view a canonical extension of an algebra in $\mathcal{A}$ is as a topological algebra, however it is concretely represented. Nevertheless we may ask when, as happens for $\mathcal{A} = \mathcal{D}$, there is an isomorphism between $\mathcal{A}_{\mathcal{T}}$ and a category, which we may denote by $\mathcal{A}^+$, which involves order structure, but does not explicitly involve topology. For this to work in the same way as it does for $\mathcal{D}$, we need the non-lattice operations of $\mathcal{A}$ to interact with the underlying order in a well-behaved way. We have the following result. It encompasses, for example, varieties of distributive modal algebras and also those with a negation operation satisfying De Morgan's laws.

Theorem 5.4. *Let* $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, *where* $\mathcal{M}$ *is a finite set of finite lattice-based algebras. Assume in addition that each non-lattice operation of arity* $\geqslant 1$ *either* (a) *preserves binary join* (*or binary meet*) *or* (b) *in each coordinate separately preserves either* $\vee$ *or* $\wedge$. *Then there is an isomorphism between* $\mathcal{A}_{\mathcal{T}}$ *and the category* $\mathcal{A}^+$ *whose objects are members of* $\mathcal{A}$ *whose lattice reducts are linked bi-algebraic lattices and whose non-lattice operations of type* (a) *preserve non-empty joins* (*meets*) *and whose operations of type* (b) *preserve non-empty joins* (*meets*) *in those coordinates in which the corresponding operations of* $\mathcal{A}$ *are required to preserve* $\vee$ ($\wedge$)*; the morphisms of* $\mathcal{A}^+$ *are* $\mathcal{A}$*-homomorphisms which preserve arbitrary non-empty joins and meets.*

Proof. At the lattice level, Proposition 3.3 reconciles the objects of $\mathcal{A}_{\mathcal{T}}$ and those of $\mathcal{A}^+$. Moreover the topology on each object of $\mathcal{A}_{\mathcal{T}}$ is the interval topology, and this coincides with the join of the Scott topology and its dual (recall Proposition 3.3).

Now consider a non-lattice operation $f\colon C^n \to C$ of type (a), where $C \in \mathcal{A}_{\mathcal{T}}$ and $n \geqslant 1$. We may, by replacing appropriate factors in C^n by their order duals, assume without loss of generality that f preserves binary join in each coordinate, that is, it is an operator; observe that $C \in \mathcal{A}_{\mathcal{T}}$ if and only if its order dual C^∂ belongs to $\mathcal{A}_{\mathcal{T}}$ and that the topology is invariant under order-reversal. A basic property of the Scott topology is that a map defined on a finite product of complete lattices is Scott continuous if and only if it is Scott continuous in each coordinate, or equivalently (in the case of an order-preserving map) if and only if it preserves directed joins in each coordinate ([24, II-2.9]). It follows that an n-ary operation which is an operator is, when interpreted on an object in $\mathcal{A}_{\mathcal{T}}$, continuous if and only if it preserves directed joins in each coordinate. Therefore the objects of $\mathcal{A}_{\mathcal{T}}$ and of $\mathcal{A}^+$ can be identified. The treatment of operations of type (a) is somewhat simpler, there being no need to consider separate continuity.

Now consider morphisms. The argument given by Davey, Haviar and Priestley [10,] for $\mathcal{A} = \mathfrak{D}$ applies equally well here. A lattice homomorphism between (the reducts of) objects in $\mathcal{A}_{\mathcal{T}}$ is continuous for the interval topology if and only if it preserves arbitrary (non-empty) meets and joins. It follows that a map is a $\mathcal{A}^+$-morphism if and only if it is an $\mathcal{A}_{\mathcal{T}}$-morphism. ■

We remark that a map between complete lattices preserves non-empty joins if and only if it preserves binary joins and directed joins, and dually for meets. Therefore the assertions in Proposition 2.4 valid for natural extensions can be seen as special cases of the claims in Theorem 5.4, but proved in a different way. Theorem 5.4 says more, since the natural extension functor from $\mathcal{A}$ to $\mathcal{A}_{\mathcal{T}}$ is not normally surjective. To see this, take for example $\mathcal{A} = \mathfrak{D}$, the variety of bounded distributive lattices. Here the image of $\mathfrak{X}_{\mathcal{T}}$ under the functor $^\flat$ is the category of profinite ordered sets, a result originally proved by Speed [33]. Not every infinite ordered set is profinite: for example $Y := \omega \oplus \omega^\partial$ does not support a topology making it into a Priestley space. If, in the notation of Fig. 1, $\mathrm{G}(Y)$ were equal to $n_{\mathfrak{D}}(A)$ for some $A \in \mathfrak{D}$, we would have $Y \cong \mathrm{D}(A)^\flat$, a contradiction.

We now pick up on the content of Theorem 5.1 as regards (full) dualities at the finite level. In Chapter VI of his classic text *Stone Spaces*, Johnstone calls a dual equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ a 'Stone-type duality' if it arises in a specific way from a dual equivalence between the full subcategories $\mathcal{C}_{\text{fin}}$ and $\mathcal{D}_{\text{fin}}$ of finite objects. The requirement is that $\mathcal{C} = \text{Ind-}\mathcal{C}_{\text{fin}}$, the Ind-completion of $\mathcal{C}_{\text{fin}}$, and $\mathcal{D} = \text{Pro-}\mathcal{D}_{\text{fin}}$, the Pro-completion of $\mathcal{D}_{\text{fin}}$;

see [28] for the categorical background and precise statements. Johnstone demonstrates how a number of well-known dualities arise this way.

Assume once more that we have the set-up of Fig. 1, and that $\mathcal{A} = \mathbb{ISP}(N_1)$ is a finitely generated variety of algebras. It is immediate that, because $\mathcal{A}$ is locally finite, $\mathcal{A}$ can be viewed as the Ind-completion of its finite members. We then ask whether $\mathcal{X}_\mathcal{T} = \text{Pro-}\mathcal{X}_{\text{fin}}$ (to simplify notation we have suppressed the subscript $\mathcal{T}$ on the right-hand side, finite structures tacitly being equipped with the discrete topology). It is obvious that every profinite limit of members of $\mathcal{X}_{\text{fin}}$ is again in $\mathcal{X}_\mathcal{T}$. For the reverse inclusion to hold we need every member of $\mathcal{X}_\mathcal{T} = \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{N}_2)$ to be profinite. On the positive side, it is true that if $\underset{\sim}{N}_2$ is an alter ego for N_1 which is a total structure, then $\mathcal{X}_\mathcal{T}$ does coincide with $\text{Pro-}\mathcal{X}_{\text{fin}}$. This fact was established by Clark, Davey, Jackson and Pitkethly [7, Corollary 2.4]. Moreover, the presence of partial operations does not automatically rule out the possibility that every member of $\mathcal{X}_\mathcal{T}$ is profinite: this is true in particular when $\mathcal{A} = \mathbb{ISP}(N_1)$ with N_1 a finite lattice-based algebra which is not injective in $\mathcal{A}$; see the discussion following Example 5.4 in [8].

However, even when a dual adjunction between $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$ gives a full duality at the finite level, it need not be true that every member of $\mathcal{X}_\mathcal{T}$ is profinite. An instructive example to demonstrate this is given by Davey [8, Section 5], building on investigations by Davey, Haviar and Willard [12]. The example views $\mathcal{A} = \mathcal{D}$ as being generated by the 3-element bounded chain. There is then an alter ego, having two unary operations, which suffices to yield a duality for $\mathcal{A}$; addition of one binary partial homomorphism provides a structure $\underset{\sim}{N}_2$ upgrading the duality to one which is full at the finite level. However not every object in $\mathcal{X}_\mathcal{T} = \mathbb{IS}^0_c\mathbb{P}^+(\underset{\sim}{N}_2)$ is profinite. In summary, the NU Strong Duality Theorem in general requires that partial operations be included in $\underset{\sim}{N}_2$ to ensure that the duality it provides between $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$ is full, and this may entail $\mathcal{X}_\mathcal{T} \subsetneq \text{Pro-}\mathcal{X}_{\text{fin}}$. Therefore a dual equivalence between $\mathcal{A}$ and a category $\mathcal{X}_\mathcal{T}$ is not always 'Stone-type' in Johnstone's sense. Likewise, with the roles of $\mathcal{A}$ and $\mathcal{X}$ interchanged, an associated dual equivalence between $\mathcal{X}$ and $\mathcal{A}_\mathcal{T}$ need not necessarily arise from lifting from the finite level by, respectively, Ind- and Pro-completion.

We have in this paper deliberately not strayed beyond the finitely generated setting. We acknowledge that, as regards direct applications, we are thereby imposing a stringent restriction: few varieties of major importance in logic are finitely generated. However the algebras in question often have reducts which lie in a finitely generated variety. This fact implicitly underpins traditional approaches to Kripke semantics for modal and intuitionistic logics. Thus we may view the diagram in Fig. 1, applied with $\mathcal{A}$ as $\mathcal{B}$, as

$\mathcal{D}$, or as some other finitely generated lattice-based variety, as providing a 'semantic platform'. On top of such a platform one may hope to build algebraic and relational semantics for a variety of logics, including many-valued logics. Some steps in this direction are taken by Vosmaer in [35, Chapter 4] in the traditional cases in which $\mathcal{A}$ is $\mathcal{B}$ or $\mathcal{D}$, and additional operations are superimposed to model modalities or Heyting implication: $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{A}^{+}$, suitably enriched, are employed to characterise certain profinite algebras in terms of Kripke frames.. More innovatively, and more in the spirit of the potential applications we have in mind for our framework, Maruyama [31] has recently shown how to adapt a natural duality for a variety generated by a finite quasiprimal algebra with a bounded lattice reduct so as to accommodate modalities.

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