Annals of Mathematical Logic 15(1978)109–160


# ITERATED ULTRAPOWERS AND PRIKRY FORCING

Patrick DEHORNOY

*Université Paris VII, Département de Mathématiques, 2, Place Jussieu, 75221 Paris Cedex 05, France*

Received 5 June 1977

## 0. Introduction

The existence of a measurable cardinal is known to be the weakest hypothesis which allows the construction a transitive ultrapower of a model of ZFC with respect to some ultrafilter belonging to this model. We thus start with an universe $N_0$ satisfying ZFC and a $\kappa$-complete ultrafilter $\mathcal{U}$ on a cardinal $\kappa$ which is in $N_0$, and construct following Gaifman and Kunen the decreasing sequence $(N_\alpha)_{\alpha\in\mathrm{On}}$ of the transitive iterated ultrapowers of $N_0$ with respect to $\mathcal{U}$, together with the corresponding elementary embeddings $i_{\alpha\beta}$ for $\alpha \leq \beta$. It has been observed that the intersection of the $\omega$ first ultrapowers of $N_0$ is a model of ZFC including $N_\omega$. Likewise if for any limit $\lambda$ we set $M_\lambda = \bigcap_{\alpha<\lambda} N_\alpha$, then $M_\lambda$ is a model of ZF including $N_\lambda$. Bukovsky showed by general methods that $M_\omega$ is a generic extension of $N_\omega$. On the other hand Prikry constructed starting with a normal ultrafilter a set of conditions which forces a measurable cardinal to become singular of cofinality $\omega$, and R. Solovay observed that, when $\mathcal{U}$ is normal, the sequence $(i_{0n}\kappa)_{n\in\omega}$ is in $M_\omega$ and is Prikry-generic over $N_\omega$. It was natural to conjecture that $M_\omega$ is exactly the Prikry-extension $N_\omega[(i_{0n}\kappa)_{n\in\omega}]$.

In this work, we prove this conjecture which has also been established independently by Bukovsky [3] by a method different from ours, and study much more generally the models $M_\lambda$ for $\lambda$ limit and their connections with Prikry extensions, without any normality hypothesis.

The main results are as follows:

**Theorem A.** *There exists a set of conditions $\mathcal{C}$ constructed from $\mathcal{U}$ such that:*

(i) *$\mathcal{C}$ reduces to Prikry' forcing just when (and only when) $\mathcal{U}$ is selective. However all the usual properties of Prikry' forcing hold for $\mathcal{C}$ in the general case;*

(ii) *$M_\omega$ is an "universal" $\mathcal{C}$-extension in the following sense: if $G$ is $N_0$-generic over $\mathcal{C}$ and $\phi(\boldsymbol{a})$ is any formula with parameters in $N_0$, then $N_0[G] \models \phi(\boldsymbol{a})$ iff $M_\omega \models \phi(i_{0\omega}\boldsymbol{a})$.*

**Theorem B.** *There exists a sequence $(\chi_\alpha)_{\alpha\in\mathrm{On}}$ of ordinals such that $\chi_\alpha = i_{0\alpha}\kappa$ when (and only when) $\mathcal{U}$ is normal, and, for any $\lambda$ limit, exactly one of the following* *holds*:

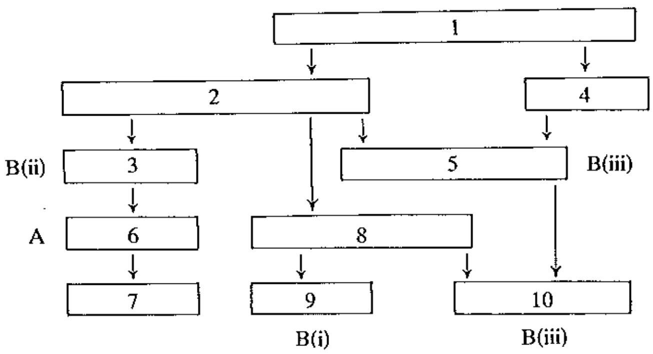
(i) $\exists\alpha<\lambda N_\alpha\models\mathrm{cf}\ \lambda>\omega$, *and* $M_\lambda=N_\lambda$.

(ii) $\lambda=\alpha+\omega$, *and* $(\chi_{\alpha+n})_{n\in\omega}$ *is* $N_\lambda$*-generic over* $i_{0\lambda}\mathscr{C}$ *and* $M_\lambda$ *is the generic extension* $N_\lambda[(\chi_{\alpha+n})_{n\in\omega}]$.

(iii) $\forall\alpha<\lambda\ \lambda\neq\alpha+\omega$, $N_\alpha\models\mathrm{cf}\ \lambda=\omega$, *and there is a set* $\mathscr{G}_\lambda$ *of* $N_\lambda$*-generic sequences over* $i_{0\lambda}\mathscr{C}$, *and a map* $i_\lambda$ *definable in* $M_\lambda$ *such that* $M_\lambda=N_\lambda[i_\lambda\restriction\mathscr{G}_\lambda]$. $M_\lambda$ *is a quasi-generic extension of* $N_\lambda$, *but no generic extension of* $M_\lambda$ *included in* $N_0$ *can satisfy* AC. *Furthermore the elements of* $\mathscr{G}_\lambda$ *are all subsequences (up to a finite number of terms) of* $(\chi_\alpha)_{\alpha\in\lambda}$.

The basic tools we employ for studying the models $M_\lambda$ are "arithmetic" properties of the composition of the embeddings $i_{\alpha\beta}$ which are developed in Chapter 2; for large values of $\lambda$ we need moreover a rather precise study of the notion of support for an element of $N_\alpha$, and this is done in Chapter 8. Chapter 3 is devoted to the proof of Theorem B(ii). We establish in Chapter 4 an useful lemma which is used in Chapter 5 to prove the part of Theorem B(iii) dealing with the axiom of choice in $M_\lambda$. The forcing conditions $\mathscr{C}$ are introduced in Section 6, and Theorem. A is proved there. Chapter 7 is devoted to a complete study of the generic sequences over $i_{0\lambda}\mathscr{C}$ which are in $M_\lambda$ when $\mathscr{U}$ is equivalent to a power of a normal ultrafilter. Finally Theorem B(i) is proved in Chapter 9, and the end of Theorem B(iii) is proved in Chapter 10.

The logical connections between various chapters are indicated in the following diagram:




I wish to thank here K. McAloon, S. Grigorieff, and A. Louveau for many stimulating conversations on the subject matter of this work.

## 1. Construction of the models

We recall the construction of the iterated ultrapowers of a model of ZFC with the help of a complete ultrafilter on a measurable cardinal. We follow the presentation and as much as possible the notations of Kunen's work, but we give a special emphasis to the introduction in Section 1.4 of the sequence $(\chi_\alpha)_{\alpha \in \mathrm{On}}$ which will provide us the basic tool for studying the ultrapowers when the ground ultrafilter is not normal. We have tried to make complete those proofs which do not appear, or appear in a different form, in [8].

1.0.

As usual, we reserve small greek letters (except $\pi$) for ordinals, or finite sequences of ordinals when set as bold face characters. We use $s$, $t$ for sequences of ordinals (of any length).

For any ordered set $X$, we let $[X]^n$ be the set of strictly increasing $n$-tuples of elements of $X$, and set $[X]^{<\omega} = \bigcup_n [X]^n$. We make no distinction between an order preserving map of an ordinal $\mu$ into another $\nu$ and its range, which is a subset of $\nu$ of order type $\mu$. This will allow us to speak of the union of two members $s$ and $t$ of $[\nu]^{<\omega}$, which is the injection whose range is the union of the ranges of $s$ and $t$, etc.

### 1.1. *Functions and subsets with finite supports*

Our ground model will be a transitive model of ZFC denoted $N_0$. We let $M$ denote an inner model of $N_0$, i.e. a class for $N_0$ which is a model of ZF (see [7]), which is transitive, and contains On. Let $I$ be any set of $M$.

If $\mu \leqslant \nu$ and $e$ is an order preserving map : $\mu \to \nu$ we can for any $s \in I^\nu$ define the composed map $se \in I^\mu$ in the usual way:

$$se(\xi) = s(e(\xi)).$$

Then let $f : I^\mu \to M$ : we construct

$$e * f : I^\nu \to M \text{ by setting}$$
$$e * f(s) = f(se).$$

We extend this definition to the subsets of $I^\mu$ looking at them as at functions from $I^\mu$ to 2, i.e. for $X$ included in $I^\mu$, we define $e * X$ included in $I^\nu$ by

$$s \in e * X \text{ iff } se \in X.$$

**Definition.** (i) $Fn_\alpha(M, I)$ is the *class* of $M$ of functions $f$ with domain $I^\alpha$ such that there exist $e \in [\alpha]^{<\omega}$ and $g$ a function with domain $I^{|e|}$ and range in $M$, with $f = e * g$.

(ii) $\mathcal{P}_\alpha(M, I)$ is the *set* in $M$ of all subsets $X$ of $I^\alpha$ such that there exist $e \in [\alpha]^{<\omega}$ and $Y$ a subset of $I^{|e|}$, with $X = e * Y$.

### 1.2. *Powers of ultrafilters*

Assume that in $M$, $\mathcal{V}$ is an ultrafilter on $I$.

**Definition.** (i) We first construct inductively $\mathcal{V}_n \subseteq \mathcal{P}(I^n)$ by: $X \in \mathcal{V}_n$ iff $\{\xi : X\xi \in \mathcal{V}\} \in \mathcal{V}_{n-1}$ where $X\xi = \{\eta : \xi^\frown \eta \in X\}$.

(ii) Then for any $\alpha$ we construct $\mathcal{V}_\alpha \subseteq \mathcal{P}_\alpha(M, I)$ by: $X = e * Y \in \mathcal{V}_\alpha$ iff $Y \in \mathcal{V}_{|e|}$.

### 1.3. *The models $N_\alpha$*

For the remainder of the paper, we assume that $\kappa$ is measurable in $N_0$, and $\mathcal{U}$ is any $\kappa$-complete (free) ultrafilter on $\kappa$.

**Theorem 1.** *There exists an unique sequence of transitive classes $(N_\alpha)_{\alpha \in \mathrm{On}}$ such that:*

(i) *$N_\alpha$ is an inner model of* ZFC *of $N_0$ including* On;

(ii) *there is a surjection $\pi_\alpha$ from $Fn_\alpha$ $(N_0, \kappa)$ onto $N_\alpha$ such that, for any formula $\Phi(v_1, \ldots, v_n)$, $N_\alpha$ satisfies $\Phi(\pi_\alpha f_1 \cdots \pi_\alpha f_n)$ iff $\{s \in \kappa^\alpha : N_0 \models \Phi(f_1(s) \cdots f_n(s))\}$* is in $\mathcal{U}_\alpha$;

(iii) *if $e : \alpha \to \beta$ is an increasing injection, the map $i_e$ from $N_\alpha$ to $N_\beta$ defined by $i_e \pi_\alpha f = \pi_\beta e * f$ is an elementary embedding;*

(iv) *if $\lambda$ is limit, then $(N_\lambda, i_{\alpha\lambda})$ is a direct limit of the system $(N_\alpha, i_{\alpha\beta})$ where $i_{\alpha\beta}$ is the embedding $i_e$ corresponding to $e = \mathrm{id}_\alpha$.*

The proof is in [8]. The well-foundedness of the models is a consequence of the closure of $\mathcal{U}$ under arbitrary countable intersections. Let us recall the proof of (iv): assume $\lambda$ limit and $x$ is a member of $N_\lambda$. There is $e$ in $[\lambda]^{<\omega}$ and $g : \kappa^{|e|} \to N_0$ such that $x = \pi_\lambda e * g$. Since $\lambda$ is limit, there is $\alpha < \lambda$ such that $e < \alpha$ (we mean: max $e < \alpha$), and so $x = \pi_\lambda \mathrm{id}_\alpha * e * g = i_{\alpha\beta} \pi_\alpha e * g$.

### 1.4. *The sequences $(\kappa_\alpha)$ and $(\chi_\alpha)$*

As usual, we set:

**Definition.** $\kappa_\alpha = i_{0\alpha}\kappa = \pi_\alpha[s \mapsto \kappa]$.

**Lemma 1.** *For all $\beta \geq \alpha$, we have $i_{\alpha\beta} \restriction V_{\kappa_\alpha} \cap N_\alpha = \mathrm{id}$.*

**Proof.** We show by induction on $\mu < \kappa_\alpha$ that $i_{\alpha\beta} \restriction V_\mu \cap N_\alpha = \mathrm{id}$. The only non-trivial case is from $\mu$ to $\mu + 1$; assume $i_{\alpha\beta} \restriction V_\mu \cap N_\alpha = \mathrm{id}$ and $x \in V_{\mu+1} \cap N_\alpha$. Since $x$ is included in $V_\mu$, for all $y \in x$ we have $y = i_{\alpha\beta} y$. Now $i_{\alpha\beta}$ is elementary, thus if $y \in x$, $i_{\alpha\beta} y \in i_{\alpha\beta} x$. Finally for $y \in x$, $y \in i_{\alpha\beta} x$, hence $x \subseteq i_{\alpha\beta} x$. Conversely assume $y \in i_{\alpha\beta} x$. Choose $e \in [\alpha]^{<\omega}$, $f \in Fn_\alpha[N_0, \kappa]$, $e' \in [\beta]^{<\omega}$, $g \in Fn_\beta[N_0, \kappa]$ such that $x = \pi_\alpha e * f$, $y = \pi_\beta e' * g$. We may clearly assume that $e = e' \cap \alpha$, and set $e' = e \cup e_1$ where $e_1 \subseteq [\alpha, \beta]$. By hypothesis, $y \in i_{\alpha\beta} x$, so

$$\{s^\frown t \in \kappa^{\alpha+\beta} : g(se^\frown te_1) \in f(se)\} \in \mathcal{U}_{\alpha+\beta},$$

or

$$\{s^\frown t \in \kappa^{|e|+|e_1|} : g(s^\frown t) \in f(s)\} \in \mathcal{U}_{|e|+|e_1|}.$$

Since $N_\alpha \models |x| < \kappa_\alpha$, we get

$$\{s \in \kappa^{|e|} : |f(s)| < \kappa\} \in \mathcal{U}_{|e|}.$$

Now by the $\kappa$-completeness of $\mathcal{U}_{|e_1|}$ there is $h : \kappa^{|e|} \to N_0$, such that

$$\{s^\frown t \in \kappa^{|e|+|e_1|} : g(s^\frown t) = h(s)\} \in \mathcal{U}_{|e|+|e_1|}.$$

This means that $y = i_{\alpha\beta}\pi_\alpha e * h$. Moreover

$$\{s \in \kappa^{|e|} : h(s) \in f(s)\} \in \mathcal{U}_{|e|},$$

so $\pi_\alpha e * h \in x$. Hence by hypothesis $i_{\alpha\beta}\pi_\alpha e * h = \pi_\alpha e * h$, i.e. $y = i_{\alpha\beta} y \in x$. We thus proved that any $y$ in $i_{\alpha\beta}x$ is in $x$, i.e. $i_{\alpha\beta}x \subseteq x$. Finally $x = i_{\alpha\beta}x$, and the lemma is proved.

**Corollary.** *If $f \in Fn_\alpha(N_0, \kappa)$ is such that for all $s$ in $\kappa^\alpha f(s) < \kappa$, then for all $\beta \geqslant \alpha$*

$$\pi_\beta[s^\frown t \mapsto f(s)] = \pi_\alpha f.$$

This enables us to set:

**Definition.** We let $\chi_\alpha$ be $\pi_{\alpha+1}[s \mapsto s(\alpha)]$, that is also $\pi_\beta[s \mapsto s(\alpha)]$ for any $\beta \geqslant \alpha + 1$.

We notice that for any $\alpha \leqslant \beta$, $\chi_\beta = i_{\alpha\beta}\chi_\alpha$, and in particular $\chi_\alpha = i_{0\alpha}\chi_0$. Let us recall:

**Definition.** $\mathcal{U}$ is said to be *normal iff* $\kappa = \pi_1 \mathrm{id}$.

We immediately notice that for any $\alpha\chi_\alpha$ is equal to $\kappa_\alpha$ just in case that $\mathcal{U}$ is normal. The sequence $(\chi_\alpha)$ is the modification of the sequence $(\kappa_\alpha)$ which is suitable for studying non-normal ultrafilters.

In particular, we get without toil the following result:

**Lemma 2.** ("normal form lemma"). *Assume $e \in [\alpha]^{<\omega}$ and $g : \kappa^{|e|} \to N_0$; then $\pi_\alpha e * g = i_{0\alpha} g(\chi_e)$, where $\chi_e = (\chi_{\alpha_1} \cdots \chi_{\alpha_n})$* if $e = (\alpha_1 \cdots \alpha_n)$.

**Proof.** We have: $\{s : (s, g(s)) \in g\} \in \mathcal{U}_{|e|}$, hence $\{s \in \kappa^\alpha : (se, g(se)) \in g\} \in \mathcal{U}_\alpha$. Thus $(\pi_\alpha[s \mapsto se], \pi_\alpha[s \mapsto g(se)]) \in \pi_\alpha[s \mapsto g]$, i.e. $(\chi_e, \pi_\alpha e * g) \in i_{0\alpha} g$, as desired.

### 1.5. *The ultrafilter $i_{0\alpha}\mathcal{U}$*

By the elementarity of $i_{0\alpha}$, $i_{0\alpha}\mathcal{U}$ is in $N_\alpha$ a $\kappa_\alpha$-complete ultrafilter on $\kappa_\alpha$. We thus are able to calculate in $N_\alpha$ the iterated ultrapowers of the universe (i.e. $N_\alpha$) with $i_{0\alpha}\mathcal{U}$. We show now that for any $\beta$ the $\beta$-th ultrapower of $N_\alpha$ with $i_{0\alpha}\mathcal{U}$ is exactly $N_{\alpha+\beta}$. This legitimates the term of iterated ultrapowers for the $N_\alpha$'s.

The next lemma is proved in [8]:

**Lemma 1.** *Assume* $\pi_\alpha f \in \mathcal{P}_n(N_\alpha, \kappa_\alpha)$; *then* $\pi_\alpha f \in i_{0\alpha}\mathcal{U}_n$ *iff* $\{s^\frown t \in \kappa^{\alpha+n} : t \in f(s)\} \in \mathcal{U}_{\alpha+n}$.

**Lemma 2.** *Assume* $N_\alpha \models e \in [\beta]^{<\omega}$ *and* dom $g = \kappa_\alpha^{|e|}$ *and* $f = e * g$, *and let* $x$ *be* $i_{\alpha\alpha+\beta}g(\chi_{\alpha+e})$. *Then* $N_\alpha$ *satisfies "*$x$ *is the image of* $f$ *in the* $\beta$*-th ultrapower of the universe by* $i_{0\alpha}\mathcal{U}$*".*

**Proof.** Let $\Pi$ be the application from $Fn_\beta(N_\alpha, \kappa_\alpha)$ to $N_0$ defined by $\Pi f = i_{\alpha\alpha+\beta}g(\chi_{\alpha+e})$ where $f$ is such that, in $N_\alpha$, $f = e * g$. By Mostowski's isomorphism theorem, we get the conclusion if we show that $\Pi$ does not depend upon the choices of $e$ and $g$, that the range of $\Pi$ is transitive in $N_0$ and that $\Pi f = \Pi f'$ (resp. $\in$) if and only if $\{s : f(s) = f'(s)\} \in (i_{0\alpha}\mathcal{U}_\beta)$ (resp. $\in$).

First notice that the third point implies the first one, and that the range of $\Pi$ is included in $N_{\alpha+\beta}$. Conversely, let $y$ be in $N_{\alpha+\beta}$. There are $e$ in $[\alpha]^{<\omega}$, $e'$ in $[\beta]^{<\omega}$ and $g : \kappa^{|e|+|e'|} \to N_0$ such that, using the normal form,

$$
\begin{aligned}
y &= i_{0\alpha+\beta}g(\chi_e{}^\frown \chi_{e'}) \\
&= i_{\alpha\alpha+\beta}[(i_{0\alpha}g(\chi_e)](\chi_{e'}),
\end{aligned}
$$

since by Section 1.4, Lemma 1, $i_{\alpha\alpha+\beta}\chi_e = \chi_e$. Now $i_{0\alpha}g(\chi_e)$ is a member of $N_\alpha$, and we have shown that $y$ is in the range of $\Pi$; this range is therefore $N_{\alpha+\beta}$, and so it is transitive.

Now assume that $f, f'$ are members of $Fn_\beta(N_\alpha, \kappa_\alpha)$ and $f = e * g$, $f' = e * g'$ in $N_\alpha$, with $g = \pi_\alpha E * G$, $g' = \pi_\alpha E * G'$. Define $F$, $F'$ by: $F(s^\frown t) = G(sE)(t(\alpha + e))$ and $F'(s^\frown t) = G'(sE)(t(\alpha + e))$ for $s^\frown t$ in $\kappa^{\alpha+\beta}$. We get:

$$
\begin{aligned}
\Pi f &= i_{\alpha\alpha+\beta}g(\chi_{\alpha+e}) \\
&= i_{\alpha\alpha+\beta}[i_{0\alpha}G(\chi_E)](\chi_{\alpha+e}) \\
&= i_{0\alpha+\beta}G(\chi_E)(\chi_{\alpha+e}) \\
&= \pi_{\alpha+\beta}[s^\frown t \mapsto G(sE)(t(\alpha+E))] \\
&= \pi_{\alpha+\beta}F.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Pi f = \Pi f' \ &\text{iff } \{s^\frown t : F(st) = F'(st)\} \in \mathcal{U}_{\alpha+\beta} \\
&\text{iff } \{s : \{t : F(s^\frown t) = F'(st)\} \in \mathcal{U}_\beta\} \in \mathcal{U}_\alpha \\
&\text{iff } \{s \in \kappa^\alpha : \{t \in \kappa^{|e'|} : G(sE)(t) = G'(se)(t)\} \in \mathcal{U}_{|e'|}\} \in \mathcal{U}_\alpha \\
&\text{iff } \{t \in \kappa^{|e'|} : g(t) = g'(t)\} \in i_{0\alpha}\mathcal{U}_{|e'|} \text{ by Lemma 1} \\
&\text{iff } \{t \in \kappa_\alpha^\beta : f(t) = f'(t)\} \in (i_{0\alpha}\mathcal{U})_\beta,
\end{aligned}
$$

and we are done since the proof would be the same for $\in$.

**Theorem 3.** (i) *The $\beta$-th ultrapower of $N_\alpha$ by $i_{0\alpha}\mathcal{U}$* is $N_{\alpha+\beta}$ *with canonical embedding $i_{\alpha\alpha+\beta}$;*

(ii) *if $\alpha \leqslant \beta$, then $N_\beta \subseteq N_\alpha$;*

(iii) *$\chi_{\alpha+\beta}$ is the value of $\chi_\beta$ calculated in $N_\alpha$.*

**Proof.** As in the proof of the preceding lemma, the $\beta$-th ultrapower of $N_\alpha$ is the range of the map $\Pi$, which we have seen to be $N_{\alpha+\beta}$. Now let $y \in N_\alpha$ and $x$ be $i_{\alpha\alpha+\beta}y$: by a special case of Lemma 2, $N_\alpha$ satisfies "$x = i_{0\beta}y$". Then, $N_\beta$ is the $(\beta-\alpha)$-th ultrapower of $N_\alpha$ if $a \leqslant \beta$, so it is included in it. Finally, we get the value of $\chi_\beta$ in $N_\alpha$ when applying Lemma 2 to the function $s \mapsto s(\beta)$ in $N_\alpha$.

1.6. *The models $M_\lambda$*

**Proposition 1.** *Let $\lambda$ be any limit ordinal and $M_\lambda$ be $\bigcap_{\alpha<\lambda} N_\alpha$. Then, $M_\lambda$ is an inner model* (*of* ZF) *of each of the $N_\alpha$ for $\alpha < \lambda$.*

**Proof.** The class $M_\lambda$ is transitive, closed under Gödel's operations, and definable in $N_\alpha$ for $\alpha < \lambda$. We get the result (see [7] Section 1.4) when showing that for any $\theta V_\theta \cap M_\lambda$ belongs to $M_\lambda$. For notice that, if $X$ is $V_\theta \cap M_\lambda$, then for any $\alpha < \lambda$, $N_\alpha$ satisfies "$X$ is $V_\theta \cap M_{\lambda-\alpha}$", so $X$ is in $N_\alpha$ for all $\alpha < \lambda$, and $X$ is in $M_\lambda$.

The study of the models $M_\lambda$ will be the main topic of this work. For the moment, let us notice that for any limit $\lambda$, $N_\lambda$ is included in $M_\lambda$ since it is included in each $N_\alpha$, $\alpha < \lambda$. More precisely, we have:

**Proposition 2.** *Assume $\lambda$ limit; then $N_\lambda$ is an inner model of $M_\lambda$.*

**Proof.** With the notations of [7] $N_0$ satisfies ZF $(M_\lambda, N_\lambda)$, and for all $\theta V_\theta \cap N_\lambda$ belongs to $M_\lambda$ since it belongs to each $N_\alpha$ for $\alpha < \lambda$. So, by Lemma 2 of Section 1.4 in [7] we get that $N_\lambda$ is an inner model (of ZF) of $M_\lambda$.

1.7. *The connection between $\mathcal{U}$ and $(\chi_\alpha)$*

We extend here well-known relation between $i_{0\alpha}\mathcal{U}$ and the sequence $(\kappa_\alpha)$ obtained in [8] for normal $\mathcal{U}$.
First recall:

**Lemma 1.** *Assume $x \subseteq N_1$ and $|x| \leqslant \kappa$; then $x \in N_1$.*

**Proof.** Let $(x_a)_{\alpha\in\kappa}$ be an enumeration of the elements of $x$; choose for each $\alpha \in \kappa$ $f_\alpha : \kappa \to N_0$, such that $x_\alpha = \pi_1 f_\alpha$. We set $F(\xi) = \{f_\alpha(\xi) : \alpha < k(\xi)\}$ where $k$ is such that $\pi_1 k = \kappa$. Then $x = \pi_1 F$.

**Corollary 2.** $V_{\kappa+1} \cap N_\alpha = V_{\kappa+1}$ *for any $\alpha$; in particular $\mathcal{P}(\kappa) \cap N_\alpha = \mathcal{P}(\kappa) \cap N_0$.*

**Proof.** If $x \in V_\kappa$, $x = i_{0\alpha}x$ by Section 1.4, Lemma 1, so $x \in N_\alpha$. If $x \in V_{\kappa+1}$, $x \subseteq V_\kappa$, hence $x \subseteq N_\alpha$ and $|x| \leqslant \kappa$, so $x \in V_{\kappa+1} \cap N_1$. Now, $\kappa + 1 < \kappa_1$, so $V_{\kappa+1} \cap N_1 = V_{\kappa+1} \cap N_\alpha$ for any $\alpha \geqslant 1$.

**Proposition 3.** *Assume $n \in \omega$, and $e \in [\alpha]^n$; then, for any $X$ in $\mathcal{P}(\kappa^n)$, $X$ is in $\mathcal{U}_n$ iff $\chi_e$ belongs to $i_{0\alpha}X$.*

**Proof.** $X \in \mathcal{U}_n$ iff $\{\xi : \xi \in X\} \in \mathcal{U}_n$ iff $\{s \in \kappa^\alpha : se \in X\} \in \mathcal{U}_\alpha$ iff $\pi_\alpha[s \mapsto se] \in \pi_\alpha[s \mapsto X]$, i.e. iff $\chi_e \in i_{0\alpha}X$.

In particular $X \subseteq \kappa$ is in $\mathcal{U}$ iff $\chi_0 \in i_{01}X$.

**Proposition 4.** *Assume $\lambda$ limit and $X \in \mathcal{P}(\kappa_\lambda) \cap N_\lambda$. Then, $X \in i_{0\lambda}\mathcal{U}$ iff $X$ contains all $\chi_\alpha$, $\alpha \in A$, for some $A$ cofinal in $\lambda$.*

**Proof.** We show that if $X$ belongs to $i_{0\lambda}\mathcal{U}$, there is a $\gamma < \lambda$ such that $X$ contains all $\chi_\alpha$, $\gamma \leqslant \alpha < \lambda$. For since $N_\lambda$ is the direct limit of the $N_\gamma$, $\gamma < \lambda$, we may choose $\gamma$ and $Y$ in $i_{0\gamma}\mathcal{U}$ such that $X = i_{\gamma\lambda}Y$. We apply the last proposition in $N_\gamma$: since $Y$ is in $i_{0\gamma}\mathcal{U}$, $\chi_{\gamma+\beta}$ belongs to $i_{\gamma\lambda}Y$ for all $\beta < \lambda - \gamma$, that is $\chi_\alpha$ belongs to $X$ for all $\alpha$, $\gamma \leqslant \alpha < \lambda$.

## 2. Combination of elementary embeddings

Here we summarize some arithmetic properties of the applications $i_{\alpha\beta}$ and their composition.

Let $x$ be any element of $N_0$ : $y = i_{0\omega}x$ is in $N_\omega$, so it is in $N_1$, and then falls in the domain of $i_{12}$. We thus may raise such questions as: what is $i_{12}(i_{0\omega}x)$?, what is $i_{1\omega_1}(i_{0\omega}x)$? etc. For the convenience, the composition of maps will be denoted only by juxtaposition in the usual inverse order, i.e. $i_{12}i_{0\omega}$ is the composition of $i_{0\omega}$ and $i_{12}$ after, $i_{12}i_{0\omega}(x) = i_{12}(i_{0\omega}(x))$.

### 2.1. *Continuity and cardinalities*

**Lemma 1.** *If $\lambda$ is limit, then $\kappa_\lambda = \sup_{\alpha<\lambda} \kappa_\alpha$.*

This is proved in [8], as well as the following results.

**Lemma 2.** (i) *For all $\gamma$, $\beta$, we have $|i_{0\gamma}\beta| \leqslant |\gamma|\,|\beta|^\kappa$;*
(ii) *If $\beta$ is a cardinal greater than $2^\kappa$, then $\beta = \kappa_\beta$.*

**Proposition 3.** *The map $i_{0\gamma}$ is continuous at $\beta$ iff* cf $\beta \neq \kappa$.

**Proof.** (i) If cf $\beta<\kappa$, let $(\alpha_\mu)_{\mu<\nu}$ a cofinal sequence in $\beta$ with $\nu<\kappa$. Then $i_{0\gamma}(\alpha_\mu)_{\mu<\nu}=(i_{0\gamma}\alpha_\mu)_{\mu<\nu}$ since $\nu<\kappa$.
Hence $i_{0\gamma}\beta=i_{0\gamma}(\sup_\mu \alpha_\mu)=\sup_\mu i_{0\gamma}(\alpha_\mu)=\sup_\mu i_{0\gamma}\alpha_\mu=\sup_{\alpha<\beta} i_{0\gamma}\alpha$.

(ii) If cf $\beta=\kappa$, let $\beta=\sup_{\mu<\kappa}\alpha_\mu$. Put $f: s\mapsto\alpha_{s(0)}$; then $\forall\mu<\kappa$, $\pi_\gamma f>i_{0\gamma}\alpha_\mu$, but on the other hand $\pi_\gamma f<i_{0\gamma}\beta$.

(iii) If cf $\beta>\kappa$, let $\xi$ be any element of $i_{0\gamma}\beta$: $\xi=\pi_\gamma e * g$ where $g:\kappa^{|e|}\to\beta$. Since cf $\beta>\kappa$, there is $\alpha<\beta$ such that im $g\subseteq\alpha$, and $\xi<i_{0\gamma}\alpha$.

2.2. *Composition*

**Proposition 1.** *Let $\alpha\leqslant\beta$ and $\gamma\leqslant\delta$ be any ordinals. Then the following diagram is correct (in the sense that the arrows effectively map the first class into the second) and commutative:*

$$\begin{array}{ccc} N_{\alpha+\gamma} & \xrightarrow{i_{\alpha\beta}} & N_{\beta+i_{\alpha\beta}\gamma} \\ {\scriptstyle i_{\alpha+\gamma\alpha+\delta}}\downarrow & & \downarrow{\scriptstyle i_{\beta+i_{\alpha\beta},\beta+i_{\alpha\beta}\delta}} \\ N_{\alpha+\delta} & \xrightarrow{i_{\alpha\beta}} & N_{\beta+i_{\alpha\beta}\delta} \end{array}$$

**Proof.** Assume that $x\in N_{\alpha+\gamma}$ and $y=i_{\alpha+\gamma\alpha+\delta}x$. Then, by Section 1.5, Theorem 3, $N_\alpha$ satisfies: "$x$ is in the $\gamma$-th ultrapower of the universe and $y$ is its image in the $\delta$-th ultrapower". Transporting that under $i_{\alpha\beta}$, we get that $N_\beta$ satisfies: "$i_{\alpha\beta}x$ is in the $i_{\alpha\beta}\gamma$-th ultrapower of the universe and $i_{\alpha\beta}y$ is its image in the $i_{\alpha\beta}\delta$-th ultrapower", that is, stated in $N_0$:

$$i_{\alpha\beta}x\in N_{\beta+i_{\alpha\beta}\gamma} \quad\text{and}\quad i_{\alpha\beta}y=i_{\beta+i_{\alpha\beta}\gamma\beta+i_{\alpha\beta}\delta}i_{\alpha\beta}x.$$

This last equality is precisely the commutativity of the diagram.

We now mention a few particular cases of the preceding simple, but useful fact.

**Corollary 2.** (i) *For any $\beta$, $\delta$, we have $i_{0\beta}i_{0\delta}=i_{0\beta+i_{0\beta}\delta}$. If moreover $\delta<\kappa$, then $i_{0\beta}i_{0\delta}=i_{0\beta+\delta}$. In particular $i_{0n}=(i_{01})^n$ and $i_{01}i_{0\omega}=i_{0\omega}$.*

(ii) *For any $\beta$, $\delta$, we have $i_{0\beta}\kappa_\delta=\kappa_{\beta+i_{0\beta}\delta}$, and $i_{0\beta}\chi_\delta=\chi_{\beta+i_{0\beta}\delta}$.*

(iii) *If $\alpha\leqslant\beta$ and $\beta+\delta=\delta$, then $i_{\alpha\beta}i_{0\delta}=i_{0i_{\alpha\beta}\delta}$.*

**Proof.** (i) Applying Proposition 1 with $\alpha=\gamma=0$, we get $i_{0\beta}i_{0\delta}=i_{\beta\beta+i_{0\beta}\delta}i_{0\beta}$, thus $i_{0\beta}i_{0\delta}=i_{0\beta+i_{0\beta}\delta}$ since for any $\varepsilon i_{\beta\varepsilon}i_{0\beta}=i_{0\varepsilon}$.

(ii) Remember that $\kappa_\delta=i_{0\delta}\kappa$ and $\chi_\delta=i_{0\delta}\chi_0$.

(iii) Applying Proposition 1 with $\gamma=0$, we get $i_{\alpha\beta}i_{0\delta}=i_{\beta\beta+i_{\alpha\beta}\delta}i_{0\beta}=i_{0\beta+i_{\alpha\beta}\delta}$. Since $\beta+\delta=\delta$, we have $i_{\alpha\beta}\beta+i_{\alpha\beta}\delta=i_{\alpha\beta}\delta$, and a fortiori $\beta+i_{\alpha\beta}\delta=i_{\alpha\beta}\delta$ since $\beta\leqslant i_{\alpha\beta}\beta$. Finally $i_{\alpha\beta}i_{0\delta}=i_{0i_{\alpha\beta}\delta}$.

**Lemma 3.** Assume $\alpha \leqslant \alpha'$: then there exists $\gamma$ such that $i_{0\alpha'} = i_{0\alpha} i_{0\gamma}$ *iff* $\alpha' - \alpha \in \mathrm{im}\ i_{0\alpha}$.

**Proof.** By the previous results $i_{0\alpha} i_{0\gamma} = i_{0\alpha + i_{0\alpha}\gamma}$, so $i_{0\alpha} i_{0\gamma} = i_{0\alpha'}$ *iff* $\alpha' - \alpha = i_{0\alpha}\gamma$.

As a consequence notice that the relation between $\alpha$ and $\alpha'$ "$\alpha \leqslant \alpha'$ and $\alpha' - \alpha \in \mathrm{im}\ i_{0\alpha}$" is an order relation.

2.3. *Limits*

**Lemma 1.** If $\lambda$ is limit, then for any $\theta$ the sequence $(i_{\alpha\lambda}\theta)_{\alpha<\lambda}$ is eventually constant.

**Proof.** If $\alpha \leqslant \alpha < \lambda$, we have $i_{\alpha\lambda}\theta = i_{\alpha'\lambda} i_{\alpha\alpha'}\theta$. Since $\theta \leqslant i_{\alpha\alpha'}\theta$, we conclude that $i_{\alpha\lambda}\theta \geqslant i_{\alpha'\lambda}\theta$. Thus the sequence $(i_{\alpha\lambda}\theta)_{\alpha<\lambda}$ is decreasing, hence it is eventually constant.

**Proposition 2.** *If $\lambda$ is limit, there is an ordinal $\varepsilon(\lambda) > \lambda$ such that for all $x$ in $N_\lambda$ the sequence $(i_{\alpha\lambda}x)_{\alpha<\lambda}$ is eventually constant and its limit is $i_{\lambda\varepsilon(\lambda)}x$.*

**Proof.** The sequence $(\lambda - \alpha)_{\alpha<\lambda}$ is decreasing, hence it is eventually constant. Choose $\alpha^{\cdot}$ and $\delta$ such that for all $\alpha \geqslant \alpha^{\cdot}$ (and $<\lambda$)$\alpha + \delta = \lambda$. Then let $x$ be a member of $N_\lambda$. There exists $\mu \geqslant \alpha^{\cdot}$ (and $<\lambda$) and $y$ a member of $N_\mu$ such that $x = i_{\mu\lambda} y = i_{\mu\mu+\delta} y$. Now for all $\alpha$ such that $\mu \leqslant \alpha < \lambda$ we have:

$$
\begin{aligned}
i_{\alpha\lambda}x &= i_{\alpha\lambda} i_{\mu\lambda} y \\
&= i_{\alpha\lambda} i_{\alpha\lambda} i_{\mu\alpha} y \quad \text{by splitting } i_{\mu\lambda} \\
&= i_{\alpha\lambda} i_{\alpha\alpha+\delta} i_{\mu\alpha} y \quad \text{by definition of } \delta \\
&= i_{\lambda\lambda + i_{\alpha\lambda}\delta} i_{\alpha\lambda} i_{\mu\alpha} y \quad \text{by Section 2.2 Proposition 1 applied with } \gamma = 0 \\
&\quad \text{and } \lambda \text{ in the place of } \beta \\
&= i_{\lambda\lambda + i_{\alpha\lambda}\delta} i_{\mu\lambda} y = i_{\lambda\lambda + i_{\alpha\lambda}\delta} x.
\end{aligned}
$$

By Lemma 1, the sequence $(i_{\alpha\lambda}\delta)_{\alpha<\lambda}$ is eventually constant. We set $\varepsilon(\lambda) = \lambda + \lim_{\alpha<\lambda} i_{\alpha\lambda}\delta$, which is also, by construction of $\delta$, $\lambda + \lim_{\alpha<\lambda} i_{\alpha\lambda}(\lambda - \alpha)$. Then assume $\alpha$ big enough to ensure that $\mu \leqslant \alpha < \lambda$ and $\lambda + i_{\alpha\lambda}\delta = \varepsilon(\lambda)$, we get:

$i_{\alpha\lambda} = i_{\lambda\varepsilon(\lambda)}x$, which is the desired result.

**Example.** For any $x$ in $N$, the sequence $(i_{n\omega}x)_{n\in\omega}$ is eventually constant and its limit is $i_{\omega\omega+\omega}x$.

2.4. *Simple ordinals*

For the convenience of the calculations in the next sections, we introduce a class of ordinals enjoying some stability properties. The result 4 below will allow us to restrict ourselves to the case of simple ordinals without loss of generality.

**Definition.** We say that $\lambda$ is *simple* if it is indecomposable (i.e. if $\alpha<\lambda$ implies $\alpha+\lambda=\lambda$) and if $\alpha<\lambda$ implies $i_{0\alpha}\lambda=\lambda$.

**Example.** The first simple ordinals are:
the indecomposable ordinals below $\kappa$: $\omega, \omega^2, \ldots, \omega_1, \ldots$
the first fixed point of the application $\theta \mapsto \kappa_\theta$, that is the supremum of $\kappa, \kappa_\kappa, \kappa_{\kappa_\kappa}$, etc....

**Lemma 1.** *If $\lambda$ is a strong limit cardinal of cofinality $\neq\kappa$, $\lambda$ is simple.*

**Proof.** Assume $\lambda>\kappa$. Since cf $\lambda\neq\kappa$, if $\alpha<\lambda$ we have $i_{0\alpha}\lambda=\sup_{\beta<\lambda} i_{0\alpha}\beta$, and $|i_{0\alpha}\beta|\leqslant|\beta|^\kappa|\alpha|<\lambda$, hence, if $\beta<\lambda$, $i_{0\alpha}\beta<\lambda$, thus $i_{0\alpha}\lambda\leqslant\lambda$, and $\lambda$ is simple.

**Lemma 2.** *If $\lambda$ is simple and $>\kappa$, then $\kappa_\lambda=\lambda$.*

**Proof.** By hypothesis $N_0\models\lambda>\kappa$. Hence for any $\alpha$, $N_\alpha\models i_{0\alpha}\lambda>\kappa_\alpha$. Hence $\lambda\geqslant \sup_{\alpha<\lambda}\kappa_\alpha=\kappa_\lambda$.

**Definition.** Assume cf $\lambda=\omega$; we say that the sequence $(\alpha_n)_{n\in\omega}$ is *good* below $\lambda$ if it is increasing and cofinal in $\lambda$, and moreover satisfies at least one of the following conditions:

(i) for all $n$ $\alpha_n$ is simple;
(ii) for all $n$ $\alpha_n<\kappa$;
(iii) for all $m$, $n$, $m<n$ $\alpha_n$ is indecomposable and $i_{0\alpha_m}\alpha_n=\alpha_{n+m}$.

**Proposition 3.** (i) *The class of simple ordinals is unbounded and $\theta$-closed for all regular $\theta\neq\kappa$.*

(ii) *If $\lambda$ is simple, then $\lambda$ is limit of simple ordinals, or* cf $\lambda=\omega$ *and there is a good sequence below $\lambda$ (the two cases do not exclude each other...).*

**Proof.** Assume $\theta$ regular $\neq\kappa$, and $\lambda=\sup_{\mu\in\theta}\alpha_\mu$ where $(\alpha_\mu)_{\mu\in\theta}$ is an increasing sequence of simple ordinals. Then $\lambda$ is simple itself: for if $\alpha<\lambda$ there is $\mu^\cdot\in\theta$ such that $\alpha<\alpha_{\mu^\cdot}$, and so $\alpha+\alpha_{\mu^\cdot}=\alpha_{\mu^\cdot}$, and a fortiori $\alpha+\lambda=\lambda$. Moreover

$$i_{0\alpha}\lambda=\sup_{\mu\in\theta} i_{0\alpha}\alpha_\mu\leqslant\sup_{\mu^\cdot\leqslant\mu<\theta} i_{0\alpha}\alpha_\mu\leqslant\sup_{\mu^\cdot\leqslant\mu<\theta} i_{0\alpha_{\mu^\cdot}}\alpha_\mu=\sup_{\mu^\cdot\leqslant\mu<\theta}\alpha_\mu=\lambda.$$

Suppose now that there exists a greatest simple ordinal below $\lambda$, say $\alpha$ (we allow $\lambda$ to be $\infty$ to prove (i)).

1*st Case.* $\lambda<\kappa$. Since $\alpha<\kappa$, $\alpha\cdot\omega$ is simple and greater than $\alpha$, so $\alpha\cdot\omega=\lambda$, and cf $\lambda=\omega$.

*2nd Case.* $\lambda > \kappa$. By Lemma 2, $\lambda = \kappa_\lambda$, thus if $\beta < \lambda$ we have $\kappa_\beta < \lambda$. We put $\alpha_0 = 0$, $\alpha_1 = \kappa_{\alpha+1}$ (where $\alpha$ is always the greatest simple below $\lambda$), and by induction $\alpha_{n+1} = i_{0\alpha_1}\alpha_n$. Clearly $\alpha_1$ is indecomposable since $\kappa$ is, and so is each of the $\alpha_n$ by the elementarity of $i_{0\alpha_1}$. Moreover the sequence $(\alpha)_{n\in\omega}$ is strictly increasing: for notice that $\alpha_{n+1} > \alpha_n$ is equivalent to $\alpha_2 > \alpha_1$. This last point we now prove: $\alpha_2 = i_{0\alpha_1}\alpha_1 = i_{0\alpha_1}\kappa_{\alpha+1} = \kappa_{\alpha_1 + i_{\alpha_1}\alpha+1} > \alpha_1 = \kappa_{\alpha+1}$, since $\alpha_1 \geq \alpha + 1 > \alpha$ implies $\alpha_1 + i_{0\alpha_1}\alpha + 1 > \alpha + 1$.

Now we check by induction on $n$ that: $\forall m\, i_{0\alpha_n}\alpha_m = \alpha_{n+m}$. This is true for $n = 1$ by definition. Then $i_{0\alpha_{n+1}}\alpha_m = i_{0 i_{0\alpha_1}\alpha_n}\alpha_m = i_{0\alpha_1 + i_{0\alpha_1}\alpha_n}\alpha_m$ since $\alpha_n$ is indecomposable, hence $i_{0\alpha_1}\alpha_n$ is, and $\alpha_1 + i_{0\alpha_1}\alpha_n$ is equal to $i_{0\alpha_1}\alpha_n$. Now recall Section 2.2 to show that $i_{0\alpha_1 + i_{0\alpha_1}\alpha_n} = i_{0\alpha_1} i_{0\alpha_n}$. Finally we have: $i_{0\alpha_{n+1}}\alpha_m = i_{0\alpha_1} i_{0\alpha_n}\alpha_m = i_{0\alpha_1}\alpha_{n+m}$ (induction hypothesis) $= \alpha_{n+m+1}$, and we are done. We finish when proving that $\lambda = \sup_n \alpha_n$: for it suffices to show that $\sup_n \alpha_n$ is simple, since the simplicity of $\lambda$ implies that for all $n$ $\alpha_n \leq \lambda$. But $\sup_n \alpha_n$ is indecomposable since each $\alpha_n$ is, and if $\alpha < \sup_n \alpha_n$ there is $n$ such that $\alpha < \alpha_{n'}$, hence $i_{0\alpha}(\sup_n \alpha_n) = \sup_n i_{0\alpha}\alpha_n \leq \sup_n i_{0\alpha_{n'}}\alpha_n = \sup_n \alpha_{n'+n} = \sup_n \alpha_n$, and we are done.

We thus have shown that, if $\lambda$ is simple and not limit of simple ordinals, $\mathrm{cf}(\lambda) = \omega$ and we constructed a good sequence below $\lambda$.

We show now that *any* limit ordinal is seen as simple when looked at from close enough.

**Proposition 4.** *For any limit $\lambda$ there exists $\alpha < \lambda$ such that for all $\beta$, $\alpha \leq \beta < \lambda$, $N_\beta$ satisfies: "$\lambda - \alpha = \lambda - \beta$ is simple"; moreover if $\lambda = \kappa_\lambda$, then $\lambda = \lambda - \alpha > \kappa_\beta$, and if $\lambda < \kappa_\lambda$, then $\lambda - \alpha < \kappa_\beta$.*

**Proof.** As in Section 2.3 first notice that the sequence $(\lambda - \gamma)_{\gamma<\lambda}$ is eventually constant since it is decreasing. Hence there is $\theta$ and $\mu$ such that for any $\gamma$, $\theta \leq \gamma < \lambda$, $\lambda = \gamma + \mu$. We claim that $\mu$ is indecomposable. For if $\nu < \mu$, we have $\theta + \nu < \theta + \mu = \lambda$, hence $\lambda = \theta + \nu + \mu = \theta + \mu$, thus $\mu = \nu + \mu$. Now the sequence $(i_{\gamma\lambda}\mu)_{\gamma<\lambda}$ is eventually constant since it is decreasing: hence there is $\alpha$ such that for every $\gamma$, $\alpha \leq \gamma < \lambda$, $i_{\gamma\lambda}\mu = i_{\alpha\lambda}\mu$. We may moreover assume that $\alpha \geq \theta$, so $\alpha + \mu = \lambda$. For any $\gamma$, $\alpha \leq \gamma < \lambda$, $i_{\gamma\lambda}\mu = i_{\alpha\lambda}\mu$, which is $i_{\gamma\lambda} i_{\alpha\gamma}\mu$, thus by the injectivity of $i_{\gamma\lambda}$ we get $\mu = i_{\alpha\gamma}\mu$. We have shown that for any $\delta < \mu$ $i_{\alpha\alpha+\delta}\mu = \mu$, which means that in $N_\alpha$ $\mu$ is fixed under each $i_{0\delta}$, $\delta < \mu$. Since $\mu$ is indecomposable, this shows that $\mu = \lambda - \alpha$ is simple in $N_\alpha$.

Now assume $\alpha \leq \beta < \lambda$: since $\lambda - \alpha$ is indecomposable, we have $\lambda - \alpha = \lambda - \beta$. Then $N_\alpha$ satisfies "$\lambda - \alpha$ is simple", so $N_\beta$ satisfies: "$i_{\alpha\beta}(\lambda - \alpha)$ is simple", but $i_{\alpha\beta}(\lambda - \alpha) = \lambda - \alpha$ since $\beta - \alpha < \lambda - \alpha$, so $N_\beta$ satisfies "$\lambda - \alpha$ is simple". Then if $\lambda = \kappa_\lambda$, $\lambda$ is indecomposable, therefore $\lambda = \lambda - \alpha$, and for any $\beta < \lambda \lambda = \kappa_\lambda > \kappa_\beta$. Finally assume $\lambda < \kappa_\lambda$: we prove that $\lambda - \alpha < \kappa_\alpha$. For $\lambda - \alpha$ is simple in $N_\alpha$, thus either $\lambda - \alpha < \kappa_\alpha$, or $N_\alpha$ satisfies "$\lambda - \alpha$ is $\kappa_{\lambda-\alpha}$", that is in $N_0$ $\lambda - \alpha = \kappa_{\alpha+(\lambda-\alpha)} = \kappa_\lambda$. But $\lambda - \alpha \leq \lambda < \kappa_\lambda$, a contradiction, so $\lambda - \alpha < \kappa_\alpha$, and the proof is finished.

## 3. The passage from $N_\omega$ to $M_\omega$

The following fact has been observed for a long time:

**Lemma 1.** *The sequences $(\kappa_n)_{n\in\omega}$ and $(\chi_n)_{n\in\omega}$ are members of $M_\omega$, but not of $N_\omega$.*

**Proof.** For any $p$ $i_{op}[(\kappa_n)_{n\in\omega}]$ is in $N_p$ since $(\kappa_n)_{n\in\omega}$ is in $N_0$ (since $\mathcal{U}$ is): but this sequence is exactly $(\kappa_n)_{n\geqslant p}$. Now $(\kappa_n)_{n\in\omega}$ is in $N_p$ as well as $(\kappa_n)_{n\geqslant p}$ since they differ only by a finite number of terms. Finally $(\kappa_n)_{n\in\omega}$ is in $M_\omega$, and by the same way $(\chi_n)_{n\in\omega}$ is in $M_\omega$.

On the other hand, they cannot be in $N_\omega$, for $\kappa_\omega=\sup_n \kappa_n=\sup_n \chi_n$ is regular in $N_\omega$, being measurable in this model.

It was then very natural to raise the question whether $M_\omega$ is precisely the model generated by $(\kappa_n)_{n\in\omega}$, or by $(\chi_n)_{n\in\omega}$, over $N_\omega$. We prove here this second point. For the connection with the first one, see chapter 7, where it is shown that the answer may be "yes" or "no" (but of course is "yes" for $\mathcal{U}$ normal since in that case $\chi_n=\kappa_n$).

When no confusion is possible, we drop the subscript "$\in\omega$" when writing $\omega$-sequences: we shall note $(\chi_n)_n$ for $(\chi_n)_{n\in\omega}$.

**Theorem 2.** *The model $M_\omega$ is exactly $N_\omega[(\chi_n)_n]$.*

**Proof.** Since $(\chi_n)_n$ is a well ordered set, $N_\omega[(\chi_n)_n]$ (which has a precise sense since $N_\omega$ is an inner model of our ground model $N_0$, see for instance [7] satisfies AC as well as $N_\omega$. By a well-known result, it suffices then to show that $M_\omega$ and $N_\omega[(\chi_n)_n]$ have the same subsets of On. By Lemma 1, $N_\omega[(\chi_n)_n]$ is included in $M_\omega$. We fix any $\theta$ and $X$ a subset of $\theta$ in $M_\omega$, and claim that $X$ belongs to $N_\omega[(\chi_n)_n]$.

Since $X$ is in $M_\omega$, it is in $N_n$ for all $n$, so there is (AC) a sequence of functions $f_n:\kappa^n\to N_0$, such that for every $n$ $X=\pi_n f_n=i_{0n}f_n(\chi_0\cdots\chi_{n-1})$ in normal form. Hence we get

$$\begin{aligned} i_{n\omega}X &= i_{n\omega}[i_{0n}f_n(\chi_0\cdots\chi_{n-1})],\\ &= i_{0\omega}f_n(\chi_0\cdots\chi_{n-1}), \end{aligned}$$

since $i_{n\omega}(\chi_0\cdots\chi_{n-1})=(\chi_0\cdots\chi_{n-1})$. Now, the sequence $(i_{0\omega}f_n)_n$ is $i_{0\omega}(f_n)_n$, so it is in $N_\omega$, and finally the sequence $(i_{n\omega}X)_n$ is in $N_\omega[(\chi_n)_n]$. It is easy to prove now that $X$ itself is in $N_\omega[(\chi_n)_n]$: in fact for any ordinal $\mu$, by Section 2.3 Proposition 2, the sequence $(i_{n\omega}\mu)_n$ is eventually constant, and its limit is $i_{\omega\omega+\omega}\mu$. But then "$\mu\in X$" is equivalent to "for $n$ big enough $i_{n\omega}\mu$ belongs to $i_{n\omega}X$", and hence to: "for $n$ big enough $i_{\omega\omega+\omega}\mu\in i_{n\omega}X$", what means that

$$X=\{\mu\in\theta\colon \exists m\ \forall n\geqslant m\ i_{\omega\omega+\omega}\mu\in i_{n\omega}X\}.$$

We are done, since $i_{\omega\omega+\omega} \upharpoonright \theta$ is in $N_\omega$, and $(i_{n\omega}X)_n$ in $N_\omega[(\chi_n)_n]$ as we saw above.

Hence $M_\omega$ and $N_\omega[(\chi_n)_n]$ have the same subsets of On, and are equal.

**Corollary 3.** *$M_\omega$ satisfies* AC.

Notice as a particular case: if $\mathcal{U}$ is normal, then $M_\omega$ is $N_\omega[(\kappa_n)_n]$. Applying Theorem 2 in $N_0$, we get:

**Corollary 4.** *For any $\rho$, $M_{\rho+\omega} = N_{\rho+\omega}[(\chi_{\rho+n})_n]$.*

## 4. Getting $N_0$ from $N_\alpha$ and $\mathcal{U}$.

We prove here that adjoining $\mathcal{U}$ to any $N_\alpha$ constructs the whole model $N_0$. This result will be used in the next section. Notice that it is obvious if $N_0 = L[\mathcal{U}]$.

4.1.

We first point out that if $\alpha < \beta$ the way from $N_\beta$ to $N_\alpha$ cannot be generic, nor quasi-generic (see [7]).

**Lemma 1.** *Assume $\alpha < \beta$; then there are arbitrary large ordinals which are beths in $N_\beta$ and are not in $N_\alpha$.*

**Proof.** We may assume $\alpha = 0$. Fix any $\gamma$ limit and bigger than $\beta$, and set $\mu = \beth_\gamma$, $\nu = (\beth_{\gamma+2^\kappa})^+$. $\mu$ is strong limit, so it is simple. We claim that $i_{0\beta}\nu = \nu$. Since $\nu$ is regular, $i_{0\beta}\nu = \sup_{\theta<\nu} i_{0\beta}\theta$, and for $\theta < \nu$ $|i_{0\beta}\theta| \leq |\beta|\,|\theta|^\kappa \leq |\beta|\ (\beth_{\gamma+2^\kappa})^\kappa = \beth_{\gamma+2^\kappa}$, so $i_{0\beta}\theta < \nu$, and $i_{0\beta}\nu \leq \nu$. Now, $N_0$ satisfies: "There are $2^\kappa$ $\beth$-cardinals between $\mu$ and $\nu$", while $N_\beta$ satisfies: "There are $2^{\kappa_\beta}$ $\beth$-cardinals between $i_{0\beta}\mu$ and $i_{0\beta}\nu$". Since $i_{0\beta}\mu = \mu$, $i_{0\beta}\nu = \nu$ and $(2^{\kappa_\beta})_{N_\beta} > 2^\kappa$, $N_0$ and $N_\beta$ cannot have the same $\beth$-numbers.

**Corollary.** *Assume $\alpha < \beta$: then $N_\alpha$ is neither a generic nor a quasi-generic extension of $N_\beta$.*

4.2.

For all $\alpha$, we can construct the inner model $N_\alpha[\mathcal{U}]$ of $N_0$. We prove that $N_\alpha[\mathcal{U}]$ is in fact $N_0$.

**Lemma 1.** *For any $\alpha$, $\mu$, $\theta$, $i_{0\theta} \upharpoonright \mu$ belongs to $N_\alpha[\mathcal{U}]$.*

**Proof.** In $N_\alpha[\mathcal{U}]$, $\mathcal{U}$ is a $\kappa$-complete ultrafilter on $\kappa$, since $\mathcal{P}(\kappa) \cap N_\alpha[\mathcal{U}] = \mathcal{P}[\kappa) \cap N_0 = \mathcal{P}(\kappa) \cap N_\alpha$. We can thus construct in $N_\alpha[\mathcal{U}]$ the iterated ultrapowers of the universe with $\mathcal{U}$. Let us write $i^*_{0\theta}$ the elementary embedding associated with the $\theta$-th such ultrapower. We shall prove that $i_{0\theta} \restriction \mathrm{On} = i^*_{o\theta} \restriction \mathrm{On}$ for every $\theta$, and this yields the lemma since for all $\theta$, $\mu$, $i^*_{0\theta} \restriction \mu$ is a set of $N_\alpha[\mathcal{U}]$. We introduce three properties:

$A_\mu$: for all $\theta$ $i_{0\theta} \restriction \mu = i^*_{o\theta} \restriction \mu$

$B_\mu$: for all $\theta$ $i_{0\theta} \restriction \mu \in N_\alpha[\mathcal{U}]$

$C_\mu$: for all $n$ $\mu^{\kappa^n} \cap N_\alpha[\mathcal{U}] = \mu^{\kappa^n} \cap N_0$

We shall prove that $(\forall \nu < \mu A_\nu) \Rightarrow B_\mu \Rightarrow C_\mu \Rightarrow A_\mu$.

1*st Step.* $\forall \nu < \mu A_\nu \Rightarrow B_\mu$

If $\mu = \nu + 1$, $i_{0\theta} \restriction \mu = i^*_{0\theta} \restriction \nu \cup (\mu, i_{0\theta}\mu) \in N_\alpha[\mathcal{U}]$

If $\mu$ is limit, $i_{0\theta} \restriction \mu = \bigcup_{\nu<\mu} i^*_{0\theta} \restriction \nu \in N_\alpha[\mathcal{U}]$

2*nd Step.* $B_\mu \Rightarrow C_\mu$:

Fix $f$: $\kappa^n \to \mu$. By hypothesis $i_{0\alpha} \restriction \mu \in N_\alpha[\mathcal{U}]$. Let $j$ be the inverse function in $N_\alpha[\mathcal{U}]$ of $i_{0\alpha} \restriction \mu$ (which is injective). Now $i_{0\alpha}f$ belongs to $N_\alpha$, hence to $N_\alpha[\mathcal{U}]$, as well as the composed map $j.\ (i_{0\alpha}f \cap \kappa^n \times \mathrm{On})$. But for $\xi$ in $\kappa^n$, $i_{0\alpha}f(\xi) = i_{0\alpha}[f(\xi)]$ since $\xi = i_{0\alpha}\xi$, and $j(i_{0\alpha}f(\xi)) = j \cdot i_{0\alpha}[f(\xi)] = f(\xi)$. Hence $f$ is $j \cdot (i_{0\alpha}f \cap \kappa^n \times \mathrm{On})$, and so is in $N_\alpha[\mathcal{U}]$.

3*rd Step.* $C_\mu \Rightarrow A_\mu$:

Fix any $\theta$. For any $f$ in $Fn_\theta(N_0, \kappa)$ with values in $\mu$, there are $e \in [\theta]^{<\omega}$ and $g \in \mu^{\kappa^{|e|}}$ such that in $N_0$, $f = e * g$. We look at two maps from $Fn_\theta(N_0, \kappa) \cap \mu^{\kappa^e}$ to $N_0$: $\pi_\theta$ and $\Pi$ such that if $f = e * g$ and $\Pi f = x$, then $N_\alpha[\mathcal{U}]$ satisfies "$x = \pi_\theta f'$ where $f' = e * g$". This has a sense since $g$ is by hypothesis in $N_\alpha[\mathcal{U}]$ while $f$ is perhaps not. The image of $\pi_\theta$ is $i_{0\theta}\mu$, so is transitive in $N_0$, the image of $\Pi$ is $i^*_{0\theta}\mu$ so is transitive in $N_\alpha[\mathcal{U}]$, hence in $N_0$. We show that these images are isomorphic: for $\pi_\theta f_1 = \pi_\theta f_2$ iff $g_1 \equiv g_2$ (mod. $\mathcal{U}_n$) (where $f_1 = e * g_1$, $f_2 = e * g_2$, and $n = |e|$), iff $N_\alpha[\mathcal{U}]$ satisfies: "$g_1 \equiv g_2$ mod. $\mathcal{U}_n$", since $\mathcal{U}_n$ is the same when calculated in $N_0$ and in $\mathcal{N}_\alpha[\mathcal{U}]$ iff $\mathcal{N}_\alpha[\mathcal{U}]$ satisfies: "$\pi_\theta e * g_1 = \pi_\theta e * g_2$", i.e. iff $\Pi f_1 = \Pi f_2$. Likewise for $\in$. This shows that $\Pi$ does not depend upon the choice of $e$ and $g$, and then by Mostowki's theorem that $\pi_\theta = \Pi$. Applying this result to constant $f$ gives what we called $A_\mu$.

**Theorem 2.** *The model* $N_\alpha[\mathcal{U}]$ *is* $N_0$ *for every* $\alpha$.

**Proof.** We show that every subset $X$ of any ordinal $\mu$ which is in $N_0$ is in $N_\alpha[\mathcal{U}]$. For $X$ is also $\{\xi \in \mu : i_{0\alpha}\xi \in i_{0\alpha}X\}$: by the last lemma $i_{0\alpha} \restriction \mu$ is in $N_\alpha[\mathcal{U}]$, and $i_{0\alpha}X$ is in $N_\alpha$, hence in $N_\alpha[\mathcal{U}]$, so $X$ itself is in $N_\alpha[\mathcal{U}]$.

4.3.

We now give various forms of the preceding results which will be of some help in the next parts.

**Theorem 1.** *Assume* $\lambda$ *limit and* $\alpha > \lambda$. *If* $M$ *is a model which includes* $N_\alpha$ *and contains an infinite subset of* $\{\chi_\beta : \beta < \lambda\}$, *it includes* $N_\lambda$.

**Proof.** Assume $M$ contains an infinite subset $R$ of $\{\chi_\beta : \beta<\lambda\}$, and let $\kappa_\gamma$ be the supremum of the $\omega$ least elements of $R$. By Section 1.7. Corollary 2, $\mathcal{P}(\kappa_\gamma)\cap N_\gamma = \mathcal{P}(\kappa_\gamma)\cap N_\alpha$, so this set belongs to $M$. Now by Section 1.7. Proposition 4, the set of $M$ defined as $\{X\in\mathcal{P}(\kappa_\gamma)\cap N_\gamma : X\cap R \text{ is infinite}\}$ is exactly $i_{0\gamma}\mathcal{U}$, which is thus a member of $M$. We now apply the Theorem 2 of last paragraph in $N_\gamma$, to get that $M$ includes $N_\gamma$; since $\gamma\leqslant\lambda$, a fortiori $M$ includes $N_\lambda$.

**Corollary 2.** *If $M$ is a model which includes $N_\alpha$ and contains an infinite bounded subset of $\{\chi_\beta : \beta<\alpha\}$, $M$ is not a generic extension of $N_\alpha$ (nor a quasi-generic one).*

**Proof.** Choose a $\lambda$ limit such that $M$ contains an infinite subset of $\{\chi_\beta : \beta<\lambda\}$ and $\lambda<\alpha$. By the last theorem, $M$ includes $N_\lambda$, so by the results of the first paragraph $M$ is not a generic extension of $N_\alpha$ since there are arbitrary large ordinals which are $\beth$'s in $N_\alpha$ and are not in $N_\lambda$, nor a fortiori in $M$.

4.4

**Theorem 1.** *For any $\alpha$, $M_\omega$ is exactly $N_\alpha[(\chi_n)_n]$.*

**Proof.** By the last results $N_\alpha[(\chi_n)_n]$ includes $N_\omega$, hence $N_\omega[(\chi_n)_n]$, which is $M_\omega$ as showed in Section 3.

## 5. The axiom of choice in $M_\lambda$.

We begin our investigation of the models $M_\lambda$ for $\lambda$ limit, and first concentrate on the case cf $\lambda=\omega$. In this chapter we prove that if $\lambda$ is not the form $\alpha+\omega$ and is "really" of cofinality $\omega$ (i.e. for all $\alpha<\lambda$, $\lambda$ is of cofinality $\omega$ in $N_\alpha$) then $M_\lambda$ does not verify AC, and in fact AC fails badly in $M_\lambda$. In Chapter 9 we will show that if $\lambda$ is not "really" of cofinality $\omega$, then $M_\lambda=N_\lambda$ and so $M_\lambda\vDash$AC. We assume throughout this chapter that cf $\lambda=\omega$.

5.1.

**Lemma 1.** *If $\lambda$ is simple, then any good sequence below $\lambda$ is in $M_\lambda$.*

**Proof.** If $\lambda<\kappa$, any sequence below $\lambda$ is in $V_\kappa$, hence in $N_\lambda$. If $\lambda>\kappa$ and $(\alpha_n)_n$ is good below $\lambda$ such that $\alpha_n$ is simple for each $n$, then for any $p$ we have: $i_{0\alpha_p}(\alpha_n)_{n>p}=(\alpha_n)_{n>p}$, so $(\alpha_n)_{n>p}$ is in $N_{\alpha_p}$, as well as $(\alpha_n)_n$, and $(\alpha_n)_n$ is in $M_\lambda$. If $\lambda>\kappa$ and $(\alpha_n)_n$ is good such that for all $m$, $n$, $m<n$, $i_{0\alpha_m}\alpha_n=\alpha_{m+n}$, then $i_{0\alpha_p}(\alpha_n)_n=(\alpha_n)_{n\geqslant p}$, so $(\alpha_n)_{n\geqslant p}$ is in $N_{\alpha_p}$, $(\alpha_n)_n$ too, and $(\alpha_n)_n$ is in $M_\lambda$.

**Proof.** Assume first that $\lambda$ is simple. If $\lambda<\kappa$ and $(\alpha_n)_n$ is any increasing cofinal sequence in $\lambda$, we claim that $(\kappa_{\alpha_n})_n$ belongs to $M_\lambda$. For any $p(\alpha_n-\alpha_p)_{n\geq p}$ is in $N_{\alpha_p}$ (since $\lambda<\kappa$), and $(\kappa_{\alpha_n})_{n\geq p}$ is in $N_{\alpha_p}$ the sequence "$(\kappa_{\alpha_n-\alpha_p})_{n\geq p}$", so $(\kappa_{\alpha_n})_{n\geq p}$ is in $N_{\alpha_p}$, as well as $(\kappa_{\alpha_n})_n$, and the claim is proved. But in $N_\lambda \kappa_\lambda$ is measurable, so it is regular, and $(\kappa_{\alpha_n})_n$ is not in $N_\lambda$.

If $\lambda\geq\kappa$, then by Chapter 2.4. Lemma 2, $\lambda=\kappa_\lambda$. Any good sequence below $\lambda$ is in $M_\lambda$ and cannot be in $N_\lambda$ since in $N_\lambda$ $\lambda$ is measurable. Now, let $\lambda$ be arbitrary; by Section 2.4. Proposition 4, there is an $\alpha<\lambda$ such that $N_\alpha$ satisfies: "$\lambda-\alpha$ is simple". So $N_\alpha$ satisfies "$M_{\lambda-\alpha}\neq N_{\lambda-\alpha}$", that is, in $N_0$, $M_\lambda\neq N_\lambda$.

5.2.

Let us recall that for two sequences $(x_n)_n$ and $(x'_n)_n$ being *eventually equal* means: $\exists n\forall m\geq n\ x_m=x'_m$. We shall note this equivalence relation by $\equiv$. We say that $(x_n)_n$ is *eventually included* in $(x'_n)_n$, or that $(x_n)_n$ is an *almost-sub-sequence* of $(x'_n)_n$ if $\exists n\forall m\geq n\ x_m\subseteq x'_m$.

**Definition.** For $\lambda$ limit, cof $\lambda$ is the set of the $\omega$-sequences which are strictly increasing, cofinal in $\lambda$ and belong to $M_\lambda$.

**Lemma 1.** *Assume* $\alpha<\lambda$: *then* $E$ *is in* cof $\lambda$ *iff* $E$ *is increasing and* $N_\alpha\models E-\alpha\in$ cof $(\lambda-\alpha)$ (*where* $E-\alpha$ *is* $(\gamma_n-\alpha)_n$ *if* $E$ *is* $(\gamma_n)_n$, *and* $\gamma-\alpha=0$ *if* $\gamma\leq\alpha$).

**Proof.** $N_\alpha\models E-\alpha\in$ cof $(\lambda-\alpha)$ iff $E-\alpha$ is increasing, cofinal in $(\lambda-\alpha)$ and belong to $M_{\lambda-\alpha}$ calculated in $N_\alpha$, i.e. to $M_\lambda$. We are done since $E$ is cofinal in $\lambda$ iff $E-\alpha$ is cofinal in $\lambda-\alpha$.

**Lemma 2.** (i) cof $\lambda\in M_\lambda$;

(ii) *If* $\lambda<\kappa_\lambda$ cof $\lambda\in N_\lambda$;

(iii) *If* $\lambda$ *is indecomposable and* cof $\lambda\neq\emptyset$, *then* $|\text{cof }\lambda/\equiv|\geq|\lambda|$;

(iv) *If* $\forall\alpha<\lambda N_\alpha\models\text{cf }\lambda=\omega$, *then* cof $\lambda\neq\emptyset$.

**Proof.** (i) By Lemma 1, cof $\lambda$ is equiconstructible to $\text{cof}^{N_\alpha}(\lambda-\alpha)$, so for all $\alpha<\lambda$ cof $\lambda$ is in $N_\alpha$.

(ii) If $\lambda<\kappa_\lambda$, there is $\alpha<\lambda$ such that $\lambda-\alpha<\kappa_\alpha$. Then $N_\alpha\models\text{cof }(\lambda-\alpha)\in V_\kappa$, id est $\text{cof}^{N_\alpha}(\lambda-\alpha)\in V_{\kappa_\alpha}\cap N_\alpha$. By Lemma 1 we get cof $\lambda\in V_{\kappa_\alpha}\cap N_\alpha$, hence cof $\lambda\in V_{\kappa_\alpha}\cap N_\lambda$.

(iii) Assume $\lambda$ indecomposable and $(\alpha_n)_n\in$cof $\lambda$; let $\gamma<\lambda$: for each $n$ $\alpha_n<\lambda$ and $\gamma<\lambda$, so $\alpha_n+\gamma<\lambda$. Thus the sequence $(\alpha_n+\gamma)_n$ is in cof $\lambda$. Clearly if $\gamma\neq\gamma'$ then we cannot have $(\alpha_n+\gamma)_n\equiv(\alpha_n+\gamma')_n$.

(iv) By Section 2.4. Proposition 4, there is $\alpha<\lambda$, such that $\lambda-\alpha$ is simple in $N_\alpha$. By hypothesis $N_\alpha\models\text{cf }\lambda=\omega$, hence $N_\alpha\models\text{cf }(\lambda-\alpha)=\omega$. By Section 2.4. Proposition 3, there exists in $N_\alpha$ a good sequence below $\lambda-\alpha$, hence by Section 5.1. Lemma 1, $\text{cof}^{N_\alpha}(\lambda-\alpha)$ is not empty. Therefore by Lemma 1, cof $\lambda$ is also not empty.

5.3.

First let us make the convention that for $E=(\gamma_n)_n$ $\chi_E$ is $(\chi_{\gamma_n})_n$. We put:

**Definition.** For $\lambda$ limit

(i) $\mathcal{G}_\lambda=\{g : g$ strictly increasing and $\exists E\in \mathrm{cof}\ \lambda\ g\equiv\chi_E\}$.

(ii) $\mathcal{N}_\lambda$ in the "*name*" *function* from $\mathcal{G}_\lambda/\equiv$ to cof $\lambda/\equiv$ which associates the class of $E$ to the class of $\chi_E$.

**Lemma 1.** *$\mathcal{G}_\lambda$ and $\mathcal{N}_\lambda$ belong to $M_\lambda$.*

**Proof.** Assume $\alpha<\lambda$, and let $d_\alpha$ the map with domain cof $\lambda/\equiv$ which associates to the class of $E$ the class of $E-\alpha$. By Section 5.2. Lemma 1, $d_\alpha$ is a bijection of cof $\lambda/\equiv$ onto $(\mathrm{cof}\,(\lambda-\alpha)/\equiv)^{N_\alpha}$. Now notice that for any $E$ in cof $\lambda\chi_{E-\alpha}$ calculated in $N_\alpha$ is $\chi_{\alpha+(E-\alpha)}$, so is eventually equal to $\chi_E$: this proves that for any $x$ in cof $\lambda/\equiv$ $\mathcal{N}_\lambda^{-1}(x)$ is also $\mathcal{N}_{\lambda-\alpha}^{-1(N_\alpha)}(d_\alpha(x))$. Since $\mathcal{N}_{\lambda-\alpha}^{(N\alpha)}$ and $d_\alpha$ are in $N_\alpha$, so is $\mathcal{N}_\lambda^{-1}$, then $\mathcal{N}_\lambda$, its range $\mathcal{G}_\lambda/\equiv$, and finally $\mathcal{G}_\lambda$. As $\alpha$ was arbitrary we are done.

**Lemma 2.** *If $\lambda<\kappa_\lambda$, $\mathcal{G}_\lambda/\equiv$ is well orderable in $M_\lambda$.*

**Proof.** If $\lambda<\kappa_\lambda$, cof $\lambda$ is in $N_\lambda$ by Section 5.2. Lemma 2, so cof $\lambda/\equiv$ is in $N_\lambda$, and since $N_\lambda$ satisfies AC cof $\lambda/\equiv$ is well orderable in $N_\lambda$. Finally the bijection $\mathcal{N}_\lambda$ of $\mathcal{G}_\lambda/\equiv$ onto cof $\lambda/\equiv$ is in $M_\lambda$ by Lemma 1, so $\mathcal{G}_\lambda/\equiv$ is well orderable in $M_\lambda$.

**Proposition 3.** *Assume that for all $\alpha<\lambda$ $N_\alpha\vDash\mathrm{cf}\ \lambda=\omega$, and that $M$ is a transitive model of* ZF *including $M_\lambda$ and containing a well-ordering of $\mathcal{G}_\lambda$. Then, if $\lambda$ is not of the form $\lambda=\rho+\omega$, there is $\alpha<\lambda$ such that $M$ includes $N_\alpha$.*

**Proof.** By 4.3. Theorem 1., it suffices to show that $M$ contains an infinite bounded subset of $\{\chi_\beta : \beta<\lambda\}$, or, equivalently, that $M$ contains an infinite subset of $\{\chi_\beta : \beta<\lambda\}$ of order type $>\omega$.

Since $M$ well orders $\mathcal{G}_\lambda$, there exists in $MG^*\subseteq\mathcal{G}_\lambda$ such that $G^*$ meets in exactly one point each class of $\mathcal{G}_\lambda$ for $\equiv$. Now $M$ well-orders also $G^*$, and on the other hand $N_0$ (which includes $M$) bijects $G^*$ onto $\mathcal{G}_\lambda/\equiv$, hence, via $\mathcal{N}_\lambda$, onto cof $\lambda/\equiv$. We now distinguish two cases:

1*st Case.* $|\mathrm{Cof}\ \lambda/\equiv|\leq 2^{\aleph_0}$.

This is possible only when $\lambda<\kappa_\lambda$. For cof $\lambda\neq\emptyset$ by Section 5.2. Lemma 2 (iv), and if $\lambda=\kappa_\lambda$ by the (iii) part of the same lemma $|\mathrm{cof}\ \lambda/\equiv|\geq|\lambda|=|\kappa_\lambda|\geq\kappa$. So by the (ii) part, we have cof $\lambda\in N_\lambda$. Since $N_\lambda$ satisfies AC, we pick $C$ in $N_\lambda$, $C\subseteq$ cof $\lambda$ such that $C$ meets in exactly one point every class of cof $\lambda$ for $\equiv$. Finally let $G=G^*$; we have:

(1) $|G|=|C|\leq 2^{\aleph_0}$ and $C\in N_\lambda$ and any member of cof $\lambda$ (resp. of $\mathcal{G}_\lambda$) is $\equiv$ to a member of $C$ (resp. of $G$).

*2nd Case.* $|\mathrm{cof}\ \lambda/\equiv| > 2^{\aleph_0}$.
In this case $|G^*| \geq (2^{\aleph_0})^+$; since $M$ well orders $G^*$ we can pick in $M$ $G \subseteq G^*$ such that $|G| = (2^{\aleph_0})^+$, so

(2) $|G| = (2^{\aleph_0})^+$ and any two members of $G$ are not $\equiv$.

Now notice this general fact: if $M'$ is a transitive model of ZF included in $N_0$ and containing $V_\kappa$, and if $x$ belongs to $M'$ and has in $M'$ a cardinal $\theta < \kappa$, then $\theta = |x|$ and any function from $x$ to $\omega$ which is in $N_0$ is in $M'$.

For fix a bijection $f' : x \to \theta$ in $M'$, and a bijection $f : x \to |x|$ in $N_0$: since $M'$ is included in $N_0 f^{-1} f'$ is in $N_0$, so $|\theta| = |x| < \kappa$, and therefore $f^{-1} f'$ is in $V_\kappa$, and thus in $M'$: hence $\theta \leq |x|$, and $\theta = |x|$. Now if $F : x \to \omega$ is in $N_0$, $F f'^{-1} : \theta \to \omega$ is in $V_\kappa$, hence in $M'$, and so is $F = (F f'^{-1}) f'$. We deduce from this that any function $G \to \omega$ is in $M$, and if (1) holds that any function $C \to \omega$ is in $N_\lambda$. Each member of $\mathscr{G}_\lambda$ is eventually included in $\{\chi_\beta : \beta < \lambda\}$ so there is a function $b : G \to \omega$ in $N_0$, hence in $M$ such that:

$$\forall g \in G \forall n \geq b(g)\ g(n) \in \{\chi_\beta : \beta < \lambda\}.$$

We put $X = \{g(m) : g \in G \text{ and } m \geq b(g)\}$ : $X$ is in $M$, since $G$ and $b$ are, $X$ is infinite and included in $\{\chi_\beta : \beta < \lambda\}$. We claim that $X$ cannot be of order type $\omega$ unless if $\lambda$ is of the form $\rho + \omega$. This will finish the proof.

First assume (1) holds and $X$ is of order type $\omega$. For each $E$ in $C$, $\chi_E$ is eventually included in $X$, so there is $a : C \to \omega$ in $N_0$, hence in $N_\lambda$, such that $\forall E \in C$ $\forall n \geq a(E)\ \chi_{E(n)} \in X$. We put $K = \{E(m) : E \in C \text{ and } m \geq a(E)\}$. $K$ is in $N_\lambda$, since $C$ and $a$ are; $K$ is infinite, and for all $\mu$ in $K$ $\chi_\mu$ is in $X$. The map $\mu \mapsto \chi_\mu$ is injective and order preserving, so if $X$ is of order type $\omega$, $K$ is also of order type $\omega$, and therefore $K$ belongs to cof $\lambda$. Say $K = \mathrm{im}(\rho_n)$. By construction every member of cof $\lambda$ is $\equiv$ to a member of $C$, hence is eventually included in $K$. In particular $(\rho_n + 1)_n$ is in cof $\lambda$, hence there is $n$ such that:

$$\forall m \geq n\ \rho_m + 1 \in K = \mathrm{im}(\rho_n)_n,$$

that is

$$\forall m \geq n\ \rho_m + 1 = \rho_{m+1}.$$

Finally we get $\lambda = \sup_m \rho_m = \rho_n + \omega$.

Now assume (2) holds:
We look at the map: $G \to \mathscr{P}(X) \times \omega$ defined by $g \mapsto (\mathrm{im}\ g \cap X, b(g))$, and claim that it is an injection: for if $b(g_1) = b(g_2) = n$ and $\mathrm{im}\ g_1 \cap X = \mathrm{im}\ g_2 \cap X$ we have $g_1''[n, \omega) = g_2''[n, \omega)$, hence $g_1 \equiv g_2$, and $g_1 = g_2$ by definition of $G$. Now $|G| > 2^{\aleph_0}$ implies $|\mathscr{P}(X)| > 2^{\aleph_0}$, and $|X| > \aleph_0$, and our initial claim is proved.

**Theorem 4.** *Assume that $\lambda$ is not of the form $\lambda = \rho + \omega$, and for all $\alpha < \lambda$ $N_\alpha \models \mathrm{cf}\ \lambda = \omega$; then no generic extension of $M_\lambda$ included in $N_0$ may well order $\mathscr{G}_\lambda$.*

In particular $M_\lambda$ does not well order $\mathscr{G}_\lambda$, and hence does not satisfy AC, and $N_\lambda$ and $M_\lambda$ have no common generic extension included in $N_0$.

**Proof.** By the previous result, any model $M$ extending $M_\lambda$ and included in $N_0$ includes some $N_\alpha$, $\alpha < \lambda$, as soon as it well orders $\mathcal{G}_\lambda$. By Section 4.1. Lemma 1, such $M$ cannot be a generic extension of $M_\lambda$.

5.4.

We show now that $N_\lambda$ and $M_\lambda$ have the same aleph's.

**Lemma 1.** *$V_{\kappa_\lambda} \cap N_\lambda = V_{\kappa_\lambda} \cap M_\lambda$ for any $\lambda$ limit.*

**Proof.** $V_{\kappa_\lambda} = \bigcup_{\alpha<\lambda} V_{\kappa_\alpha}$, and for $\alpha < \lambda$

$$V_{\kappa_\alpha} \cap N_\lambda \subseteq V_{\kappa_\alpha} \cap M_\lambda \subseteq V_{\kappa_\alpha} \cap N_\alpha = V_{\kappa_\alpha} \cap N_\lambda,$$

whence the desired equality holds.

**Corollary 2.** *For any $\lambda$ limit, $N_\lambda$ and $M_\lambda$ have the same aleph's below $\kappa_\lambda$.*

5.5.

We now follow Bukovsky's method in [3] to prove the same equality above $\kappa_\lambda$, $\lambda$ being of cofinality $\omega$.

**Lemma 1.** *Assume $x \in N_\alpha$ and $|x|^{N_\alpha} \leq i_{0\alpha}\mu$. Then there exists $y \in N_0$, $|y| \leq \kappa \cdot \mu$ such that $x \subseteq i_{0\alpha} y$.*

**Proof.** Set $x = \pi_\alpha e * g$, $g : \kappa^{|e|} \to N_0$. Since $|x|$ in $N_\alpha$ is $\leq i_{0\alpha}\mu$, we may assume that for all $\xi$ in $\kappa^{|e|}$ $|g(\xi)| \leq \mu$. Then set $y = \bigcup_{\xi \in \kappa^{|e|}} g(\xi)$. Clearly $x \subseteq i_{0\alpha} y$.

**Lemma 2.** *Assume $f \in N_\alpha$, $f$ functional and for all $x$ in* dom $f$ $|f(x)|^{N_\alpha} \leq \kappa_\alpha$. *Then there exists a function $g$ such that:* $\forall x \in \text{dom } g$ $|g(x)| \leq \kappa$ *and* dom $f \subseteq$ dom $i_{0\alpha} g$ *and* $\forall x \in \text{dom } f$ $f(x) \subseteq [i_{0\alpha} g](x)$.

**Proof.** Set $f = \pi_\alpha e * F$ *with* $F : \kappa^{|e|} \to N_0$.
For all $\xi$ in $\kappa^{|e|} F(\xi)$ is a function and

$$\forall x \in \text{dom } F(\xi)\ |F(\xi)(x)| \leq \kappa.$$

We define $g$, a function with domain $\cup_\xi \text{dom } F(\xi)$, by

$$g(x) = \bigcup_{\substack{\xi \in \kappa^{|e|} \\ x \in \text{dom } F(\xi)}} F(\xi)(x).$$

Clearly for all $x$ in dom $g$ $|g(x)| \leq \kappa$. Now assume that $x \in \text{dom } f$: write $x = \pi_\alpha E * X$ with $X : \kappa^{|E|} \to N_0$. We may assume that $E \supseteq e$, say $E = E'e$ (looking at $e$ and $E$ as at injections into $\alpha$). Since $x \in \text{dom } f$, we have:

$$\{\eta \in \kappa^{|E|} : X(\eta) \in \text{dom } F(\eta E')\} \in \mathcal{U}_{|E|},$$

hence $\{\eta : X(\eta) \in \text{dom } g\} \in \mathcal{U}_{|E|}$, and so $x \in \text{dom } i_{0\alpha} g$.

Finally suppose that $(x, y) \in f$ and $(x, z) \in i_{0\alpha} g$ : with $x = \pi_\alpha E * X$, $y = \pi_\alpha E * Y$, $z = \pi_\alpha E * Z$ and $E = E'e$. By hypothesis

$$\{\eta : (X(\eta), Y(\eta)) \in F(\eta E') \text{ and } (X(\eta), Z(\eta)) \in g\}$$

is in $\mathcal{U}_{|E|}$, so $\{\eta : Y(\eta) \subseteq Z(\eta)\}$ is in $\mathcal{U}_{|E|}$, and $y \subseteq z$, the lemma is proved.

**Proposition 3.** *Assume that $f$ is a function in $M_\lambda$ with domain and range included in $N_\lambda$. There there exist in $N_\lambda$ functions $g$ and $g_0, g_1, g_2, \ldots$ such that*

(i) $\forall x \in \operatorname{dom} f\ \exists n\ x \in \operatorname{dom} g_n$ *and* $f(x) = g_n(x)$;

(ii) $\forall x \in \operatorname{dom} f\ x \in \operatorname{dom} g$ *and* $f(x) \in g(x)$ *and in* $N_\lambda$ $|g(x)| \leqslant \kappa_\lambda$.

With the notations of [3] (ii) means that $\mathrm{Apr}_{N\lambda, M\lambda}(\kappa_\lambda^+)$ holds.

**Proof.** Fix an increasing cofinal sequence $(\alpha_n)_n$ below $\lambda$. Since $f$ is in $N_{\alpha_n}$ as well as any set-restriction of $i_{\alpha_n}\lambda$, we define a function $f_n$ in $N_{\alpha_n}$ by $f_n = \{(x, y) : f(i_{\alpha_n\lambda} x) = i_{\alpha_n\lambda} y\}$. Now assume that $(x, y) \in f$ : $x$, $y$ are in $N_\lambda$, so there are $n$, $x'$, $y'$ such that $(x, y) = (i_{\alpha_n\lambda} x', i_{\alpha_n\lambda} y')$, and $(x, y) \in f$ implies $(x', y') \in f_n$, thus $(x, y) \in i_{\alpha_n\lambda} f_n$. We set $g_n = i_{\alpha_n\lambda} f_n$, and (i) is proved.

(ii) With the same notations, we apply Lemma 2 to the function $x \mapsto \{f_n(x)\}$, hence getting $g'_n$ in $N_0$ such that: $\forall x \in \operatorname{dom} g'_n, |g'_n(x)| \leqslant \kappa$, and $\forall x \in \operatorname{dom} f_n x \in \operatorname{dom} i_{0\alpha_n} g'_n$ and $f_n(x) \in i_{0\alpha_n} g'_n(x)$. Now we put $g'(x) = \bigcup_n g'_n(x)$, defined for $x$ being in $\bigcup_n \operatorname{dom} g'_n$, and finally $g = i_{0\lambda} g'$. Clearly for all $x$ in dom $g$ $|g(x)|^{N_\lambda} \leqslant \kappa_\lambda$. If $x$ is in dom $f$, it is in dom $g_n$ for some $n$, i.e. in $i_{\alpha_n\lambda}$ dom $f_n$, hence in $i_{0\lambda}$ dom $g'_n$, and finally in dom $g$. Moreover we have then: $f(x) = g_n(x) \in i_{0\lambda} g'_n(x)$ hence $f(x) \in g(x)$, and this finishes the proof.

**Theorem 4.** *For $\lambda$ of cofinality $\omega$, $M_\lambda$ and $N_\lambda$ have the same aleph's.*

**Proof.** By induction on $\theta$: by Section 5.4. Corollary 2. the result is proved for $\theta \leqslant \kappa_\lambda$. Now assume $\theta$ is an $\aleph > \kappa_\lambda$ in $M_\lambda$ and $N_\lambda$ and $\delta < (\theta^+)^{M_\lambda}$: there is a surjection $f : \theta \to \delta$ in $M_\lambda$. By Prop. 3 (ii). there is $g : \theta \to \mathcal{P}(\delta)$ in $N_\lambda$ such that for all $\mu < \theta$ $g(\mu)| \leqslant \kappa_\lambda < \theta$ and $f(\mu) \in g(\mu)$. Since $f$ is surjective we have in $N_\lambda$ $\delta \subseteq \bigcup_{\mu \in \theta} g(\mu)$, hence $|\delta| \leqslant \theta \cdot \theta = \theta$.

Notice that Prop. 3(i) implies the following "covering lemma" holds between $N_\lambda$ and $M_\lambda$ ($\lambda$ being always of cofinality $\omega$).

**Proposition 5.** *Assume that $X \in M_\lambda$ and $X \subseteq N_\lambda$. Then there is in $M_\lambda$ a sequence $(X_n)_n$ of members of $N_\lambda$ such that for all $n$ $|X_n| \leqslant |X|$ and $X \subseteq \bigcup_n X_n$.*

**Proof.** If $X$ belongs to $M_\lambda$ and is included in $N_\lambda$, it is well orderable in $M_\lambda$; fix a bijection in $M_\lambda$ $f$: $\theta \to X$, and apply Proposition 3 (i) to get in $N_0$ a sequence $(g_n)_n$

of functions with domain $\subseteq \theta$ such that $f \subseteq \bigcup_n g_n$, defined by $y = f_n(x)$ iff $i_{\alpha_n\lambda}y = f(i_{\alpha_n\lambda}x)$ and $g_n = i_{\alpha_n\lambda}f_n$. Applying Section 2.2. Proposition 1, Section 2.3. Proposition 2., we get that, for $n$ big enough to ensure that

$$i_{\alpha_n\lambda}i_{\alpha_n\lambda} = i_{\lambda\varepsilon(\lambda)}i_{\alpha_n\lambda},\ y = g_n(x) \quad \text{iff} \quad i_{\lambda\varepsilon(\lambda)}y = i_{\alpha_n\lambda}f(i_{\lambda\varepsilon(\lambda)}x).$$

Now since $f$ is in $M_\lambda$, the sequence $(i_{\alpha_n\lambda}f)_n$ is in $M_\lambda$, and the same holds for the sequence $(g_n)_n$ since any set-restriction of $i_{\lambda\varepsilon(\lambda)}$ belongs to $M_\lambda$. Finally we set $X_n = g_n''\theta$.

**Corollary 6.** *For any* $\theta$, $\mathrm{cf}^{M_\lambda}\theta = \mathrm{cf}^{N_\lambda}\theta$, *or* $\mathrm{cf}^{M_\lambda}\theta = \omega$.

**Remark.** Using the methods of [4] instead of those of [3] as made in Proposition 3(i) one shows more precisely that for any $\theta$ either $\mathrm{cf}^{N_\lambda}\theta = \mathrm{cf}^{M_\lambda}\theta$ or $\mathrm{cf}^{N_\lambda}\theta = \kappa_\lambda$ and $\mathrm{cf}^{M_\lambda}\theta = \omega$.

## 6. An extension of Prikry's forcing

In [11] Prikry introduced a notion of forcing to change to $\omega$ the cofinality of a measurable cardinal with the help of a normal ultrafilter. We investigate here the connection between the extensions as studied in the previous chapter and Prikry's ones. In fact we show that $M_\omega$ is a Prikry extension of $N_\omega$ when $\mathcal{U}$ is normal, or at least selective, and we define an extended forcing to get the same result for general $\mathcal{U}$. Analogous results hold for $M_\lambda$, $\lambda$ limit of cofinality $\omega$. But the main result is perhaps that $M_\omega$ is an "universal" extension for the extended Prikry forcing in the sense that to show some property in any Prikry extension it suffices to show this property in $M_\omega$. This enables us to prove without any technical forcing work all the results listed by Prikry, and to generalize them to the extended forcing.

6.1.

We first recall Prikry's construction.

**Definition.** $\mathcal{P}_{N_0}(\mathcal{U})$, or $\mathcal{P}$, is the set of *ordered pairs* $(s, X)$ with $s$ in $[\kappa]^{<\omega}$ and $X$ in $\mathcal{U}$ ordered by $(s^\frown t, Y) < (s, X)$ iff $t \in [X]^{<\omega}$ and $Y \subseteq X$.

To study $\mathcal{P}$-extensions, Prikry often uses the following Rowbottom style property (see for instance [5, Chapter 8]): "for any $f$: $[\kappa]^{<\omega} \to \nu < \kappa$ there is $X$ in $\mathcal{U}$ such that $|f''[X]^{<\omega}| \leqslant \aleph_0$." On the other when we try to show that the sequence $(\chi_n)_n$ is $N_\omega$-generic over $i_{0\omega}\mathcal{P}$, we must assume that:

"for any $Y$ in $\mathcal{U}_n$ there is $X$ in $\mathcal{U}$ such that $[X]^n \subseteq Y$".

Chapter 7 will be devoted to the study of such properties for $\mathcal{U}$. For the moment, let us simply notice that the second property implies the first one, and set:

**Definition.** $\mathcal{U}$ is said to be *selective iff* for all $Y$ in $\mathcal{U}_n$ there is $X$ in $\mathcal{U}$ such that $[X]^n \subseteq Y$.

It is well known that normal ultrafilters are selective. Now, we remark that the following holds for any complete ultrafilter $\mathcal{U}$: "for any $f$: $[\kappa]^{<\omega} \to \nu < \kappa$, there is a sequence $(X_l)_l$ with $X_l$ in $\mathcal{U}_l$ such that $|f'' \cup_l X_l| \leq \aleph_0$". This suggests replacing in Prikry's conditions the unique member of $\mathcal{U}$ by a sequence $(X_l)_l$ with $X_l$ in $\mathcal{U}_l$.

And in fact it turns out that all the proofs below can be carried out for such a set of conditions. Nevertheless, we shall prefer a slight different presentation using trees which makes the proofs easier to read.

6.2.

Our set of conditions will be as follows:

**Definition.** (i) $\mathcal{C}_{\aleph_0}(\mathcal{U})$, or $\mathcal{C}$, is the set of the trees $T$ on $[\kappa]^{<\omega}$ such that

(a) $T$ is closed under inclusion: $s \in T$ and $t \subseteq s$ implies $t \in T$;

(b) there is a member of $T$, called $s_T$ ("trunk" of $T$) satisfying: $\forall s \in T\ s \subseteq s_T$ or $S_T \subseteq s$ and if $S_T \subseteq s$ then $\{\xi: s^\frown \xi \in T\} \in \mathcal{U}$. We order $\mathcal{C}$ by inclusion.

(ii) We set $\mathcal{C}^\circ = \{T \in \mathcal{C}: s_T = \emptyset\}$.

(iii) For $T$ in $\mathcal{C}$ and $s$ we define $T^s$ in $\mathcal{C}$ to be $\{t \in T;\ t \subseteq s$ or $s \subseteq t\}$.

Let us first show that $\mathcal{C}$ behaves as a kind of $\kappa$-complete and normal filter.

**Lemma 1.** *Assume $(T_x)_{x \in X}$ is a family of members of $\mathcal{C}$:*

(i) *If $|X| < \kappa$ and $S_{T_x} = \emptyset$ for $x$ in $X$, then $\bigcap_{x \in X} T_x$ is in $\mathcal{C}$;*

(ii) *If there is an $s$ such that $X = \{t \in [\kappa]^{<\omega}: s \subseteq t\}$ and for each $t \supseteq s$ $s_{T_t} = t$, then there is $T$ in $\mathcal{C}$ such that $s_T = s$ and for all $t \supseteq s$ $T^t \subseteq T_t$.*

**Proof.** (i) Assume $t \in \bigcap_{x \in X} T_x$: then $\{\xi: t^\frown \xi \in \bigcap_{x \in X} T_x\}$ is $\bigcap_{x \in X} \{\xi: t^\frown \xi \in T_x\}$, so it belongs to $\mathcal{U}$ by the $\kappa$-completeness.

(ii) We construct $T$ by induction on levels: first put $s$ and all $t \subseteq s$ in $T$. Now assume $t \supseteq s$ has been put in $T$ in such a way that for all $t'$, $s \subseteq t' \subseteq t$, $t$ belongs to $T_{t'}$; for each $t'$, $s \subseteq t' \subseteq t$, $T^t_{t'}$ belongs to $\mathcal{C}$, and $s_{T^t_{t'}} = t$, so using (i) (but only the $\omega$-completeness of $\mathcal{U}$), we define $\{\xi: t^\frown \xi \in T\}$ to be $\bigcap_{s \subseteq t' \subseteq t} T^t_{t'}$. Clearly the induction hypothesis is satisfied, since for any $\xi$ $t^\frown \xi$ belongs to $T_{t \cap \xi}$. Finally if $t^\frown u$ belongs to $T$, it belongs to $T_t$, which means that $T^t \subseteq T_t$, and the lemma is proved.

*As a particular case of (i) notice that* $T$ and $T'$ are compatible iff $S_T$ and $S_{T'}$, are. Notice that if we put on $\kappa$ the discrete topology and on $\kappa^\omega$ the product topology then to each tree $T$ on $[\kappa]^{<\omega}$ is associated a closed subset of $\kappa^\omega$, the set of its branches $[T]$, defined by $s \in [T]$ iff all initial segments of $s$ belong to $T$. Let us call $\mathcal{U}^\omega$ the set of $[T]$ for $T$ in $\mathcal{C}^\circ$. Lemma 6 says that $\mathcal{U}^\omega$ is the basis of a

$\kappa$-complete filter on the closed subsets of $\kappa^\omega$. Moreover $\mathcal{U}^\omega$ is an extension of $\mathcal{U}_\omega$ which is a $\kappa$-complete ultrafilter on the subsets of $\kappa^\omega$ with finite support, which are precisely the clopen ones. Finally it is easy to present $\mathcal{U}^\omega$ as a kind of inverse limit of the $\mathcal{U}_n$, the arrows being the trace maps introduced below.

The next lemma shows that forcing with $\mathscr{C}$ is the same as Prikry's forcing with sequences $(X_l)_l$ where $X_l$ is a member of $\mathcal{U}_l$.

**Lemma 2.** *Assume $(X_l)_l$ is a sequence such that for all $l X_l \in \mathcal{U}_l$; then there exists $T$ in $\mathscr{C}^\circ$ which is included in $\bigcup_l X_l$.*

**Proof.** We define for $X \subseteq \kappa^m$ and $n \leqslant m$ the *trace* of $X$ on $\kappa^n$ by $\mathrm{tr}_{mn} X = \{\xi \in \kappa^n : X^\xi \in \mathcal{U}_{m-n}\}$. Now assume that for each $n$ $X_n$ is a member of $\mathcal{U}_n$, and define: $Y_n = \bigcap_{m \geqslant n} \mathrm{tr}_{mn} X_m$. For each $n$ $Y_n$ is a member of $\mathcal{U}_n$ included in $X_n$, for, if $X_m$ belongs to $\mathcal{U}_m$, then $\mathrm{tr}_{mn} X_m$ belongs to $\mathcal{U}_n$ for each $n \leqslant m$, and $\mathrm{tr}_{mm} X_m = X_m$ (by convention). We claim that for all $k \leqslant l$ $Y_k \subseteq \mathrm{tr}_{lk} Y_l$: for notice that for any sequence $(A_\mu)_{\mu<\nu}$ of subsets of $\kappa^l$ with $\nu < \kappa$ we have

$$\mathrm{tr}_{lk}\Big(\bigcap_{\mu<\nu} A_\mu\Big) = \bigcap_{\mu<\nu} \mathrm{tr}_{lk} A_\mu$$

since $\mathcal{U}_{l-k}$ is $\kappa$-complete. So fix $k$, and arguing by induction assume $Y_k \subseteq \mathrm{tr}_{lk} Y_l$; we get

$$\begin{aligned} \mathrm{tr}_{l+1k}\, Y_{l+1} &= \bigcap_{m \geqslant l+1} \mathrm{tr}_{l+1k}\ \mathrm{tr}_{ml+1} X_m \\ &= \bigcap_{m \geqslant l+1} \mathrm{tr}_{mk} X_m \\ &\supseteq \bigcap_{m \geqslant l} \mathrm{tr}_{mk} X_m = Y_k, \end{aligned}$$

and the claim is proved. We now construct $T \in \mathscr{C}^\circ$ by setting:

$$s \in T \text{ iff } \forall s' \subseteq s\ s' \in Y_{|s'|}.$$

Assume $s$ is in $T$: first if $s' \subseteq s$, then $s'$ is in $T$ also. Next $s^\frown\xi$ is in $T$ iff $s^\frown\xi$ is in $Y_{|s|+1}$: but $s$ is in $T$, so $s$ is in $Y_{|s|}$, hence in $\mathrm{tr}_{|s|+1|s|} Y_{|s|+1}$, that is $\{\xi : s^\frown\xi \in Y_{|s|+1}\} \in \mathcal{U}$. So $T$ is in fact in $\mathscr{C}^\circ$; now for any $l$, if $s \in T \cap \kappa^l$, $s$ belongs to $Y_l$, hence to $X_l$, and we are done.

As a corollary of the last lemma, it is easy to show that a closed subset $X$ of $\kappa^\omega$ is in the filter generated by $\mathcal{U}^\omega$ if and only if for all $l$ the restriction of $X$ to $\kappa^l$ (i.e. the set of restrictions to $\kappa^l$ of elements of $X$) is in $\mathcal{U}_l$. It can be also shown that $\mathcal{U}^\omega$ and $\mathcal{U}_n \otimes \mathcal{U}^\omega$ generate the same filter on $\kappa^\omega$. Finally we show that $\mathscr{C}^\circ$ behaves like a selective filter, having the following Rowbottom-style property:

**Lemma 3.** *Let $f$ be any function: $[\kappa]^{<\omega} \to \nu < \kappa$; then there is $T$ in $\mathscr{C}^\circ$ such that $f$ is constant on each level of $T$, that is, for all $l$ $|f''(T \cap \kappa^l)| = 1$.*

**Proof.** By the completeness of $\mathscr{V}$, construct a sequence $(X_l)_l$ such that for all $l$ $X_l$ is in $\mathscr{U}_l$ and $|f''X_l|=1$. Then apply the last lemma.

6.3.

We now turn to study forcing with $\mathscr{C}$. We first investigate the connection with Prikry's forcing.

**Definition.** (i) For $s$ in $[\kappa]^{<\omega}$ and $X$ in $\mathscr{U}$, we define a member $T_{s,x}$ of $\mathscr{C}$ by $T_{s,X}=s^\frown[X]^{<\omega}=\{t: t\subseteq s \text{ or } t=s^\frown t' \text{ for some } t'\in X^{|t'|}\}$.

(ii) $\mathscr{P}'$ is the set of $T_{s,X}$ for $s$ in $[\kappa]^{<\omega}$ and $X$ in $\mathscr{U}$.

**Lemma 1.** (i) *$\mathscr{P}'$ is isomorphic to $\mathscr{P}$;*

(ii) *$\mathscr{P}'$ is dense in $\mathscr{C}$ iff $\mathscr{U}$ is selective.*

**Proof.** (i)$T_{t,Y}\subseteq T_{s,X}$ iff $t=s^\frown u$, $u\in[X]^{<\omega}$ and $Y\subseteq X$, that is iff $(t,Y)\leqslant(s,X)$ in $\mathscr{P}$.

(ii) Assume $\mathscr{U}$ selective, and let $T$ be any member of $\mathscr{C}$: for each $l$ $\{t\in \kappa^l: s_T{}^\frown t\in T\}\in\mathscr{U}_l$. So by selectivity there is $X_l$ in $\mathscr{U}$ such that $[X_l]^l\subseteq \{t\in\kappa: s_T{}^\frown t\in T\}$. Put $X=\bigcap_l X_l$, $X$ is in $\mathscr{U}$ and $s_T{}^\frown[X]^{<\omega}\subseteq T$, i.e. $T_{s_t,X}\subseteq T$, and so $\mathscr{P}'$ is dense in $\mathscr{C}$.

Conversely, assume $\mathscr{P}'$ is dense and let $Y$ be any member of $\mathscr{U}_n$; we define $(X_l)_l$ by $X_{kn}=[Y]^k$ and $X_{kn+m}=X_{kn}\times\kappa^m$ for $m=1,\ldots,n-1$. For each $l$ $X_l$ is in $\mathscr{U}_l$, so by Section 6.2 Lemma 2 there is $T$ in $\mathscr{C}^\circ$ such that for all $l$ $T\cap\kappa^l\subseteq X_l$. Since $\mathscr{P}'$ is dense there are $s$ and $X$ such that $T_{s,X}\subseteq T$; choose $l\geqslant n$ such that $|s|+l$ is multiple of $n$, say $|s|+l=kn$: we have $T_{s,X}\cap\kappa^{kn}\subseteq T\cap\kappa^{kn}\subseteq X_{kn}=[Y]^k$, hence $s^\frown[X]^l\subseteq[Y]^k$, so $[X]^n\subseteq Y$; this shows that $\mathscr{U}$ is selective.

6.4.

We now get characterizations for $\mathscr{C}$-generic sets.

**Definition.** (i) For $\mathscr{A}\subseteq\mathscr{C}$, we set $g(\mathscr{A})=\bigcup\{s_T: T\in\mathscr{A}\}$;

(ii) For $g:\omega\to\kappa$ (not necessarily in $N_0$) we set $P(g)=\{T\in\mathscr{C}: \forall n\, g\restriction n\in T\}$.

The following is as it should be:

**Lemma 2.** *Assume $P$ is generic over $\mathscr{C}$; then*

(i) *$g(P)$ is a cofinal subset of $\kappa$ of order type $\omega$.*

(ii) *$P$ is exactly $P(g(P))$, and in particular $P$ and $g(P)$ are equiconstructible over $N_0$.*

**Proof.** (i) First fix $\alpha<\kappa$: the set $\{T$: the last term of $s_T$ is $\geqslant\alpha\}$ is dense in $\mathscr{C}$, so $g(P)$ is cofinal in $\kappa$. Now let $\alpha\in g(P)$, and $T$ in $P$ such that $\alpha\in s_T$. For any $T'$ in $P$,

$T$ and $T'$ are compatible, so $S_T \cap \alpha \supseteq s_{T'} \cap \alpha$, and $g(P) \cap \alpha = s_T \cap \alpha$ is finite, so $g(P)$ is of order type $\omega$.

(ii) Assume $T$ belongs to $P$ and $|s_T| = n$. Fix any $K$: the set $D_k = \{T'; |s_{T'}| \geq n+k\}$ is dense in $\mathscr{C}$; take $T'$ in $P \cap D_k$, and then $T'' \subseteq T \cap T'$, $s_{T'} = g(P) \upharpoonright n+m$ with $m \geq k$, and $s_{T''} = g(P) \upharpoonright n+m^\frown t$; since $T'' \subseteq T$, we get $g(P) \upharpoonright n+m \in T$, and a fortiori $g(P) \upharpoonright n+k \in T$: this proves, since $k$ was arbitrary, that $P \subseteq P(g(P))$.

Conversely, assume $T$ belongs to $P(g(P))$. Say $s_T = g \upharpoonright n$; we shall write $s$ for $s_T$. We define a map $B$ which associates to certain trees $T'$ in $\mathscr{C}$ another tree $B(T')$ in $\mathscr{C}$ as follows: assume $|s_{T'}| > n$ and $\max s = g(n-1) \leq \max s_{T'}$ we set $t \in B(T')$ iff $t \subseteq s$ or $t \supseteq s$ and $s_{T'} \upharpoonright n^\frown t \upharpoonright [n, |t|) \in T'$. So $B(T')$ is a copy of $T'$ with $g \upharpoonright n$ replacing the $n$ first elements of $s_{T'}$. Now let $D = \{T'; |s_{T'}| > n$ and $\max s \leq \max s_{T'}$ and $(s_{B(T')} \in T \Rightarrow B(T') \subseteq T)\}$; we claim that $D$ is dense, for if $|s_{T'}| > n$, $\max s \leq \max s_{T'}$ and $s_{B(T')} \in T$, $T^{s_{B(T')}}$ is in the range of $B$, say $T^{s_n(T')} = B(T'')$ and $T^* = T' \cap T''$ satisfies: $T^* \subseteq T'$, $|s_{T^*}| \geq |s_T| > n$, $\max s \leq \max s_{T'} \leq \max s_{T^*}$, and $B(T^*) \subseteq B(T'') \subseteq T$.

Now let $T'$ in $D \cap P$, and say $s_{T'} = g \upharpoonright n+k$. Since $T$ belongs to $P(g(P))$, $g \upharpoonright n+k$, which is $g \upharpoonright n s_{T'} \upharpoonright [n, |s_{T'}|)$, belongs to $T$, so we have:

$$\forall t \in T'(|t| \geq n \Rightarrow g \upharpoonright n^\frown t \upharpoonright [n, |t|) \in T),$$

which is, since $s_{T'} \upharpoonright n = g \upharpoonright n$, $T' \subseteq T$. By hypothesis $T'$ is in $P$ and $P$ is generic, so $T$ is in $P$, and we proved that $P(g(P)) \subseteq P$

The next lemma can be stated "Every open subset of $\kappa^\omega$ for the topology generated by $\mathscr{C}$ has the Ramsey property" and will enable us to give characterizations of $\mathscr{C}$-generic sequences as made by Mathias in [10] for Prikry's forcing.

**Lemma 3.** *Assume $D$ is an open dense subset of $\mathscr{C}$; then for each $s$ in $[\kappa]^{<\omega}$ there is $T$ in $\mathscr{C}$ and $k \in \omega$ such that $s_T = s$ and for all $t$ in $T \cap \kappa^{|s|+k}$ $T^t$ belongs to $D$.*

**Proof.** For $t \supseteq s$ we choose $T_t$ in $\mathscr{C}$ such that $s_{T_t} = t$ and $T_t$ belongs to $D$ if there exists such $T$. Applying Lemma 7, we get $T'$ in $\mathscr{C}$ such that $s_{T'} = s$ and for every $t \supseteq s$ $T'^t \subseteq T_t$. Since $D$ is open, we have that, if $t \supseteq s$ and there is a $T$ in $\mathscr{C}$ such that $s_T = t$ and $T$ belongs to $D$, then $T'^t$ belongs to $D$. We now define $f: [\kappa]^{<\omega} \to 3$ by

$$f(t) = 0, \quad \text{if } t \notin T' \text{ or } t \not\supseteq s$$
$$f(t) = 1, \quad \text{if } t \in T',\ t \supseteq s \text{ and } T'^t \in D$$
$$f(t) = 2, \quad \text{if } t \in T',\ t \supseteq s \text{ and } T'^t \notin D.$$

By 6.1 Lemma 3, there is a $T$ in $\mathscr{C}$ such that $T \subseteq T'$, $s_T = s$ and $f$ is constant on each level of $T$. Since $D$ is dense, there is $T^* \subseteq T$, $T^*$ in $D$.

Let $t^*$ be $s_{T^*}$. Since $T^* \subseteq T'$, we get $t^* \in T'$ and $t^* \supseteq s$.

Moreover, $T^*$ is in $D$, so by construction of $T'$ $T'^{t^*}$ is in $D$, so $f(t^*) = 1$. Set $|t^*| = |s| + k$, and let $t$ be any member of $T \cap \kappa^{|s|+k}$: by construction of $T$, $f$ is constant on the $|s|+k$-th level of $T$, so $f(t)=1$, and so $T''$ belongs to $D$. Since $D$ is open, a fortiori $T^t$ belongs to $D$ and we are done.

**Lemma 4.** *The set $P(g)$ is generic over $\mathscr{C}$ iff the following condition on g holds in $\mathcal{N}_0$:*
(*) For every $\varphi$: $[\kappa]^{<\omega}\to\mathscr{U}$ there is $n$ such that $\forall m\geq n\ g(m)\in\varphi(g\restriction m)$.

**Proof.** Assume $P(g)$ generic, and let $\varphi$ any map: $[\kappa]^{<\omega}\to\mathscr{U}$. Let $D=\{T: \forall t\supseteq s_T(t^\frown\xi\in T\Rightarrow \xi\in\varphi(t))\}$: we claim that $D$ is dense. For we construct for any $T$ a subtree $T'$ of $T$ in $D$, with $s_{T'}=s_T$, by induction on levels: $t^\frown\xi\in T'$ iff $t^\frown\xi\in T$ and $\xi\in\varphi(t)$. Let $T$ be a member of $D\cap P(g)$, and $s_T=g\restriction n$; for any $m\geq n$ $g\restriction m^\frown g(m)$ belongs to $P(g)$, so $g(m)\in\varphi(g\restriction m)$.

Conversely, assume g satisfies (*). First if $T$ and $T'$ are in $P(g)$, so is $T\cap T'$, and if $T$ is in $P(g)$ and $T'$ includes $T$, then $T'$ is in $P(g)$. Now assume $D$ is an open dense subset of $\mathscr{C}$: we claim that $D$ meets $P(g)$. By Lemma 3, there are maps $\phi$: $[\kappa]^{<\omega}\to\mathscr{C}$ and $k:[\kappa]^{<\omega}\to\omega$ such that for all $s$ $s_{\phi(s)}=s$ and for all $t$ in $\phi(s)\cap\kappa^{|s|+k(s)}$ $\phi(s)^t$ is in $D$. Now define $\varphi$: $[\kappa]^{<\omega}\to\mathscr{U}$ by:

$$\varphi(s)=\{\xi: \forall s'\subseteq s\ s^\frown\xi\in\phi(s')\}.$$

We apply to $\varphi$ the hypothesis on g, getting $n$ such that $\forall m\geq n\ g(m)\in\varphi(g\restriction m)$. Let $T$ defined by: $s\in T$ *iff* $s\subseteq g\restriction n$ or $g\restriction n\subseteq s$ and for all $m$, $n\leq m<|s|$, $s(m)\in\varphi(s\restriction m)$. We claim that $T$ belongs to $\mathscr{C}$ and $s_T=g\restriction m$. For if $s\in T$ and $s'\subseteq s$, then $s'\in T$, and for $s$ in $T$, $\{\xi: s^\frown\xi\in T\}=\varphi(s)$ belongs to $\mathscr{U}$. Moreover for all $m\geq n$ $g\restriction m\in T$, so $T$ is in $P(g)$.

Now suppose $t\in T$, $|t|=p+1>n$: by hypothesis $t(p)\in\varphi(t\restriction p)$, that is by construction of $\varphi\forall s'\subseteq t\restriction p\ t\restriction p^\frown t(p)\in\phi(s')$. In particular $t=t\restriction p^\frown t(p)\supseteq g\restriction n$, so $t\in\phi(g\restriction n)$, and hence $T\subseteq\phi(g\restriction n)$. Let $m$ be $k(g\restriction n)$: for all $t$ in $\phi(g\restriction n)\cap\kappa^{n+m}\phi(g\restriction n)^t$ belongs to $D$; a fortiori since $D$ is open, for all $t$ in $T\cap\kappa^{n+m}$, $T^t$ belongs to $D$. In particular $g\restriction n+m$ belongs to $T$, since $T$ is in $P(g)$, so $T^{g\restriction n+m}$ is in $D$, and it is obviously in $P(g)$. Thus $P(g)$ is $\mathscr{C}$-generic.

Finally, we give to the last result a more handy form:

**Theorem 5.** *$P$ is $\mathcal{N}_0$-generic over $\mathscr{C}$ iff $P$ is $P(g)$ where g satisfies the following condition over $\mathcal{N}_0$:*
(**) *For every decreasing $\psi$: $\kappa\to\mathscr{U}$ there is $n$ such that: $\forall m\geq n\ g(m)\in\psi(g(m-1))$.*

As a particular case we get back Mathias' result about Prikry's forcing: "If $\mathscr{U}$ is normal, then $P$ is $\mathscr{P}$-generic iff $P=P'(g)$ (with the obvious definition of $P'(g)\cdots$) and every member of $\mathscr{U}$ contains almost all terms of g". For notice that in the case of normality any family $\psi$: $\kappa\to\mathscr{U}$ may be replaced by the constant one whose value is the diagonal intersection of im $\psi$.

**Proof of Theorem.** In light of Lemma 12, we just have to show that conditions $(*)$ and $(**)$ are equivalent.
$(*)$ implies $(**)$: starting from $\psi$, define $\varphi$: $[\kappa]^{<\omega}\to\mathcal{U}$ by $\varphi(s)=\psi(\max s)$.
$(**)$ implies $(*)$: starting from $\varphi$, define $\psi$: $\kappa\to\mathcal{U}$ decreasing by $\psi(\xi)=\bigcap_{\max s\leqslant\xi}\varphi(s)$, using the $\kappa$-completeness of $\mathcal{U}$. Now if g satisfies $(**)$, we get $n$ such that $\forall m\geqslant n\ g(m)\in\psi(g(m-1))$; but $\max g\restriction m\leqslant g(m-1)$, so $g(m)\in\varphi(g\restriction m)$, and g satisfies $(*)$.

6.5.

In [3] Bukovský has shown by Vopenka's general methods that $M_\omega$ is a generic extension of $N_\omega$. We show here that the forcing involved is exactly $\mathscr{C}$, as the sequence $(\chi_n)_n$ turns out to be $N_\omega$-generic over $i_{0\omega}\mathscr{C}$.

We set $\mathscr{C}_\alpha=i_{0\alpha}\mathscr{C}=\mathscr{C}_{N_\alpha}(i_{0\alpha}\mathcal{U})$. Then we have the following general result:

**Theorem 1.** *Assume* cf $\lambda=\omega$ *and* $g\in\mathscr{G}_\lambda$, *then* $P(g)$ *is* $N_\lambda$*-generic over* $\mathscr{C}_\lambda$.

This result can be easily deduced from Section 6.4. Theorem 5. Now since the generalized Prikry forcing $\mathscr{C}$ has originally been constructed just to allow the proof of Theorem 1 to work even for non-normal ultrafilters, we prefer to give here this proof.

**Proof of theorem 1.** Let g be a member of $\mathscr{G}_\lambda$; there is $(\gamma_n)_n$ in cof $\lambda$ and $n_0$ such that for $n\geqslant n_0$ $g(n)=\chi_{\gamma_n}$. Let $D$ be any dense subset of $\mathscr{C}_\lambda$. There is $\alpha<\lambda$ and $\Delta$ dense subset of $\mathscr{C}_\alpha$ such that $D=i_{\alpha\lambda}\Delta$. We may assume that $\alpha=\gamma_{n_1}$ with $n_1\geqslant n_0$. We set

$$S_n(\Delta)=\{s\in\kappa_\alpha^n:\exists T\in\Delta\ s_T=g\restriction n_1{}^\frown s\}.$$

Assume that for all $n$ $S_n(\Delta)$ is not in $i_{0\alpha}\mathcal{U}_n$. Then for all $n$ $\kappa_\alpha^n\backslash S_n(\Delta)$ is in $i_{0\alpha}\mathcal{U}_n$, so by 6.2 lemma 2., there is $T^\circ$ in $\mathscr{C}_\alpha^\circ$ such that for all $n$ $T^\circ\cap\kappa_\alpha^n\cap S_n(\Delta)$ is empty. We define $T^*$ in $\mathscr{C}_\alpha$ by $s\in T^*$ iff $s\subseteq g\restriction n_1$ or $s=g\restriction n_1{}^\frown t$ and $t\in T^\circ$ ("$T^*=g\restriction n_1{}^\frown T^\circ$").

Assume $T\subseteq T^*$: $s_T\supseteq s_{T^*}$, say $s_T=g\restriction n_1{}^\frown t$. By construction $t$ is in $T^\circ$, so it is not in $S_{|t|}(\Delta)$, hence there is no $T''$ in $\Delta$ with $s_{T''}=s_T$: in particular $T$ is not in $\Delta$, and this contradicts the density of $\Delta$. Hence there is $n_2$ such that $S_{n_2}(\Delta)\in i_{0\alpha}\mathcal{U}_{n_2}$. By Section 1.7. Proposition 3, this implies:

$$(\chi_{\gamma_{n_1}},\ldots,\chi_{\gamma_{n_1+n_2-1}})\in i_{\alpha\gamma_{n_1+n_2}}S_{n_2}(\Delta),$$

which is $S_{n_2}(i_{\alpha\gamma_{n_1+n_2}}\Delta)$. Thus there is $T$ in $i_{\alpha\gamma_{n_1+n_2}}\Delta$, such that $s_T=g\restriction n_1{}^\frown(\chi_{\gamma_{n_1}},\ldots,\chi_{\gamma_{n_1+n_2-1}})$, i.e. $s_T=g\restriction n_1+n_2$. Now let $T'$ be $i_{\gamma_{n_1+n_2}\lambda}T$: $T'$ belongs to $\mathscr{C}_\lambda$; moreover $s_{T'}=i_{\gamma_{n_1+n_2}\lambda}s_T=g\restriction n_1+n_2$, and $T\in i_{\alpha\gamma_{n_1+n_2}}\Delta$ implies $T'\in i_{\alpha\lambda}\Delta=D$. It remains to verify that $T'$ belongs to $P(g)$; fix any $m$, we show that

$g \upharpoonright n_1+n_2+m$ belongs to $T'$. For notice that

$$\{s \in \kappa^m_{\gamma_{n_1+n_2}} : g \upharpoonright n_1+n_2{}^\frown s \in T\}$$

belongs to $i_{0\gamma_{n_1+n_2}}\mathcal{U}_m$; so by Section 1.7. Proposition 3.

$$g \upharpoonright n_1+n_2{}^\frown(\chi_{\gamma_{n_1+n_2}}, \ldots, \chi_{\gamma_{n_1+n_2+m-1}}) \in i_{\gamma_{n_1+n_2}\gamma_{n_1+n_2+m}} T,$$

i.e. $g \upharpoonright n_1+n_2+m \in i_{\gamma_{n_1+n_2}\gamma_{n_1+n_2+m}} T$, thus $g \upharpoonright n_1+n_2+m \in T'$. We constructed $T'$ in $D \cap P(g)$, and so $P(g)$ is generic over $\mathcal{C}_\lambda$.

**Corollary 2.** *If $\mathcal{U}$ is selective, then for all g in $\mathcal{G}_\lambda$, $P'(g)$ is N-generic over $\mathcal{P}_\lambda$.*

6.6.

We see now how the preceding result gives us with the help of section 3 a very powerful way to study Prikry's extensions.

We recall some notations: if $C$ is a partially ordered set in $M$, $\mathcal{B}=\mathcal{B}(C)$ the complete Boolean algebra associated to $C$, we let $M^{\mathcal{B}}$ be the corresponding Boolean model; and if $G$ is $C$-generic over $M$, for $x \in M[G]$, we choose an element $\mathring{x}$ of $M^{\mathcal{B}}$ whose value in $M[G]$ is $x$, and say it a name for $x$; we note $\mathrm{Val}^M_G$ the evaluation map $\mathring{x} \mapsto x$.

**Theorem 1.** *Let $G$ be any $\mathcal{C}$-generic set over $N_0$, $\phi(v_1 \cdots v_n)$ any formula and $x_1 \cdots x_n \in N_0[G]$. Assume that for any g in $\mathcal{G}_\omega$ which contains almost all $\chi_n$ we have $M_\omega \models \phi(\mathrm{Val}^{N_\omega}_{P(g)} i_{0\omega}\mathring{x}_1, \ldots, \mathrm{Val}^{N_\omega}_{P(g)} i_{0\omega}\mathring{x}_n)$; then $N_0[G] \models \phi(x_1, \ldots, x_n)$.*

**Proof.** Assume the hypothesis. We claim that $\{T \in \mathcal{C}_\omega : T \Vdash \phi(i_{0\omega}\mathring{x}_1, \ldots, i_{0\omega}\mathring{x}_n)\}$ is dense in $\mathcal{C}_\omega$. For assume $T \in \mathcal{C}_\omega$ and no $T' \subseteq T$ forces $\phi(i_{0\omega}\mathring{x}_1, \ldots, i_{0\omega}\mathring{x}_n)$. Then $T \Vdash \neg\phi(i_{0\omega}\mathring{x}_1, \ldots, i_{0\omega}\mathring{x}_n)$. Since $T$ is in $N_\omega$, there is $p$ and $T^*$ in $\mathcal{C}_p$ such that $T = i_{p\omega}T^*$, and we may assume that $\max s_T < \kappa_p$. For every $n$, the set $\{t \in \kappa^n_p;\ s_T{}^\frown t \in T^*\}$ is in $i_{0p}\mathcal{V}_n$, so by Section 1.7. Proposition 3, $s_T{}^\frown(\chi_p, \ldots, \chi_{p+n-1}) \in i_{pp+n}T^*$, thus $s_T{}^\frown(\chi_p, \ldots, \chi_{p+n-1}) \in T$. This proves that, if we set $g = s_T{}^\frown(\chi_n)_{n \geq p}$, then $T \in P(g)$. Now $g$ contains almost all the $\chi_n$, so $N_\omega[P(g)] = M_\omega$, and since $T \in P(g)$ and $T \Vdash \neg\phi$, we have

$$M_\omega \models \neg\phi(\mathrm{Val}^N_{P(g)} i_{0\omega}\mathring{x}_1, \ldots, \mathrm{Val}^N_{P(g)} i_{0\omega}\mathring{x}_n),$$

contradicting the hypothesis.

Hence the claim is proved; and $i_{0\omega}$ being an elementary embedding we conclude that $\{T \in \mathcal{C}_0 : T \Vdash \phi(\mathring{x}_1, \ldots, \mathring{x}_n)\}$ is dense in $\mathcal{C}_0$, so for any $G$ $\mathcal{C}_0$-generic, we have

$$N_0[G] \models \phi(x_1, \ldots, x_n).$$

**Corollary 2.** *Let $\phi$ be any closed formula and G any $\mathcal{C}$-generic set over $N_0$: then $N_0[G] \models \phi$ iff $M_\omega \models \phi$. More generally if $X_1 \cdots x_n$ belong to $N_0$, $N_0[G] \models \phi(x_1 \cdots x_n)$ iff $M_\omega \models \phi(i_{0\omega}x_1, \ldots, i_{0\omega}x_n)$.*

**Application.** Let $\mathcal{U}$ a $\kappa$-complete ultrafilter on $\kappa$ in a model $N_0$ and $N'$ any $\mathcal{C}_{N_0}(\mathcal{U})$-generic extension of $N_0$. Then

(i) Cardinalities are preserved;

(ii) $V_\kappa \cap N' = V_\kappa \cap N_0$;

(iii) each set of ordinals in $N'$ can be covered in $N'$ by the union of $\omega$ sets of $N_0$ of the same cardinality;

(iv) If $P(g)$ is $N_0$-generic over $\mathcal{C}$, and $g'$ is any almost-sub-sequence of $g$, then $P(g')$ is $\mathcal{N}_0$-generic over $\mathcal{C}$;

(v) if moreover $\mathcal{U}$ is selective, all these results hold for Prikry's forcing.

Notice that these results are very easy to establish, since we need only the facts that $M_\omega = N_\omega[(\chi_n)_n]$, which uses a little part of Chapter 2, and that $(\chi_n)_n$ is $N_\omega$-generic over $\mathcal{C}_\omega$.

## 7. Normal and selective ultrafilters

In this part, we exactly determine the $N_\lambda$-generic sets over $\mathcal{C}_\lambda$ which belong to $M_\lambda$ (for $\lambda$ of cofinality $\omega$) when $\mathcal{U}$ is equivalent (in the sense of [8]) to some finite power of a normal ultrafilter. It is easy to show from Section 6.3. Theorem 5. that for $\mathcal{U}$ normal, the union of two Prikry generic sets is still generic. This will fail for $\mathcal{C}$ forcing in general.

7.1.

We first recall well-known facts about normal ultrafilters.

**Definition.** Assume $\mathcal{V}$ is a $\kappa$-complete ultrafilter on $\kappa$ and $h : \kappa^p \to \kappa$. We set: $h * \mathcal{V} = \{X \subseteq \kappa : h^{-1}X \in \mathcal{V}_p\}$.

**Lemma 1.** (i) if $\pi_p^{\mathcal{V}} h \geqslant \kappa$, then $h * \mathcal{V}$ is a $\kappa$-complete free ultrafilter on $\kappa$;

(ii) if $\pi_p^{\mathcal{V}} h_1 = \pi_p^{\mathcal{V}} h_2$, then $h_1 * \mathcal{V} = h_2 * \mathcal{V}$;

(iii) if $h'$: $\kappa \to \kappa$, then $h' * (h * \mathcal{V}) = h'h * \mathcal{V}$;

(iv) for any $f : \kappa \to \kappa$, $\pi_1^{h*\mathcal{V}} f$ is included in the transitive closure of $\pi_p^{\mathcal{V}} fh$;

(v) if moreover $h$ is injective, $\pi_1^{h*\mathcal{V}} f$ is exactly $\pi_p^{\mathcal{V}} fh$.

**Proof (iv) and (v).** Assume $X$ transitive, and look at the two following maps from ${}^\kappa X$ to $N_0$:

$$\Pi_1 : f \mapsto \Pi_1^{h*\mathcal{V}} f \text{ and } \Pi_2 : f \mapsto \Pi_p^{\mathcal{V}} fh.$$

By the definition of $h * \mathcal{V}$, the ranges of $\Pi_1$ and $\Pi_2$ are isomorphic, and moreover the range of $\Pi_1$ is transitive: we get (iv) applying Mostowski's theorem. Further, if $h$ is injective, the map $f \mapsto fh$ from ${}^\kappa X$ into itself is surjective, and the image of $\Pi_2$ is transitive as well as the one of $\Pi_1$, so $\Pi_1 = \Pi_2$.

It is clear from lemma 1 that normal ultrafilters exist whenever complete ones do: take for $h$ any function such that $\pi_1^{\mathcal{V}} h = \kappa$ ($\mathcal{V}$ any complete ultrafilter on $\kappa$). By (iv), $\pi_1^{h*\mathcal{V}} \mathrm{id}_\kappa \subseteq \pi_1^{\mathcal{V}} h = \kappa$, hence $h * \mathcal{V}$ is normal.

**Definition.** Assume $\mathcal{V}$ is included in $\mathcal{P}(\kappa)$ and $(X_\xi)_{\xi \in [\kappa]^n}$ is a family of members of $\mathcal{V}$; we define the *diagonal intersection* of the family, $\Delta_\xi X_\xi$, by $\eta \in \Delta_\xi X_\xi$ *iff* $\forall \xi < \eta \, \eta \in X_\xi$ (where $\xi < \eta$ means max $\xi < \eta$).

**Lemma 2.** *Assume $\mathcal{V}$ normal on $\kappa$ and for all $\xi$ in $[\kappa]^n$ $X_\xi$ is in $\mathcal{V}$; then $\Delta_\xi X_\xi$ is in $\mathcal{U}$.*

**Proof.** Define $f : \kappa \to [\kappa]^n$ by $f(\eta) =$ the first $\xi$ with $\xi < \eta$ and $\eta \notin X_\xi$ if it exists, $(\eta, \ldots, \eta + n - 1)$ if not. For all $\xi$ in $[\kappa]^n$, $X_\xi$ is in $\mathcal{U}$, so $\{\eta : f(\eta) = \xi\}$ is not in $\mathcal{U}$, and $\pi_1^{\mathcal{V}} f$ is not $\xi$. In particular if $f_{n-1}$ is the last component of $f$, $\pi_1^{\mathcal{V}} f_{n-1} \geqslant \kappa$, i.e. $\pi_1 f_{n-1} \geqslant \pi_1$ id, so

$$\{\eta : f_{n-1}(\eta) = \eta + n - 1\} \in \mathcal{U}, \text{ thus}$$
$$\{\eta : \exists \xi < \eta \ \eta \notin X_\xi\} \notin \mathcal{U}, \text{ and its complement } \Delta_\xi X_\xi \text{ is in } \mathcal{V}.$$

**Lemma 3.** *For $\mathcal{V}$ being a $\kappa$-complete ultrafilter on $\kappa$, the following are equivalent:*

(i) *$\mathcal{V}$ is selective,* i.e. *for any $Y$ in $\mathcal{V}_n$ there is $X$ in $\mathcal{V}$ such that $[X]^n \subseteq Y$;*

(ii) *for any $Y$ in $\mathcal{V}_2$, there is $X$ in $\mathcal{V}$ such that $[X]^2 \subseteq Y$;*

(iii) *for every $f : \kappa \to \kappa$ such that $\pi_1^{\mathcal{V}} f \geqslant \kappa$, there are $f_1$, $f_2 : \kappa \to \kappa$ such that $\pi_1 f = \pi_1 f_1 = \pi_1 f_2$ and $f_1$ is bijective und $f_2$ strictly increasing;*

(iv) *there is a bijection $h$ such that $h * \mathcal{V}$ is normal.*

**Proof.** (i)$\Rightarrow$(ii); (ii)$\Rightarrow$(iii): let $f$: $\kappa \to \kappa$ such that $\pi_1 f \geqslant \kappa$; for every $\xi$ in $\kappa$, the set $\{\eta : f(\eta) > f(\xi)\}$ is in $\mathcal{V}$, so $\{\xi^\frown \eta : f(\xi) > f(\eta)\}$ is in $\mathcal{V}_2$; by (ii) there is $X$ in $\mathcal{V}$ such that for $\xi < \eta$, $\xi$, $\eta$ in $X$, $f(\xi) < f(\eta)$. We can then modify $f$ on a set of zero-measure to make it bijective or increasing.

(iii)$\Rightarrow$(iv). Let $f$ be any function $\kappa \to \kappa$ such that $\pi_1 f = \kappa$: there exists $f_1$ bijective such that $\pi_1 f = \pi_1 f_1 = \kappa$, so $f_1 * \mathcal{V}$ is normal.

(iv)$\Rightarrow$(i). We first show that (i) hold for normal $\mathcal{V}$, and then show (i) is preserved under increasing injections. Assume $\mathcal{V}$ normal and we argue by induction on $n$: assume $Y$ in $\mathcal{V}_{n+1}$. By definition $\{\xi : Y^\xi \in \mathcal{V}\} \in \mathcal{V}_n$, so by induction hypothesis there is $Z$ in $\mathcal{V}$ such that $[Z]^n \subseteq \{\xi : Y^\xi \in \mathcal{V}\}$. Set $Y_\xi = Y^\xi$ if $\xi$ is in $[Z]^n$, $Y_\xi = \kappa$ if not. By Lemma 2, $\Delta_\xi Y_\xi$ is in $\mathcal{U}$; set $X = Z \cap \Delta_\xi Y_\xi$, and let $\xi^\frown \eta \in [X]^{n+1}$: $\xi \in [X]^n$, thus $\xi \in [Z]^n$, hence $Y_\xi = Y^\xi \in \mathcal{V}$. Since $\xi < \eta$ and $\eta \in X$, $\eta \in Y_\xi = Y^\xi$, so $\xi^\frown \eta \in Y$. We are done since $[X]^{n+1} \subseteq Y$. Now assume (i) holds for $\mathcal{V}$ and $h$ is strictly increasing; let $Y \in (h * \mathcal{V})_n$: the set $\{(\eta_0 \cdots \eta_{n-1}) : (h(\eta_0) \cdots h(\eta_{n-1})) \in Y\}$ is in $\mathcal{V}_n$, so there is $Z$ in $\mathcal{V}$ such that for $(\eta_0 \cdots \eta_{n-1})$ in $[Z]^n$ $(h(\eta_0) \cdots h(\eta_{n-1}))$ is in $Y$. Let $X$ be $h''Z : X$ is in $h * \mathcal{V}$, and if $(\xi_0 \cdots \xi_{n-1})$ is in $[X]^n$, there is $(\eta_0 \cdots \eta_{n-1})$ in $[Z]^n$ such that $\xi_i = h(\eta_i)$, $i = 0, \ldots, n-1$, so $(\xi_0, \ldots, \xi_{n-1})$ is in $Y$, and this finishes the proof.

7.2.

We now prove that if $\mathcal{U}$ is selective then $\mathcal{G}_\lambda$ is exactly the set of $N_\lambda$-generic sets over $\mathcal{C}_\lambda$ in $M_\lambda$.

**Lemma 1.** *Let $f:\kappa\to\kappa$ be such that $\{\xi: f(\xi)\neq\xi\}\in\mathcal{U}$. Then there exists $X$ in $\mathcal{U}$ such that $f^{-1}X\notin\mathcal{U}$.*

The proof is well-known (without any completeness).

**Lemma 2.** *If $\mathcal{U}$ is selective and $X\in i_{0\alpha}\mathcal{U}$, there exists $Y\in\mathcal{U}$ and $\gamma<\alpha$ such that $(i_{0\alpha}Y)^{>\chi_\gamma}\subseteq X$, where $Z^{>\mu}$ means $\{\zeta\in Z:\zeta>\mu\}$.*

**Proof.** Write $X=\pi_\alpha e * f$ where $f:\kappa^{|e|}\to\mathcal{U}$. Define $\xi^\frown\eta\in Z$ iff $\eta\in f(\xi): Z\in\mathcal{U}_{n+1}$, so there is $Y$ in $\mathcal{U}$ such that $[Y]^{n+1}\subseteq Z$. For all $\xi$ in $[Y]^n$ $Y^{>\xi}\subseteq f(\xi)$, so

$$\pi_\alpha[s\mapsto Y]^{>\pi_\alpha[s\mapsto s(e(n-1))]}\subseteq \pi_\alpha[s\mapsto f(se)],$$

that is $(i_{0\alpha}Y)^{>\chi_{e(n-1)}}\subseteq X$.

**Lemma 3.** *Assume $\lambda$ limit and $\mu\notin\{\chi_\alpha:\alpha<\lambda\}$. Then if $\mathcal{U}$ is selective there is $X\in\mathcal{U}$ such that $\mu\notin i_{0\lambda}X$.*

**Proof.** There is $\alpha<\lambda$ such that $\kappa_\alpha\leqslant\mu<\kappa_{\alpha+1}$, and $\mu\neq\chi_\alpha$. Applying Lemma 1 in $N_\alpha$, we get $Y$ in $i_{0\alpha}\mathcal{U}$ such that $\mu\notin i_{\alpha\alpha+1}Y$. Apply 2 to get $X$ in $\mathcal{U}$ and $\gamma<\alpha$ such that $(i_{0\alpha}X)^{>\chi_\gamma}\subseteq Y$; since $\gamma<\alpha$, $\chi_\gamma<\kappa_\alpha$, and $i_{\alpha\alpha+1}\chi_\gamma=\chi_\gamma$. Thus $\mu\notin i_{\alpha\alpha+1}Y$ implies $\mu\notin(i_{0\alpha+1}X)^{>\chi_\gamma}$, hence $\mu\notin i_{0\alpha+1}X$ since $\mu>\chi_\gamma$, and $\mu<\kappa_{\alpha+1}$ implies $\mu=i_{\alpha+1\lambda}\mu\notin i_{0\lambda}X$.

**Proposition 4.** *Assume $g\in M_\lambda$ and $P(g)$ is $N_\lambda$-generic over $\mathcal{C}_\lambda$, $\lambda$ of cofinality $\omega$. Then if $\mathcal{U}$ is selective, $g\in\mathcal{G}_\lambda$.*

**Proof.** Assume that $\forall m\exists n\geqslant m$ $g(n)\notin\{\chi_\alpha:\alpha<\lambda\}$. By the results of Section 6.6. every subsequence of $g$ is still generic, so we may assume that for all $n$ $g(n)\notin\{\chi_\alpha:\alpha<\lambda\}$. By Lemma 3 choose for each $n$ $X_n$ in $\mathcal{U}$ such that $g(n)\notin i_{0\lambda}X_n$. Set $X=\bigcap_n X_n$, $X\in\mathcal{U}$ since $g\in N_0$, and for all $n$ $g(n)\notin i_{0\lambda}X$: this contradicts the criterion Section 6.4. Theorem 5. Hence if $P(g)$ is generic over $\mathcal{C}_\lambda$, there is a sequence $(\gamma_n)_n$ increasing and cofinal in $\lambda$ such that $g$ is eventually equal to $(\chi_{\gamma_n})_n$. If moreover $g$ is in $M_\lambda$, $(\gamma_n)_n$ is in $M_\lambda$, hence in Cof $\lambda$, and $g$ is in $\mathcal{G}_\lambda$.

**Corollary 5.** *If $\mathcal{U}$ is* selective,

(i) *$\mathcal{G}_\lambda$ is in $\mathrm{ODN}_\lambda^{M_\lambda}$ for any $\lambda$ of cofinality $\omega$;*

(ii) *For any $\mathcal{P}$-generic extension $N_0[g]$ of $N_0$, the generic sequences over $\mathcal{P}$ which are in $N_0[g]$ are exactly the almost-sub-sequences of $g$.*

**Proof.** $\mathcal{G}_\lambda$ is the set of $N_\lambda$-generic sequences over $\mathcal{C}_\lambda$ in $M_\lambda$, so it is ordinal definable in $M_\lambda$ from $\kappa$ and $i_{0\lambda}\mathcal{U}$. In particular the $N_\omega$-generic sequences over $\mathcal{C}_\omega$ in $M_\omega$ are the almost-sub-sequences of $(\chi_n)_n$, which by Section 6.6. Theorem 1. gives (ii).

7.3.

In this paragraph, we assume that $\mathcal{V}$ is in $N_0$ a second $\kappa$-complete ultrafilter on $\kappa$ such that $\mathcal{U} = h * \mathcal{V}_p$ for some bijection $h : \kappa^p \to \kappa$. We investigate the connection between $\mathcal{C}(\mathcal{U})$- and $\mathcal{C}(\mathcal{V})$-generic sets. Notations $\xi$, $\eta$ will refer to members of $\kappa$.

For simplicity constructions related to $\mathcal{V}$ will be noted with a $^*$, while constructions related to $\mathcal{U}$ will be noted without any superscript. Applying Section 7.1. Lemma 1 (v), we get in the same way:

**Lemma 1.** (i) *For any* $g : \kappa^n \to N_0$ *and* $(e_1 \cdots e_n) \in [Y]^{<\omega}$

$$\pi_\gamma e * g = \pi^*_{p\gamma}[s \mapsto g(h(s(pe_1) \cdots s(pe_1 + p - 1)), \ldots, h(s(pe_n) \cdots s(pe_n + p - 1)))].$$

(ii) *In particular* $N_\gamma = N^*_{p\gamma}$, *and* $i_{0\gamma} = i^*_{0p\gamma}$ *for any* $\gamma$; $N_\lambda = N^*_\lambda$, $M_\lambda = M^*_\lambda$ *and* $i_{0\lambda} = i^*_{0\lambda}$ *for limit* $\lambda$.

**Remark.** This lemma enables us to construct a counterexample to the conjecture $M_\omega = N_\omega[(\kappa_n)_n]$ when $\mathcal{U}$ is not assumed normal. For assume $\mathcal{V}$ normal and $\mathcal{U} = h * \mathcal{V}_2$. Then $M_\omega = M^*_\omega = N_\omega[(\kappa_n)_n]$, and $\kappa_n = i_{on}\kappa = i^*_{02n}\kappa = \kappa^*_{2n}$, so $N_\omega[(\kappa_n)_n] = N_\omega[(\kappa^*_{2n})_n] = N^*_\omega[(\kappa^*_{2n})_n]$, and it is easy to prove that $N^*_\omega[(\kappa^*_{2n})_n] \neq N^*_\omega[(\kappa^*_n)_n]$.

One would hope to establish a correspondence between $\mathcal{C}$- and $\mathcal{C}^*$-generic sequence applying the criterion Section 6.4. Theorem 5. However, it turns out that for $p > 1$, it is simpler to establish a correspondence between $\mathcal{C}$ and $\mathcal{C}^*$ directly as follows.

**Definition.** (i) For $Y \subseteq \kappa^p$, we set $Y \in \mathcal{V}'_p$ iff $Y \subseteq [\kappa]^p$ and $s^\frown t \in Y$ implies $Y^s \in \mathcal{V}_{p-|s|}$.

(ii) For $X \subseteq \kappa$, we set $X \in \mathcal{U}'$ iff $h^{-1}X \in \mathcal{V}'_p$.

**Lemma 2.** (i) *Assume* $T \in \mathcal{C}^*$ *and* $s \supseteq s_T$. *Then* $\{\xi : s^\frown\xi \in T\}$ *belongs to* $\mathcal{V}'_p$.

(ii) *For any* $Y$ *in* $\mathcal{V}_p$, *there is* $Y'$ *in* $\mathcal{V}'_p$, $Y' \subseteq Y$, *and the same holds for* $\mathcal{U}$ *and* $\mathcal{U}'$.

**Proof.** (i) If $T \in \mathcal{C}^*$, $s \supseteq s_T$ and $s^\frown\xi \in T$, $\xi$ is strictly increasing by definition. Then for any $t \supseteq s$ and any length $n$ $\{t' \in \kappa^n : t^\frown t' \in T\}$ is in $\mathcal{V}_n$ if $t$ is in $T$, so $\{\xi : s^\frown\xi \in T\}$ is in $\mathcal{V}'_p$.

(ii) Like for Section 6.2. Lemma 2, set $s \in X'$ iff $\forall s' \subseteq s\ s' \in \mathrm{tr}_{p|s'|} X$. Then $X'$ is in $\mathcal{V}_p$ and for $s$ in $X'$ and $s' \subseteq s$ $X^{s'}$ is in $\mathcal{V}_{p-|s'|}$, so $X'$ is in $\mathcal{V}'_p$.

**Lemma 3.** (i) *If $X$ is in $\mathcal{U}$ and $\theta<\kappa$, there is $X'\subseteq X$ such that $X'\in\mathcal{U}'$ and for $\xi$ in $X'$ $h^{-1}(\xi)$ is in $[\kappa]^p$ and* $\min h^{-1}(\xi)>\theta$;

(ii) *If $Y$ is in $\mathcal{V}_p$ and $\theta<\kappa$, there is $Y'\subseteq Y$ such that $Y'\subseteq\mathcal{V}'_p$ and for $\boldsymbol{\xi}$ in $Y'$ $h(\boldsymbol{\xi})>\theta$.*

**Proof.** (i) Since $[\kappa]^p$ is in $\mathcal{V}_p$, $h''[\kappa]^p$ is in $\mathcal{U}$; moreover $\{\boldsymbol{\xi}:\min\boldsymbol{\xi}>\theta\}$ is in $\mathcal{V}_p$, so $\{\xi:\xi\in X$ and $h^{-1}(\xi)\in[\kappa]^p$ and $\min h^{-1}(\xi)>\theta\}$ is in $\mathcal{U}$. We then apply Lemma 1 to get $X'\in\mathcal{U}'$.

(ii) For any $\theta<\kappa$ $\{\boldsymbol{\xi}:h(\boldsymbol{\xi})\leq\theta\}$ is not in $\mathcal{V}_p$ since $h$ is injective, so $\{\boldsymbol{\xi}:\boldsymbol{\xi}\in Y$ and $h(\boldsymbol{\xi})>\theta\}$ is in $\mathcal{V}_p$. We then apply Lemma 1 to get $Y'\in\mathcal{V}'_p$.

**Definition.** We construct $A$ and $A^*$, two subsets of $[\kappa]^{<\omega}$ by:

(i) $s\in A$ *iff* $s=(\xi)$ and $h^{-1}(\xi)\in[\kappa]^p$ or $s=s'^\frown(\eta,\xi)$ and $h^{-1}(\xi)\in[\kappa]^p$ and $\max h^{-1}(\eta)<\min h^{-1}(\xi)$.

(ii) $s\in A^*$ *iff* $|s|=kp$ with $k\geq2$, say $s=s'^\frown\boldsymbol{\eta}^\frown\boldsymbol{\xi}$ and $h(\boldsymbol{\eta})<h(\boldsymbol{\xi})$ or $|s|$ is not of the form $kp$ with $k\geq2$.

We then construct $\mathcal{C}_2\subseteq\mathcal{C}_1\subseteq\mathcal{C}$ and $\mathcal{C}_2^*\subseteq\mathcal{C}_1^*\subseteq\mathcal{C}^*$ by:

(iii) $T\in\mathcal{C}_1$ *iff* for all $s\supseteq s_T$ in $T$ $\{\xi:s^\frown\xi\in T\}$ is in $\mathcal{U}'$ and if moreover $s\neq s_T$ then $s\in A$;

(iv) $T\in\mathcal{C}_1^*$ *iff* $|s_T|$ is multiple of $p$ and if $s\supseteq s_T$ in $T$ and $|s|$ is multiple of $p$, then $\{\boldsymbol{\xi}:s^\frown\boldsymbol{\xi}\in T\}$ is in $\mathcal{V}'_p$ and if $s\supseteq s_T$ in $T$ then $s\in A^*$.

(v) $T\in\mathcal{C}_2$ *iff* $T\in\mathcal{C}_1$ and $T\subseteq A$;

$T\in\mathcal{C}_2^*$ *iff* $T\in\mathcal{C}_1^*$ and $T\subseteq A^*$.

**Lemma 4.** (i) *$\mathcal{C}_1$ is dense in $\mathcal{C}$;*

(ii) *$\mathcal{C}_1^*$ is dense in $\mathcal{C}^*$.*

**Proof.** (i) Let $T$ be any member of $\mathcal{C}$; we construct $T'\subseteq T$ by induction on levels. First $s_{T'}=s_T$; assume $s$ has been put in $T'$; let $\eta$ be the last term of $s$, $\theta$ be $\max h^{-1}(\eta)$ and $X$ be $\{\xi;\ s^\frown\xi\in T\}$. We apply to $X$ and $\theta$ the Lemma 3 getting $X'\subseteq X$, and set $s^\frown\xi\in T'$ *iff* $\xi\in X'$; $\{\xi:s^\frown\xi\in T'\}$ is in $\mathcal{U}'$, and for all $s^\frown\xi$ in $T'$, $h^{-1}(\xi)\in[\kappa]^p$ and $\min h^{-1}(\xi)>\max h^{-1}(S)$ by construction, so $s^\frown\xi$ is in $A$. Thus $T'$ is in $\mathcal{C}_1$, which is dense in $\mathcal{C}$.

(ii) Let $T$ be any member of $\mathcal{C}^*$: we construct $T'\subseteq T$ by induction on groups of $p$ levels. First choose $s_0\supseteq s_T$ such that $|s|$ is multiple of $p$; we set $s_{T'}=s_0$. Now assume $s$ has been put in $T'$; let $\boldsymbol{\eta}$ be the last $p$ terms of $s$, $\theta$ be $h(\boldsymbol{\eta})$ and $Y=\{\boldsymbol{\xi}:s^\frown\boldsymbol{\xi}\in T\}$. We apply to $Y$ and $\theta$ the Lemma 3 (ii) getting $Y'$, and we set $s^\frown\boldsymbol{\xi}\in T'$ iff $\boldsymbol{\xi}\in Y'$. Now $Y'$ is in $\mathcal{V}'_p$, and for all $s^\frown\boldsymbol{\eta}^\frown\boldsymbol{\xi}$ in $T'h(\boldsymbol{\xi})>h(\boldsymbol{\eta})$ by construction, so $s^\frown\boldsymbol{\eta}^\frown\boldsymbol{\xi}$ is in $A^*$. Thus $T'$ is in $\mathcal{C}_1^*$, which is dense in $\mathcal{C}^*$.

**Proposition 5.** (i) *If $P(g)$ is generic over $\mathcal{C}$, then there is $n$ such that $\forall m\geq n$ $h^{-1}(g(m))\in[\kappa]^p$ and* $\max h^{-1}(g(m-1))<\min h^{-1}(g(m))$;

(ii) *If $P^*(g)$ is generic over $\mathcal{C}^*$, then there is $n$ such that $\forall m\geq n$* $h(g\restriction[mp,(m+1)p))>h(g\restriction([m-1)p,mp))$.

**Proof.** Assume $T$ is in $P(g)\cap\mathscr{C}_1$, and take $n$ such that $s_T=g\restriction(n-1)$; then for all $s$ in $T$ such that $s\supseteq s_T$, $s$ is in $A$. In particular for $m\geqslant n$ $g\restriction m$ is in $A$, which gives (i).
The proof is analogous for (ii).

**Corollary 6.** (i) *If $P(g)$ is generic over $\mathscr{C}$, then there is $g'$ eventually equal to $g$ such that $P(g')\cap\mathscr{C}_1\subseteq\mathscr{C}_2$.*

(ii) *If $P^*(g)$ is generic over $\mathscr{C}^*$, then there is $g'$ eventually equal to $g$ such that $P^*(g')\cap\mathscr{C}_1^*\subseteq\mathscr{C}_2^*$.*

**Proof.** (i) First fix in $N_0$ a sequence $s:\omega\to\kappa$ such that for all $n$ $s\restriction n$ is in $A$ (construct it inductively). Certainly $s$ is bounded below $\kappa$, as well as $\bigcup\{h^{-1}(s(n))\colon n\in\omega\}$. Since $P(g)$ is generic over $\mathscr{C}$, $g$ is cofinal in $\kappa$, and so is $\cup\{h^{-1}(g(n))\colon n\in\omega\}$ since for any $\theta<\kappa$ the set $\{T\in\mathscr{C}:\forall s\supseteq s_T\ \forall\xi\ s^\frown\xi\in T\Rightarrow\min h^{-1}(\xi)>\theta\}$ is dense in $\mathscr{C}$. By Proposition 5 (i) there is $n$ such that $\forall m\geqslant n\ h^{-1}(g(m))\in[\kappa]^p$ and $\max h^{-1}(g(m-1))<\min h^{-1}(g(m))$, and since $\cup\{h^{-1}(g(n)):n\in\omega\}$ is cofinal in $\kappa$ we may assume that $\max\ h^{-1}(s(n-1))<\min\ h^{-1}(g(n))$. Then we set $g'=s\restriction n^\frown g\restriction[n,\omega)$: $P(g')$ is generic over $\mathscr{C}$ as well as $P(g)$ is, and for all $m$ $g'\restriction m$ is in $A$, and $P(g')$ is included in $\mathscr{C}_2\cup\mathscr{C}\setminus\mathscr{C}_1$. The proof is parallel for $\mathscr{C}^*$.

**Definition.** (i) For $T$ in $\mathscr{C}_2^*$ we define $H^+(T)$ as follows: if $s\in\kappa^{kp}$, say $s=\xi_0{}^\frown\cdots{}^\frown\xi_{k-1}$, we set $\boldsymbol{h}(s)=h(\xi_0),\ldots h(\xi_{k-1}))$, and put $H^+(T)=\boldsymbol{h}''(T\cap\bigcup_k\kappa^{kp})$.

(ii) For $T$ in $\mathscr{C}_2$ we define $H^-(T)$ to be the closure under inclusion of $\boldsymbol{h}^{-1}T$.

**Claim.** $H^+$ is an isomorphism from $\mathscr{C}_2^*$ onto $\mathscr{C}_2$ and $H^-$ its inverse. For, if $T$ is in $\mathscr{C}_2^*$, $H^+(T)$ is included in $[\kappa]^{<\omega}$ since if $s^\frown\xi^\frown\eta\in T$ $h(\xi)<h(\eta)$. Then $H^+(T)$ is closed under inclusion since $T$ is. Set $s'_T=\boldsymbol{h}(s_T)$ (since $|s_T|=kp$): then if $s\in H^+(T)$ $s\supseteq s'_T$ or $s\subseteq s'_T$. If moreover $s\supseteq s'_T$, $s=\boldsymbol{h}(t)$ with $t\in T$, so $\{\eta:t^\frown\eta\in T\}\in\mathscr{V}'_p$, and $\{\xi:s^\frown\xi\in H^+(T)\}\in\mathscr{U}'$. So $H^+(T)$ belongs to $\mathscr{C}$. Now assume $s^\frown\xi\in H^+(T)$; $s^\frown\xi=\boldsymbol{h}(t^\frown\eta)$, so $h^{-1}(\xi)\in[\kappa]^p$ and

$$\min\ h^{-1}(\xi)=\min\ \eta>\max\ t=\max\ \boldsymbol{h}^{-1}(s),$$

so $s^\frown\xi\in A$, and $H^+(T)\in\mathscr{C}_2$.

Now assume $T$ is in $\mathscr{C}_2$: $H^-(T)$ is a subset of $[\kappa]^{<\omega}$ for if $s^\frown\xi^\frown\eta\in T$, then $h^{-1}(\xi)\in[\kappa]^p$, $h^{-1}(\eta)\in[\kappa]^p$ and $\max h^{-1}(\xi)<\min h^{-1}(\eta)$ since $s^\frown\xi^\frown\eta\in A$; $h^-(T)$ is closed under subsequence, and since $T$ is, the closure adds no sequence of length multiple of $p$. Set $s'_T=\boldsymbol{h}^{-1}(s_T)$: then $|s'_T|$ is multiple of $p$ and if $s\in H^-(T)$ $s\subseteq s'_p$ or $s\supseteq s'_T$. If $s\supset s'_T$, $s$ in $H^-(T)$ and $|s|$ is multiple of $p$, then $s=\boldsymbol{h}^{-1}(t)$ with $t\in T$: so $\{\eta:t^\frown\eta\in T\}\in\mathscr{U}'$, thus $\{\xi:s^\frown\xi\in H^-(T)\}\in\mathscr{V}'_p$; hence $H^-(T)$ is in $\mathscr{C}^*$. Assume $s^\frown\xi\in H^-(T)$ and $|s|$ multiple of $p:s^\frown\xi=\boldsymbol{h}^{-1}(t^\frown\boldsymbol{h})$, thus $\max t<\eta$, hence $h$(the last $p$ terms of $s)=t<\eta=h(\xi)$: so $s^\frown\xi\in A^*$, and $H^-(T)$ belongs to $\mathscr{C}_2^*$.

Finally it is obvious that $H^+$ is order preserving, and $H^-$ is the inverse of $H^+$: the claim is thus proved.

**Definition.** We extend to $[\kappa]^\omega$ the definition of $\boldsymbol{h}$ by $\boldsymbol{h}(g)(l) = \boldsymbol{h}(g \restriction [lp, (l+1)p))$.

**Theorem 7.** *$P(g)$ is generic over $\mathscr{C}$ iff there exists $g^*$ such that $P^*(g^*)$ is generic over $\mathscr{C}^*$ and $g$ is eventually equal to $\boldsymbol{h}(g^*)$.*

**Proof.** Assume $P(g)$ generic over $\mathscr{C}$: by the Corollary 6, there exists $g'$ such that $g$ is eventually equal to $g'$ and $P(g') \cap \mathscr{C}_1 \subseteq \mathscr{C}_2$. We define $g^*$ by $g^* \restriction kp = \boldsymbol{h}^{-1}(g' \restriction k)$. By the choice of $g'$, $g^*$ is in $[\kappa]^\omega$, and we prove that $P^*(g^*)$ is generic over $\mathscr{C}^*$. For let $D^*$ be a dense open subset of $\mathscr{C}^*$, and $D_2^*$ be $D^* \cap \mathscr{C}_2^*$. We set $D = H^{+\prime\prime} D_2^* \cup \mathscr{C}_1 \backslash \mathscr{C}_2$, and claim that $D$ is dense in $\mathscr{C}$. For $\mathscr{C}_1$ is dense in $\mathscr{C}$ and, if $T$ is in $\mathscr{C}_2$, $H^-(T)$ is in $\mathscr{C}_2^*$; since $D^*$ is dense in $\mathscr{C}^*$ there is $T^*$ in $D_1^* T^* \subseteq H^-(T)$; since $\mathscr{C}_1^*$ is dense in $\mathscr{C}^*$ there is $T_1^*$ in $\mathscr{C}_1^*$, $T_1^* \subseteq T^*$. Since $D^*$ is open, $T_1^*$ is in $D^*$. Moreover $\mathscr{C}_2^*$ is open in $\mathscr{C}_1^*$, hence $H^-(T) \in \mathscr{C}_2^*$ and $T_1^* \in \mathscr{C}_1^*$, $T_1^* \subseteq H^-(T)$ implies $T_1^* \in \mathscr{C}_2^*$, that is $T_1^* \in D_2^*$. So $H^+(T_1^*) \in D$, and is included in $H^+(H^-(T)) = T$. The claim is proved. Now since $P(g')$ is generic over $\mathscr{C}$, there exists $T' \in P(g') \cap D$. We have then $T' \in P(g') \cap \mathscr{C}_1$, hence $T' \in \mathscr{C}_2$, so $T' = H^+(T^*)$ for some $T^*$ in $D_2^*$. We claim that $T^* \in P^*(g^*)$; for fix any $k$; $g' \restriction k \in T'$, so $\boldsymbol{h}^{-1}(g' \restriction k) \in T^* = H^-(T')$. But $\boldsymbol{h}^{-1}(g \restriction k)$ is $g^* \restriction kp$. Finally we proved that $P^*(g^*)$ meets $D^*$, and so $P^*(g^*)$ is generic over $\mathscr{C}^*$.

Conversely assume $P^*(g^*)$ generic over $\mathscr{C}^*$. By Corollary 6, there exists $g^{*\prime}$ eventually equal to $g^*$ and such that $P^*(g^{*\prime}) \cap \mathscr{C}_1^* \subseteq \mathscr{C}_2^*$. We set $g = \boldsymbol{h}(g^{*\prime})$: $g$ is increasing by the choice of $g^{*\prime}$, is eventually equal to $\boldsymbol{h}(g^*)$, and we prove that $P(g)$ is generic over $\mathscr{C}$. For let $D$ be a dense open subset of $\mathscr{C}$, $D_2$ be $D \cap \mathscr{C}_2$, $D^* = H^{-\prime\prime} D_2 \cap \mathscr{C}_1^* \backslash \mathscr{C}_2^*$. As precedently $D^*$ is dense since $D$ is dense open; since $P^*(g^{*\prime})$ is generic over $\mathscr{C}^*$ there is $T^* \in P^*(g^{*\prime}) \cap D^*$. Then $H^+(T^*)$ is in $P(g) \cap D$, and $P(g)$ is generic over $\mathscr{C}$.

7.4.

We apply the preceding result to generic sequences over $\mathscr{C}_\lambda$. First notice that by Section 7.3. Lemma 1. $\mathscr{C}_\lambda^* = i_{0\lambda}^* \mathscr{C}^*$ is also $i_{0\lambda} \mathscr{C}^*$, a member of $N_\lambda = N_\lambda^*$, for any $\lambda$ limit. Fix $\lambda$ of cofinality $\omega$.

By Section 7.3. Lemma 1. for $\gamma < \lambda$ $\chi_\gamma$, which is $\pi_\lambda[s \mapsto s(\gamma)]$, is also $\pi_\lambda^*[s \mapsto h(s(p\gamma) \cdots s(p\gamma + p - 1))]$, that is $i_{0\lambda} h(\chi_{p\gamma}^*, \ldots, \chi_{p\gamma+p-1}^*)$. By Section 6.5. Theorem 1. applied to $\mathscr{C}_\lambda^*$, we see that for any $(\gamma_n)_n$ in cof $\lambda$ $(\chi_{\gamma_n}^*)_n$ is $N_\lambda$-generic over $\mathscr{C}_\lambda^*$. Thus the previous theorem yields:

**Theorem 1.** *Assume $\mathscr{U} = h * \mathscr{V}_p$, $h$ bijective and $\mathscr{V}$ $\kappa$-complete, and $\lambda$ limit cofinality $\omega$.*

Define $\gamma < \lambda$ $\chi_{p\gamma+k}^*$ to be the $k$-th term of $i_{0\lambda} h^{-1}(\chi_\gamma)$, and let $\mathscr{G}_\lambda^h$ be the set of sequences which are eventually equal to some sequence $\boldsymbol{h}((\chi_{\gamma_n}^*)_n)$ with $(\gamma_n)_n$ in cof $\lambda$. Then

(i) $\mathscr{G}_\lambda^h$ is in $M_\lambda$;

(ii) For each $g$ in $\mathscr{G}_\lambda^h$, $P(g)$ is $N_\lambda$-generic over $\mathscr{C}_\lambda$;

(iii) If moreover $\mathscr{V}$ is normal, $\mathscr{G}_\lambda^h$ is exactly the set of the $N_\lambda$-generic sequences over $\mathscr{C}_\lambda$ which are in $M_\lambda$.

**Proof.** (i) By Section 5.2. Lemma 1 $\mathscr{G}_\lambda^*$ is in $M_\lambda^* = M_\lambda$, and $\mathscr{G}_\lambda^h$ is the set of the sequences which are eventually equal to some $i_{0\lambda}\boldsymbol{h}(g)$ with $g$ in $\mathscr{G}_\lambda^*$, by the actual definition of $\chi_\gamma^*$ which is as we noticed the same as precedently. Since $i_{0\lambda}h$ is in $N_\lambda$, $\mathscr{G}_\lambda^h$ is in $M_\lambda$.

(ii) and (iii) result from Section 7.3.

**Corollary 2.** *If $\mathscr{U}$ is equivalent to the $p$-th power of some $\kappa$-complete ultrafilter with $p \geqslant 2$, then if $g$ is any $N_0$-generic sequence over $\mathscr{C}$:*

(i) *There are in $N_0[g]$ $N_0$-generic sequences over $\mathscr{C}$ which are not eventually included in $g$;*

(ii) *There are in $N_0[g]$ two $N_0$-generic sequences over $\mathscr{C}$ whose union is not generic over $\mathscr{C}$.*

**Proof.** It suffices to look at $M_\omega$. Members of $\mathscr{G}_\omega$ correspond in $\mathscr{G}_\omega^h$ to those sequences $(l_n)_n$ in cof $\omega$ which satisfy $l_{pn+k} = pl'_n + k$ for $k = 1, \ldots, p-1$, $(l'_n)_n$ being itself in cof $\omega$; then it is clear that $\boldsymbol{h}((\chi_n^*)_{n\geqslant 1})$, which is generic over $\mathscr{C}_\omega$, is not an almost-sub-sequence of $(\chi_n)_n$, which is $\boldsymbol{h}((\chi_n^*)_{n\geqslant 0})$, and that the union of $(\chi_n)_n$ and $\boldsymbol{h}((\chi_n^*)_{n\geqslant 1})$ is not generic over $\mathscr{C}_\omega$ since it cannot satisfy the necessary condition given by Section 7.3. Proposition 5.

**Corollary 3.** *If $N_0 = L[\mathscr{U}]$, then there is $p$ and $h: \kappa^p \to \kappa$ such that for any $\lambda$ of cofinality $\omega$, the set of $N_\lambda$-generic sequences over $\mathscr{C}_\lambda$ which are in $M_\lambda$ is exactly $\mathscr{G}_\lambda^h$.*

**Proof.** Kunen has shown that if $N_0 = L[\mathscr{U}]$, then $\mathscr{U}$ is equivalent to some power of a normal ultrafilter.

## 8. Support theory

For any $\alpha$ and $x$ in $N_\alpha$, there are many ways representing $x$ as the image under $\pi_\alpha$ of a finite support function. We show now that for each $x$ in $N_\alpha$ there is a minimum finite subset of $\alpha$ which is a support for functions representing $x$; and of course we call this minimum subset *the* support of $x$. We also treat the connection between the supports at levels $\alpha$ and $\beta$ of an element of $N_\alpha$ for $\beta \leqslant \alpha$. The results obtained in Section 8.2. suffice for Chapter 9. The rather complicated machinery of Section 8.3. is needed for Chapter 10.

### 8.1. *The existence of the support*

**Lemma 1.** *Assume $\alpha$, $x$, $e_1$, $e_2$, $g_1$, $g_2$ are such that $x = \pi_\alpha e_1 * g_1 = \pi_\alpha e_2 * g_2$. Then there is $g$: $\kappa^{|e_1 \cap e_2|} \to N_0$ such that $x = \pi_\alpha(e_1 \cap e_2) * g$.*

**Proof.** The proof is exactly what one may expect. One must simply pay attention to the notations.

So assume $e_1 \neq e_2$ and let $\mu$ the greatest ordinal which is in one, but not both, of $e_1$ and $e_2$, say $\mu \in e_1 \backslash e_2$. Put $e_0 = e_1 \backslash \{\mu\}$. We claim that $g_0$ exists: $\kappa^{|e_0|} \to N_0$ such that $x = \pi_\alpha e_0 * g_0$.

We set $e = e_1 \cup e_2$, $p_1 = |e_1 \cap \mu|$, $p = |e \cap \mu|$ and $q = |e_1| - p_1 - 1 = |e| - p - 1$. Finally note $E_1$, $E_2$ the injections $p_1 + q \to p + q$, $p_2 + q \to p + q$ respectively such that $e_1 = eE_1$ and $e_2 = eE_2$, and also their restrictions to $p_1$ or $p_2$.
Now by hypothesis $\{s \in \kappa^\alpha : g_1(se_1) = g_2(se_2)\} \in \mathcal{U}_\alpha$, i.e.

$$\{s \in \kappa^{|e|} : g_1(sE_1) = g_2(sE_2)\} \in \mathcal{U}_{|e|}.$$

This means that $X \in \mathcal{U}_p$, where $X = \{\xi \in \kappa^p : X_\xi \in \mathcal{U}\}$ and

$$X_\xi = \{\eta \in \kappa : \{\zeta \in \kappa^q : g_1(\xi E_1 {}^\frown \eta {}^\frown \zeta) = g_2(\xi E_2 {}^\frown \zeta)\} \in \mathcal{U}_q\},$$

since when $s = \xi {}^\frown \eta {}^\frown \zeta$ $g_1(sE_1) = g_1(\xi E_1 {}^\frown \eta {}^\frown \zeta)$ and $g_2(sE_2) = g_2(\xi E_2 {}^\frown \zeta)$. Put $Y = \{\xi E_1 : \xi \in X\}$. If $\xi$ is in $X$, $\xi E_1$ is in $Y$, so $\xi$ is in $E_1 * Y$, hence $E_1 * Y$ is in $\mathcal{U}_p$, and $Y$ is in $\mathcal{U}_{p_1}$.

For $\boldsymbol{\sigma}$ in $\kappa^{p_1}$, we set

$\xi(\boldsymbol{\sigma})$ = any element of $X$ such that $\boldsymbol{\sigma} = \xi(\boldsymbol{\sigma})E_1$, if $\boldsymbol{\sigma} \in Y$, $(0, \ldots, 0)$ if not.

$\eta(\boldsymbol{\sigma})$ = any element of $X_{\xi(\boldsymbol{\sigma})}$ if $\boldsymbol{\sigma} \in Y$, 0 if not.

Finally, we define $g_0$: $\kappa^{p+q} \to N_0$ by

$$g_0(\boldsymbol{\sigma} {}^\frown \zeta) = g_1(\boldsymbol{\sigma} {}^\frown \boldsymbol{\eta}(\boldsymbol{\sigma}) {}^\frown \zeta).$$

For $\boldsymbol{\sigma}$ in $Y$, $\xi(\boldsymbol{\sigma})$ is in $X$, hence for any $\eta$ in $X_{\xi(\boldsymbol{\sigma})}$

$$\{\zeta \in \kappa^q : g_1(\boldsymbol{\sigma} {}^\frown \eta {}^\frown \zeta) = g_2(\xi(\boldsymbol{\sigma}) E_2 {}^\frown \zeta)\} \in \mathcal{U}_q,$$

thus

$$\{\xi \in \kappa^q : g_1(\boldsymbol{\sigma} {}^\frown \eta {}^\frown \zeta) = g_1(\boldsymbol{\sigma} {}^\frown \boldsymbol{\eta}(\boldsymbol{\sigma}) {}^\frown \zeta) = g_0(\boldsymbol{\sigma} {}^\frown \zeta)\} \in \mathcal{U}_q,$$

hence

$$Y \subseteq \{\boldsymbol{\sigma} : \{\eta : \{\zeta : g_1(\boldsymbol{\sigma} {}^\frown \eta {}^\frown \zeta) = g_0(\boldsymbol{\sigma} {}^\frown \zeta)\} \in \mathcal{U}_q\} \in \mathcal{U}\},$$

i.e.

$$\{\boldsymbol{\sigma} {}^\frown \eta {}^\frown \zeta \in \kappa^{|e_1|} : g_1(\boldsymbol{\sigma} {}^\frown \eta {}^\frown \zeta) = g_0(\boldsymbol{\sigma} {}^\frown \zeta)\} \in \mathcal{U}_{p_1},$$

and

$$x = \pi_\alpha e_1 * g_1 = \pi_\alpha e_0 * g_0.$$

**Definition.** For any $\alpha$ and $x \in N_\alpha$, we let $e_\alpha(x)$ be the least subset $e$ of $\alpha$ such that there exists $g : \kappa^{|e|} \to N_0$ satisfying $x = \pi_\alpha e * g = i_{0\alpha} g(\chi_e)$. We call $e_\alpha(x)$ the $\alpha$-*support* of $x$. In particular $e_\alpha(x)$ is empty iff there exists $y$ in $N_0$ such that $x = i_{0\alpha} y$.

By the very definition of $e_\alpha(x)$, we get:

**Lemma 2.** *If $x \in N_\alpha$ and $x$ can be written $x = i_{0\alpha}g(\chi_e)$, then $e_\alpha(x) \subseteq e$.*

8.2. *Upwards connection between supports*

**Proposition 1.** *Assume $\alpha \leqslant \beta$ and $\beta - \alpha \in \operatorname{im} i_{0\alpha}$. If $x \in N_\beta$ and if moreover for each $\mu$ in $e_\beta(x) \cap [\alpha, \beta)$, we have $\mu - \alpha \in \operatorname{im}\, i_{0\alpha}$, then $e_\alpha(x) \subseteq e_\beta(x)$.*

**Proof.** Let $\gamma$ be such that $i_{0\beta} = i_{0\alpha} i_{0\gamma}$ (by Section 2.2. Lemma 3) and $\mu_1, \ldots, \mu_n$ such that $\chi_{e_\beta(x)\cap[\alpha,\beta)} = i_{0\alpha}(\chi_{\mu_1} \cdots \chi_{\mu_n})$ (which exist by the hypothesis on $e_\beta(x)$, by Section 2.2 Lemma 3. and because $\chi_\mu = i_{0\mu}\chi_0$). Then if $x = \pi_\beta e_\beta(x) * f = i_{0\beta} f(\chi_{e_\beta(x)})$, we have:

$$\begin{aligned} x &= i_{0\beta} f(\chi_{e_\beta(x)\cap\alpha}{}^\frown i_{0\alpha}(\chi_{\mu_1} \cdots \chi_{\mu_n})) \\ &= i_{0\alpha} i_{0\gamma} f(\chi_{e_\beta(x)\cap\alpha}{}^\frown i_{0\alpha}(\chi_{\mu_1} \cdots \chi_{\mu_n})) \\ &= i_{0\alpha}[i_{0\gamma} F(\chi_{\mu_1} \cdots \chi_{\mu_n})](\chi_{e_\beta(x)\cap\alpha}) \end{aligned}$$

where we set $F(\xi)(\eta) = f(\eta^\frown \xi)$, so

$$x = i_{0\alpha}[i_{0\gamma} F(\chi_{\mu_1} \cdots \chi_{\mu_n}) \cap \kappa^m \times N_0](\chi_{e_\beta(e)\cap\alpha})$$

since $\chi_{e_\beta(x)\cap\alpha}$ is in $\kappa_\alpha^m$, so in dom $i_{0\alpha}[i_{0\gamma}F(\chi_{\mu_1} \cdots \chi_{\mu_n}) \cap \kappa^m \times N_0]$.

But $i_{0\gamma}F(\chi_{\mu_1} \cdots \chi_{\mu_n}) \cap \kappa^m \times N_0$ (where $m = |e_\beta(x) \cap [\alpha, \beta)|$) is a function $\kappa^m \to N_0$, so we have written $x$ in the form $x = i_{0\alpha} g(\chi_{e_\beta(x)\cap\alpha})$ and by 8.1. Lemma 2. $e_\alpha(x) \subseteq e_\beta(x) \cap \alpha$, thus $e_\alpha(x) \subseteq e_\beta(x)$.

**Corollary 2.** (i) *If $\beta - \alpha < \kappa$, then for all $x$ in $N_\beta$ $e_\alpha(x) \subseteq e_\beta(x)$;*

(ii) *If $\beta - \alpha \in \operatorname{im}\, i_{0\alpha}$ and $x$ in $N_\beta$ is such that $e_\beta(x) \subseteq \alpha$, then $e_\alpha(x) \subseteq e_\beta(x)$.*

8.3. *Downwards connection between supports*

Our aim is now to prove a converse result for supports, i.e. an inclusion of $e_\beta(x) \cap \alpha$ in $e_\alpha(x)$ for $\alpha \leqslant \beta$ and $x$ in $N_\beta$. We need first establish a few lemmas.

**Lemma 1.** *Assume $x \in N_\gamma$ and let $\theta$, $\theta'$, $\delta$ be such that $\theta \leqslant \gamma$, $\theta \leqslant \theta'$ and $i_{\theta\theta'} i_{0\gamma} = i_{0\delta}$; then $\chi_{e_\delta(i_{\theta\theta'}x)} = i_{\theta\theta'}(\chi_{e_\gamma(x)})$.*

**Proof.** Write $x = i_{0\gamma} g(\chi_{e_\gamma(x)})$. Then $i_{\theta\theta'}x = i_{\theta\theta'}[i_{0\gamma} g(\chi_{e_\gamma(x)})] = i_{0\delta} g(i_{\theta\theta'}\chi_{e_\gamma(x)})$, and Section 8.1. Lemma 2 proves that $\chi_{e_\delta(i_{\theta\theta'}x)} \subseteq i_{\theta\theta'}\chi_{e_\gamma(x)}$. Conversely since $i_{\theta\theta'}\chi_{e_\gamma(x)} \subseteq \operatorname{im}\, i_{\theta\theta'}$ and $\chi_{e_\delta(i_{\theta\theta'}x)} \subseteq i_{\theta\theta'}\chi_{e_\gamma(x)}$, we have $\chi_{e_\delta(i_{\theta\theta'}x)} \subseteq \operatorname{im}\, i_{\theta\theta'}$, say $\chi_{e_\delta(i_{\theta\theta'}x)} = i_{\theta\theta'}\chi_e$ (since $\chi_\mu$ can only be the image of another $\chi$ under $i_{\theta\theta'}$). Now if $i_{\theta\theta'}x = i_{0\delta} g'(\chi_{e_\delta(i_{\theta\theta'}x)}) = i_{0\delta} g'(i_{\theta\theta'}\chi_e)$, we get $i_{\theta\theta'}x = i_{\theta\theta'}[i_{0\gamma} g'(\chi_e)]$, so $x = i_{0\gamma} g'(\chi_e)$, and finally $e \supseteq e_\gamma(x)$, hence $\chi_{e_\delta(i_{\theta\theta'}x)} \supseteq i_{\theta\theta'}\chi_{e_\gamma(x)}$.

**Lemma 2.** *Assume $x \in N_\mu$: then $x \in \operatorname{im}\, i_{\mu\mu+1}$ iff $i_{\mu\mu+1}x = i_{\mu+1\ \mu+2}x$.*

**Proof.** We argue in $N_\mu$, and may assume $\mu=0$. By Section 2.2. Corollary 2. if $x\in \text{im } i_{01}$ then $i_{01}x=i_{12}x$. Conversely assume $x\notin \text{im } i_{01}$: then $0\in e_1(x)$, so by Lemma 1, $1\in e_2(i_{01}x)$ since $i_{01}i_{01}=i_{02}$ implies $\chi_{e_2(i_{01}x)}=i_{01}\chi_{e_1}(x)$, hence $e_2(i_{01}x)=1+e_1(x)$. But Lemma 1 implies also that $\chi_{e_2(i_{12}x)}=i_{12}\chi_{e_1}(x)$ hence $e_2(i_{12}x)=e_1(x)$, and in particular $1\notin e_2(i_{12}x)$, so we cannot have $i_{01}x=i_{12}x$.

**Lemma 3.** *Assume $\mu<\gamma$: then $i_{\mu\mu+1}i_{0\gamma}=i_{\mu+1\mu+2}i_{0\gamma}$ iff $\gamma-\mu\in \text{im } i_{\mu\mu+1}$.*

**Proof.** By 2.2 Proposition 1. we have:

$$i_{\mu\mu+1}i_{0\gamma}=i_{0\mu+1+i_{\mu\mu+1}(\gamma-\mu)},$$
$$i_{\mu+1\mu+2}i_{0\gamma}=i_{0\mu+2+i_{\mu+1\mu+2}(\gamma-(\mu+1))}.$$

If $\gamma-\mu=n<\omega$, $i_{\mu\mu+1}n=i_{\mu+1\mu+2}n=n$, so $i_{\mu\mu+1}i_{0\gamma}=i_{\mu+1\mu+2}i_{0\gamma}=i_{0\gamma+1}$. If $\gamma-\mu\geq\omega$, $1+i_{\mu\mu+1}(\gamma-\mu)=i_{\mu\mu+1}(\gamma-\mu)$ and $2+i_{\mu+1\mu+2}(\gamma-(\mu+1))=i_{\mu+1\mu+2}(\gamma-\mu)$, and using that $\mu+\delta=\mu+\delta'$ iff $\delta=\delta'$ we get that $i_{\mu\mu+1}i_{0\gamma}=i_{\mu+1\mu+2}i_{0\gamma}$ *iff* $i_{\mu\mu+1}(\gamma-\mu)=i_{\mu+1\mu+2}(\gamma-\mu)$, that is by Lemma 2 iff $\gamma-\mu\in \text{im } i_{\mu\mu+1}$.

**Lemma 4.** *Assume $\mu<\gamma$, $\gamma-\mu\in \text{im } i_{\mu\mu+1}$ and $x\in N_\gamma$; then $i_{\mu\mu+1}x=i_{\mu+1\mu+2}x$ iff $\chi_{e_\gamma(x)}$ is included in* im $i_{\mu\mu+1}$ *(which implies that $\mu\notin e_\gamma(x)$ since $\chi_\mu\notin \text{im } i_{\mu\mu+1}$).*

**Proof.** By Lemma 3 there is $\delta$ such that $i_{\mu\mu+1}i_{0\gamma}=i_{\mu+1\mu+2}i_{0\gamma}=i_{0\delta}$. Write $x=i_{0\gamma}f(\chi_{e_\gamma(x)})$. Then $i_{\mu\mu+1}x=i_{0\delta}f(i_{\mu\mu+1}\chi_{e_\gamma(x)})$ and $i_{\mu+1\mu+2}x=i_{0\delta}f(i_{\mu+1\mu+2}\chi_{e_\gamma(x)})$, so $i_{\mu\mu+1}x=i_{\mu+1\mu+2}x$ *iff* $i_{\mu\mu+1}\chi_{e_\gamma(x)}=i_{\mu+1\mu+2}\chi_{e_\gamma(x)}$, that is, by Lemma 3, *iff* $\chi_{e_\gamma(x)}\subseteq \text{im } i_{\mu\mu+1}$.

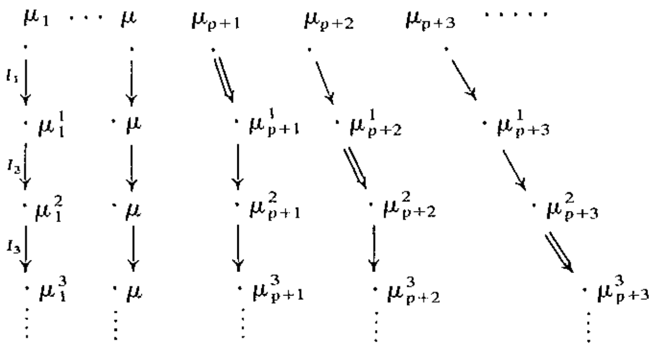
Our aim is to show that, under certain conditions, $\mu$ cannot belong to $e_\gamma(x)$ and not to $e_\delta(x)$ for $x$ in $N_\gamma\cap N_\delta$. Lemma 4 shows that $\mu$ cannot belong to $e_\gamma(x)$ if in the same time $\chi_{e_\delta(x)}$ is included in im $i_{\mu\mu+1}$. We now turn to prove that if $\mu$ does not belong to $e_\delta(x)$, then, up to changing $x$ a little $\chi_{e_\delta(x)}$ may be assumed to be included in im $i_{\mu\mu+1}$, which gives the result.

**Lemma 5.** *Assume $\lambda$ is simple and $\mu_1<\cdots<\mu_p<\mu<\mu_{p+1}<\cdots<\mu_n<\lambda$. Then there is a map $I$ which is a composition of elementary embeddings $i_{\varepsilon\varepsilon'}$, with $\mu_{p+1}\leq\varepsilon\leq\varepsilon'<\lambda$ which satisfies:*

(i) $I\chi_\mu=\chi_\mu$;
(ii) for $i=1\cdots n$ $I\chi_{\mu_i}\in \text{im } i_{\mu\mu+1}$;
(iii) $I\, i_{0\lambda}=i_{0\lambda}$.

**Proof.** We construct $I$ by induction from $p+1$ to $n$. We know that $i_{\mu\mu+1}\chi_{\mu_{p+1}}$ is a $\chi$, say $i_{\mu\mu+1}\chi_{\mu_{p+1}}=\chi_{\mu^1_{p+1}}$: put $I_1=i_{\mu_{p+1}\mu^1_{p+1}}$. We then have $I_1\chi_\mu=\chi_\mu$ since $\mu<\mu_{p+1}$, $I_1\chi_{\mu_i}=\chi_{\mu_i}\in \text{im } i_{\mu\mu+1}$ if $i\leq p$, and $I_1\chi_{\mu_{p+1}}=\chi_{\mu^1_{p+1}}\in \text{im } i_{\mu\mu+1}$ by construction. We now iterate this operation. First we set $I_1\chi_{\mu_q}=\chi_{\mu^1_q}$ for $q=1\cdots n$, and assume that $i_{\mu\mu+1}\chi_{\mu^1_{p+2}}=\chi_{\mu^2_{p+2}}$. We put $I_2=i_{\mu^1_{p+2}\mu^2_{p+2}}$. We have $I_2I_1\chi_\mu=\chi_\mu$, $I_2I_1\chi_{\mu_i}=\chi_{\mu_i}$ for $i\leq p$,

$I_2 I_1 \chi_{\mu_{p+1}} = I_1 \chi_{\mu_{p+1}}$ since $\mu_{p+1} < \mu_{p+2} \leqslant \mu^1_{p+2}$, so $I_2 I_1 \chi_{\mu_{p+1}}$ is in im $i_{\mu\mu+1}$, and $I_2 I_1 \chi_{\mu_{p+2}} = I_2 \chi_{\mu^1_{p+2}} = \chi_{\mu^2_{p+2}}$ is in im $i_{\mu\mu+1}$ by construction, and so on.




We set $I = I_{n-p} \cdot \cdots \cdot I_2 I_1$. Clearly $I\, i_{0\lambda} = i_{0\lambda}$, since $\lambda$ is simple.

**Lemma 7.** *Assume* $\lambda \leqslant \theta$, $\lambda$ *simple, and* $\theta$ *such that*: $\forall \gamma < \lambda\ \gamma + \theta = i_{0\lambda}\theta = \theta$. *Then if* $y \in N_\theta$, *we have* $e_\theta(y) \cap \lambda \subseteq e_\lambda(y)$.

**Proof.** Assume $\mu < \lambda$ is such that $\mu \in e_\theta(y)$ and $\mu \notin e_\lambda(y)$. We apply Lemma 5 to $e_\lambda(y) = \{\mu_1 \cdots \mu_n\}$ to get $I$, a composition of maps $i_{\varepsilon\varepsilon'}$ with $\varepsilon \leqslant \varepsilon' < \lambda$ such that $I\chi_\mu = \chi_\mu$ and $I\chi_{e_\lambda(y)} \in \text{im } i_{\mu\mu+1}$. We have $I\, i_{0\lambda} = i_{0\lambda}$, and by the hypothesis on $\theta$, $I\, i_{0\theta} = i_{0\theta}$ too, since $i_{\varepsilon\varepsilon'} i_{0\theta} = i_{0\varepsilon' + i_{\varepsilon\varepsilon'}(\theta - \varepsilon)}$ by Section 2.2. Proposition 1 and $\theta = \varepsilon' + i_{\varepsilon\varepsilon'}(\theta - \varepsilon)$ since $\varepsilon \leqslant \varepsilon' < \lambda$. We put $z = Iy$, and get by Lemma 1:

$$\chi_{e_\lambda(z)} = I\chi_{e_\lambda(y)} \subseteq \text{im } i_{\mu\mu+1} \tag{$*$}$$

and

$$\chi_{e_\theta}(z) = I\chi_{e_\theta(y)}, \text{ so } \mu \in e_\theta(z) \tag{$**$}$$

since $\lambda$ is simple, $\lambda - \mu \in \text{im } i_{0\mu}$, and moreover $\theta = \theta - \mu = i_{0\mu}\theta$ since $\mu < \lambda$: we may then apply Lemma 4. The assertion $(*)$ proves that $i_{\mu\mu+1}z = i_{\mu+1\mu+2}z$ and $(**)$ that $i_{\mu\mu+1}z \neq i_{\mu+1\mu+2}z$, a contradiction, so we get that every $\mu$ in $e_\theta(y) \cap \lambda$ is in $e_\lambda(y)$.

**Proposition 8.** *Assume* $\alpha \leqslant \beta$ *and* $\beta - \alpha \in \text{im } i_{0\alpha}$. *Then if* $x \in N_\beta$, *we have* $e_\beta(x) \cap \alpha \subseteq e_\alpha(x)$.

**Proof.** Assume $\beta - \alpha = i_{0\alpha}\delta$, and choose $\lambda$ simple $> \beta$. We put $y = i_{\alpha\lambda}x$, and $\theta = \lambda + i_{\alpha\lambda}(\beta - \alpha)$. First notice that $i_{\alpha\lambda}(\beta - \alpha) = i_{\alpha\lambda} i_{0\alpha}\delta = i_{0\lambda}\delta$. Now we calculate: $i_{\alpha\lambda} i_{0\beta} = i_{\alpha\lambda} i_{\alpha\beta} i_{0\alpha} = i_{\lambda\lambda + i_{\alpha\lambda}(\beta-\alpha)} i_{\alpha\lambda} i_{0\alpha} = i_{0\theta}$. So by Lemma 1, we get: $e_\alpha(x) = e_\lambda(y)$ and $\chi_{e_\theta(y)} = i_{\alpha\lambda}\chi_{e_\beta(x)}$, hence $e_\beta(x) \cap \alpha = e_\theta(y) \cap \alpha$. Now let $\gamma < \lambda$: we have $\gamma + \theta = \theta$

since $\theta > \lambda$ and $\lambda$ is simple, and then $i_{0\gamma}\theta = i_{0\gamma}(\lambda + i_{0\lambda}\delta) = i_{0\gamma}\lambda + i_{0\gamma}i_{0\lambda}\delta = \lambda + i_{0\lambda}\delta = \theta$ since $\lambda$ is simple. Therefore, $\lambda$, $\theta$ satisfy the conditions of Lemma 7, which shows that: $e_\beta(x) \cap \alpha = e_\theta(y) \cap \alpha \subseteq e_\lambda(y) \cap \alpha = e_\alpha(x)$.

**Corollary 9.** (i) *If $\alpha \leqslant \beta$ and $\beta - \alpha < \kappa$, then, if $x \in N_\beta$, we have $e_\alpha(x) = e_\beta(x) \cap \alpha$.*

(ii) *If $\alpha \leqslant \beta$ and $\beta - \alpha \in \operatorname{im} i_{0\alpha}$, then, if $x \in N_\beta$ and $e_\beta(x) \subseteq \alpha$, we have $e_\alpha(x) = e_\beta(x)$.*

This results from Section 8.2. Proposition 1. and Section 8.3. Proposition 8.

**Remark.** These results are in some sense the best possible: for let $x = \chi_\kappa = i_{0\kappa}\chi_0$. Clearly $e_\kappa(x) = \emptyset$, but for no $\alpha < \kappa$ we have $e_\alpha(x) = \emptyset$ because if it would be the case we should have $i_{01}x = x$, which is false since $i_{01}\chi_\kappa = \chi_{\kappa_1}$. On the other hand we can easily check that for $\alpha > \kappa$ $e_\alpha(x) = \{\kappa\}$.

Let us finally mention this last result:

**Proposition 10.** *Assume $\lambda$ simple and $x \in N_\lambda$. Then the followings are equivalent:*

(i) *for all $\alpha < \lambda$ $i_{0\alpha}x = x$;*

(ii) *there exists $y$ such that $x = i_{0\lambda}y$;.*

(iii) $i_{0\lambda}x = i_{\lambda\varepsilon(\lambda)}x$.

**Proof.** By Section 2.2. Proposition 1 (ii) implies (i) and (iii). Conversely, assume $x \in N_\lambda$ and (i) holds: for all $\alpha < \lambda$ $e_\alpha(x) = \emptyset$. By 8.3. we have $e_\lambda(x) \cap \alpha \subseteq e_\alpha(x)$ hence $e_\lambda(x) \cap \alpha = \emptyset$ for all $\alpha$, so $e_\lambda(x) = \emptyset$, which is (ii). Now assume $x \in N_\lambda$ and (ii) holds: by 8.3 Lemma 1 $e_{\varepsilon(\lambda)}(i_{\lambda\varepsilon(\lambda)}x) = e_\lambda(x)$. But also $\gamma_{e_{\varepsilon(\lambda)}}(i_{0\lambda}x) = i_{0\lambda}\chi_{e_\lambda(x)}$, hence $e_{\varepsilon(\lambda)}(i_{0\lambda}x) \cap \lambda = \emptyset$. Then (iii) implies that $\emptyset = e_{\varepsilon(\lambda)}(i_{0\lambda}x) \cap \lambda = e_{\varepsilon(\lambda)}(i_{\lambda\varepsilon(\lambda)}x) = e_\lambda(x)$.

### 8.4. *Partial supports*

Assume $\alpha \leqslant \beta$ and $x \in N_\beta$; then $N_\alpha$ satisfies: "$x$ belongs to the $(\beta - \alpha)$-th ultrapower of the universe", so in $N_\alpha x$ has a $(\beta - \alpha)$-support. It is determined by the following:

**Proposition 1.** *Assume $\alpha \leqslant \beta$ and $x \in N_\beta$; let $e = e_\beta(x) \cap [\alpha, \beta)$. Then $N_\alpha$ satisfies: "$e - \alpha$ is the $(\beta - \alpha)$-support of $x$".*

**Proof.** First write $x = i_{0\beta}f(\chi_{e_\beta(x)\cap\alpha}{}^\frown\chi_e) = i_{\alpha\beta}[i_{0\alpha}g(\chi_{e_\beta(x)\cap\alpha})](\chi_e)$ where $g(\xi)(\eta) = f(\xi^\frown\eta)$. Then $i_{0\alpha}g(\chi_{e_\beta(x)\cap\alpha})$ is a function in $N_\alpha$, and this writing shows that $e - \alpha$ includes the $(\beta - \alpha)$-support of $x$ in $N_\alpha$ since $\chi_e$ is $\chi_{e-\alpha}$ calculated in $N_\alpha$.

Conversely, if $N_\alpha \models x = i_{0\beta-\alpha}F(\chi_{e'})$, then in $N_0$ we have $x = i_{\alpha\beta}F(\chi_{\alpha+e'})$ with $F$ in $N_\alpha$. Write $F = i_{0\alpha}G(\chi_{e''})$: we get

$$
\begin{aligned}
x &= i_{\alpha\beta}[i_{0\alpha}G(\chi_{e''})](\chi_{\alpha+e'}) \\
&= i_{0\beta}G(\chi_{e''})(\chi_{\alpha+e'}),
\end{aligned}
$$

and this shows that $\alpha = e' \supseteq e_\beta(x) \cap [\alpha, \beta)$: this finishes the proof.

## 9. The passage from $N_\lambda$ to $M_\lambda$, $\lambda$ of cofinality $>\omega$

We show here that if $\lambda$ is of cofinality $>\omega$ then $M_\lambda$ is exactly $N_\lambda$.

9.1.

**Lemma 1.** *Assume $\lambda$ limit and $X \in M_\lambda$ such that*
(ii) $X \subseteq N_\lambda$;
(ii) *the sequence $(i_{\alpha\lambda}X)_{\alpha<\lambda}$ is eventually constant. Then $X \in N_\lambda$.*

**Proof.** By 2.3. Proposition 2. there exists $\varepsilon(\lambda)>\lambda$ such that, for all $x$ in $N_\lambda$, the sequence $(i_{\alpha\lambda}x)_{\alpha<\lambda}$ is eventually constant and its limit is $i_{\lambda\varepsilon(\lambda)}x$. Put $X'=\{x \in N_\lambda : i_{\lambda\varepsilon(\lambda)}x \in \lim_{\alpha<\lambda} i_{\alpha\lambda}X\}$. Any set-restriction of $i_{\lambda\varepsilon(\lambda)}$ is in $N_\lambda$, and $\lim_{\alpha<\lambda} i_{\alpha\lambda}X$ is in $N_\lambda$ (since by hypothesis it is some $i_{\alpha_0\lambda}X$), so $X'$ belongs to $N_\lambda$. We claim that $X'=X$. For let $\alpha.$ such that for $\alpha \geq \alpha.$ $i_{\alpha\lambda}X = i_{\alpha.\lambda}X$, and let $x \in N_\lambda$. There is $\alpha \geq \alpha.$ such that $i_{\alpha\lambda}x = i_{\lambda\varepsilon(\lambda)}x$: so $i_{\lambda\varepsilon(\lambda)}x \in i_{\alpha.\lambda}X$ is equivalent to $i_{\alpha\lambda}x \in i_{\alpha\lambda}X$, therefore to $x \in X$.

Now to prove that for cf $\lambda>\omega$ $M_\lambda$ is equal to $N_\lambda$, it suffices to show that for any $X$ in $M_\lambda$ the sequence $(i_{\alpha\lambda}X)_{\alpha<\lambda}$ is eventually constant in that case. Then we can argue by induction on the rank or simply use the well known result which ensures that two models of ZF are equal as soon as they have the same sets of ordinals and at least one of them satisfies AC.

9.2.

**Lemma 1.** *Assume $\lambda$ simple and* cf $\lambda>\omega$. *Then there is a sequence $(\alpha_\mu)_{\mu<\mathrm{cf}\lambda}$ which is*
(i) *strictly increasing, continuous and cofinal in $\lambda$,*
(ii) *such that, if* cf $\mu \neq \kappa$, *then $\alpha_\mu$ is simple. Such a sequence we shall call a* good *sequence below $\lambda$.*

**Proof.** Let $(\gamma_\mu)_{\mu<\nu}$ any strictly increasing cofinal sequence under $\lambda$. We put $\alpha_0=\gamma_0$ and if $\mu$ is limit $\alpha_\mu = \sup_{\theta<\mu}\alpha_\theta$; if $\mu=\theta+1$, then $\alpha_\mu$ is the least simple ordinal greater than $\alpha_\theta$ and $\gamma_\mu$. By 2.4. Proposition 3. the existence of such an ordinal is ensured. Then $\alpha_\mu$ is simple for any successor $\mu$, so it is simple for all $\mu$ of cofinality $\neq\kappa$ by the closure properties of simple ordinals.

**Lemma 2.** *Assume $\lambda$ simple,* cf $\lambda>\omega$ *and $(\alpha_\mu)_{\mu<\nu}$ is a good sequence below $\lambda$. Then if $X$ is in $M_\lambda$, there is a* finite *subset $E$ of $\lambda$ such that*

$$\forall \mu<\nu\ e_{\alpha_\mu}(X) \subseteq E.$$

**Proof.** Set, for $\mu \leq \nu$, $E_\mu = \bigcup_{\theta<\mu} e_{\alpha_\theta}(X)$. We show by induction on $\mu$, that $E_\mu$ is the finite, and with $E_\nu$ we shall be done. If $\mu=\theta+1$, then $E_\mu \subseteq E_\theta \cup e_{\alpha_\mu}(X)$ is finite if $E_\theta$ is. If $\mu$ is limit and cf $\mu=\omega$, choose a sequence $(\theta_n)_n$ increasing and cofinal below $\mu$. Then $\alpha_\mu = \sup_n \alpha_{\theta_n}$ since $(\alpha_\mu)_{\mu<\nu}$ is good. Now $e_{\alpha_\mu}(X)$ is finite, so there is certainly $n$ such that $e_{\alpha_\mu}(X) \subseteq \alpha_{\theta_n}$ and so for each $m \geqslant n$ $e_{\alpha_\mu}(X) \subseteq \alpha_{\theta_m}$. Since $\alpha_\mu$ is simple, by 8.2. Corollary 2. we have for $m \geqslant n$ $e_{\alpha_{\theta_m}}(X) \subseteq e_{\alpha_\mu}(X)$, and so $E_\mu \subseteq E_{\theta_n} \cup e_{\alpha_\mu}(X)$ is finite if $E_{\theta_n}$ is.

If $\mu$ is limit and cf $\mu > \omega$, look at the function $\mu \to \omega$ defined by $\theta \mapsto |E_\theta|$: this function is increasing by construction, and takes its values in $\omega$ by induction hypothesis. It is then eventually constant since cf $\mu > \omega$. So is therefore the sequence $(E_\theta)_{\theta<\mu}$, and $E_\mu = \bigcup_{\theta<\mu} E_\theta$ is finite as the $E_\theta$'s are.

**Lemma 3.** *Assume $\lambda$ simple and* cf $\lambda > \omega$; *then if $X$ is in $M_\lambda$, the sequence $(i_{\alpha\lambda}X)_{\alpha<\lambda}$ is eventually constant.*

**Proof.** Choose a good sequence $(\alpha_\mu)_{\mu<\nu}$ below $\lambda$. There is $\alpha < \lambda$ and $E$ finite subset of $\alpha$ such that for all $\mu < \nu$ $e_{\alpha_\mu}(X) \subseteq E$. Now assume that $\theta < \nu$ and $\alpha_0 \geqslant \alpha$: we show that $i_{\alpha\alpha_\theta}X = X$. For since $e_{\alpha_{\theta+1}}(X) \subseteq E$ there is g in $N_0$ such that $X = i_{0\alpha_{\theta+1}}g(\chi_E)$, and $i_{\alpha\alpha_\theta}X = i_{\alpha\alpha_\theta} i_{0\alpha_{\theta+1}} g(i_{\alpha\alpha_\theta}\chi_E)$. Now $\alpha_{\theta+1}$ is simple and $\alpha_\theta < \alpha_{\theta+1}$, so $i_{\alpha\alpha_\theta} i_{0\alpha_{\theta+1}} = i_{0\alpha_{\theta+1}}$, and $E \subseteq \alpha$, so $i_{\alpha\alpha_\theta}\chi_E = \chi_E$, and we get that $i_{\alpha\alpha_\theta}X = X$.

Fix $\beta$ arbitrary such that $\alpha \leqslant \beta < \lambda$: there is a successor $\theta$ such that $\beta \leqslant \alpha_\theta < \lambda$, and so $X = i_{\alpha\alpha_\theta}X = i_{\alpha\beta} i_{\alpha\alpha_\theta} X = i_{\alpha\beta}X$ since $\alpha_\theta$ is simple. Consequently, $i_{\alpha\lambda}X = i_{\beta\lambda}$ $i_{\alpha\beta}X = i_{\beta\lambda}X$, and we are done.

**Theorem 4.** *If there exists $\alpha < \lambda$ such that $N_\alpha \vDash$* cf $\lambda > \omega$, *then $M_\lambda = N_\lambda$.*

**Proof.** Assume first $\lambda$ simple and cf $\lambda > \omega$: by 9.1. Lemma 1. and by Lemma 3 any $X$ of $M_\lambda$ which is included in $N_\lambda$ belongs to $N_\lambda$, and therefore $M_\lambda = N_\lambda$. Now for any $\lambda$ if $N_\alpha \vDash$ cf $\lambda > \omega$, then $N_\beta \vDash$ cf $\lambda > \omega$ for $\beta \geqslant \alpha$ since $N_\beta \subseteq N_\alpha$. By 2.4. Proposition 4. there is a $\beta \geqslant \alpha$ such that $\lambda - \beta$ is simple in $N_\beta$, and $N_\beta \vDash \mathrm{cf}(\lambda - \beta) =$ cf $\lambda > \omega$. Therefore $N_\beta \vDash M_{\lambda-\beta} = N_{\lambda-\beta}$, that is, in $N_0$, $M_\lambda = N_\lambda$.

**Remark.** This proves that for cf $\lambda > \omega M_\lambda = N_\lambda$. But for instance we have cf $\kappa_\omega = \omega$ and nevertheless $M_{\kappa_\omega} = N_{\kappa_\omega}$ for in $N_\omega$ $\kappa_\omega$ is regular. Notice that the hypothesis of Theorem 4 are conserve to those of 5.3. Theorem 4, so we have already proved that for any $\lambda$ limit exactly one among the followings can happen:

(i) $M_\lambda = N_\lambda$;
(ii) $\lambda = \rho + \omega$, and $M_\lambda = N_\lambda[(\chi_{\rho+n})_n]$, a $\mathscr{C}_\lambda$ generic extension:
(iii) $M_\lambda \neq \neg$AC.

## 9.3. *Ultrafilters on ordinals* $> \kappa$

Let $\kappa'$ be a cardinal bigger than $\kappa$: we study in this section the connection between the sets which are $\kappa'$-complete ultrafilters and those which are such in $N_{\kappa'}$.

**Lemma 1.** *Assume that $\mathscr{V}$ is a $\kappa^+$-complete ultrafilter on a simple ordinal $\theta$, and $\{\eta < \theta\colon i_{0\alpha}\eta = \eta\}$ belongs to $\mathscr{V}$, where $\alpha < \theta$. Then $\mathscr{V} \cap N_\alpha = i_{0\alpha}\mathscr{V}$.*

**Proof.** (Kunen). Assume $X \in i_{0\alpha}\mathcal{V}$: $X$ is $\pi_\alpha e * f$ where $f$: $\kappa^{|e|} \to \mathcal{V}$. Since $\mathcal{V}$ is $\kappa^+$-complete, $Y = \bigcap_{\xi \in \kappa} |e|\ f(\xi)$ belongs to $\mathcal{V}$, and $i_{0\alpha} Y \subseteq X$. Now the set $\{\eta \in \theta$: $\eta = i_{0\alpha}\eta$ and $n \in Y\}$ belongs to $\mathcal{V}$ also, and is included in $i_{0\alpha}Y$, hence in $X$. Therefore $X$ belongs to $\mathcal{V}$.

**Lemma 2.** *Assume that $\mathcal{V}$ is a $\kappa'$-complete ultrafilter on a simple ordinal $\theta$, and $\{\eta < \theta : i_{0\alpha}\eta = \eta\}$ belongs to $\mathcal{V}$, where $\kappa'$ is a cardinal and $\alpha < \kappa'$. Then, for every function $f : \theta \to \theta$, there exists in $N_\alpha$ a function $g : \theta \to \theta$ such that $\{\eta < \theta : f(\eta) = g(\eta)\}$ is in $\mathcal{V}$.*

**Proof.** For all $\eta < \theta$ $f(\eta)$ belongs to $N_\alpha$, so it can be written as $f(\eta) = \pi_\alpha e_\eta * f_\eta$ where $e_\eta$ is a finite subset of $\alpha$. Since $|[\alpha]^{<\omega}| = |\alpha| < \kappa'$ and $\mathcal{V}$ is $\kappa'$-complete, there exist $Z$ in $\mathcal{V}$, which we may assume included in $\{\eta < \theta$: $i_{0\alpha}\eta = \eta\}$, and $e$ a finite subset of $\alpha$ such that for all $\eta$ in $Z$ $f(\eta) = \pi_\alpha e * f_\eta$, where $f_\eta$: $\kappa^{|e|} \to \theta$. We now put for $\xi$ in $\kappa^{|e|}$. $F(\xi) = \{(\eta, f_\eta(\xi)) : \eta \in Z\} \cup \{(\eta, 0) : \eta \notin Z\}$, and $g = \pi_\alpha e * F$. Clearly, $g$ is a function from $i_{0\alpha}\theta = \theta$ into itself, and for $\eta$ in $Z$ $g(\eta)$ is equal to $f(\eta)$.

**Proposition 3.** *Assume that $\kappa'$ is a regular cardinal $> \kappa$, and $\mathcal{V}$ is a $\kappa'$-complete ultrafilter on a simple ordinal $\theta$. Then $\mathcal{V} \cap N_{\kappa'}$, and therefore is in $N_{\kappa'}$ a $\kappa'$-complete ultrafilter on $\theta$.*

**Proof.** We have only to show that for all $\alpha < \kappa'$ $\mathcal{V} \cap N_\alpha$ belongs to $N_\alpha$, for, if $\mathcal{V} \cap N_\alpha \in N_\alpha$, we have also $\mathcal{V} \cap N_{\kappa'} = (\mathcal{V} \cap N_\alpha) \cap N_{\kappa'}^{(N_\alpha)} \in N_\alpha$, hence $\mathcal{V} \cap N_{\kappa'}$ belongs to $M_{\kappa'}$ , which is $N_{\kappa'}$ by 9.2. Theorem 4. Now fix $\alpha < \kappa'$, and let $Z$ be $\{\eta < \theta$: $i_{0\alpha}\eta = \eta\}$. Since $\theta$ is simple, $Z$ has the same cardinality as $\theta$. Choose $X$ in $\mathcal{V}$ such that $|\theta| = |X| = |\theta \backslash X|$, and a bijection $f$: $\theta \to \theta$ such that $f''X = Z$. Then $f * \mathcal{V}$ is a $\kappa'$-complete ultrafilter on $\theta$, and $Z$ belongs to $f * \mathcal{V}$. Hence by Lemma 1 $(f * \mathcal{V}) \cap N_\alpha$ belongs to $N_\alpha$. Moreover by Lemma 2 there exists in $N_\alpha$ a function $g$ such that $\{\eta < \theta$: $g(\eta) = f^{-1}(\eta)\}$ belongs to $f * \mathcal{V}$. By 7.1. Lemma 1 we get: $g * f(* \mathcal{V}) = f^{-1} * (f * \mathcal{V}) = f^{-1} f * \mathcal{V} = \mathcal{V}$, and since $g$ belongs to $N_\alpha$

$$g * ((f * \mathcal{V}) \cap N_\alpha) = (g * (f * \mathcal{V})) \cap N_\alpha = \mathcal{V} \cap N_\alpha \text{ belongs to } N_\alpha \text{ as requested.}$$

**Proposition 4.** *Assume that $\mathcal{F}$ is a $\kappa$-complete filter on $\kappa$ and $\kappa'$ is a regular cardinal $\geqslant \kappa'''$.*

(i) *$i_{0\kappa'}\mathcal{F}$ is a basis of a $\kappa^+$-complete filter on $\kappa$.*

(ii) *$i_{0\kappa'}\mathcal{F}$ is a basis of a $(2^\kappa)^+$-complete filter just in case that there exists $n$ and $f$: $\kappa^n \to \kappa$ such that $\mathcal{F} \subseteq f * \mathcal{U}_n$. In that case $i_{0\kappa'}\mathcal{F}$ is a basis of a $\kappa'$-complete ultrafilter on $\kappa'$.*

**Proof.** (i) Assume that $(Y_\alpha)_{\alpha<\kappa}$ is a sequence of sets in $i_{0\kappa'}\mathcal{F}$. Write $Y_\alpha = \pi_{\kappa'} e_\alpha * f_\alpha$ where $e_\alpha$ is a finite subset of $\kappa'$ and $f_\alpha$: $\kappa^{|e_\alpha|} \to \mathcal{F}$. Since cf $\kappa' > \kappa$, there is

$\gamma<\kappa'$ such that for all $\alpha<\kappa$ $e_\alpha\subseteq\gamma$. We set for $\xi$ in $\kappa$:

$$F(\xi)=\bigcap_{\substack{\alpha<\xi\\ s<\xi}} f_\alpha(s)$$

for $s$ of convenient length. For all $\xi$ in $\kappa$ $F(\xi)$ is in $\mathcal{F}$. Let $Y$ be $\pi_{\kappa'}\{y\}*F:Y$ belongs to $i_{0\kappa'}\mathcal{F}$. But let $\alpha$ be fixed: $\{s^\frown\xi:F(\xi)\subseteq f_\alpha(s)\}\supseteq\{s^\frown\xi;\ \alpha<\xi$ and $s<\xi\}$, so this set belongs to $\mathcal{U}_{|e_\alpha|+1}$, and since $e_\alpha\subseteq\gamma$ $\{t\in\kappa^{\kappa'}:F(t(\gamma))\subseteq f_\alpha(se_\alpha)\}\in\mathcal{U}_{\kappa'}$, i.e. $\pi_{\kappa'}\{\gamma\}*F\subseteq\pi_{\kappa'}e_\alpha*f_\alpha$, or $Y\subseteq Y_\alpha$. This proves that $\bigcap_{\alpha<\kappa}Y_\alpha$ is in the filter generated in $N_0$ by $i_{0\kappa'}\mathcal{F}$, and so $i_{0\kappa'}\mathcal{F}$ is the basis of a $\kappa^+$-complete filter on $\kappa'$.

(ii) Let $(Y_\alpha)_{\alpha<2^\kappa}$ be an enumeration of $\mathcal{F}:\bigcap_{\alpha<2^\kappa}Y_\alpha$ is empty (since $\mathcal{F}$ is of course assumed non principal). Assume that $\bigcap_{\alpha<2^\kappa}i_{0\kappa'}Y_\alpha$ is not empty: there exists thus $e\subseteq\kappa'$ and $f:\kappa^{|e|}\to\kappa$ such that $\pi_{\kappa'}e*f\in i_{0\kappa'}Y_\alpha$ for all $\alpha<2^\kappa$: but this means that $\{\xi$: $f(\xi)\in Y_\alpha\}$ belongs to $\mathcal{U}_{|e|}$, and so that $Y_\alpha$ belongs to $f*\mathcal{U}_{|e|}$. Finally we get $\mathcal{F}\subseteq f*\mathcal{U}_{|e|}$.

Conversely, let us prove that $i_{0\kappa'}\mathcal{U}_n$ is the basis of a $\kappa'$-complete filter on $N_0$: assume that $\gamma<\kappa'$ and for $\alpha<\gamma$ $Y_\alpha$ is in $i_{0\kappa'}\mathcal{U}_n$. We have for each $\alpha$: $Y_\alpha=i_{\gamma_\alpha\kappa'}X_\alpha$ with $\gamma_\alpha<\kappa'$ and $X_\alpha$ in $i_{0\gamma_\alpha}\mathcal{U}_n$ hence for all $\beta$ between $\gamma_\alpha$ and $\kappa'$ $(\chi_\beta,\ldots,\chi_{\beta+n-1})$ belongs to $Y_\alpha$. Now since $\gamma<\operatorname{cf}\kappa'=\kappa'$, there exists $\beta<\kappa'$ such that for all $\alpha<\gamma$ $\gamma_\alpha<\beta$, and so $(\chi_\beta,\ldots,\chi_{\beta+n-1})$ belongs to all $Y_\alpha$, $\alpha<\gamma$, and $\bigcap_{\alpha<\gamma}Y_\alpha$ is not empty. Finally notice that $i_{0\kappa'}(f*\mathcal{U}_n)$ is $i_{0\kappa'}f*i_{0\kappa'}\mathcal{U}_n$, so for any $n$ and $f$, $i_{0\kappa'}(f*\mathcal{U}_n)$ is the basis of a $\kappa'$-complete filter.

## 10. The passage from $N_\lambda$ to $M_\lambda$ in cofinality $\omega$.

When one tries to extend to any $\lambda$ of cofinality $\omega$ the result of Chapter 3 two difficulties arise:

(1) to define for any element of $M_\lambda$ a kind of infinite support in $\mathcal{G}_\lambda$ to play the role of $(\chi_n)_n$;

(2) to prove the equality of two models without AC.

We assume until Theorem 8 that $\lambda$ is simple of cofinality $\omega$, and that $(\alpha_n)_n$ is a fixed good sequence below $\lambda$.

### 10.1. *The support of an element of $M_\lambda$*

**Definition.** For $X$ in $M_\lambda$ we set: $E_\lambda(X)=\bigcup_n e_{\alpha_n}(X)$.

We need the machinery of Chapter 8 to show that $E_\lambda(X)$ is of order type $\omega$ and belongs to $M_\lambda$.

**Lemma 1.** *For all $X$ in $M_\lambda$, $E_\lambda(X)$ is finite or it belongs to* cof $\lambda$.

**Proof.** It is clear from the definition of a good sequence in Section 2.4, that if $m<n$ then $\alpha_n-\alpha_m\in\operatorname{im} i_{0\alpha_m}$, so applying 8.3. Proposition 8 we get for any $X$ in

$M_\lambda$ and $m \leq n$ $e_{\alpha_n}(X) \cap_{\alpha_m} \subseteq e_{\alpha_m}(X)$, and therefore $E_\lambda(X) \cap \alpha_m = \bigcup_{k \leq m} e_{\alpha_k}(X)$. So $E_\lambda(X)$ is finite, or it is of order type $\omega$ and cofinal in $\lambda$. Moreover $E_\lambda(X)$ is $\bigcup_{k \leq m} e_{\alpha_k}(X) \cup \bigcup_{n>m} (e_{\alpha_n}(X) \cap [\alpha_m, \lambda))$. By 8.4. Proposition 1. $e_{\alpha_n}(X) \cap [\alpha_m, \lambda]$ is the support of $X$ when viewed as an element of the $(\alpha_n - \alpha_m)$-th ultrapower of $N_{\alpha_m}$. Since $(\alpha_n)_n$ is a good sequence, $(\alpha_n - \alpha_m)_{n>m}$ belongs to $N_{\alpha_m}$, as well as $\bigcup_{n>m}(e_{\alpha_n}(X) \cap [\alpha_m, \lambda))$, and so $E_\lambda(X)$ belongs to $N_{\alpha_m}$, and finally to $M_\lambda$.

**Lemma 2.** *If $X$ is in $M_\lambda$, then:*

(*i*) *if $X$ is included in $N_\lambda$, then $X$ belongs to $N_\lambda[\chi_{E_\lambda(X)}]$;*

(ii) *in any case, the sequence $(i_{\alpha_n\lambda}X)_n$ belongs to $N_\lambda[\chi_{E_\lambda(X)}]$ (and hence to $N_\lambda[\mathscr{G}_\lambda]$).*

**Proof.** There exists in $N_0$ a sequence $(g_n)_n$ of functions such that for every $n$ $X = i_{0\alpha_n} g_n(\chi_{E_\lambda(X)\cap\alpha_n})$ since $e_{\alpha_n}$ is included in $E_\lambda(X) \cap \alpha_n$. Hence $i_{\alpha_n\lambda}X = i_{0\lambda}g_n(\chi_{E_\lambda(X)\cap\alpha_n})$. The sequence $(i_{0\lambda}g_n)_n$ is $i_{0\lambda}(g_n)_n$, so it belongs to $N_\lambda$. Now we notice that the sequence $(\alpha_n)_n$ does not belong necessarily to $N_\lambda$, but if the arguments of $i_{0\lambda}g_n$ are $p_n$-tuples of ordinals, $\chi_{E_\lambda(X)\cap\alpha_n}$ is the set of the $p_n$ first members of $\chi_{E_\lambda(X)}$. It is then clear that $(i_{\alpha_n\lambda}X)_n$ is constructed from $N_\lambda$ and $\chi_{E_\lambda(X)}$. Now recall 2.3. Proposition 2: there exists $\varepsilon(\lambda)$ such that for all $x$ in $N_\lambda$ the sequence $(i_{\alpha_n\lambda}x)_n$ is eventually constant with limit $i_{\lambda\varepsilon(\lambda)}x$. So as for 9.1. Lemma 1. we have if $X$ is included in $N_\lambda$:

$$X = \{x \in N_\lambda : \exists m \forall n \geq m\ i_{\lambda\varepsilon(\lambda)}x \in i_{\alpha_n\lambda}X\},$$

and get that $X$ is in $N_\lambda[\chi_{E_\lambda(X)}]$ since $i_{\lambda\varepsilon(\lambda)}$ and $(i_{\alpha_n\lambda}X)_n$ are there.

Notice that we proved that $M_\lambda$ and $N_\lambda[\mathscr{G}_\lambda]$ have the same sets of ordinals, but of course we cannot conclude anything except for the case $\lambda = \omega$.

10.2. *The map $i_\lambda$*:

**Definition.** (i) For any sequence of ordinals $(\sigma_n)_n$ we denote by $\overline{\lim}_n \sigma_n$ the least $\sigma$ such that

$$\exists p \forall q \geq p\ \sigma_q \leq \sigma;$$

(ii) For any sequence $(x_n)_n$ we denote by $(x_n)_{(n)}$ and $(x_n)_{((n))}$ the *sets* of sequences which are eventually equal to $(x_n)_n$ and take their values in $V_\sigma \cap N_\lambda$ and $V_\sigma \cap N_{\varepsilon(\lambda)}$ respectively, where $\sigma$ is $\overline{\lim}_n\, rkx_n$ and $\varepsilon(\lambda)$ is as provided by 2.3. Proposition 2, i.e. when $\lambda$ is simple $\varepsilon(\lambda) = \lambda + i_{\alpha_n\lambda}\lambda$ for any $n$.

Notice that if $(x_n)_n$ takes its values in $N_\lambda$, then $(x_n)_{(n)} = (x'_n)_{(n)}$ iff $(x_n)_n \equiv (x'_n)_n$.

**Definition.** For $X$ in $M_\lambda$, we set $i_\lambda X = (i_{\alpha_n\lambda}X)_{(n)}$.

**Lemma 1.** (i) The range of $i_\lambda$ is included in $N_\lambda[\mathscr{G}_\lambda]$;

(ii) For all $A$ in $M_\lambda$, $i_\lambda \restriction A$ belongs to $M_\lambda$;

(iii) The map $i_\lambda$ is injective.

**Proof.** (i) is clear from 10.1. Lemma 2. Now fix $A$ in $M_\lambda$. The map with domain $A: x \mapsto (i_{\alpha_n\lambda}x)_{n\geqslant m}$ is in $N_{\alpha_m}$, since it is in this model the map $x \mapsto (i_{\alpha_n-\alpha_m\lambda}x)_{n\geqslant m}$ and $(\alpha_n)_n$ is good, hence belongs to $M_\lambda$. Then notice that the rank of an element of $N_\lambda$ is the same in $N_0$, in $N_{\alpha_m}$ and in $N_\lambda$. So the map $x \mapsto (i_{\alpha_n\lambda}x)_{(n)}$ is in $N_{\alpha_m}$, and (ii) is proved. Finally $i_\lambda x = i_\lambda x'$ iff $(i_{\alpha_n\lambda}x)_{(n)} = (i_{\alpha_n\lambda}x')_{(n)}$ iff $\exists p \forall q \geqslant p\ i_{\alpha_q\lambda}x = i_{\alpha_q\lambda}x'$ iff $x = x'$, and (iii) is proved.

We now give another elementarity property of $i_\lambda$ in the form that will be needed for the end of the proof.

**Lemma 2.** *For any* $x$, $y$, $z$ *define* $(x, y)(\varepsilon)z$ *by*:

$$\forall (x_n)_n \in x \forall (y_n)_n \in y \forall (z_n)_n \in z \exists p \forall q \geqslant p (x_n, y_n) \in z_n.$$

*Then for any* $X$, $Y$, $Z$ *in* $M_\lambda$ *the following are equivalent*:

(*i*) $(X, Y) \in Z$;
(ii) $(i_\lambda X, i_\lambda Y)(\varepsilon) i_\lambda Z$.

**Proof.** $(X, Y) \in Z$ is equivalent for any $q$ to $(i_{\alpha_q\lambda}X, i_{\alpha_q\lambda}Y) \in i_{\alpha_q\lambda}Z$. Now let $(x_n)_n$, $(y_n)_n$, $(z_n)_n$ be in $i_\lambda X$, $i_\lambda Y$, $i_\lambda Z$ respectively: there is $p$ such that for all $q \geqslant p$ $x_q = i_{\alpha_q\lambda}X$, $y_q = i_{\alpha_q\lambda}Y$, $z_q = i_{\alpha_q\lambda}Z$, and we get the conclusion of the lemma.

The next lemma is a straightforward adaptation of a well known argument when AC holds.

**Lemma 3.** Assume that $M$ is a model of ZF, $N_\lambda[\mathcal{G}_\lambda] \subseteq M \subseteq M_\lambda$ and there is in $M$ a definable class $\mathcal{O}$ *such that* $\mathcal{O}$ includes the range of $i_\lambda$ and $\mathcal{O} \times \mathcal{O}$ has the same subsets in $M_\lambda$ and in $M$: then $M_\lambda = M$.

**Proof.** The class $\mathcal{O}$ replaces the ordinals, and the map $i_\lambda$ replaces the injection of any set into the ordinals provided by AC. Let $A$ be any transitive set of $M_\lambda$. We define in $M_\lambda$ a subset $Z$ of $\mathcal{O} \times \mathcal{O}$ by $(X, Y) \in Z$ iff $\exists X' \in A\ \exists Y' \in A\ X = i_\lambda X'$ and $Y = i_\lambda Y'$ and $X' \in Y'$ (this is possible since $i_\lambda \restriction A$ belongs to $M_\lambda$ by Lemma 1(ii)). By hypothesis $Z$ belongs to $M$. Moreover $Z$ is extensional and well-founded, so it collapses in $M$ onto the $\varepsilon$-relation on a transitive set which can be nothing but $A$, and $A$ is in $M$.

**Lemma 4.** *Assume that* $M$ *is a model of* ZF, $N_\lambda[\mathcal{G}_\lambda] \subseteq M \subseteq M_\lambda$, *and* $\mathcal{O}$ *is a definable class of* $M$ *such that the restriction of* $i_\lambda$ *to* $\mathcal{O}$ *is definable in* $M$: *then* $\mathcal{O} \times \mathcal{O}$ *has the same subsets in* $M$ *and in* $M_\lambda$.

**Proof.** Let $Z$ be any subset of $\mathcal{O} \times \mathcal{O}$ in $M$. We set $Z^* = \{(X, Y) \in \mathcal{O} \times \mathcal{O} : (i_\lambda X, i_\lambda Y)(\varepsilon) i_\lambda Z\}$. By hypothesis $Z^*$ belongs to $M$ since $i_\lambda \restriction \mathcal{O}$ is definable in $M$ and by Lemma 1 (i) $i_\lambda Z$ belongs to $N_\lambda[\mathcal{G}_\lambda]$, hence to $M$. Now by Lemma 2 $Z^*$ is exactly $Z$.

Our way is now clear: to prove that $M = M_\lambda$ we have to construct in $M$ a class $\mathcal{O}$ such that:

(i) $\mathcal{O}$ includes the range of $i_\lambda$;
(ii) $i_\lambda \restriction \mathcal{O}$ is definable in $M$.

We now construct such an $\mathcal{O}$ in $N_\lambda[i_\lambda \restriction \mathcal{G}_\lambda]$.

**Definition.** (i) $\mathcal{H}$ is the class of triples $(g, (p_n)_n, (f_n)_n)$ such that

(a) g belongs to $\mathcal{G}_\lambda$;
(b) $(p_n)_n$ is an increasing unbounded sequence of natural numbers;
(c) $|\mathrm{im}\ g \cap \chi_{\alpha_n}| = p_n$;
(d) $(f_n)_n$ belongs to $N_\lambda$ and for each $n$ $f_n$ is a function with domain $\kappa_\lambda^{p_n}$.

(ii) $\mathcal{O}$ is the class of $X$ which are of the form $X = (f_n(g \restriction p_n))_{(n)}$ for some $(g, (p_n)_n, (f_n)_n)$ in $\mathcal{H}$.

(iii) $\mathcal{T}$ is the binary relation defined by: $(X, Y) \in \mathcal{T}$ *iff*: $\exists (g, (p_n)_n, (f_n)_n) \in \mathcal{H}$ $\exists (G_t)_t \in i_\lambda g$ $X = (f_n(g \restriction p_n))_{(n)}$ and $Y = ((i_{\lambda\varepsilon(\lambda)} f_n(G_t \restriction p_n))_{((n))})_{(t)}$.

It is clear from this definition that the following holds:

**Lemma 5.** *$\mathcal{H}$ and $\mathcal{O}$ are definable classes of $N_\lambda[\mathcal{G}_\lambda]$ and $\mathcal{T}$ is a definable class of $N_\lambda[i_\lambda \restriction \mathcal{G}_\lambda]$.*

**Lemma 6.** *For any $X$ in $M_\lambda$, $i_\lambda X$ belongs to $\mathcal{O}$.*

**Proof.** We take $\chi_{E_\lambda(X)}$ for g. Then there exists a sequence $(f'_n)_n$ of functions such that for all $n$ $X = i_{0\alpha_n} f'_n(\chi_{E_\lambda(X)} \cap \chi_{\alpha_n})$, and so $i_{\alpha_n\lambda} X = i_{0\lambda} f'_n(\chi_{E_\lambda(X)} \cap \chi_{\alpha_n})$. We set $(f_n)_n = i_{0\lambda}(f'_n)_n$ and $p_n =$ the cardinality of any member of the domain of $f_n$. Clearly $(g, (p_n)_n, (f_n)_n)$ is in $\mathcal{H}$ and $i_\lambda X = (f_n(g \restriction p_n))_{(n)}$.

**Lemma 7.** *Assume that $X \in \mathcal{O}$: then $(X, Y) \in \mathcal{T}$* iff *$Y = i_\lambda X$.*

**Proof.** Let $(g, (p_n)_n, (f_n)_n)$ be any member of $\mathcal{H}$ such that $X = (f_n(g \restriction p_n))_{(n)}$, and let $(G_t)_t$ be any member of $i_\lambda g$. We show that $i_\lambda X = ((i_{\lambda\varepsilon(\lambda)} f_n(G_t \restriction p_n))_{((n))})_{(t)}$. For there is $r$ such that:

(i) there exists $(f'_n)$ in $N_{\alpha_r}$ such that $(f_n)_n = i_{\alpha\lambda}(f'_n)_n$;
(ii) for all $t \geq r$ $G_t$ is $i_{\alpha_t\lambda} g$. Now for $t \geq r$ we have:

$$i_{\alpha_t\lambda} X = i_{\alpha_t\lambda}((f_n(g \restriction p_n))_{(n)}) = (i_{\alpha_t\lambda} f_n(i_{\alpha_t\lambda} g \restriction p_n))_{((n))}$$

since the definition ensures that $i_{\alpha_t\lambda}((x_n)_{(n)}) = (i_{\alpha_t\lambda} x_n)_{((n))}$, then

$$i_{\alpha_t\lambda} X = (i_{\alpha_t\lambda} i_{\alpha_r\lambda} f'_n(G_t \restriction p_n))_{((n))} = (i_{\lambda\varepsilon(\lambda)} i_{\alpha_r\lambda} f'_n(G_t \restriction p_n))_{((n))}$$

since by Proposition 1 $i_{\alpha_t\lambda} i_{\alpha_r\lambda} = i_{\lambda\varepsilon(\lambda)} i_{\alpha_r\lambda}$, and so $i_{\alpha_t\lambda} X = (i_{\lambda\varepsilon(\lambda)} f_n(G_t \restriction p_n))_{((n))}$. Finally we get $i_\lambda X = (i_{\alpha_t\lambda} X)_{(t)} = ((i_{\lambda\varepsilon(\lambda)} f_n(G_t \restriction p_n))_{((n))})_{(t)}$, and the lemma is proved.

We thus have proved:

**Theorem 8.** *If $\lambda$ is simple and* cf $\lambda = \omega$, *then $M_\lambda$ is exactly $N_\lambda[i_\lambda \restriction \mathcal{G}_\lambda]$.*

As in the previous chapters the way for extending this result to any $\lambda$ such that $\alpha < \lambda$ $N_\alpha \models \text{cf } \lambda = \omega$ is clear from 2.4. Proposition 4. Moreover there is an obvious bijection from $\mathcal{G}_\lambda$ onto $\mathcal{G}_{\lambda-\alpha}^{(N_\alpha)}$ for any $\alpha < \lambda$. Then choose $\alpha < \lambda$ such that $\lambda - \alpha$ is simple in $N_\alpha$: we apply Theorem 8 in $N_\alpha$, and get with an easy computation:

**Theorem 9.** *If for all $\alpha < \lambda$ $N_\alpha$ satisfies*: "cf $\lambda = \omega$", *then there exists in $M_\lambda$ a definable functional class $i_\lambda$ such that $M_\lambda = N_\lambda[i_\lambda \restriction {}_\lambda]$. Moreover $rk(i_\lambda | \mathcal{G}_\lambda) = \kappa_{\varepsilon(\lambda)} + 6$, where $\varepsilon(\lambda) = \lambda + \inf_{\alpha<\lambda} i_{\alpha\lambda}\lambda$.*

10.3.

We now apply general forcing results of [7].

**Proposition 1.** *For any limit $\lambda$, $N_\lambda$ is* $(\text{HODN}_\lambda)^{M_\lambda}$ *and* $(\text{HODN}_\lambda)^{N_\lambda[\mathcal{G}_\lambda]}$.

**Proof.** Assume $\lambda$ simple. To prove the first result, it suffices to show that any subset $A$ of On which is in $(\text{ODN}_\lambda)^{M_\lambda}$ is in $N_\lambda$ (since $N_\lambda$ satisfies AC), and so, using 9.1. Lemma 1. it suffices to show that the sequence $(i_{\alpha\lambda}A)_{\alpha<\lambda}$ is eventually constant. By hypothesis there is a term $t$ of the language of set theory and elements $a_1 \cdots a_n$ of $N_\lambda$ such that $M_\lambda \models A = t(a_1 \cdots a_n)$. Since $a_1 \cdots a_n$ are in $N_\lambda$ there is $\alpha < \lambda$ such that for all, $\alpha \leqslant \beta < \lambda$, and for all $j = 1 \cdots n$, $a_j = i_{\alpha\beta}a_j$. Now the model $N_\alpha$ satisfies (since $\lambda$ is simple): "$A$ is the value of the term $t$ calculated in $M_\lambda$ at the sets $a_1 \cdots a_n$". Hence for all $\beta$, $\alpha \leqslant \beta < \lambda$, the model $N_\beta$ satisfies: "$i_{\alpha\beta}A$ is the value of the term $t$ calculated in $M_\lambda$ at the sets $i_{\alpha\beta}a_1 = a_1, \ldots, i_{\alpha\beta}a_n = a_n$", and then $M_\lambda \models i_{\alpha\beta}A = t(a_1 \cdots a_n)$, therefore $A = i_{\alpha\beta}A$. Finally $A \in N_\lambda$, and $N_\lambda = (\text{HODN}_\lambda)^{M_\lambda}$. The proof is similar for $(\text{HODN}_\lambda)^{N_\lambda[\mathcal{G}_\lambda]}$ using the fact that, when $_\lambda$ is simple $N_\lambda[\mathcal{G}_\lambda]$ is the same when calculated in any model $N_\alpha$ for $\alpha < \lambda$.

Now applying Theorem 9 of Section 10.2 and 9.3 Theorem 1 of [7] we obtain:

**Theorem 2.** *If for all $\alpha < \lambda$ $N_\alpha$ satisfies*: "cf $\lambda = \omega$", *$M_\lambda$ is a quasi generic extension of $N_\lambda$* (i.e. *$M_\lambda$ and $N_\lambda$ have a common generic extension*). *Moreover $M_\lambda$ is a generic extension of $N_\lambda[\mathcal{G}_\lambda]$ (possibly a trivial one,* i.e. *$M_\lambda$ equal to), which is itself a quasi generic extension of $N_\lambda$.*

10.4.

We finish with a special form of 10.2 Theorem 9. for the case where $\lambda$ is "little". First recall that we proved in 5.3. Lemma 2. that, if $\lambda < \kappa_\lambda$, then $\mathcal{G}_\lambda / \equiv$ is well-orderable in $M_\lambda$. Is it necessarily well-orderable in $N_\lambda[\mathcal{G}_\lambda]$? We have the following partial answer to this question.

**Theorem 1.** *If for all $\alpha < \lambda$ $N_\alpha$ satisfies "*cf $\lambda = \omega$*" and if moreover $\lambda < \kappa_\lambda$, then $M_\lambda$ is the least model including $N_\lambda[\mathscr{G}_\lambda]$ and containing a well ordering of $\mathscr{G}_\lambda/\equiv$.*

**Proof.** We may assume that $\lambda$ is simple and so $<\kappa$. If a model $M$, $N_\lambda[\mathscr{G}_\lambda] \subseteq M \subseteq M_\lambda$, bijects $\mathscr{G}_\lambda/\equiv$ onto some ordinal $\mu$, necessarily $\mu < \kappa$, and so $M$ contains every bijection from $\mathscr{G}_\lambda/\equiv$ onto a set of $N_\lambda$ which is in $N_0$, and in particular the name function $\mathscr{N}_\lambda$ in Section 5.3. Therefore $M$ contains a function: $g \mapsto [g]$ from $\mathscr{G}_\lambda$ to cof $\lambda$ such that for all $g$ in $\mathscr{G}_\lambda g \equiv \chi_{[g]}$ (recall that cof $\lambda$ belongs to $N_\lambda$ in this case). It remains to show that $i_\lambda | \mathscr{G}_\lambda$ can be calculated from this map [ ]. Now recall that for $\alpha < \lambda$ and $\mu < \lambda$ we have $i_{\alpha\lambda}\chi_\mu = \chi_\mu$ if $\mu < \alpha$, and $i_{\alpha\lambda}\chi_\mu = \chi_{\lambda + i_{\alpha\lambda}(\mu - \alpha)}$ if not, hence in this last case since $\lambda < \kappa$ by hypothesis $i_{\alpha\lambda}\chi_\mu = \chi_{\lambda + (\mu - \alpha)}$.

Let always $(\alpha_n)_n$ be the good sequence involved in the definition of $i_\lambda$, and let $g$ be any member of $\mathscr{G}_\lambda$. There exists $r$ such that:

$$\text{im } g \backslash \chi_{\alpha_r} = \text{im } \chi_{[g]} \backslash \chi_{\alpha_r} \text{ since } g \equiv \chi_{[g]}.$$

Then for all $t \geqslant r$, we have (identifying $g$ and im $g$):

$$\begin{aligned} i_{\alpha_t\lambda} g &= i_{\alpha_t\lambda}(g \cap \chi_{\alpha_t}) \cup i_{\alpha_t\lambda}(g \backslash \chi_{\alpha_t}) \\ &= i_{\alpha_t\lambda}(g \cap \chi_{\alpha_t}) \cup i_{\alpha_t\lambda}(\chi_{[g]} \backslash \chi_{\alpha_t}) \\ &= (g \cap \chi_{\alpha_t}) \cup (\chi_{[g]} \cap [\chi_{\alpha_r}, \chi_{\alpha_t})) \cup \chi_{\lambda + ([g] - \alpha_t)} \end{aligned}$$

where

$$\chi_{\lambda + ([g] - \alpha_t)} = \{\chi_{\lambda + (\mu - \alpha_t)} : \mu \in [g] \text{ and } \mu \geqslant \alpha_t\}.$$

Finally

$$i_{\alpha_t\lambda} g = (g \cap \chi_{\alpha_t}) \cup \chi_{\lambda + ([g] - \alpha_t)},$$

and

$$i_\lambda g = ((g \cap \chi_{\alpha_t}) \cup \chi_{\lambda + ([g] - \alpha_t)})_{(t)},$$

so clearly $i_\lambda \restriction \mathscr{G}_\lambda$ can be computed from $N_\lambda$, $\mathscr{G}_\lambda$, $(\alpha_n)_n$ and the map [ ]. The theorem is thus proved.

Notice finally that this proof does not extend to the case $\lambda = \kappa_\lambda$, for it is not even clear that in this case cof $\lambda$ belongs to $N_\lambda[\mathscr{G}_\lambda]$ or that $\mathscr{G}_\lambda/\equiv$ is well orderable in $M_\lambda$ (this is probably false).

## References

[1] W. Boos, Lectures on large cardinal axioms, Proceedings of Kiel Summer School, Springer Lecture Notes N° 499.

[2] L. Bukovsky, Changing cofinality of a measurable cardinal (an alternative proof) CMUC (1973) 689–697.

[3] L. Bukovsky, To appear.

[4] P. Dehornoy, Iterated ultrapowers and Magidor's forcing, to appear.

[5] F. Drake, Set Theory, (North-Holland, Amsterdam, 1974).
[6] H. Gaifman, Pushing up the measurable cardinal, Proceedings of 1967 UCLA Summer Institute.
[7] S. Grigorieff, Intermediate submodels and generic extensions, Annals of Mathematics 101 (3) 447–490.
[8] K. Kunen, Some application of iterated ultrapowers in set theory, Annals of Math. Logic 1 (1970) 179–227.
[9] K. Kunen, J. B. Paris, Boolean extensions and measurable cardinals, Annals of Math. Logic 2 (1971) 359–379.
[10] A. Mathias, On sequences generic in the sense of Prikry, Unpublished.
[11] K. Prikry, Changing measurable into accessible cardinals, Dissertationes Math. n°68 (1970).