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Discussive Logic
A Short History of the First Paraconsistent Logic

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Abstract

In this paper we present an overview, with historical and critical remarks, of two articles by S. Jaśkowski ([20, 21] 1948 and [22, 23] 1949), which contain the oldest known formulation of a paraconsistent logic. Jaśkowski has built the logic – he termed *discussive* ($\mathbf{D}_2$) – by defining two new connectives and by introducing a modal translation map from $\mathbf{D}_2$ systems into Lewis' modal logic **S5**. Discussive systems, for their formal details and their original philosophical justification, have attracted discrete attention among experts. Indeed, in what follows, after having introduced Jaśkowski's methodology of building $\mathbf{D}_2$ and his main philosophical motivations for providing such a system, we will explore some of the main contributions to the development of $\mathbf{D}_2$.

1 The Origins of Discussive Logic

Throughout this paper we will consider the following classical connectives, $\sim$ *(negation),* $\wedge$ *(conjunction),* $\vee$ *(disjunction),* $\supset$ *(material implication), plus the modal operators,* $\Box$ *(necessary) and* $\Diamond$ *(possible). All*

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additions and changes will be explicitly stated and explained.
$\Gamma, \Delta, \Sigma, \ldots$ *and* $A, B, C, \ldots$ *denote sets of formulas and formulas, respectively.* $p, q, r \ldots$ *stand for propositional variables..*

1.1 The first discussive system

S. Jaśkowski (1906-1965)[1] is the author of several important logical and mathematical studies. To cite some of them, Jaśkowski is usually acknowledged as one of the inventors of the natural deduction calculus (accomplishing this work almost at the same time of G. Gentzen) and as the proponent of the first paraconsistent logic known as «discussive» (or «discursive») logic[2]. In [21] (which corresponds to the English translation of Jaśkowski's original article [20], published in 1948), the logician proposed a logic which should capture situations where discussants are in conflict. Jaśkowski's main idea was to consider a discussant's statement, p, as inherently consistent, but potentially incoherent with some other discussant's proposition. With this in mind, Jaśkowski focused his attention on a classically valid law, namely *ex contradictione quodlibet [sequitur]* ((ECQ), «from a contradiction everything [follows]») – $p \supset (\sim p \supset q)$ – claiming that it should not be generally valid. His strategy, in order to invalidate (ECQ), has been that of getting rid of the classical connective of material implication, i.e., $\supset$, in favour of so-called «discussive implication», i.e., $\rightarrow_d$. Lewis' modal logic **S5** has played a fundamental role in the formulation of such discussive systems, so, let's recall the definition of **S5**:

Definition 1.1. **S5** is axiomatized is follows:

If A is a theorem of **PC**, then A is a theorem of **S5**.

$$\Box(A \supset B) \supset (\Box A \supset \Box B) \tag{K}$$

$$\Box A \supset A \tag{T}$$

$$\Diamond A \supset \Box\Diamond A \tag{5}$$

and the following rules:

[1] For biographical informations one can consider [27, 16, 19]. For synthetic introductions to Jaśkowski's discussive logic, see, for example, [40, 42].

[2] Jaśkowski denoted this logic by $\mathbf{D}_2$, where the label '2' indicates that we are dealing with the 'two-valued discussive sentential calculus'.

$$\frac{A \quad A \supset B}{B}\ \text{MP} \qquad\qquad \frac{A}{\Box A}\ \text{Nec}$$

Finally, we say that a modal logic $\mathbf{L}$ is of $\mathbf{S5}$-type iff $\mathbf{L} \subseteq \mathbf{S5}$[3].

Thanks to Lewis' modal system, Jaśkowski established the definition of discussive implication in the following way: $p \rightarrow_d q \stackrel{\text{def}}{=} \Diamond p \supset q$, validating thus the discussive version of *modus ponens*:

$$\frac{A \quad A \rightarrow_d B}{B}\ \text{MP}_d$$

Additionally, we can get also the definition of «discussive bi-implication», $p \leftrightarrow_d q \stackrel{\text{def}}{=} (\Diamond p \supset q) \wedge (\Diamond q \supset \Diamond p)$. Notice that, so defined, both, $\rightarrow_d$ and $\leftrightarrow_d$, are asymmetric connectives. One might wonder what the $\Diamond$ operator is meant to represent in a discussive framework. According to Jaśkowski's own perspective:

> To bring out the nature of the theses of such a system it would be proper to precede each thesis by the reservation: "in accordance with the opinion of one of the participants in the discussion" or "for a certain admissible meaning of the terms used". Hence the joining of a thesis to a discussive system has a different intuitive meaning than has assertion in an ordinary system. *Discussive assertion* includes an implicit reservation of the kind specified above, which [...] has its equivalent in $\Diamond$ [21, 43].

In a latest note, [23] (the English translation of the 1949 paper [22]), Jaśkowski proposed to substitute from the set of connectives also classical conjunction in favour of "discussive conjunction" and chose the following definition: $p \wedge_d q \stackrel{\text{def}}{=} p \wedge \Diamond q$. With this additional connective, then Jaśkowski defined again discussive bi-implication in the following manner: $p \leftrightarrow_d q \stackrel{\text{def}}{=} (p \rightarrow_d q) \wedge_d (q \rightarrow_d p)$. So, in sum, to prove discussive formulas, i.e., formulas including discussive connectives, Jaśkowski suggested to transform such formulas accordingly to their modal definitions and to prove the resulting modal formula in $\mathbf{S5}$. In more rigorous terms:

[3] As known, $\mathbf{S5}$ has several equivalent axiomatization; for instance, one can employ (4) $(\Box A \supset \Box\Box A)$ and (B) $(A \supset \Box\Diamond A)$ instead of axiom (5).

Definition 1.2. $\mathbf{D}_2$ is the system whose language $\mathscr{L}$ includes the following set of connectives $\mathsf{S} = \{\sim, \vee, \wedge_d, \rightarrow_d, \leftrightarrow_d\}$. Take a function $\tau : \mathsf{Form}_{\mathbf{D}_2} \mapsto \mathsf{Form}_{\mathbf{S5}}$ such that, for any A, $B \in \mathsf{Form}_{\mathbf{D}_2}$:

$$\begin{aligned}
&\tau(p) = p \\
&\tau(\sim A) = \sim\tau(A) \\
&\tau(A \vee B) = \tau(A) \vee \tau(B) \\
&\tau(A \wedge_d B) = \tau(A) \wedge \Diamond\tau(B) \\
&\tau(A \rightarrow_d B) = \Diamond\tau(A) \supset \tau(B) \\
&\tau(A \leftrightarrow_d B) = (\Diamond\tau(A) \supset \tau(B)) \wedge \Diamond(\Diamond\tau(B) \supset \tau(A))
\end{aligned}$$

Let $\Diamond\Gamma \stackrel{\text{def}}{=} \{\Diamond\tau(A_1), \ldots, \Diamond\tau(A_n) \mid A_1, \ldots, A_n \in \Gamma\}$, then for all $\Gamma \subseteq \mathsf{Form}_{\mathbf{D}_2}$ and $B \in \mathsf{Form}_{\mathbf{D}_2}$, we set:

$$\Gamma \models_{\mathbf{D}_2} B \text{ iff } \Diamond\Gamma \models_{\mathbf{S5}} \Diamond\tau(B).$$

In other words, a formula B is said to be a *discussive consequence* of a set of premises $\{A_1, \ldots, A_n\}$ just in case $\Diamond\tau(B)$ follows from the set $\{\Diamond\tau(A_1), \ldots, \Diamond\tau(A_n)\}$ in **S5**. Following Jaśkowski:

> [...] if a thesis A is recorded in a discussive system, its intuitive sense ought to be interpreted so as if it were preceded by the symbol $\Diamond$, that is, the sense: "it is possible that A". This is how an impartial arbiter might understand the theses of the various participants in the discussion [21, 43].

The motivation behind this quote and Definition 1.2 can be intuitively explained with the following example. If we take formulas including $\rightarrow_d$ and replace it simply accordingly to τ we will obtain a great number of **S5** invalid formulas. In this case, even the identity, $A \rightarrow_d A$, if transformed in $\Diamond A \supset A$, turns out to be **S5**-invalid. However, many of this negative results can be avoided, if we prefix $\Diamond$ to every modally translated formula. For example, $A \rightarrow_d A$, if translated as follows $\Diamond(\Diamond A \supset A)$, turns out to be **S5**-valid.

Observation 1. To see the paraconsistent character of $\mathbf{D}_2$ consider that already in [21], the discussive version of (ECQ), $A \rightarrow_d (\sim A \rightarrow_d B)$, was not included as a theorem of $\mathbf{D}_2$. To see this, consider always the modal translation of (ECQ), i.e., $\Diamond(\Diamond A \supset (\Diamond{\sim}A \supset B))$, which is not valid in **S5**. Consequently to the rejection of (ECQ), the existence of contradictory statements, $\Diamond A$ and $\Diamond{\sim}A$, is possible without that their presence entails the 'overfilling' (triviality) of the system. However, the logic is not paraconsistent with respect to conjuncted contradictions, indeed, $\Diamond(\Diamond(A \wedge {\sim}A) \supset B)$ is still a theorem of **S5** . Moreover, notice that in this framework $\wedge$ adjunction fails (i.e., $A \wedge B$ cannot be inferred from A and B) and, for this specific reason, the $\{{\sim}, \vee, \wedge, \rightarrow_d\}$-fragment of $\mathbf{D}_2$ is usually classified among the *non-adjunctive* approaches to paraconsistent logics:

> [...] discussive logic represents an ideology that is, to my mind, the most appropriate one for paraconsistency. To put it informally: at the very core of paraconsistency lies not negation, but conjunction. [...] With respect to inconsistency tolerating calculi, this connective seems to be the most important one [45, 487].

Nonetheless, in [23], thanks to the presence of discussive conjunction, adjunction can be successfully restated in the system. The discussive version of the law of non contradiction (LNC), ${\sim}(A \wedge_d {\sim}A)$, remains a valid law. To see this consider always the **S5** invalid formula $\Diamond(A \wedge \Diamond{\sim}A)$. Finally, the discussive version of conjunctive (ECQ), $(A \wedge_d {\sim}A) \rightarrow_d B$, is no longer valid, making, thus, $\mathbf{D}_2$ paraconsistent also with respect to conjuncted contradictions.

Observation 2. Jaśkowski's definition of $\wedge_d$ and $\rightarrow_d$ are not the only ones available and, indeed, experts considered different variants, such as:

$$A \wedge_d^l B \stackrel{\text{def}}{=} \Diamond A \wedge B$$

$$A \wedge_d^s B \stackrel{\text{def}}{=} \Diamond A \wedge \Diamond B$$

$$A \rightarrow_d^s B \stackrel{\text{def}}{=} \Diamond A \supset \Diamond B$$

As one can easily see, the introduction of these new connectives tries to recover the asymmetry present in Jasśkowski's original proposal. Anyway, notice that the formulas $\Diamond(A \wedge \Diamond B)$, $\Diamond(\Diamond A \wedge B)$ and $\Diamond(\Diamond A \wedge \Diamond B)$

are all equivalent in **S5**, while $\diamond(\diamond A \supset B)$ and $\diamond(\diamond A \supset \diamond B)$ are already equivalent in **S4** (a subset of **S5**). Moreover, as known since [21], $\mathbf{D}_2$ is a paraconsistent extension of the $\{\vee, \wedge, \supset\}$-fragment of classical logic. In other words, the discussive operators in $\mathbf{D}_2$ behave just like their classical counterparts. Interestingly, however, if we consider also an enriched language which includes a negation connective, the discussive logics generated by these new operators will no longer coincide with the $\{\sim, \vee, \wedge, \supset\}$-fragment of classical logic.

> It is not true thus that different translation clauses 'would have just the same consequences' [...]. Different choices of discussive conjunction and discussive implication would in fact define logics distinct from $\mathbf{D}_2$ [28, 215].

This is a struggling point. Indeed, as we will see in section 2.3, some notable problems arise in the formulation and comparison of axiomatic systems including different discussive connectives and negation.

1.2 Jaśkowski's Philosophical Motivations

In his celebrated *Metaphysics*, Aristotle claimed that «the most indisputable of all beliefs is that contradictory statements are not at the same time true» ([3, Γ, 1011b13–14]), establishing, thus, – in a crystal clear way for the first time in the history of philosophy – one of the most celebrated and debated logical, psychological and ontological laws, i.e., the so-called law of non-contradiction (LNC). Roughly, Aristotle was convinced that the principle for which two opposite propositions, usually, one the negation of the other, cannot both be true at the same time had a very special status. Indeed, (LNC) corresponds, according to the Greek philosopher, to the most certain principle, which has a triple valence: it is a law of human rationality and reasoning (logic), it is a law governing reality (ontology) and, finally, it is a law concerning human beliefs (psychology). The discussions continued and, finally, during the middle ages, the debates on contradictions reached another fundamental turning point. An unknown author, usually acknowledged under the pseudonymous of Pseudo-Scotus, defined for the first time the principle of *ex contradictione quodlibet [sequitur]* in a commentary to Aristotle's *Analytica Priora* [43]. Importantly, William of Soissons, during the XII century, proposed the first known proof of the aforementioned principle and it is documented that already during the XIV century logicians knew about its existence and accepted (ECQ) as true[4]. However, the birth and the growing interest towards formal logical systems, strictly matched to philosophical considerations and objectives, has led some philosophers and logicians to re-consider also the validity and the truth of (LNC) and (ECQ). Jaśkowski has been among them. Indeed, in the first paragraphs of his celebrated 1949 article he develops a brief survey concerning the most important philosophical positions which, according to his reading, have provided some motivations to accept the presence of contradictory sentences (especially, Hegel and Marx)[5]. For instance, with respect to empirical sciences, Jaśkowski wrote:

> [...] it is known that the evolution of the empirical disciplines is marked by periods in which the theorists are unable to explain the results of experiments by a homogenous and consistent theory, but use different hypotheses, which are not always consistent with one another, to explain the various groups of phenomena. This applies, for instance, to physics in its present-day stage. Some hypotheses are even termed working hypotheses when they result in certain correct predictions, but have no chance to be accepted for good, since they fail in some other cases [21, 37].

The theoretical solution, according to Jaśkowski, is the following:

> we have to take into account the fact that in some cases we have to do with a system of hypotheses which, if subjected to a too consistent analysis, would result in a contradiction between themselves or with a certain accepted law, but which we use in a way that is restricted so as not to yield a self evident falsehood [21, 37].

Indeed, in the paragraphs were he begins to elaborate more formally his ideas, Jaśkowski distinguishes very strictly between «inconsistent» and

[4] Importantly, the works by William of Soissons have not been preserved, however a witness of his work is contained in John of Salisbury's *Metalogicon*.

[5] For more philosophical details on the consequences of adopting a paraconsistent point of view, one might consider [41].

«trivial» system. The first notion is linked to the presence, within the logical system under consideration, of two theses, one the negation of the other (p and $\sim p$); the second concept, instead, asserts that in a system it is possible to derive any formula if there is a couple of contradictory statements. So, as obvious, systems in which every proposition is derivable have no practical significance, since everything can be asserted. So, finally:

> [...] the task is to find a system of the sentential calculus which: (1) when applied to the inconsistent systems would not always entail their overfilling, (2) would be rich enough to enable practical inference, (3) would have an intuitive justification [21, 38].

Jaśkowski did not further elaborate his philosophical considerations, but, nowadays, scholars provided – by taking inspiration directly from Jaśkowski's brief suggestions – some interesting philosophical applications of $\mathbf{D}_2$ (for example, to the foundations of physical theories, to the notion of pragmatic (or partial) truth [10, 14], to the formal study of belief structures and argumentation schemes [17]).

2 The Development of Discussive Logic

Discussive systems have attracted discrete attention and various experts contributed to their development[6]. Our aim, in what follows, is to systematize and explain some of the main works concerning Jaśkowski's discussive logic. To keep the presentation as much as possible self-contained, we will restrict our attention to three distinct, even if connected, paths. More precisely, we will focus our attention on:

- §2.1 the connections between discussive logic and modal systems;
- §2.2 a family of logics, called "**J**" systems;
- §2.3 the "direct" axiomatizations of $\mathbf{D}_2$, i.e., those systems which include axioms for discussive connectives.

[6] At the best of our knowledge, one previous attempt in that direction was made by Ciuciura in [4] from 1999. Nonetheless, in what follows, we wish to consider also alternative approaches towards discussive systems and enrich our considerations by commenting more recent works.

2.1 Connections to Modal Logics

2.1.1 Early developments

The tradition of modal studies connected to $\mathbf{D}_2$ started already in 1968 thanks to a paper by N. da Costa [11] and continued uninterrupted throughout the years. Roughly said:

> Besides non-adjunctiveness, another common obsession of discussivists concerns the alleged 'modal character' of $\mathbf{D}_2$ [28, 217].

Early remarkable results have been provided by J. Kotas in [24] from 1974. First of all, let's fix the next definition:

Definition 2.1. Let $\heartsuit \in \{\Box, \Diamond\}$. A $\heartsuit$-counterpart of a modal system $\mathbf{M}$ is defined as follows: $\heartsuit^n(\mathbf{M}) = \{A \mid \heartsuit^n A \in \mathbf{M}\}$, for $n \geq 1$.

With respect to Jaśkowski's $\mathbf{D}_2$, Kotas elaborated an axiomatization having as primitive connectives only $\sim$, $\supset$, $\Box$. We will denote this system $\mathbf{D}_2^K$, where 'K' stands for Kotas. The axioms of $\mathbf{D}_2^K$ are:

$$\Box(A \supset ({\sim}A \supset B)) \tag{K1}$$
$$\Box((A \supset B) \supset ((B \supset C) \supset (A \supset C))) \tag{K2}$$
$$\Box(({\sim}A \supset A) \supset A) \tag{K3}$$
$$\Box(\Box(A \supset B) \supset (\Box A \supset \Box B)) \tag{K4}$$
$$\Box(\Box A \supset A) \tag{K5}$$
$$\Box({\sim}\Box A \supset \Box{\sim}\Box A) \tag{K6}$$
$$\text{Substitution} \tag{Sub}$$

$$\frac{\Box A \qquad \Box(A \supset B)}{\Box B}\ \Box\text{MP} \qquad \frac{\Box A}{\Box\Box A}\ \text{R4} \qquad \frac{\Box A}{A}\ \text{Den} \qquad \frac{{\sim}\Box{\sim}A}{A}\ \text{Dep}\Box$$

As usual, if we want to add the possibility operator, we can define it: $\Diamond A \stackrel{\text{def}}{=} {\sim}\Box{\sim}A$. Notice that by having $\Diamond$ as a defined connective, Dep$\Box$ may be substituted by:

$$\frac{\Diamond A}{A}\ \text{Dep}$$

An important achievement of [24] is the presentation of the following equivalences between **S5**-type systems and various combinations of axioms and rules of $\mathbf{D}_2^K$:

K1-K6	(Sub)	(□MP)	(R4)	(Den)	(Dep□)/(Dep)	Equivalent System
✓	✓	✓	✓	-	-	□**S5**
✓	✓	✓	✓	✓	-	**S5**
✓	✓	✓	✓	✓	✓	◇**S5**

Notice that, according to the table above, Kotas proved that $\mathbf{D}_2^K$ is equivalent to $\Diamond\mathbf{S5}$. This result allowed him, finally, to prove that $\mathbf{D}_2^K$ is finitely axiomatizable. To obtain his results, Kotas relied on two different Jaśkowski- style translation functions. Take τ of Definition 1.2 and substitute the clauses for $\wedge_d$ and $\rightarrow_d$ with the following ones:

$$\tau^*(A \wedge_d B) = \sim(\sim\tau^*(A) \vee \Box\sim\tau^*(B))$$
$$\tau^*(A \rightarrow_d B) = (\sim\Box\sim\tau^*(A) \supset \tau^*(B)$$

In addition, consider a map τ_1 such that $\mathsf{Form}_{\Diamond\mathbf{S5}} \mapsto \mathsf{Form}_{\mathbf{D}_2^K}$. For any $A, B \in \Diamond\mathbf{S5}$:

$$\tau_1(p) = p$$
$$\tau_1(\sim A) = \sim\tau_1(A)$$
$$\tau_1(A \supset B) = \sim\tau_1(A) \vee \tau_1(B)$$
$$\tau_1(\Box A) = \sim((\sim p \vee p) \wedge_d \tau_1(A))$$

First of all, the equivalence between $\mathbf{D}_2^K$ and $\Diamond\mathbf{S5}$ follows also thanks to the introduction of two additional connectives [24, 197], [46, 37], namely:

$$A \multimap B \stackrel{\text{def}}{=} \Box(A \supset B) \qquad (\multimap)$$
$$A \rightharpoonup B \stackrel{\text{def}}{=} \sim((\sim p \vee p) \wedge_d \sim(\sim A \vee B)) \qquad (\rightharpoonup)$$

In particular, Kotas showed that the interpretation τ turns the implication $\rightharpoonup$ in the strict implication $\multimap$, and the interpretation τ_1 turns the implication $\multimap$ in $\rightharpoonup$. Collecting all this together, Kotas proved that:

1. The translations maps τ and τ_1 establish that $\mathbf{D}_2^K$ and $\Diamond\mathbf{S5}$ are equivalent. In other words, if $\models_{\mathbf{D}_2^K} A$ then $\models_{\Diamond\mathbf{S5}} \tau(A)$ and if $\models_{\Diamond\mathbf{S5}} B$ then $\models_{\mathbf{D}_2^K} \tau_1(B)$, [24, 198-199].

2. $\mathbf{D}_2^K$ is a finitely axiomatizable system [24, 199].

Along these lines of studies, the polish logician T. Furmanowski [18] published a paper concerning the smallest modal system whose $\Diamond$- counterpart coincides with discussive logic. So, by starting from Kotas' axiomatization K1-K5, Furmanowski defined $\Diamond\mathbf{S4}$, i.e., the $\Diamond$- counterpart of **S4**. As usual, by adding axiom K6 to the axiomatization, we get $\Diamond\mathbf{S5}$. In particular, in [18], what's interesting, with respect to these systems, is the equality between $\Diamond\mathbf{S4}$ and $\Diamond\mathbf{S5}$. This result is obtained by showing that both inclusions, (i) $\Diamond\mathbf{S4} \supseteq \Diamond\mathbf{S5}$ and (ii) $\Diamond\mathbf{S5} \supseteq \Diamond\mathbf{S4}$, are satisfied. The latter inclusion is trivial since it is well-known that $\mathbf{S5} \supseteq \mathbf{S4}$. For (i), instead, we need to show that the axioms K1-K5 and the rules of inferences of [24] constitute a complete axiomatization of $\Diamond\mathbf{S4}$ ([18, 39]) and, secondly, to prove that the characteristic axiom of $\Diamond\mathbf{S5}$ K6 is also a formula of $\Diamond\mathbf{S4}$ ([18, 41]). This equality states that, for any A, $\models_{\Diamond\mathbf{S4}} A$ just in case $\models_{\Diamond\mathbf{S5}} A$. So, roughly, the quality of modality in $\Diamond\mathbf{S4}$ is the same as in $\Diamond\mathbf{S5}$. From this result and the axiomatizations of $\Diamond\mathbf{S4}$ and $\Diamond\mathbf{S5}$, Furmanowski proved that, for any system **S** such that, $\mathbf{S4} \subseteq \mathbf{S} \subseteq \mathbf{S5}$: $\models_{\mathbf{S}} \Diamond A$ if and only if $\models_{\Diamond\mathbf{S5}} \Diamond A$. At this point, with this background, Furmanowski defined Jaśkowski's discussive logic by starting from such a system **S**:

Definition 2.2. Let $\mathbf{D}(\mathbf{S})$ be a discussive system as based on a modal system **S**, such that $\mathbf{S4} \subseteq \mathbf{S} \subseteq \mathbf{S5}$:

$$\mathbf{D}(\mathbf{S}) = \{A \in \mathsf{Form}_{\mathbf{D}(\mathbf{S})} \mid \Diamond\tau(A) \in \mathbf{S}\}$$

Take Jaśkowski's translation map τ. Then: $\models_{\mathbf{D}(\mathbf{S})} A$ iff $\models_{\mathbf{S}} \Diamond\tau(A)$.

Notice that, if $\mathbf{S} = \mathbf{S5}$, then $\mathbf{D}(\mathbf{S5}) = \mathbf{D}_2$. From this fact, and by the previous result for which, for any system $\mathbf{S4} \subseteq \mathbf{S} \subseteq \mathbf{S5}$, it holds that $\models_{\mathbf{S}} \Diamond A$ if and only if $\models_{\Diamond\mathbf{S5}} \Diamond A$, we may conclude that, for any such modal system **S**: $\mathbf{D}(\mathbf{S}) = \mathbf{D}_2$.

2.1.2 Recent developments

The tradition of modal studies connected to Jaśkowski's logic continued and largely increased. Recently, the gigantic work of M. Nasieniewski and A. Pietruszczak in [35, 36, 37] contributed to the development of the weakest regular modal logic[7] (denoted by $\mathbf{rS5}^{\mathrm{M}}$) that defines $\mathbf{D}_2$. In [35], the authors analyse $\mathbf{S5}^{\mathrm{M}}$, i.e., a normal modal logic presented previously by J. Perzanowski. Let $\mathbf{L}$ be any modal logic such that $\mathbf{L}$ defines $\mathbf{D}_2$ iff $\mathbf{D}_2 = \{A \in \mathsf{Form}_{\mathbf{D}_2} \mid \Diamond\tau(A) \in \mathbf{L}\}$. We denote with $\Diamond\mathbf{NS5}$ the set of all normal logics from $\Diamond\mathbf{S5}$. By having this in mind and by following the authors of [35], let's introduce the system $\mathbf{S5}^{\mathrm{M}}$ with the following axioms:[8]

$$\Box p \supset \Diamond p \tag{D}$$
$$\Diamond\Box(\Diamond\Box p \supset \Box p) \tag{ML5}$$
$$\Diamond\Box(\Box p \supset p) \tag{MLT}$$

and the rule:

$$\frac{\Diamond\Diamond A}{\Diamond A}\ \mathrm{RM}^2_1$$

A preliminary result is that $\mathbf{S5}^{\mathrm{M}}$ is the smallest logic in $\Diamond\mathbf{NS5}$ [35, 199] but, also, that $\mathbf{S5}^{\mathrm{M}}$ is the smallest normal logic defining $\mathbf{D}_2$.
Starting from $\mathbf{S5}^{\mathrm{M}}$, the authors consider $\mathbf{rS5}^{\mathrm{M}}$, which is the smallest regular logic which contains (MLT) and (RM^2_1). As expected, $\mathbf{rS5}^{\mathrm{M}} \in \Diamond\mathbf{RS5}$ and, moreover, it constitutes the smallest logic belonging to $\Diamond\mathbf{RS5}$. With respect to discussive logic, Nasieniewski and Pietruszczak aimed at showing that $\mathbf{rS5}^{\mathrm{M}}$ is the smallest regular (non-normal) modal logic defining Jaśkowski's $\mathbf{D}_2$. To do this, the author of [36] consider again the function τ of Definition 1.2 together with the following map, labelled τ_2. Let τ_2 be a map such that $\mathsf{Form}_{\mathbf{rS5}^{\mathrm{M}}} \mapsto \mathsf{Form}_{\mathbf{D}_2}$. For any formula $A, B \in \mathbf{rS5}^{\mathrm{M}}$:

$$\tau_2(p) = p$$
$$\tau_2(\sim A) = \sim\tau_2(A)$$
$$\tau_2(A \vee B) = \tau_2(A) \vee \tau_2(B)$$

[7] As usual, we define a regular modal logic $\mathbf{L}$ as a set of modal formulas satisfying the following conditions: (i) $\mathbf{PC} \subseteq \mathbf{L}$, (ii) $\Diamond p \leftrightarrow \sim\Box\sim p \in \mathbf{L}$ and (iii) $\mathbf{L}$ is closed under *modus pones* for $\supset$, under the regularity rule $(A \wedge B) \supset C/(\Box A \wedge \Box B) \supset \Box C$, and under uniform substitution A/A', where A' is the result of uniform substitution of propositional variables in A. Moreover, $\mathbf{L}$ is said to be normal if $\mathrm{K} \in \mathbf{L}$ and $\mathrm{Nec} \in \mathbf{L}$.

[8] Notice that in all normal and regular modal logics axiom (D) can be equivalently formulated as $\Diamond(p \supset p)$.

$$\tau_2(A \wedge B) = {\sim}({\sim}\tau_2(A) \vee {\sim}\tau_2(B))$$
$$\tau_2(A \supset B) = {\sim}\tau_2(A) \vee {\sim}\tau_2(B)$$
$$\tau_2(A \leftrightarrow B) = {\sim}({\sim}({\sim}\tau_2(A) \vee \tau_2(B)) \vee {\sim}({\sim}\tau_2(B) \vee \tau_2(A)))$$
$$\tau_2(\Diamond A) = (p \vee {\sim}p) \wedge_d \tau_2(A)$$
$$\tau_2(\Box A) = {\sim}\tau_2(A) \rightarrow_d {\sim}(p \vee {\sim}p)$$

With this in mind, we are able to introduce $\mathbf{D}_2$ as follows:

Definition 2.3. Let $\mathbf{L}$ be any modal logic such that:

$$\mathbf{D}(\mathbf{L}) \stackrel{\text{def}}{=} \{A \in \mathsf{Form}_{\mathbf{D}_2} \mid \Diamond\tau(A) \in \mathbf{L}\}$$

Then: $\mathbf{L}$ defines $\mathbf{D}_2$ iff $\mathbf{D}(\mathbf{L}) = \mathbf{D}_2$.

So, for any modal logic $\mathbf{L}$ such that, if $\mathbf{L} \in \Diamond\mathbf{S5}$ then $\mathbf{L}$ defines $\mathbf{D}_2$. Additionally, $\mathbf{rS5}^{\mathrm{M}} \in \Diamond\mathbf{RS5}$ and $\mathbf{S5}^{\mathrm{M}} \in \Diamond\mathbf{NS5}$. For $\Diamond\mathbf{RS5}$ and $\Diamond\mathbf{NS5}$ being subsets of $\Diamond\mathbf{S5}$, we get that $\mathbf{rS5}^{\mathrm{M}} \in \Diamond\mathbf{S5}$ and $\mathbf{S5}^{\mathrm{M}} \in \Diamond\mathbf{S5}$. So, $\mathbf{rS5}^{\mathrm{M}}$ and $\mathbf{S5}^{\mathrm{M}}$ both define $\mathbf{D}_2$ and, hence, $\mathbf{D}_2 = \mathbf{D}(\mathbf{rS5}^{\mathrm{M}}) = \mathbf{D}(\mathbf{S5}^{\mathrm{M}})$. In other words, $\mathbf{rS5}^{\mathrm{M}}$ is the regular version of the smallest normal modal logic $\mathbf{S5}^{\mathrm{M}}$ such that (i) $\mathbf{rS5}^{\mathrm{M}} \subsetneq \mathbf{S5}^{\mathrm{M}}$ and (ii) every theorem beginning with $\Diamond$ of $\mathbf{rS5}^{\mathrm{M}}$ is also a theorem of $\mathbf{S5}^{\mathrm{M}}$ [35, 204]. So, finally, collecting together all these results, we get the main *desiderata* of [35]: $\mathbf{rS5}^{\mathrm{M}}$ is the smallest regular non-normal modal logic defining $\mathbf{D}_2$.
Additionally, in [36], the authors showed that $\mathbf{rS5}^{\mathrm{M}}$ can be axiomatized without the rule of inference (RM^2_1) and that it is the smallest regular logic which contains the following theorems:

$$\Box p \supset \Diamond\Box\Box p \tag{4_s}$$
$$\Box p \supset \Diamond\Box p \tag{5_c}$$

In other terms, $\mathbf{rS5}^{\mathrm{M}} = \mathbf{C4_s5_c}$. Moreover, (i) if $\mathbf{rS5}^{\mathrm{M}}$ contains (4_s) and (MLT), we get that $\mathbf{rS5}^{\mathrm{M}} = \mathbf{C4_s}(\mathbf{MLT})$. Finally, (ii) $\mathbf{rS5}^{\mathrm{M}} = \mathbf{C5_c}(\mathbf{RM^2_1})$ iff it contains (5_c) and is closed under the rule (RM^2_1) [36, 49].
In [37], Nasieniewski and Pietruszczak gave a Kripke semantics for the smallest regular modal logic $\mathbf{rS5}^{\mathrm{M}}(= \mathbf{C4_s5_c})$. The paper contains specific frame conditions for $\mathbf{rS5}^{\mathrm{M}}$ and completeness results. Let's begin with the next definition:

Definition 2.4. A frame for regular modal logic $\mathbf{rS5}^{\mathrm{M}}(=\mathbf{C4_s5_c})$ is a triple $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}} = \langle W, \mathcal{R}, N\rangle$, where W is the set of *worlds*, $N \subseteq W$ consists of *regular worlds* and $\mathcal{R}$ is the accessibility relation[9]. Furthermore, $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}} = \langle W, \mathcal{R}, N\rangle$ satisfies the following conditions:

$$\forall w \in N, \exists u \in N(w\mathcal{R}u \wedge \forall x \in W(u\mathcal{R}x \Rightarrow w\mathcal{R}x)) \tag{Fr1}$$

$$\forall w \in N, \exists u \in N(w\mathcal{R}u \wedge \forall x \in W(\exists y \in N(u\mathcal{R}y \wedge y\mathcal{R}x) \Rightarrow w\mathcal{R}x)) \tag{Fr2}$$

$(\mathsf{5_c})$ is valid in frames satisfying (Fr1) [37, 177] and $(\mathsf{4_s})$ is valid if the frame satisfies (Fr2) [37, 178]. Notice that both frame conditions constitute strengthening of seriality [37, 179]. Finally, as usual:

Definition 2.5. A model $\mathcal{M}_{\mathbf{rS5}^{\mathrm{M}}} = \langle W, \mathcal{R}, N, v\rangle$ for $\mathbf{rS5}^{\mathrm{M}}(=\mathbf{C4_s5_c})$ is based on a frame $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}}$ and on a valuation $v : \mathsf{Form}_{\mathbf{rS5}^{\mathrm{M}}} \times W \to \{0, 1\}$ such that for any $A \in \mathsf{Form}_{\mathbf{rS5}^{\mathrm{M}}}$ and $w \in W$:

$$v(\Box A) = 1 \quad \text{iff} \quad w \in N \text{ and } \forall x \in \mathcal{R}(w),\ v(A, x) = 1$$

$$v(\Diamond A) = 0 \quad \text{iff} \quad w \notin N \text{ or } \exists x \in \mathcal{R}(w),\ v(A, x) = 1$$

where $\mathcal{R}(w) \stackrel{\text{def}}{=} \{x \in W \mid w\mathcal{R}x\}$.
A formula A is true in a model $\mathcal{M}_{\mathbf{rS5}^{\mathrm{M}}}$ iff $v(A, w = 1)$ for any $w \in W$. A formula A is valid in a given frame $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}}$ iff it is true in all models $\mathcal{M}_{\mathbf{rS5}^{\mathrm{M}}}$ based on the aforementioned frame.

In sum, the authors of [35, 36, 37] provided both an axiomatic system and a possible worlds semantics for the regular version of **S5** and, consequently, defined discussive logic on that formal basis[10]. From the perspective of Jaśkowski's $\mathbf{D}_2$, the work of Nasieniewski and Pietruszczak is interesting since it shows, not only that there other normal modal logics different from **S5** defining discussive logic, but that there are also non-normal regular versions of **S5** which define $\mathbf{D}_2$.

[9] If we let $W = N$, then we get the pair $\langle W, \mathcal{R}\rangle$, which corresponds to a frame for normal modal logics.

[10] Notice, finally, that we have restricted our attention just to some of the papers that Nasieniewski, Pietruszczak and collaborators devoted to $\mathbf{D}_2$. For more on their work see our conclusive remarks.

2.2 The 'J' Systems

Remarkably,

> [t]he year 1967 was a turning point in the development of the discussive logic. Newton C.A. da Costa and Lech Dubikajtis met in Paris and gradually commenced the development of the logic [4, 10].

Indeed, as said above, in a paper from 1968 [11], da Costa and Dubikajtis presented the first modal-type axiomatization of $\mathbf{D}_2$. The **S5**-type system they proposed, known as **J**, has become famous in the context of discussive systems. As remarked by the authors, **J** has several axiomatizations[11] and, in what follows, we will refer to the axiomatic system presented in [10] from 1995. Interestingly, **J**, and in particular some of its extensions, have been applied to philosophical problems, such as to the debate on the underlying logic of scientific theories. However, before turning to the philosophical applications of **J** and related systems, let's introduce them. **J** is the system composed by the following axioms and rules [10, 45]:

If A is a theorem of **S5**, then $\Box A$ is a theorem of **J**.

$$\frac{\Box A \qquad \Box(A \supset B)}{\Box B}\ \Box\text{MP} \qquad \frac{\Box A}{A}\ \text{Den} \qquad \frac{\Diamond A}{A}\ \text{Dep} \qquad \frac{\Box A}{\Box\Box A}\ \text{R4}$$

J has been introduced in the literature as another $\Diamond$-counterpart of **S5** and, indeed, $\models_{\mathbf{J}} A$ iff $\models_{\mathbf{S5}} \Diamond A$. Starting from **J**, da Costa and Doria presented a first-order variant of it, denoted $\mathbf{J}^*$, by adding the universal quantifier $\forall$ among the connectives. As usual, the existential quantifier can be defined $\exists x A \stackrel{\text{def}}{=} \sim\forall x \sim A$. Before, defining $\mathbf{J}^*$, it is useful to recall the axiomatic system for $\mathbf{S5Q}^=$ (quantified **S5** with identity):

If A is a theorem of **PC**, then A is an theorem of $\mathbf{S5Q}^=$.

All axioms of **S5** (Definition 1.1), plus :

$$x = x \tag{Id1}$$

[11] For other synthetic reconstructions one can also consider [4, 46].

$$x = y \supset (A(x) \leftrightarrow B(x)) \tag{Id2}$$
$$\forall x A(x) \supset A(t), \tag{$\forall$1}$$

where t is either a variable free for x in $A(x)$ or an individual constant. And the following rule:

$$\frac{A \supset B(x)}{A \supset \forall x B(x)}\ \text{R}\forall$$

Now, the language of $\mathbf{J}^*$ coincides the language of $\mathbf{S5Q}^=$ and, indeed, $\mathbf{J}^*$ is introduced as follows:

If A is an theorem of $\mathbf{S5Q}^=$, then $\Box A$ is a theorem of $\mathbf{J}^*$

The axioms and rules of $\mathbf{J}$, plus:

$$\frac{\Box(A \supset B(x))}{\Box(A \supset \forall x B(x))}\ \text{R}\Box\forall$$

where, in the rule (R$\Box\forall$), x is not free in A.
Notice that, differently from Jaśkowski's papers, da Costa and Doria considered left-discussive conjunction. Roughly, by adding both, $\wedge_d^l$ and $\rightarrow_d$, to $\mathbf{J}$ and $\mathbf{J}^*$, the paraconsistent character of such systems. Indeed, the following formulas, governing the 'explosion' of logical systems, are not valid neither in $\mathbf{J}$ nor in $\mathbf{J}^*$. Let $\wedge$ be classical conjunction:

$$A \rightarrow_d (B \rightarrow_d A \wedge B)$$
$$((A \wedge B) \rightarrow_d C) \rightarrow_d (A \rightarrow_d (B \rightarrow_d C))$$
$$A \rightarrow_d (\sim A \rightarrow_d B)$$
$$(A \rightarrow_d \sim A) \rightarrow_d B$$

Furthermore, let $\Gamma \stackrel{\text{def}}{=} \{A \mid \Gamma \vdash_{\mathbf{J}^*} A\}$. As usual, if Γ is the set of all formulas, then Γ is trivial. If not, Γ is non-trivial; if we have a formula A such that both $\Gamma \vdash_{\mathbf{J}^*} A$ and $\Gamma \vdash_{\mathbf{J}^*} \sim A$, then Γ is inconsistent. If not, Γ is consistent. With respect to these definitions, the two authors – who where interested in modelling situations in which scientists may reason through inconsistent sets of sentences, considered as "working hypothesis" [10, 46] – showed that their $\mathbf{J}$-systems allow to deal with inconsistent and non-trivial sets of premises. In other words, da Costa and Doria proved that $\mathbf{J}$ and $\mathbf{J}^*$ are paraconsistent logics.

2.2.1 D_2, $\mathbf{J}^*$ & the foundations of physics

Recall that Jaśkowski believed that "the evolution of the empirical disciplines is marked by periods in which [...] the results of experiments [...] are not always consistent with one another" [21]. Accordingly, the inconsistent results are to be considered as 'working hypothesis', i.e., as sentences that are taken *as if they were true* to inspect their respective consequences and establish which one describes more accurately scientific phenomena. da Costa and Doria tried to make sense of Jaśkowski's idea by elaborating a variant of $\mathbf{J}^*$ which can be used as underlying logic for physical theories. The starting point has been represented by the (formal) conceptions of physical structure and theory, due to M.L. Dalla Chiara and G. Toraldo di Francia (see [15, 10, 14]). First of all, a 'physical structure' $\mathcal{A}$ is a set-theoretic structure of the following form:

$$\mathcal{A} = \langle M, S, < Q_0, Q_1, \ldots, Q_n >, \rho \rangle$$

where, M represents a set of mathematical structures. Notice, the authors of [15] aimed at modelling physical concepts, such as vector spaces, as set-theoretic structures, by taking the axioms of **ZF** set theory. Secondly, S is a set of «physical situations», i.e., a set of physical states assumed by a physical system in a certain time interval. In other words, S is the element of the physical structure that 'mirrors' the physical theory that $\mathcal{A}$ is trying to capture. Each Q_k $(0 \leq k \leq n)$ is an «operationally defined quantity» whose domain of definition is some $S_1 \subseteq S$. As a convention, let Q_0 denote time. To be clear, if we wish to measure a quantity Q_k of a physical system in a state $s \in S$ at a time t_k, with $1 \leq k \leq n$, we get an interval $I(k, t_k)$ of the real number line $\mathbb{R}$. So, if we measure time, i.e., Q_0, the result we obtain is a «time interval». t and t_k represent time instants and we express, in $\mathscr{L}$, the «acceptable values» of Q_k at t_i as $q_k(t_i)$. So, in a certain sense, *all values* in a interval $I(k, t_k)$ are «appropriate values» for the measurement of the quantity Q_k of the physical system in a state $s \in S$. Finally, ρ associates mathematical structures of M to their physically meaningful quantity.

To see how this framework is supposed to work, as usual, let $A(t, q_k(t_k))$ be a formula whose only free variables are those one expressing time instants, (t and t_k). Formally, $\models_s A(t, q_k(t_k))$ means that a formula A, in a certain interval of time, is true for a physical state s if there are values t^0 and q_k^0 (of Q_k) in the interval $I(t, t_k)$, with $1 \leq k \leq n$, such that $A(t^0, q_k^0)$ is true in s. Now, let $\models_{\mathcal{A}} A(t, q_k(t_k))$ denote that A is true in the physical structure $\mathcal{A}$. If we obtain t in I_t and q_k in $I(t, t_k)$, so that $\sim A(t, q_k(t_k))$ is also true in $\mathcal{A}$, then the physical theory captured by $\mathcal{A}$ is paraconsistent. In other words, as one might have expected, with respect to $\mathcal{A}$, we get a paraconsistent physical theory whenever $\models_{\mathcal{A}} A(t, q_k(t_k))$ and $\models_{\mathcal{A}} \sim A(t, q_k(t_k))$.[12]. At this point, da Costa and Doria aimed at demonstrating that:

> [...] the underlying logic of a physical theory in Dalla Chiara and di Francia approach is most adequately represented by Jaśkowski's discussive logic [10, 57].

and, more precisely, by $\mathbf{J}^{**}$. This system is similar to $\mathbf{J}^{*}$, but imposes some more restrictions on bound variables [10, 14]. Take again $\mathbf{S5Q}^{=}$ and let $\uplus A \stackrel{\text{def}}{=} \forall x_n \, A(x_n)$ be denoting a formula A preceded by a sequences of universal quantifiers so that all variables in A are bound. $\mathbf{J}^{**}$ is constituted by the following axioms and rules:

If A is an instance of a theorem of $\mathbf{S5Q}^{=}$, then $\Box \uplus A$ is a theorem of $\mathbf{J}^{*}$.

$$\Box \uplus (\Box(A \supset B) \supset (\Box A \supset \Box B)) \qquad (\mathrm{J}_1^{**})$$

$$\Box \uplus (\Box A \supset A) \qquad (\mathrm{J}_2^{**})$$

$$\Box \uplus (\Diamond A \supset \Box \Diamond A) \qquad (\mathrm{J}_3^{**})$$

$$\Box \uplus (\forall x A(x) \supset A(t)) \qquad (\mathrm{J}_4^{**})$$

$$\Box \uplus (x = x) \qquad (\mathrm{J}_5^{**})$$

[12] To be clear, consider the following example due to [14, 849-850]. Take Newton's second law: $F = m \cdot a$. The variables appearing in the equation corresponds to the physical quantities to be measured: «force» (F), «mass» (m) and «acceleration» (a). If we take a state $s \in S$, their values stand in the following three intervals $I(F_1, F_2) \subseteq \mathbb{R}$, $I(m_1, m_2) \subseteq \mathbb{R}$ and $I(a_1, a_2) \subseteq \mathbb{R}$. When we are able to find three real numbers $p_1 \in I(F_1, F_2)$, $q_1 \in I(m_1, m_2)$ and $r_1 \in I(a_1, a_2)$, such that $p_1 = q_1 \cdot r_1$, then it holds that $\models_s F = m \cdot a$. Likewise, if we encounter the opposite situation, namely we find three real numbers, in their respective intervals, such that $p_2 \neq q_2 \cdot r_2$, also these three real numbers can be considered as acceptable values for solving the equation. So, $\models_s \sim(F = m \cdot a)$ and, hence, Newton's second law, in the very same physical situation s, is both, true and false. In this case, for the same situation s, Newton's law is a proposition C, such that $\models_s C$ and $\models_s \sim C$. However, $\models_s C \wedge \sim C$ does not hold, since it would mean to find three real numbers p, q, r, in their respective intervals, for which the conjunction $p = q \cdot r \wedge p \neq q \cdot r$ holds.

$$\Box \uplus (x = y \supset (A(x) \leftrightarrow A(y))) \qquad (J_6^{**})$$

$$\frac{\Box \uplus A \qquad \Box \uplus (A \supset B)}{\Box \uplus B}\ \uplus\Box\text{MP} \qquad \frac{\Box \uplus A}{A}\ \uplus\text{Den} \qquad \frac{\Box \uplus A}{\Box \uplus \Box A}\ \uplus\text{R4}$$

$$\frac{\Diamond \uplus A}{A}\ \uplus\text{Dep} \qquad \frac{\Box \uplus (A \supset B(x))}{\Box \uplus (A \supset \forall x B(x))}\ \text{R}\uplus\Box\forall$$

So: $\models_{\mathbf{J}^{**}} A$ iff $\models_{\mathbf{S5Q}^{=}} \Diamond \uplus A$. Notice that vacuous quantification can be introduced/eliminated in any formula.

The only difference between $\mathbf{J}^{**}$ and $\mathbf{J}^{*}$ concerns the applications: the first one is more suitable than the second one to handle with, since there's no problem on the discussive interpretation of the free variables. Accordingly, a physical theory, denoted, $\mathcal{T}$, extends the notion of physical structure and, in sum, it is composed by the following elements:

1. A formal language $\mathscr{L}$.
2. A set of axioms $\mathscr{A}$ expressed in $\mathscr{L}$ such that $\mathscr{A} = \mathscr{A}_L \cup \mathscr{A}_M \cup \mathscr{A}_P$, where $\mathscr{A}_L, \mathscr{A}_M$ and $\mathscr{A}_P$ are, respectively, the set of logical, mathematical and physical axioms.
3. A language $\mathscr{L}_0 \subset \mathscr{L}$. The logic $\mathscr{L}_0$, used to deal with the mathematical structures of $\mathcal{T}$, is classical and, hence, $\mathscr{A}_M$ includes all classically valid formulas.
4. The axioms of $\mathbf{J}^{**}$ are included in $\mathscr{A}_L$ to deal with inconsistent sets of premises.
5. $\mathscr{A}_M$ must contain all axioms for the structures of M.
6. $\mathscr{A}_P$ contains all "physically motivated sentences".

So, finally, for A being a theorem of $\mathcal{T}$, then it holds that: if A is formulated in $\mathscr{L}_0$, then A is closed under classical consequence relation. Furthermore, from the perspective of inconsistent theories: for all $A \in \mathcal{T}$, A is closed under $\mathbf{J}^{**}$-consequence relation.

Notice that terms of $\mathscr{L}_0$ cannot refer to the quantities Q_k, but exclusively to mathematical structures of M, which are totally classical. More precisely, exactly the quantities Q_k induce the language to be paraconsistent. Indeed, if we are given a formula B such that its terms refer to some of the Q_k, generally, it can result that both, B and $\sim B$ are true in $\mathcal{T}$. Consequently, both sentences should be included in $\mathscr{A}_P$. Here's exactly the paraconsistent character of the definition of truth, i.e., in a physical theory $\mathcal{T}$, for some state $s \in S$ and a formula B, we can reach both, $\models_s B$ and $\models_s \sim B$. As said above, the acceptance of pairs of contradictory statements, such as B and $\sim B$, is meant to mirror those situations in which two inconsistent sentences are taken to be true with the aim to inspect their respective consequences and chose which one provides a more accurate description of the scientific phenomena under consideration. Of course, this does not mean that: $\models_s B \wedge \sim B$.

2.3 Introducing Discussive Connectives

In the previous discussion we have left apart the centrality of discussive connectives in formulating Jaśkowski's discussive logic in favour of an analysis principally focused on the development of the connections between $\mathbf{D}_2$ and modal systems. In what follows, we reverse the perspective by analysing some of the major attempts to give axioms to Jaśkowski's $\mathbf{D}_2$, without relying on translations and by considering directly a language including $\wedge_d$, $\rightarrow_d$ instead of $\wedge$, $\supset$. The challenge of providing such an axiomatization, usually known as 'Jaśkowski's problem' [46, 42], has been stated by N. da Costa already in 1975 [9, 14]:

> Is it possible to formulate a **natural** and **simple** axiomatization for $\mathbf{D}_2$ employing $\rightarrow_d, \wedge_d, \vee$ and $\sim$ as the only primitive connectives?

According to [25], the first non modal axiomatization of $\mathbf{D}_2$ has been proposed by Furmanowski but has never been published before Kotas' paper from 1975 [46]. It is worth having a look at Furmanowski's work not only for its historical importance, but also for the originality of the proposed axioms. Let $A, B, C, \ldots$ be formulas and let $\bot \stackrel{\text{def}}{=} \sim(A \vee \sim A)$. The discussive logic $\mathbf{D}_2^F$ is axiomatized by the following axioms:

$$\sim(A \supset (\sim A \supset B)) \rightarrow_d \bot \tag{F1}$$

$$(A \supset B) \supset ((B \supset C) \supset (A \supset C)) \rightarrow_d \bot \tag{F2}$$

$$\sim((\sim A \supset B) \supset A) \rightarrow_d \bot \tag{F3}$$

$$\sim((\sim A \supset B) \supset A) \rightarrow_d B \tag{F4}$$

$$\sim((\sim(A \supset B) \rightarrow_d \bot) \rightarrow_d ((\sim A \rightarrow_d \bot) \supset (\sim B \rightarrow_d \bot))) \rightarrow_d \bot \tag{F5}$$

$$\sim(\sim\sim(\sim A \supset \bot) \vee \sim\sim(\sim A \rightarrow_d \bot)) \rightarrow_d \bot \tag{F6}$$

$$(\sim(A \supset B) \rightarrow_d C) \rightarrow_d ((\sim A \rightarrow_d C) \rightarrow_d (\sim B \rightarrow_d C)) \tag{F7}$$

$$(\sim A \rightarrow_d \bot) \rightarrow_d A \tag{F8}$$

$$(A \rightarrow_d B) \rightarrow_d (\sim(A \rightarrow_d B) \rightarrow_d B) \tag{F9}$$

$$\sim(\sim\sim A \rightarrow_d B) \rightarrow_d A \tag{F10}$$

Notice that, $\mathbf{D}_2^F$ is still 'impure' in the sense that, even though, Furmanowski did not include the modal operators, he still kept the presence of two conditionals, including the material one. So, strictly speaking, accordingly to [9], $\mathbf{D}_2^F$ cannot be regarded as a solution to 'Jaśkowski's problem'. In 1977 [12, 13] da Costa and Dubikajtis presented the first complete axiomatization of discussive logic including directly discussive connectives in the axiom schemata. In particular, da Costa and Dubikajtis [12] presented some axioms including $\rightarrow_d$ and $\wedge_d^l$. From now on, we will denote the discussive logic so formalized by $\mathbf{D}_2^l$, where 'l' indicates the presence of $\wedge_d^l$, instead of Jaśkowski's $\wedge_d$. So, the discussive logic $\mathbf{D}_2^l$ is axiomatized as follows

$$A \rightarrow_d (B \rightarrow_d A) \tag{Ax1}$$

$$(A \rightarrow_d (B \rightarrow_d C)) \rightarrow_d ((A \rightarrow_d B) \rightarrow_d (A \rightarrow_d C)) \tag{Ax2}$$

$$(A \wedge_d^l B) \rightarrow_d A \tag{Ax3}$$

$$(A \wedge_d^l B) \rightarrow_d B \tag{Ax4}$$

$$A \rightarrow_d (B \rightarrow_d (A \wedge_d^l B)) \tag{Ax5}$$

$$A \rightarrow_d (A \vee B) \tag{Ax6}$$

$$B \rightarrow_d (A \vee B) \tag{Ax7}$$

$$(A \rightarrow_d C) \rightarrow_d ((B \rightarrow_d C) \rightarrow_d (A \vee B) \rightarrow_d C) \tag{Ax8}$$

$$A \rightarrow_d \sim\sim A \tag{Ax9}$$

$$\sim\sim A \rightarrow_d A \tag{Ax10}$$

$$((A \rightarrow_d B) \rightarrow_d A) \rightarrow_d A \tag{Ax11}$$

$$\sim(A \vee \sim A) \rightarrow_d B \tag{Ax12}$$

$$\sim(A \vee B) \rightarrow_d \sim(B \vee A) \tag{Ax13}$$

$$\sim(A \vee B) \rightarrow_d (\sim B \wedge_d^l \sim A) \tag{Ax14}$$

$$\sim(\sim\sim A \vee B) \rightarrow_d \sim(A \vee B) \tag{Ax15}$$

$$(\sim(A \vee B) \rightarrow_d C) \rightarrow_d ((\sim A \rightarrow_d B) \vee C) \tag{Ax16}$$

$$\sim((A \vee B) \vee C) \rightarrow_d \sim(A \vee (B \vee C)) \tag{Ax17}$$

$$\sim((A \rightarrow_d B) \vee C) \rightarrow_d (A \wedge_d^l \sim(B \vee C)) \tag{Ax18}$$

$$\sim((A \wedge_d^l B) \vee C) \rightarrow_d (A \rightarrow_d \sim(B \vee C)) \tag{Ax19}$$

$$\sim(\sim(A \vee B) \vee C) \rightarrow_d (\sim(\sim A \vee C) \vee \sim(\sim B \vee C)) \tag{Ax20}$$

$$\sim(\sim(A \rightarrow_d B) \vee C) \rightarrow_d (A \rightarrow_d \sim(\sim B \vee C)) \tag{Ax21}$$

$$\sim(\sim(A \wedge_d^l B) \vee C) \rightarrow_d (A \wedge_d^l \sim(\sim B \vee C)) \tag{Ax22}$$

$$\frac{A \qquad A \rightarrow_d B}{B}\ \text{MP}_d$$

Remark 2.6. $\mathbf{D}_2^l$ includes the following set of connectives into its language $\{\sim, \vee, \wedge_d^l, \rightarrow_d\}$, where the only difference, as said, with $\mathbf{D}_2$ is the presence of left-discussive conjunction. Notice that, even though $\mathbf{D}_2^l$ constitutes a complete axiomatization [12, 54], from the perspective of [9], it might be still considered only as a 'partial' solution to 'Jaśkowski's problem'. Indeed, this time the 'impurity' of the axioms is not linked to the inclusion of other connectives than the discussive ones, plus $\sim$ and $\vee$, but to the presence of $\wedge_d^l$. As remarked above (Observation 2), the interaction of $\sim$ with different discussive operators defines logics distinct from Jaśkowki's $\mathbf{D}_2$. Indeed, strictly speaking, since Jaśkowski's $\mathbf{D}_2$ included right-discussive conjunction, $\mathbf{D}_2^l$ can be considered only as a variation of $\mathbf{D}_2$.

More recently, J. Alama and H. Omori [44] presented a complete and sound axiomatization for discussive logic, including Jaśkowski's right-discussive conjunction (denoted $\mathbf{D}_2^r$). The starting point of [44] are the axioms of $\mathbf{D}_2^l$. The only necessary change to get $\mathbf{D}_2^r$, is to drop Ax19 and Ax22 in favour of:

$$\sim((A \wedge_d B) \vee C) \rightarrow_d (B \rightarrow_d \sim(A \vee C)) \tag{Ax19$'$}$$

$$\sim(\sim(A \wedge_d B) \vee C) \rightarrow_d (\sim(\sim A \vee C) \wedge_d B) \tag{Ax22$'$}$$

Moreover, in the axioms involving conjunction, one simply needs to replace $\wedge_d^l$ with $\wedge_d$. Of course, Ax19 and Ax22 of $\mathbf{D}_2^l$ mirrored the behaviour of negated left-discussive conjunction. Ax19$'$ and Ax22$'$ absolve the same job, but with respect to right-discussive conjunction. Both axioms are $\mathbf{D}_2$-valid if and only if their modally translated versions are **S5**-valid, i.e., just in case the following formulas are valid in **S5**, accordingly to τ (see 1.2):

$$\Diamond(\Diamond{\sim}((A \wedge \Diamond B) \vee C) \supset (\Diamond B \supset {\sim}(A \vee C)))$$
$$\Diamond(\Diamond{\sim}({\sim}(A \wedge \Diamond B) \vee C) \supset ({\sim}({\sim}A \vee C) \wedge \Diamond B))$$

Following the changes proposed in [44], it seems that $\mathbf{D}_2^r$ accomplishes, at least, the task of finding a correct and complete axioms system for Jaśkowski's discussive logic. At this point, it might be naturally asked if $\mathbf{D}_2^r$ goes even further and gives a positive and definitive answer to 'Jaśkowski's problem'. Up to now it seems to be the best candidate.
We wish to strengthen this idea by considering briefly two other axiomatizations for $\mathbf{D}_2$, both elaborated by J. Ciuciura in [6, 8]. First of all, consider again a set of connectives including lef-discussive conjunction and the axiomatic system proposed in [6] (denoted $\mathbf{D}_2^C$). Take Ax1-Ax8, plus MP_d, of $\mathbf{D}_2^l$, and add the following axioms:

$$A \vee (A \rightarrow_d B) \tag{C1}$$
$$A \rightarrow_d {\sim}({\sim}(A \vee B) \wedge_d^l {\sim}B \wedge_d^l {\sim}A) \tag{C2}$$
$${\sim}({\sim}(A \vee B) \wedge_d {\sim}B \wedge_d^l {\sim}A) \rightarrow_d {\sim}({\sim}(A \vee B \vee C) \wedge_d^l {\sim}C \wedge_d^l {\sim}B \wedge_d^l {\sim}A) \tag{C3}$$
$${\sim}({\sim}(A \vee B \vee C) \wedge_d^l {\sim}C \wedge_d^l {\sim}B \wedge_d^l {\sim}A) \rightarrow_d \rightarrow_d {\sim}({\sim}(A \vee B \vee C) \wedge_d^l {\sim}B \wedge_d^l {\sim}C \wedge_d^l {\sim}A) \tag{C4}$$
$${\sim}({\sim}(A \vee B) \wedge_d^l {\sim}B \wedge_d^l {\sim}A) \rightarrow_d ((A \vee {\sim}B) \rightarrow_d A) \tag{C5}$$
$${\sim}({\sim}(A \vee B \vee C) \wedge_d^l {\sim}C \wedge_d^l {\sim}B \wedge_d^l {\sim}A) \rightarrow_d ((A \vee B \vee {\sim}C) \rightarrow_d (A \vee B)) \tag{C6}$$
$${\sim}({\sim}(A \vee B \vee C) \wedge_d^l {\sim}C \wedge_d^l {\sim}B \wedge_d^l {\sim}A) \rightarrow_d \rightarrow_d ({\sim}({\sim}(A \vee B \vee {\sim}C) \wedge_d^l {\sim}{\sim}C \wedge_d^l {\sim}B \wedge_d^l {\sim}A) \rightarrow_d {\sim}({\sim}B \wedge_d^l {\sim}A)) \tag{C7}$$
$${\sim}({\sim}A \wedge_d^l {\sim}B) \rightarrow_d (A \vee B) \tag{C8}$$
$$(A \vee {\sim}{\sim}B) \rightarrow_d (A \vee B) \tag{C9}$$
$$(A \vee B) \rightarrow_d (A \vee {\sim}{\sim}B) \tag{C10}$$

As usual, the consequence relation $\vdash_{\mathbf{D}_2^C}$ is determined by the axioms Ax1-Ax8, C1-C10 and by the rule MP$_d$. Additionally, to prove soundness and completeness results, Ciuciura proposed a possible world semantics for $\mathbf{D}_2^C$, in which all elements are identical to those of Definition 2.5, except that $W = N$ and that we include the following clauses:

$$v(A \wedge_d^l B, w) = 1 \quad \text{iff} \quad \exists x \in \mathcal{R}(w),\ v(A, x) = 1 \text{ and } v(B, w) = 1$$
$$v(A \rightarrow_d B, w) = 1 \quad \text{iff} \quad \forall x \in \mathcal{R}(w),\ v(A, x) = 0 \text{ or } v(B, w) = 1$$

Since $\mathbf{D}_2^C$ relies on an equivalence relation between worlds, the accessibility relation may be not explicitly stated in the clauses. In any case, these changes will not affect their meaning, [6, 239-240.]. Importantly, Ciuciura aimed at proving soundness and completeness of $\mathbf{D}_2^C$, but, unfortunately, in [44, 1171], it was proved that in $\mathbf{D}_2^C$ there is (at least) one unprovable formula. The point is struggling, since the formula in question, i.e., $\sim(A \vee \sim A) \rightarrow_d B$, is valid according to the Jaśkowski-style translation τ of Definition 1.2. Consequently, one might naturally doubt whether $\mathbf{D}_2^C$ *is*, in some sense, an axiomatization of Jaśkowski's discussive logic in the sense of [9], given also the presence of $\wedge_d^l$ instead of $\wedge_d$. However, in an another paper [8], Ciuciura restated the presence of right-discussive conjunction among the connectives and provided an axiomatic system for it. We denote Ciuciura's second axiomatization by $\mathbf{D}_2^{C*}$. Take again Ax1-Ax8 (replacing $\wedge_d^l$ with $\wedge_d$ in Ax3, Ax4, Ax5) and MP$_d$ of $\mathbf{D}_2^l$, plus the following axioms:

$$A \vee (A \rightarrow_d B) \tag{C1*}$$

$$\sim(\sim A \wedge_d \sim\sim A \wedge_d \sim(A \vee \sim A)) \tag{C2*}$$

$$\sim(\sim A \wedge_d \sim B \wedge_d \sim(A \vee B)) \rightarrow_d \sim(\sim A \wedge_d \sim B \wedge_d \sim C \wedge_d \sim(A \vee B \vee C)) \tag{C3*}$$

$$\sim(\sim A \wedge_d \sim B \wedge_d \sim C \wedge_d \sim(A \vee B \vee C)) \rightarrow_d \rightarrow_d \sim(\sim A \wedge_d \sim C \wedge_d \sim B \wedge_d \sim(A \vee C \vee B)) \tag{C4*}$$

$$\sim(\sim A \wedge_d \sim B \wedge_d \sim C \wedge_d \sim(A \vee B \vee C)) \rightarrow_d ((A \vee B \vee \sim C) \rightarrow_d (A \vee B)) \tag{C5*}$$

$$\sim(\sim A \wedge_d \sim B) \rightarrow_d (A \vee B) \tag{C6*}$$

$$(A \vee (B \vee \sim B)) \rightarrow_d \sim(\sim A \wedge_d \sim(B \vee \sim B)) \tag{C7*}$$

As in the case of $\mathbf{D}_2^C$, to prove soundness and completeness, Ciuciura proposed a possible worlds semantics, but dropping out the clause for $\wedge_d^l$ in favour of the following one for $\wedge_d$:

$$v(A \wedge_d B, w) = 1 \quad \text{iff} \quad \exists x \in \mathcal{R}(w),\ v(A, w) = 1 \text{ and } v(B, x) = 1$$

Some criticism has been moved against Ciuciura's $\mathbf{D}_2^{C*}$. J. Alama [1] noticed that if we take the axioms Ax1-Ax22 of da Costa's and Dubikajtis' $\mathbf{D}_2^l$, in comparison to the ones of Ciuciura, we will get a troublesome situation: the two axiomatizations share some theses (Ax1-Ax8), while some others are respectively unprovable. Technically, if we encounter this situation, the two logics under considerations are said to be «orthogonal». In this specific case [1, 4-8]:

Proposition 1. $\mathbf{D}_2^{C*} \nvdash$ Ax9, Ax12, Ax13, Ax14, Ax15, Ax16, Ax17, Ax18, Ax19, Ax20, Ax21, Ax22.

At this point, consequently, it might be naturally asked whether $\mathbf{D}_2^{C*}$ corresponds to a restriction of $\mathbf{D}_2^l$. The answer is no, since there is (at least) one axiom of $\mathbf{D}_2^{C*}$ which is $\mathbf{D}_2^l$-unprovable [1, 11]:

Proposition 2. $\mathbf{D}_2^l \nvdash$C5*

In sum, $\mathbf{D}_2^{C*}$. and $\mathbf{D}_2^l$, one with respect to the other, are not complete axiomatizations and, moreover, they ought to be called as orthogonal, i.e., they overlap and each one has theorems which are not formulas of the other. Finally, also the addition of new axioms still confirms that Ciuciura's axiomatization $\mathbf{D}_2^{C*}$ is an incomplete system of axioms [44, 1168.].

Notice , finally, that $\mathbf{D}_2^{C*}$ also fails to be an axiomatization Jaśkowski's $\mathbf{D}_2$, in the sense that there are $\mathbf{D}_2$-valid formulas, that are unprovable in $\mathbf{D}_2^{C*}$ [44, 1167-1170], namely[13]:

$$A \rightarrow_d \sim\sim A$$
$$\sim(A \vee \sim A) \rightarrow_d B$$
$$\sim(A \vee B) \rightarrow_d \sim(B \vee A)$$
$$\sim(\sim\sim A \vee B) \rightarrow_d \sim(A \vee B)$$

[13] Notice that those $\mathbf{D}_2^{C*}$ unprovable formulas correspond to Ax9, Ax12, Ax13 and Ax15 of both, $\mathbf{D}_2^r$ and $\mathbf{D}_2^l$.

Remark 2.7. In conclusion, all these considerations lead us in doubting that $\mathbf{D}_2^C$ and $\mathbf{D}_2^{C*}$ did provide a solution to 'Jaśkowksi's problem'. Furthermore, given the presence of both, Observation 2 and of Proposition 2, also $\mathbf{D}_2^l$ seems to be far from providing a solution. Nonetheless, $\mathbf{D}_2^r$, as elaborated in [44], seems to be an adequate candidate to settle positively the problem raised in [9].

3 Conclusive remarks

We have selected some of the perspectives under which discussive systems can be considered and, for the sake of brevity, we have chosen to explain and discuss just some of the main contributions present in the literature. For example, we have analysed how 'Jaśkowski's problem' might be solved, given the axiomatic systems we discussed. Nonetheless, many other works could have been considered (to cite a few of them, see [25, 9, 46, 30]). J. Perzanowski, in the critical notes to [23, 59], showed how to define 'discussive negation', i.e., $\sim_d A \overset{\text{def}}{=} \Diamond{\sim}A$. Interestingly, the equivalence between $\Diamond{\sim}A$ and ${\sim}\Box A$, makes, in fact, $\sim_d$ equal to 'un-necessity'. However, there are only few articles considering these kind of extensions of the set of discussive connectives. Remarkably, in [7], there's an axiomatization of discussive logic including also $\sim_d$ among the connectives, but, unfortunately, this attempt has some problems (see, [44, 1178-1179]). Hence, the challenge of developing an axiomatization for $\mathbf{D}_2$, including also $\sim_d$, is still open.
As remarked several times, Jaśkowski's logic has attracted discrete attention and many other research paths have been inaugurated. For instance, there has been some interest in developing discussive logic by getting rid of classical **S5**, in favour of other non-classical systems (see, among others [26, 5, 2]). Additionally, the work of connecting $\mathbf{D}_2$ to modal logics (especially, the articles by M. Nasieniewski and colleagues) increased (for example, [38, 31, 39]). Among their gigantic work, it's worth mentioning the proposal of an 'adaptive' (inconsistency-tolerant) version of discussive logic (see [32, 33, 34] and [29]).
From a more philosophical perspective, instead, one might find another interesting application of discussive logic to the philosophy of sciences in [10], where, in addition to the applications of $\mathbf{J}^{**}$ to the foundations

of physical theories, the authors propose also a theory of 'pragmatic' (or 'partial') truth. The intuition underlying their idea, roughly, is that, with respect to inconsistent informations, scientists work with such informations *as if they were true*, and do not take them to be true *simpliciter*. Also in this case, $\mathbf{J}^*$ and $\mathbf{J}^{**}$ show their usefulness in modelling reasoning with inconsistent sets of premises. Importantly, in [17], the authors – by taking inspiration from Jaśkowski's main motivation to build $\mathbf{D}_2$ – propose a four-valued discussive logic ($\mathbf{D}_4$) with the aim of capturing situations in which discussants put forward inconsistent opinions. Roughly, this work includes a 'doxastic' variant of discussive logic, allows to distinguish among different agents, each one with its respective set of beliefs, and models (through a function) the agents' capabilities (e.g., perception, expert-supplied knowledge, communication, discussion). The idea is that a reasoner, that starts from a lack of informations, can – in the process of acquiring more data – reach either support or refutation of such data. However, if there's an overload of informations, the reasoner may reach both, truth and falsity, i.e., inconsistent data.

In conclusion, as said, this overview is not exhaustive and, indeed, our aim was to indicate just some of the most interesting directions that discussive logic oriented researches have taken, by starting from Jaśkowski's papers. We think that thanks to its historical importance as the first known formulation of a paraconsistent logic and to its subsequent developments, discussive logic is still an interesting and vital field of investigation.

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