Fusion, fission, and Ackermann's truth constant in relevant logics: A proof-theoretic investigation

Fabio De Martin Polo*
Ruhr University Bochum
demartinpolofabio@hotmail.it

Abstract

The aim of this paper is to provide a proof-theoretic characterization of relevant logics including fusion and fission connectives, as well as Ackermann's truth constant. We achieve this by employing the well-established methodology of labelled sequent calculi. After having introduced several systems, we will conduct a detailed proof-theoretic analysis, show a CUT-admissibility theorem, and establish soundness and completeness. The paper ends with a discussion that contextualizes our current work within the broader landscape of the proof theory of relevant logics.

1 Introduction and aim

Relevant logics are a recognized family of non-classical logics designed to tackle paradoxes of material and strict implication. According to relevantists, the symbol $\rightarrow$ represents a more refined and philosophically motivated concept of conditional. Early proponents, like Anderson and Belnap, argued that a valid conditional requires a strong connection between the antecedent and consequent, where both are relevant to each other. These logics have garnered significant attention among logicians, leading to the application of various formal structures to provide detailed and systematic characterizations. In this paper, I extend the proof-theoretic investigation of relevant logics initiated in [13] (see also [29]). I explore the inclusion of additional operators in relevant systems, considering the following:

1. The incorporation of *fusion*, denoted as $\circ$, as a primitive binary operator. This study is primarily influenced by the work in [61, pp. 365–366] (Section 2).
2. The inclusion of *fission*, i.e., intensional disjunction, following some remarks expressed in [61, 56].

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3. The inclusion of *connexive fusion*, denoted as $\bullet$, in relevant systems, as proposed by R. Routley. The motivation behind this addition is rooted in the idea that "connexive and relevant logics are one and the same" [59, p. 393] (Section 3).
4. The introduction of *Ackermann's truth constant*, denoted as $\mathtt{t}$, within the framework of relevant logics. Specifically, I study a conservative extension of Anderson and Belnap's logic of entailment $\mathbf{E}$, named $\mathbf{E}^{\mathbf{t}}$, as discussed in [61, pp. 407–424, Appendix 1] (Section 4).

All logics resulting from the above additions are defined using reduced Routley-Meyer frames and models, which are structures featuring a ternary relation among states and a distinguished state interpreted as the *real* or *actual* world (see [60, 61, 63, 22]). By employing these semantic structures, I establish corresponding labelled sequent calculi based on the methodology applied to many other non-classical logics (e.g., [41, 42, 16, 24]). To achieve this, I convert the semantic clauses and frame conditions of each logic under consideration into well-constructed schematic rules, demonstrating that the resulting calculi exhibit various desirable properties (Sections 5 and 6). The paper ends by offering a comparison with other proof-theoretic techniques used to characterize relevant logics, as well as by identifying potential avenues for future research (Section 7).

2 Fusion and fission in relevant logics

The analysis of the connective usually referred to as *fusion*, denoted $\circ$, has played a central role in the formal and philosophical development of relevant logics. In [61, p. 366], amongst other, the authors explain that:

> "[...] fusion can be added conservatively to L sentential systems, and is a useful connective to introduce for algebraic and Gentzenisation exercises."

Let's start by considering a base system of relevant logic $\mathbf{B}$, including $\circ$ among its connectives, denoted $\mathbf{B}^{\circ}$. As usual, $p, q, \ldots$ and $A, B, \ldots$ denote atomic and compound formulas, respectively. The set of formulas is defined recursively in the standard way.
At the level of Hilbert systems, $\mathbf{B}^{\circ}$ contains all instances of the following axioms and it is closed under the following rules ("$\Rrightarrow$" is employed as a rule-forming operator distinct from both the sequent arrow "$\Rightarrow$" and the meta-level symbol " $\Longrightarrow$ "):

(A1) $A \to A$
(A2) $A_1 \land A_2 \to A_i$
(A3) $(A \to B) \land (A \to C) \to (A \to (B \land C))$
(A4) $A_i \to (A_1 \lor A_2)$
(A5) $(A \to C) \land (B \to C) \to ((A \lor B) \to C)$
(A6) $A \land (B \lor C) \to (A \land B) \lor (A \land C)$
(A7) $\sim\sim A \to A$

(R1)	$A, A \to B \Rrightarrow B$
(R2)	$A, B \Rrightarrow A \wedge B$
(R3)	$A \to B \Rrightarrow (C \to A) \to (C \to B)$
(R4)	$A \to B \Rrightarrow (B \to C) \to (A \to C)$
(R5)	$A \to B \Rrightarrow {\sim}B \to {\sim}A$
(Port°)	$(A \circ B) \to C \Lleftarrow\Rrightarrow A \to (B \to C)$

Observation 1. To be precise, within the full system **R**, fusion can be precisely defined as $A \circ B =_{\mathsf{df}} {\sim}(A \to {\sim}B)$. However, it is important to note that this definition does not hold in weaker relevant logics (as discussed, e.g., in [56, 15]). Hence, the discussions we will engage with may be particularly valuable for researchers and practitioners working with weaker relevant logics.

The two-ways rule "portation", (Port°), encodes the fundamental feature of $\circ$. Indeed, as highlighted in [61, p. 385], "fusion accomplishes 'conjunction' of nested implications". For the semantic characterization of relevant logics including $\circ$, we will use so-called reduced Routley-Meyer frames and models. Within the context of such a frame semantics, fusion and implication are interconnected through the shared ternary relation R. Implication is associated with the universal condition, while fusion corresponds to the existential condition. Let's start by introducing the semantics for the base logic $\mathbf{B}^\circ$ [61, Ch. 4 and pp. 365-366].

Definition 2.1. A *reduced* $\mathbf{B}^\circ$ *frame* $\mathscr{F}$ is a quadruple $\langle 0, K, R, * \rangle$, where K is a set of states (possible worlds, points), $0 \in K$ (the *real* or *actual world*), $R \subseteq K^3$, and $* : K \mapsto K$. We impose the following constraints on R:[1]

(p1)	$R0aa$.
(p2)	$a = a^{**}$.
(p3)	If $R0ad$ and $Rdbc$, then $Rabc$.
(p3′)	If $R0ad$ and $Rbca$, then $Rbcd$.
(p4)	If $R0ab$, then $R0b^*a^*$.
(p5)	If $R0ab$ and $R0bc$, then $R0ac$.

We recall that (p3′) is the frame condition added specifically to deal with the rule (Port°).

Definition 2.2. A *reduced* $\mathbf{B}^\circ$ *model* $\mathscr{M}$ is a pair $\langle \mathscr{F}, v \rangle$, where $\mathscr{F}$ is a reduced frame and v is a function, called *valuation*, such that $v : \mathsf{At} \mapsto \wp(K)$, and subject to the so-called *heredity condition*. For $p \in \mathsf{At}$ and $a, b \in K$:

(AtHer) If $R0ab$ and $a \in v(p)$, then $b \in v(p)$.

Finally, the valuation v is extended to the whole language as follows. For all $A, B \in \mathsf{Frm}$ and $a \in K$:

[1]Relations of the form $R0ab$ and $R0ab \wedge R0ba$ can be shortened by using the notation $a \leq b$ and $a = b$, respectively. However, given that the latter symbols are precisely defined in terms of the ternary accessibility relation, we can employ only R to characterize relevant logics.

$\mathscr{M}, a \Vdash p$	iff	$a \in v(p)$, for $p \in \mathsf{At}$
$\mathscr{M}, a \Vdash {\sim}A$	iff	$\mathscr{M}, a^* \nVdash A$
$\mathscr{M}, a \Vdash A \wedge B$	iff	$\mathscr{M}, a \Vdash A$ and $\mathscr{M}, a \Vdash B$
$\mathscr{M}, a \Vdash A \vee B$	iff	$\mathscr{M}, a \Vdash A$ or $\mathscr{M}, a \Vdash B$
$\mathscr{M}, a \Vdash A \circ B$	iff	$\exists b, c \in K$, s.t. $Rbca$ and $\mathscr{M}, b \Vdash A$ and $\mathscr{M}, c \Vdash B$
$\mathscr{M}, a \Vdash A \to B$	iff	$\forall b, c \in K$, if $Rabc$ and $\mathscr{M}, b \Vdash A$, then $\mathscr{M}, c \Vdash B$

Finally:
A formula B is *satisfied* in a model $\mathscr{M} = \langle \mathscr{F}, v \rangle$ iff $\mathscr{M}, 0 \Vdash B$.
"*A entails B in $\mathscr{M}$*" iff, for all $a \in K$, if $a \Vdash A$, then $a \Vdash B$.
A formula B is *valid* in a frame $\mathscr{F} = \langle K, 0, *, R \rangle$ iff, for all valuations v, the formula B is satisfied in $\mathscr{M}$.

Finally, two important, standard lemmas are the following ones (see, among others, [56, 58, 15]):

Heredity Lemma. *If $R0ab$ and $\mathscr{M}, a \Vdash A$, then $\mathscr{M}, b \Vdash A$.*

Verification Lemma. *A entails B in a given model $\mathscr{M}$ iff $A \to B$ is satisfied in that model, i.e., for all $a \in K$, ($\mathscr{M}, a \Vdash A \implies \mathscr{M}, a \Vdash B$) iff $\mathscr{M}, 0 \Vdash A \to B$.*

Observation 2. In the preceding definitions, we introduced "reduced" models for relevant logics (see, for example, [63, 22]). These models differ from those referred to as "non-reduced" (or "unreduced") models (see, for example, [60, 61]).[2] There are several key differences to consider. Let $\mathscr{F}'$ and $\mathscr{M}'$ denote non-reduced frames and models, respectively. $\mathscr{F}'$ is defined as the structure $\langle K, 0, T, *, R \rangle$, where 0 is taken to be a subset of K, rather than a singleton, and T is a distinct element $T \in 0$ called the "designated situation". The members of 0 are referred to as "regular situations". A model $\mathscr{M}'$ is defined as the structure $\langle \mathscr{F}', v \rangle$. Finally, satisfaction in a model is defined with respect to regular situations, i.e., A is satisfied in a model $\mathscr{M}'$ if $\mathscr{M}', x \Vdash A$ for all $x \in 0$. Validity on $\mathscr{F}'$ is defined as before.

In relevant logics, an additional intensional connective, known as fission or intensional disjunction, denoted by the symbol "+", can also be examined. Its presence within the Anderson and Belnap tradition in relevant logic stems from the criticism directed at the inference ${\sim}A, A \vee B \Rrightarrow B$, which is regarded as a fallacy of relevance (see [5, p. 19]). As a solution to preserve the relevance criterion while upholding disjunctive syllogism, Anderson and Belnap proposed a version of it that incorporates the connective $+$, for which addition principles do not hold, rather than its extensional counterpart $\vee$. Taking a slightly different perspective, Stephen Read [56] argues that the problem lies not with the rule of disjunctive syllogism itself, but rather with our understanding of disjunction in natural language, and suggests that we should interpret natural language disjunctions as fissions. In a previous work [55], Read explores the question of whether the distinction between the two

[2] The original Routley-Meyer models for relevant logics were the reduced ones, whereas the introduction of unreduced models happened later, driven by the interest in exploring weaker relevant systems. (Thanks to an anonymous reviewer for contributing to the clarification of this point.)

notions of disjunction, $\vee$ and $+$, can be observed in natural reasoning. He puts forth a positive stance on this matter, contending that his analysis corroborates the relevantist standpoint that *or* is ambiguous. Note that Mares raises doubts about the possibility of considering natural language disjunction as fission. According to Mares, the formal treatment of disjunction, particularly within the framework of ternary frame semantics, may have limited similarity to our intuitive understanding of natural language disjunction (see, [33, pp. 617–619]). Before delving into technical aspects, let's consider the following example to gain an informal understanding of intensional disjunction:

> "Intuitively, the intensional variety of *or* would be one requiring "relevance" between the disjuncts. On the intensional reading, "A or B" would entail that A and B are so related that we are entitled to say "If A had not been true, B would have been true" or "If B had not been true, A would have been true" or the like. A disjunction like *Either Napoleon was born in Corsica or else the number of the beast is perfect* clearly fails to have this property and therefore is of the truth-functional kind. Whereas *That is either Drosophilia Melanogaster or D. virilis, Im not sure which* appears to entail that if it is not the one then it is the other, and thus is of the intensional kind."[3] [2, p. 17]

Formally, the choice of treating fission as either a primitive or a defined connective can be made. If one opts for the former and formalizes it as $A + B$, the following frame condition and semantic clause may be considered (see, e.g., [33, p. 618]):

(p3″) If $R0da$ and $Rbca$, then $Rbcd$.

$(+^1)$ $\mathscr{M}, a \Vdash A + B$ iff $\forall b, c \in K$, if $Rbca$, then either $\mathscr{M}, b \Vdash A$ or $\mathscr{M}, c \Vdash B$.

More standardly, by following the presentations in [61, pp. 286, 361] and [56, p. 72], we can consider fission defined in terms of negation and implication, that is $A+B =_{\mathsf{df}} {\sim}A \to B$. The semantic condition for $+$ is precisely derived from its definition, i.e.:

$(+^2)$ $\mathscr{M}, a \Vdash A + B$ iff $\forall b, c \in K$, if $Rabc$, then either $\mathscr{M}, b^* \Vdash A$ or $\mathscr{M}, c \Vdash B$.

Another notion of fission, which is equivalent to the previous one in **R** but differs from it in weaker systems, is defined as $A \oplus B =_{\mathsf{df}} {\sim}({\sim}A \circ {\sim}B)$ (see [61, p. 361], [56, p. 53]). The corresponding semantic condition can be given as follows:

$(\oplus)$ $\mathscr{M}, a \Vdash A \oplus B$ iff $\forall b, c \in K$, if $Rbca^*$, then either $\mathscr{M}, b^* \Vdash A$ or $\mathscr{M}, c^* \Vdash B$.

Where, again, the clause is derived from the semantic definitions of $\circ$ and $\sim$.

[3] The mentioned example was also discussed in Anderson and Belnap's [5, pp. 176–177].

2.1 Labelled sequent systems

In this part of the paper, we shall define Gentzen-style calculi for $\mathbf{B}^\circ$, and extensions thereof, by using the methodology of labelled sequent calculi. We extend the vocabulary of sequents with a bunch of labels $(a, b, c, \dots, x, y, z \dots)$ denoting states in Routley-Meyer semantics and, intuitively, use the notation $a : A$ to express the forcing relation $a \Vdash A$ via sequents.[4]

Definition 2.3. Let K be a set of labels, including a distinguished label denoted 0. For any $A \in \mathsf{Frm}$ and labels $a, b, c \in K$, the set of well-formed formulas consists of (i) labelled formulas of the form $a : A$ and (ii) relational atoms of the form $Rabc$.[5] Given two multisets Γ, Δ of labelled formulas and relational atoms, a labelled sequent is an object of the following form: $\Gamma \Rightarrow \Delta$.

The labelled rules of our sequent systems are subject to the following *closure condition*. Consider a rule $\mathbf{r}$ of the following form:

$$\frac{A, B_1, \dots, B_n, B_{n+1}, B_{n+1}, \Gamma \Rightarrow \Delta}{B_1, \dots, B_n, \Gamma \Rightarrow \Delta}\ \mathbf{r}$$

Applying the closure condition on $\mathbf{r}$ means to substitute the multiple occurrence B_{n+1}, B_{n+1} with a single one to obtain a rule $\mathbf{r}^*$ of the following shape:

$$\frac{A, B_1, \dots, B_n, B_{n+1}, \Gamma \Rightarrow \Delta}{B_1, \dots, B_n, \Gamma \Rightarrow \Delta}\ \mathbf{r}^*$$

The base labelled calculus for $\mathbf{B}^\circ$, termed $\mathbf{G3rB}^\circ$, is built as follows.

Initial sequents: $R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$ (possibly a^*, b^*).
Logical rules:

$$\frac{\Gamma \Rightarrow \Delta, a^* : A}{a : {\sim}A, \Gamma \Rightarrow \Delta}\ L{\sim} \qquad \frac{a^* : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : {\sim}A}\ R{\sim}$$

$$\frac{a : A, a : B, \Gamma \Rightarrow \Delta}{a : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, a : A \quad \Gamma \Rightarrow \Delta, a : B}{\Gamma \Rightarrow \Delta, a : A \wedge B}\ R\wedge$$

$$\frac{a : A, \Gamma \Rightarrow \Delta \quad a : B, \Gamma \Rightarrow \Delta}{a : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, a : A, a : B}{\Gamma \Rightarrow \Delta, a : A \vee B}\ R\vee$$

$$(b, c \text{ fresh})\ \frac{Rbca, b : A, c : B, \Gamma \Rightarrow \Delta}{a : A \circ B, \Gamma \Rightarrow \Delta}\ L\circ$$

$$\frac{Rbca, \Gamma \Rightarrow \Delta, a : A \circ B, b : A \quad Rbca, \Gamma \Rightarrow \Delta, a : A \circ B, c : B}{Rbca, \Gamma \Rightarrow \Delta, a : A \circ B}\ R\circ$$

[4] The relationship between the two forcing expressions will be explicitly presented on pp. 32ff, where the proof of soundness is provided.

[5] In this section, the development of our desired labelled calculi solely relies on relational atoms of the form $Rabc$. However, as we proceed to the subsequent sections, where Routley-Meyer semantics are extended to accommodate additional relations between states, we will also introduce new relational symbols in our calculi.

$$\frac{Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad c : B, Rabc, a : A \to B, \Gamma \Rightarrow \Delta}{Rabc, a : A \to B, \Gamma \Rightarrow \Delta}\ L\to$$

$$(b, c \text{ fresh})\ \frac{Rabc, b : A, \Gamma \Rightarrow \Delta, c : B}{\Gamma \Rightarrow \Delta, a : A \to B}\ R\to$$

Relational rules for R:

$$\frac{R0aa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R1} \qquad \frac{R0aa^{**}, R0a^{**}a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R2}$$

$$\frac{Rabc, R0ad, Rdbc, \Gamma \Rightarrow \Delta}{R0ad, Rdbc, \Gamma \Rightarrow \Delta}\ \text{R3} \qquad \frac{Rbcd, R0ad, Rbca, \Gamma \Rightarrow \Delta}{R0ad, Rbca, \Gamma \Rightarrow \Delta}\ \text{R3}'$$

$$\frac{R0b^*a^*, R0ab, \Gamma \Rightarrow \Delta}{R0ab, \Gamma \Rightarrow \Delta}\ \text{R4} \qquad \frac{R0ac, R0ab, R0bc, \Gamma \Rightarrow \Delta}{R0ab, R0bc, \Gamma \Rightarrow \Delta}\ \text{R5}$$

Observation 3. Initial sequents are stated in their weakening-absorbing form and express, at the calculus level, the heredity condition for atomic formulas. As the heredity rules represent forms of contraction, namely:

$$\frac{b : p, R0ab, a : p, \Gamma \Rightarrow \Delta}{R0ab, a : p, \Gamma \Rightarrow \Delta}\ \text{AtHer-l} \qquad \frac{R0ab, \Gamma \Rightarrow \Delta, b : p, a : p}{R0ab, \Gamma \Rightarrow \Delta, b : p}\ \text{AtHer-r}$$

it is more advantageous to adopt a system where these rules can be shown to be height-preserving admissible rather than primitive (Proposition 5.7). This is why we incorporate heredity into the axioms. Furthermore, given Proposition 5.2, the generalized version of AtHer can be derived using (admissible) cut and contraction (consider always Prop. 5.7 below). Rules $R\circ$ and $L\to$ include copies of the principal formulas in the premises to prove that contraction is height-preserving admissible (Lemma 5.5). Rules $L\circ$ and $R\to$ are subject to the eigenvariable condition, i.e., each application of those rules requires the introduction of fresh (i.e., not previously used) labels. Finally, all relational rules are obtained by converting the conditions of $\mathbf{B}^\circ$ frames into schematic rules according to the methodology developed and used, amongst others, in [41, 16] for modal and intermediate logics.

Furthermore, if one intends to incorporate fission as a primitive connective in a proof system, the semantic clause ($+^1$) and the frame condition ($\mathsf{p3}''$) stated on page 5 can be converted into the following labelled rules and relational rule, respectively:

$$\frac{b : A, Rbca, a : A + B, \Gamma \Rightarrow \Delta \qquad c : B, Rbca, a : A + B, \Gamma \Rightarrow \Delta}{Rbca, a : A + B, \Gamma \Rightarrow \Delta}\ L+^1$$

$$(b, c \text{ fresh})\ \frac{Rbca, \Gamma \Rightarrow \Delta, b : A, c : B}{\Gamma \Rightarrow \Delta, a : A + B}\ R+^1 \qquad \frac{Rbcd, R0da, Rbca, \Gamma \Rightarrow \Delta}{R0da, Rbca, \Gamma \Rightarrow \Delta}\ \text{R3}''$$

Alternatively, if fission is defined in terms of $\to$ and $\sim$, or in terms of $\circ$ and $\sim$, the semantic clause ($+^2$), or ($\oplus$), can be transformed into the following two sets of left and right rules, respectively:

$$\frac{b^* : A, Rabc, a : A + B, \Gamma \Rightarrow \Delta \qquad c : B, Rabc, a : A + B, \Gamma \Rightarrow \Delta}{Rabc, a : A + B, \Gamma \Rightarrow \Delta}\ L{+}^2$$

$$(b, c \text{ fresh})\ \frac{Rabc, \Gamma \Rightarrow \Delta, b^* : A, c : B}{\Gamma \Rightarrow \Delta, a : A + B}\ R{+}^2$$

$$\frac{b^* : A, Rbca^*, a : A \oplus B, \Gamma \Rightarrow \Delta \qquad c^* : B, Rbca, a : A \oplus B, \Gamma \Rightarrow \Delta}{Rbca^*, a : A \oplus B, \Gamma \Rightarrow \Delta}\ L\oplus$$

$$(b, c \text{ fresh})\ \frac{Rbca^*, \Gamma \Rightarrow \Delta, b^* : A, c^* : B}{\Gamma \Rightarrow \Delta, a : A \oplus B}\ R\oplus$$

It is important to note that we shall not include these two latter sets of rules as primitive in our intended labelled calculi. As demonstrated in Proposition 5.7, rules for the two defined notions of fission are admissible.

Modular extensions. Hilbert systems for some common stronger relevant logics can be obtained by the addition of axioms to the system for $\mathbf{B}^\circ$. Likewise, frames for $\mathbf{B}^\circ$ can be enriched to capture stronger relevant logics by adding some further constraints on R. In what follows, we list some axioms and the conditions needed to validate them.
(Some of these frame conditions appeal to the standard definitions, $Rabcd ::= \exists x(Rabx \wedge Rxcd)$ and $Ra(bc)d ::= \exists x(Raxd \wedge Rbcx)$)

(A8)	$(A \to B) \to ({\sim}B \to {\sim}A)$	(p6)	$Rabc \implies Rac^*b^*$
(A9)	$(A \to B) \wedge (B \to C) \to (A \to C)$	(p7)	$Rabc \implies Ra(ab)c$
(A10)	$(A \to B) \to ((B \to C) \to (A \to C))$	(p8)	$Rabcd \implies Rb(ac)d$
(A11)	$(A \to B) \to ((C \to A) \to (C \to B))$	(p9)	$Rabcd \implies Ra(bc)d$
(A12)	$(A \to (A \to B)) \to (A \to B)$	(p10)	$Rabc \implies Rabbc$
(A13)	$(A \wedge (A \to B)) \to B$	(p11)	$Raaa$
(A14)	$(A \to {\sim}A) \to {\sim}A$	(p12)	Raa^*a
(A15)	$(A \to (B \to C)) \to (B \to (A \to C))$	(p13)	$Rabcd \implies Racbd$
(A16)	$A \to ((A \to B) \to B)$	(p14)	$Rabc \implies Rbac$
(A17)	$A \vee {\sim}A$	(p15)	$R00^*0$
(A18)	$((A \to A) \to B) \to B$	(p16)	$Ra0a$
(A19)	$A \to (A \to A)$	(p17)	$Rabc \implies (R0ac \vee R0bc)$

From the point of view of labelled systems, it is possible to obtain calculi for stronger relevant logics by converting further frame conditions into relational rules. For example, to obtain a calculus for $\mathbf{DW}^\circ$, it suffices to convert condition (p6) into the schematic rule:

$$\frac{Rac^*b^*, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \text{R6}$$

and add it to the rules for $\mathbf{G3rB}^\circ$ displayed above.

3 Fusion in relevant connexive logics à la Routley

A less standard approach to understanding the notion of fusion was proposed by Routley in [59]. To differentiate it from the connectives $\circ$ and $\wedge$, Routley used the symbol $\bullet$ to represent fusion, and we will adopt this notation hereafter. In his 1978 article, Routley introduced frames and models that allowed him to capture connexive logics while preserving the relevant implication.[6] He stated:

> "[...] if antecedent and consequent enjoy a meaning connexion then they are relevant in meaning to one another, and if they are relevant in meaning to one another then they have through the relevance relation a connexion in meaning. Thus the general classes of connexive and relevant logics are one and the same." [59, 393, Emphasis mine]

Connexivity is typically associated with the presence of specific principles within a system. Routley discussed three such principles, namely:

Aristotle's thesis	$\sim(A \to \sim A)$	(Ar)
Boethius' thesis	$(A \to B) \to \sim(A \to \sim B)$	(Bo)
Strawson's thesis	$\sim((A \to B) \bullet (A \to \sim B))$	(St)

The choice of using $\bullet$ instead of $\circ$ or $\wedge$ is related to a common feature in many formalizations of connexive logics: the rejection of "simplification" principles. Specifically, laws such as $A_1 \bullet A_2 \to A_i$ are required to fail. Routley's argument, based on the "cancellation account of negation", explains this as follows:

> "[...] $(A \bullet \sim A) \to A$ and $(A \bullet \sim A) \to \sim A$ fail. For $\sim A$ cancels out A, so that the conjoined content of $A \bullet \sim A$ is less than that of A and $\sim A$. But implication requires content inclusion, so these (degenerate) examples of Simplification fail. This explains, in a sketchy way, the character of the connexivist argument against Simplification. The same argument explains why $(A \bullet \sim A)$ does not imply $\sim(A \bullet \sim A)$, and provides a basis for an argument for Aristotle's thesis." [59, p. 395]

After introducing a Hilbert calculus for a basic system of relevant connexive logic (referred to as $\mathbf{B}^{\bullet}$), Routley proceeds to introduce specific frames and models by stating:

> "Connexive modellings do not differ from relevant modellings as to the implication connective: thus the pure entailment theories are the same: It is only when negation and conjunction are introduced that marked differences begin to emerge." [59, p. 398]

In constructing these models, Routley introduced two new elements: a ternary relational symbol S, distinct from the relevance relation R, to define the truth condition for $\bullet$, and the co-called "generation relation" denoted by G, which indicates that a

[6] To place Routley's work in relation to other important works on connexive logics, refer to Observation 4 provided below.

formula A generates a situation b, meaning that everything holding in b is implied by A. The inclusion of G in the proposed structures validates connexive principles [59, pp. 398–399].

By turning to the formal details of [59], let's introduce frames and models for relevant connexive $\mathbf{B}^{\bullet}$ as follows[7]:

Definition 3.1. A *reduced* $\mathbf{B}^{\bullet}$ *frame* $\mathscr{F}$ is a sextuple $\langle 0, K, R, S, *, \mathsf{G}\rangle$, where $K, 0, R$ and $*$ are as before. $S \subseteq K^3$ is an additional relational symbol and G is a relation on formulas and worlds (i.e. the *generation* relation). Finally, we impose the following constraints on R and S:

(p1), (p2), (p3), (p4) of Def. 2.1, plus:

(p6) If $R0ab$ and $Scda$, then $Scdb$.

Definition 3.2. A *reduced* $\mathbf{B}^{\bullet}$ *model* $\mathscr{M}$ is a pair $\langle \mathscr{F}, v\rangle$, where $\mathscr{F}$ is a reduced frame and v is a function, called *valuation*, such that $v : \mathsf{At} \mapsto \wp(K)$, and subject to the so-called *heredity condition* (see Def. 2.2). The valuation v is extended to the whole language as follows. For all $A, B \in \mathsf{Frm}$ and $a \in K$:

$\mathscr{M}, a \Vdash p$	iff	$a \in v(p)$, for $p \in \mathsf{At}$;
$\mathscr{M}, a \Vdash {\sim}A$	iff	$\mathscr{M}, a^* \nVdash A$;
$\mathscr{M}, a \Vdash A \bullet B$	iff	$\exists b, c \in K$, s.t. $Sbca$ and $\mathscr{M}, b \Vdash A$ and $\mathscr{M}, c \Vdash B$;
$\mathscr{M}, a \Vdash A \to B$	iff	$\forall b, c \in K$, if $Rabc$ and $\mathscr{M}, b \Vdash A$, then $\mathscr{M}, c \Vdash B$.

We include the following clause for G:

(Gen) If $A\mathsf{G}a$, then $\mathscr{M}, a \Vdash A$.

Finally, satisfaction in a model $\mathscr{M}$ and validity on a frame $\mathscr{F}$, as well as the notion of entailment, are as in Definition 2.2.

In order to validate connexive principles, Routley suggested to add further constraints on $\mathbf{B}^{\bullet}$ frames. Specifically, we consider the following additions:

for Aristotle's thesis:	$\exists y(R0^*yy^* \wedge A\mathsf{G}y)$	(F.Ar)
for Boethius' thesis:	$\exists s, t, u(Rast \wedge Ra^*su \wedge A\mathsf{G}s \wedge R0tu^*)$	(F.Bo)
for Strawson's thesis:	$Sbc0^* \implies \exists y, z(Rbyz \wedge Rcyz^* \wedge A\mathsf{G}y)$	(F.St)

Observation 4. Routley's approach to connexivity is relatively uncommon in the subject. However, C. Mortensen and R. Brady have given serious consideration to Routley's approach and proposed additional modifications to his original ideas. Specifically, Mortensen [40] defines Ar as the condition that for every model $\mathscr{M}$, the set $C_A =_{\mathsf{df}} \{w : \mathscr{M}, w \vDash A \text{ and } \mathscr{M}, w \nvDash {\sim}A\}$ is non-empty. He claims that this characterization aligns with the idea expressed by Ar and emphasizes that for a self-inconsistent proposition A, if A is denied, the set C_A must be empty. Brady [11], instead, presented another relational semantics for relevant logic $\mathbf{B}$ extended with either Ar or Bo. The semantics introduced a non-empty subset of worlds

[7] Similarly to the previous section, our focus will be on reduced models. However, it should be noted that Routley [59] also explores non-reduced models and other types of semantic structures in his work.

$0 \subset K$, including a distinguished element T. The extended model structures include a function mapping sets of worlds, including propositions, i.e., interpretations of formulas, to sets of worlds. However, besides the approaches using ternary relational frames, the debate on connexivity is wide and it's challenging to summarize it under a single definition, as there exist various divergent approaches.
The origins of connexive logic can be traced back to the works of Richard Angell and Storrs McCall.[8] Specifically, Angell's [6] main goal was to create a formal system that embodies what he termed the *principle of subjunctive contrariety*, which asserts the incompatibility of statements like "*If p were true, then q would be true*" and "*If p were true, then q would be false*". Building upon Angell's work, McCall [35] introduced the terminology *connexive* logic and extensively studied Angell's formal system. Additionally, McCall [36] explored the potential of using connexive implication to capture all valid moods of Aristotle's syllogistic within a first-order language. In recent years, instead, we have seen the emergence of several different notable programs, including for example:

- C. Pizzi's notion of consequential implication, which is defined using familiar modal concepts and strongly motivated by historical and philosophical inquiries into ancient notions of conditionals (see, e.g., [50, 51, 52, 53]).
- H. Wansing's constructive connexive logic called **C** developed by working in a bilateral setting and modifying the falsity condition of implication as given for Nelson's constructive logic **N4** (see, e.g., [72, 26, 73]). This approach is sometimes referred to as the *Bochum plan* and, from its perspective, connexive logic can be seen as contributing to the exploration of roads to contra-classicality (see, especially, [17]).
- N. Francez's *poly-connexivity* which modifies the familiar falsity conditions of conjunctions and disjunctions, in addition to the falsity clause for intuitionistic implication (see, e.g., [20, 21]). Interestingly, this approach also discusses how to modify the base logic to switch from constructive to relevant systems. Drawing inspiration from Wansing's approach, Francez combined relevant logic **R** with connexive principles. For that purpose, he devised a system of natural deduction that effectively merges the relevant conditional with connexive theses (see [19]).

[8] On the one hand, it is true that connexive logics are characterized by formulas that reference two ancient philosophers, namely Aristotle and Boethius. On the other hand, it remains unclear whether the origins of connexivity can legitimately be traced back to ancient times. Aristotle wrote: "[...] when B is not large, the same B must necessarily be large (which is impossible)" (*Prior Analytics*, 57b14-16). By formalizing the quote with contemporary methods, one obtains the formula usually referred to as Aristotle's Thesis, namely $\sim(\sim B \to B)$, and/or $\sim(B \to \sim B)$ (see [74]). A similar strategy occurred with Boethius' work *De Syllogismo Hypothetico*, where he argues that "*Si est A, non est B*" is the negation of "*Si est A, est B*", that is $(A \to B) \to \sim(A \to \sim B)$ in contemporary notation. The utilization of this formal approach in the examination of ancient texts has led to the current convention among logicians to view Aristotle and Boethius as historical points of reference within the domain of connexive logics. However, it's important to note that historians of logic often cast various doubts and hold differing perspectives regarding whether Aristotle and Boethius were conscious advocates of connexivity, both philosophically and logically. For divergent historical reconstructions, see, e.g., Łukasiewicz [32] and McCall [35], as well as Lenzen [30, 31], Weiss [76], and Ruge [62] for more contemporary sources.

Other important and interesting approaches to connexivity can be found in the following surveys [47, 74, 21].

3.1 Labelled sequent systems

A labelled calculus based on the models just introduced can be found in Table 1.[9] We remark that both $R\rightarrow$ and $L\bullet$ are subject to the eigenvariable restriction, and that R6 is the result of converting frame condition (p6) into a rule. Finally, notice that the premises of both rules, $R\bullet$ and $L\rightarrow$, are stated in their contraction-absorbing form. Last but not least, we introduce a rule specific rule for the generation relation G and show that it is height-preserving admissible:

Proposition 3.1. *The following rule:*

$$\frac{A\mathsf{G}a, a : A, \Gamma \Rightarrow \Delta}{A\mathsf{G}a, \Gamma \Rightarrow \Delta}\ L\mathsf{G}$$

is height-preserving admissible.

Proof. As the proof requires other definitions and results to be introduced in Section 5, we spell out the details in Proposition 5.7. □

To show that our calculus is adequate to deal with the connexive principles mentioned above, we prove the next result:

Proposition 3.2. G3rB$^\bullet$, *extended by the addition of the following rules:*

$$\textit{(y fresh)}\ \frac{R0^*yy^*, A\mathsf{G}y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{AR} \qquad \textit{(s, t, u fresh)}\ \frac{Rast, Ra^*su, A\mathsf{G}s, R0tu^*, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{BO}$$

$$\textit{(y, z fresh)}\ \frac{Rbyz, Rcyz^*, A\mathsf{G}y, Sbc0^*, \Gamma \Rightarrow \Delta}{Sbc0^*, \Gamma \Rightarrow \Delta}\ \mathrm{ST}$$

is a connexive system. More precisely:

1. **G3rB$^\bullet$** $+ \mathrm{AR} \vdash \Rightarrow 0 : {\sim}(A \rightarrow {\sim}A)$;
2. **G3rB$^\bullet$** $+ \mathrm{BO} \vdash \Rightarrow 0 : (A \rightarrow B) \rightarrow {\sim}(A \rightarrow {\sim}B)$;
3. **G3rB$^\bullet$** $+ \mathrm{ST} \vdash \Rightarrow 0 : {\sim}((A \rightarrow B) \bullet (A \rightarrow {\sim}B))$.

We remark that AR, BO and ST correspond to the conversion into schematic rules of (F.Ar), (F.Bo) and (F.St), respectively.

Proof. We proceed by root-first derivation. For ${\sim}(A \rightarrow {\sim}A)$ and $(A \rightarrow B) \rightarrow {\sim}(A \rightarrow {\sim}B)$ see Table 3, and for ${\sim}((A \rightarrow B) \bullet (A \rightarrow {\sim}B))$ consider the derivation in Table 4. □

Finally, Routley observes that:

[9] The notion of labelled sequent is as in Definition 2.3. Also the closure condition applies in this case.

Initial sequents:

$$R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$$

(possibly a^*, b^*)

Logical rules:

$$\frac{\Gamma \Rightarrow \Delta, a^* : A}{a : {\sim}A, \Gamma \Rightarrow \Delta}\ L{\sim} \qquad \frac{a^* : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : {\sim}A}\ R{\sim}$$

$$\text{(}b, c\text{ fresh)}\ \frac{Sbca, b : A, c : B, \Gamma \Rightarrow \Delta}{a : A \bullet B, \Gamma \Rightarrow \Delta}\ L\bullet$$

$$\frac{Sbca, \Gamma \Rightarrow \Delta, a : A \bullet B, b : A \qquad Sbca, \Gamma \Rightarrow \Delta, a : A \bullet B, c : B}{Sbca, \Gamma \Rightarrow \Delta, a : A \bullet B}\ R\bullet$$

$$\frac{Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad c : B, Rabc, a : A \to B, \Gamma \Rightarrow \Delta}{Rabc, a : A \to B, \Gamma \Rightarrow \Delta}\ L \to$$

$$\text{(}b, c\text{ fresh)}\ \frac{Rabc, b : A, \Gamma \Rightarrow \Delta, c : B}{\Gamma \Rightarrow \Delta, a : A \to B}\ R \to$$

Relational rules for R and S:

$$\frac{R0aa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R1} \qquad \frac{R0aa^{**}, R0a^{**}a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R2}$$

$$\frac{Rabc, R0ad, Rdbc, \Gamma \Rightarrow \Delta}{R0ad, Rdbc, \Gamma \Rightarrow \Delta}\ \text{R3} \qquad \frac{R0b^*a^*, R0ab, \Gamma \Rightarrow \Delta}{R0ab, \Gamma \Rightarrow \Delta}\ \text{R4}$$

$$\frac{Scdb, R0ab, Scda, \Gamma \Rightarrow \Delta}{R0ab, Scda, \Gamma \Rightarrow \Delta}\ \text{R6}$$

Table 1: **G3rB$^\bullet$**

> "The semantics given suggests two reductions, of S to R or vice versa, both of which should also be resisted. [Such reductions] would make connexive logic a, perhaps *bizarre*, branch of relevant logic." [59, 410, Emphasis mine]

The reduction that Routley has in mind ca be spelled out by identifying the relevant relation R and the newly introduced one S. This would make connexive fusion, $\bullet$, identical to relevant fusion, $\circ$, and would allow one to validate the following rule:

$$(\mathsf{Port}^\bullet) \quad (A \bullet B) \to C \Longleftarrow\!\!\Longrightarrow A \to (B \to C)$$

As a result, by reducing R to S and viceversa, one misses Routley's leading idea to interpret and characterize $\bullet$ as a proper type of connexive fusion connective, different from both, $\circ$ and $\wedge$. However, if the temptation to identify R and S, cannot be resisted, we introduce the following notion to cope with the resulting logics:[10]

Definition 3.3. We say that:

1. A Hilbert-system for $\mathbf{B}^\bullet$ is *Routley-bizarre* iff it is closed under the two-ways rule $(\mathsf{Port}^\bullet)$.
2. A reduced $\mathbf{B}^\bullet$ frame is *Routley-bizarre* iff it satisfies the following frame constraint: $Sabc \Longleftrightarrow Rabc$ (biz).

At the calculus level we can cope with the notion of *Routley-bizarre* by converting each direction of (biz) into a relational rule. Indeed, by adding such newly constructed relational rules to $\mathbf{G3rB}^\bullet$, we're allowed to extend the notion of *Routley-bizarre* to labelled calculi for connexive relevant logics as follows:

Proposition 3.3. $\mathbf{G3rB}^\bullet$*, extended by the following two rules:*

$$\frac{Rabc, Sabc, \Gamma \Rightarrow \Delta}{Sabc, \Gamma \Rightarrow \Delta}\ \textsc{biz}_1 \qquad \frac{Sabc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \textsc{biz}_2$$

is Routley-bizarre, i.e., the two-ways rule $(\mathsf{Port}^\bullet)$ *is admissible.*

Before displaying the proof, we recall that a sequent-style rule:

$$\frac{\mathcal{P}_1 \ldots \mathcal{P}_n}{\mathcal{C}}\ \mathbf{r}$$

is admissible in a sequent calculus $\mathbf{L}$, if $\vdash_{\mathbf{L}} \mathcal{P}_1, \ldots, \vdash_{\mathbf{L}} \mathcal{P}_n$, together imply $\vdash_{\mathbf{L}} \mathcal{C}$. Finally, let's move to the proof:

Proof. $(\Rrightarrow)$ We first show that, in $\mathbf{G3rB}^\bullet + \textsc{biz}_1 + \textsc{biz}_2$, if $\Rightarrow 0 : A \bullet B \to C$, then $\Rightarrow 0 : A \to (B \to C)$.

[10] The terminology is inspired by the words used by Routley himself in the previously mentioned quote.

$$
\dfrac{\dfrac{\dfrac{\Rightarrow 0 : A \bullet B \to C \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_1\\ Sacd, R0dd, 0 : A \bullet B \to C, Racd, c : B, a : A \Rightarrow d : C, d : A \bullet B\end{array}}{R0dd, 0 : A \bullet B \to C, Racd, c : B, a : A \Rightarrow d : C, d : A \bullet B}\,\text{BIZ}_2 \qquad R0dd, d : C, \mathcal{A} \Rightarrow d : C}{R0dd, 0 : A \bullet B \to C, Racd, c : B, a : A \Rightarrow d : C}\,L\to}{0 : A \bullet B \to C, Racd, c : B, a : A \Rightarrow d : C}\,\text{R1}}{Racd, c : B, a : A \Rightarrow d : C}\,\text{CUT}\ (\text{Thm. 5.6})}{a : A \Rightarrow a : B \to C}\,R\to\ (c, d \text{ fresh})}{\Rightarrow 0 : A \to (B \to C)}\,\text{VER}_2\ (\text{Prop. 5.7})
$$

where δ_1 is concluded by:

$$
\dfrac{\dfrac{R0aa, Sacd, R0dd, a : A, \mathcal{A}'' \Rightarrow d : C, d : A \bullet B, a : A}{Sacd, R0dd, a : A, \mathcal{A}'' \Rightarrow d : C, d : A \bullet B, a : A} \qquad \dfrac{R0cc, Sacd, c : B, \mathcal{A}' \Rightarrow d : C, d : A \bullet B, c : B}{Sacd, c : B, \mathcal{A}' \Rightarrow d : C, d : A \bullet B, c : B}\,\text{R1}}{Sacd, R0dd, 0 : A \bullet B \to C, Racd, c : B, a : A \Rightarrow d : C, d : A \bullet B}\,R\bullet
$$

where $\mathcal{A}' = R0dd, 0 : A \bullet B \to C, Racd, a : A$, and $\mathcal{A}'' = 0 : A \bullet B \to C, Racd, c : B$.

($\Leftarrow$) We finish the proof by showing that, if $\Rightarrow 0 : A \to (B \to C)$, then $\Rightarrow 0 : A \bullet B \to C$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 0 : A \to (B \to C) \qquad \dfrac{\dfrac{R0bb, b : A, \mathcal{A} \Rightarrow a : C, b : A \qquad \begin{array}{c}\vdots\,\delta_1\\ b : B \to C, R0bb, Rbca, c : B, \mathcal{A}' \Rightarrow a : C\end{array}}{R0bb, 0 : A \to (B \to C), Rbca, Sbca, b : A, c : B \Rightarrow a : C}\,L\to}{0 : A \to (B \to C), Rbca, Sbca, b : A, c : B \Rightarrow a : C}\,\text{R1}}{Rbca, Sbca, b : A, c : B \Rightarrow a : C}\,\text{CUT}\ (\text{Thm. 5.6})}{Sbca, b : A, c : B \Rightarrow a : C}\,\text{BIZ}_1}{a : A \bullet B \Rightarrow a : C}\,L\bullet\ (b, c \text{ fresh})}{\Rightarrow 0 : (A \bullet B) \to C}\,\text{VER}_2\ (\text{Prop. 5.7})
$$

where, $\mathcal{A} = 0 : A \to (B \to C), Rbca, Sbca, c : B$ and $\mathcal{A}' = 0 : A \to (B \to C), Sbca, b : A$, and δ_1 is concluded by:

$$
\dfrac{\dfrac{R0cc, Rbca, c : B, \mathcal{A}'' \Rightarrow a : C, c : B}{Rbca, c : B, \mathcal{A}'' \Rightarrow a : C, c : B}\,\text{R1} \qquad \dfrac{R0aa, a : C, Rbca, c : B, \mathcal{A}'' \Rightarrow a : C}{a : C, Rbca, c : B, \mathcal{A}'' \Rightarrow a : C}\,\text{R1}}{b : B \to C, R0bb, Rbca, c : B, \mathcal{A}' \Rightarrow a : C}\,L\to
$$

where $\mathcal{A}'' = 0 : A \to (B \to C), Sbca, b : A, b : B \to C, R0bb$. We remark that all applications of CUT and VER_2 are admissible, as it will be shown in Section 5. ☐

Finally, modular extensions of **G3rB$^\bullet$** can be constructed by converting other frame conditions (see list stated in [59, pp. 397–398]) into relational rules. Notice that the methodology we briefly introduced at the end of last section works also in this case.

4 Ackermann's truth constant in relevant logics: the case of $\mathbf{E}^{\mathsf{t}}$

In this section, I'll consider the proof theory for relevant logics including so-called *Ackermann's truth constant.* To be precise, W. Ackermann, in his 1956 paper [1], introduced a so-called *falsity* (or *negative truth*) *constant* ($\curlywedge$), sometimes read as "the absurd", in order to define the modal operators of impossibility, necessity and possibility, as $\curlywedge \to A$, $\sim A \to \curlywedge$ and $\sim(A \to \curlywedge)$, respectively. In their 1959 paper [4], Anderson and Belnap proved that the addition of $\curlywedge$ is conservative, and that the aforementioned modal notions can be actually defined without relying on $\curlywedge$. These considerations motivated Anderson and Belnap in preferring the $\curlywedge$-free fragment of their modified, albeit theorem-wise identical, version of Ackermann's logic, i.e., the well-known logic of entailment **E**. More precisely, the Anderson and Belnap tradition in relevant logics (see [5]) takes $\curlywedge$ (or its positive counterpart, namely t, that is, basically, shorthand for $\sim\curlywedge$) as a constant of convenience. In other words, if it can be added conservatively to a logic, it simplifies the presentation and/or proofs. Accordingly, the authors of [61, Ch. 5] considered several semantic structures to formalize the behaviour of Ackermann's constant t and, mostly, focused their attention on the logic **E** extended by t (see especially, [61, Appendix 1, pp. 407-424]).
The strategy that we will explore, following the work done in [61], consists in the addition of a unary predicate on states interpreted, roughly, as a *possibility* predicate. In other words, if we denote such a predicate with the letter P and let a be a world in K, then $\mathsf{P}a$ tells us that a is a *logically possible* state:

> "P can be construed, without too much distortion as a possibility predicate [...] it shows that t marks out the regular worlds, i.e. those where the theorems hold". [61, p. 351]
> "A situation a in K has property P iff each proposition which is necessarily true holds in a, i.e., a is a possible 'world'." [61, p. 407]

So, P selects regular (or, normal) worlds, i.e., those worlds at which all theorems are true under all interpretations.
Before entering the specific details of such models, let us draw a distinction between the cases which make t either *distinctive* or *non-distinctive.* The characteristic rules, which make t *distinctive*, are:

(Rt1) $\mathsf{t} \to A \Rrightarrow A$
(Rt2) $A \Rrightarrow \mathsf{t} \to A$

At the model-theoretic level, the authors of [61], discuss two possible semantic clauses that may be formulated for t, depending on whether it is distinctive or not. After some inspection, the authors notice that the two different clauses for t, can be always condensed into a single one. Nevertheless, we remark that, in all these cases, reduced structures face some limitations. Roughly, only a limited amount of, rather weak, relevant logics can be successfully characterized by reduced modellings. A notable exclusion is the well-known logic $\mathbf{E}^{\mathsf{t}}$, for which another solution was introduced.

From a closer perspective, $\mathbf{E}^{\mathsf{t}}$ contains all instances of the following axioms and is closed under the following rules [61, p. 408]:

(T1)	$(\mathsf{t} \to A) \to A$	(T2)	$\mathsf{t} \to (A \to A)$
(A2)	$A_1 \wedge A_2 \to A_i$	(A3)	$(A \to B) \wedge (A \to C) \to (A \to (B \wedge C))$
(A4)	$A_i \to (A_1 \vee A_2)$	(A5)	$(A \to C) \wedge (B \to C) \to ((A \vee B) \to C)$
(A7)	$\sim\sim A \to A$	(A6′)	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$
(A14)	$(A \to \sim A) \to \sim A$	(A8)	$(A \to B) \to (\sim B \to \sim A)$
(R1)	$A, A \to B \Rrightarrow B$	(A10)	$(A \to B) \to ((B \to C) \to (A \to C))$
(R2)	$A, B \Rrightarrow A \wedge B$	(A12)	$(A \to (A \to B)) \to (A \to B)$
(Rt2)	$A \Rrightarrow \mathsf{t} \to A$		

$\mathbf{E}^{\mathsf{t}}$ is a conservative extension of Anderson and Belanp's logic of entailment $\mathbf{E}$, and to get the axiomatization of $\mathbf{E}$, it suffices to replace (T1) and (T2) by the following two axioms:

(AxE1) $((A \to A) \to B) \to B$
(AxE2) $\Box A \wedge \Box B \to \Box(A \wedge B)$

where $\Box A =_{\mathsf{df}} (A \to A) \to A$. As it was pointed out by the authors of [61], "part of the point of the t-reformulation of $\mathbf{E}$ is to dispose of the intially troublesome [modal] axiom", that is (AxE2). As the authors of [61, p. 407] remark:

> "The chief innovation in the semantical analysis of $\mathbf{E}$ we offer, [...] is the introduction of a new property P of situations. [...]
> P is required to cope with $\mathbf{E}$ in view of the way $\mathbf{E}$ introduces necessity as part of the logic of entailment, most obviously through the postulates $\Box A \to A$ and $\Box A \wedge \Box B \to \Box(A \wedge B)$ and through the (derived) rule of necessitation ." (Notation adapted)

Reduced frames and models for $\mathbf{E}^{\mathsf{t}}$ were introduced as follows [61, p. 411]:

Definition 4.1. A $\mathbf{E}^{\mathsf{t}}$ *reduced frame* is a quintuple $\langle 0, K, \mathsf{P}, R, * \rangle$, where 0, K, and R are as in Definition 2.1. Finally, P denotes a property on K, called the *possibility predicate*. R and P are subject to the following conditions:

(p1) $R0aa$.
(p2) $a = a^{**}$.
(p6) If $Rabc$, then Rac^*b^*.
(p8) If $Rabcd$, then $Rb(ac)d$.
(p10) If $Rabc$, then $Rabbc$.
(p12) Raa^*a.
(t1) $\exists x(\mathsf{P}x$ and $Raxa)$.
(t2) If $\mathsf{P}x$ and $Rxab$, then $R0ab$.
(t3) If $R0bx$ and $Raxc$, then $Rabc$.

Definition 4.2. An $\mathbf{E}^{\mathsf{t}}$ *reduced model* is a structure $\langle \mathscr{F}, v \rangle$, where $\mathscr{F}$ is an $\mathbf{E}^{\mathsf{t}}$ frame and v is as in Definition 2.2. We add the new condition for t:

$\mathscr{M}, a \Vdash \mathsf{t}$ iff $\exists b \in K$, such that $R0ba$ and $\mathsf{P}b$.

Initial sequents: $R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$ (possibly a^*, b^*)
Rules for t:

$$\frac{}{\mathsf{P}b, R0ba, \Gamma \Rightarrow \Delta, a : \mathsf{t}}\ R\mathsf{t} \qquad (b \text{ fresh})\ \frac{\mathsf{P}b, R0ba, a : \mathsf{t}, \Gamma \Rightarrow \Delta}{a : \mathsf{t}, \Gamma \Rightarrow \Delta}\ L\mathsf{t}$$

Logical rules:

$$\frac{\Gamma \Rightarrow \Delta, a^* : A}{a : {\sim}A, \Gamma \Rightarrow \Delta}\ L{\sim} \qquad \frac{a^* : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : {\sim}A}\ R{\sim}$$

$$\frac{a : A, a : B, \Gamma \Rightarrow \Delta}{a : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, a : A \quad \Gamma \Rightarrow \Delta, a : B}{\Gamma \Rightarrow \Delta, a : A \wedge B}\ R\wedge$$

$$\frac{a : A, \Gamma \Rightarrow \Delta \quad a : B, \Gamma \Rightarrow \Delta}{a : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, a : A, a : B}{\Gamma \Rightarrow \Delta, a : A \vee B}\ R\vee$$

$$\frac{Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \quad c : B, Rabc, a : A \to B, \Gamma \Rightarrow \Delta}{Rabc, a : A \to B, \Gamma \Rightarrow \Delta}\ L\to$$

$$(b, c \text{ fresh})\ \frac{Rabc, b : A, \Gamma \Rightarrow \Delta, c : B}{\Gamma \Rightarrow \Delta, a : A \to B}\ R\to$$

Relational rules for R **and** P:

$$\frac{R0aa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{R1} \qquad \frac{R0aa^{**}, R0a^{**}a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{R2}$$

$$\frac{Rac^*b^*, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \mathrm{R6} \qquad (y \text{ fresh})\ \frac{Rbyd, Racy, Rabx, Rxcd, \Gamma \Rightarrow \Delta}{Rabx, Rxcd, \Gamma \Rightarrow \Delta}\ \mathrm{R8}$$

$$(x \text{ fresh})\ \frac{Rabx, Rxbc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \mathrm{R10} \qquad \frac{Raa^*a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{R12}$$

$$(x \text{ fresh})\ \frac{\mathsf{P}x, Raxa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{T1} \qquad \frac{R0ab, \mathsf{P}x, Rxab, \Gamma \Rightarrow \Delta}{\mathsf{P}x, Rxab, \Gamma \Rightarrow \Delta}\ \mathrm{T2}$$

$$\frac{Rabc, R0bx, Raxc, \Gamma \Rightarrow \Delta}{R0bx, Raxc, \Gamma \Rightarrow \Delta}\ \mathrm{T3}$$

Table 2: $\mathbf{G3rE}^{\mathsf{t}}$

Finally, satisfaction in a model $\mathscr{M}$ and validity on a frame $\mathscr{F}$, as well as the notion of entailment, are as in Definition 2.2.

Based on this model, we construct the corresponding labelled proof system, termed $\mathbf{G3rE^t}$ (Table 2, p. 18).[11] T1, T2 and T3 are (t1), (t2) and (t3) converted into rule form, respectively. We remark that other useful relational rules can be considered as well. For example:

$$\frac{Rabd, R0bc, Racd, \Gamma \Rightarrow \Delta}{R0bc, Racd, \Gamma \Rightarrow \Delta}\ \mathrm{R5'}$$

can be obtained by an application of T3. The same holds for:

$$\frac{R0ac, R0ab, R0bc, \Gamma \Rightarrow \Delta}{R0ab, R0bc, \Gamma \Rightarrow \Delta}\ \mathrm{R5}$$

which can be obtained by an application of R5′. More generally speaking, all relational rules, obtained by converting the frame conditions listed on [61, p. 411], are derivable by means of the relational rules taken as primitive in Table 2. We observe that, instead of $R\mathsf{t}$, it is possible to incorporate another type of initial sequents, namely structures of the following form: $\mathsf{P}b, R0ba, \Gamma \Rightarrow \Delta, a : \mathsf{t}$. However, to maintain a certain symmetry in the construction of the calculus, we made the choice of incorporating the 0-premise rule $R\mathsf{t}$ alongside $L\mathsf{t}$, which represents the rule-version of the semantic clause for t (Definition 4.2), instead of introducing additional initial sequents.[12]

Observation 5. The section started with an examination of Ackermann's falsity constant and its impact on the debate surrounding relevant logics. It was noted that the standard approach considers truth as a primitive concept, while defining falsity as the negation of truth, i.e., $\mathsf{f} =_{\mathsf{df}} {\sim}\mathsf{t}$.[13] I followed this convention and, as a result, the two following admissible rules for f can be considered:

$$(b \text{ fresh})\ \frac{\mathsf{P}b, R0ba^*, a^* : \mathsf{t}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : \mathsf{f}}\ R\mathsf{f} \qquad\qquad \frac{}{a : \mathsf{f}, \mathsf{P}b, R0ba^*, \Gamma \Rightarrow \Delta}\ L\mathsf{f}$$

By proceeding root-first then we get the following derivations:

$$(b \text{ fresh})\ \cfrac{\cfrac{\cfrac{\mathsf{P}b, R0ba^*, a^* : \mathsf{t}, \Gamma \Rightarrow \Delta}{a^* : \mathsf{t}, \Gamma \Rightarrow \Delta}\ L\mathsf{t}}{\Gamma \Rightarrow \Delta, a : {\sim}\mathsf{t}}\ R{\sim}}{\Gamma \Rightarrow \Delta, a : \mathsf{f}}\ \mathsf{df.f} \qquad\qquad \cfrac{\cfrac{\cfrac{}{\mathsf{P}b, R0ba^*, \Gamma \Rightarrow \Delta, a^* : \mathsf{t}}\ R\mathsf{t}}{a : {\sim}\mathsf{t}, \mathsf{P}b, R0ba^*, \Gamma \Rightarrow \Delta}\ L{\sim}}{a : \mathsf{f}, \mathsf{P}b, R0ba^*, \Gamma \Rightarrow \Delta}\ \mathsf{df.f}$$

[11] Definition 2.3 and the closure condition always apply.

[12] Let me also emphasize that in the context of labelled calculi based on Negri's methodology, it is quite common to use rules for constants, such as t or $\bot$, rather than introducing additional initial sequents apart from those that include atomic formulas (see, e.g., [45, 46]).

[13] While historically the falsity constant has been denoted as $\curlywedge$, one can use the more contemporary notation and represent it with the small case letter f.

Additionally, one might consider a Routley-Meyer semantic presentation of Ackermann's system from [1], known as Π'', which includes the original $\curlywedge$, and attempt to convert its semantic definition, along with the frame conditions, into well-constructed sequent-style rules. A semantics of this nature has been developed in [37]. However, delving into the examination of this approach within our current discussion would require introducing certain modifications and explanations that would make this section excessively lengthy. Therefore, a comprehensive investigation of the proof theory stemming from the semantic frameworks discussed in [37] will be temporarily set aside.

As remarked above, $\mathbf{E}^{\mathsf{t}}$ conservatively extends Anderson and Belnap's logic **E**. Accordingly, all **E** theorems are also $\mathbf{E}^{\mathsf{t}}$ theorems. However, as the modality $\Box$ is difficult to use in proofs[14], another, more standard modal notion can be defined by relying on the truth constant t:

(df■) $\quad \blacksquare A = \mathsf{t} \to A.$

Accordingly, (T1), (T2) and (Rt2) can be reformulated as $\blacksquare A \to A$, $\blacksquare(A \to A)$ and $A \Rrightarrow \blacksquare A$, respectively. As remarked by Meyer:

> "[...] '$\to$' [is] primitive and '■' [is] contextually defined by [df■]. ([df■] is essentially the definition of '■' of Ackermann's [1], modified by the insights of [Anderson and Belnap's] [4].) [They] prefer the essentially equivalent $\Box A = (A \to A) \to A$ [...]. My view is that the sentential constant [t] is well-motivated and ought to be introduced in the name of elegance; theirs seems to be that it is superfluous and ought to be thrown out in the name of Ockham." [38, p. 196] (Notation adapted)

In virtue of our observations, we can prove the following result:

Proposition 4.1. *If A is a theorem of Anderson and Belnap's* **E**, *then* $\mathbf{G3rE}^{\mathsf{t}} \vdash \Rightarrow 0 : A$.

Proof. We proceed by showing that (AxE1) and (AxE2) can be derived in $\mathbf{G3rE}^{\mathsf{t}}$.
$\mathbf{G3rE}^{\mathsf{t}} \vdash \Rightarrow 0 : ((A \to A) \to B) \to B$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0cd, Rxcd, c : A, \mathsf{P}x, Raxa, \mathcal{A} \Rightarrow b : B, d : A}{Rxcd, c : A, \mathsf{P}x, Raxa, \mathcal{A} \Rightarrow b : B, d : A}\,\mathrm{T2}}{\mathsf{P}x, Raxa, \mathcal{A} \Rightarrow b : B, x : A \to A}\,R\to\,(c, d \text{ fresh}) \qquad a : B, R0ab, \mathcal{A}' \Rightarrow b : B}{\mathsf{P}x, Raxa, R0ab, a : (A \to A) \to B \Rightarrow b : B}\,L\to}{R0ab, a : (A \to A) \to B \Rightarrow b : B}\,\mathrm{T1}\,(x \text{ fresh})}{\Rightarrow 0 : ((A \to A) \to B) \to B}\,R\to\,(a, b \text{ fresh})$$

[14]Mares and Standefer remark that: "The operator $\Box$ is extremely difficult to use in proofs. [...]. Written in primitive notation [(AxE2)] is $(((A \to A) \to A) \wedge ((B \to B) \to B)) \to (((A \wedge B) \to (A \wedge B)) \to (A \wedge B))$. Using this formula to prove other modal theses can be quite difficult" [34, pp. 701-702] (Notation adapted).

where $\mathcal{A} = R0ab, a : (A \to A) \to B$ and $\mathcal{A}' = \mathsf{P}x, Raxa, a : (A \to A) \to B$.
$\mathbf{G3rE^t} \vdash \Rightarrow 0 : \blacksquare A \wedge \blacksquare B \to \blacksquare(A \wedge B)$

$$
\dfrac{
\dfrac{\dfrac{}{\mathsf{P}x, R0xc, Racd, c : \mathsf{t}, \mathcal{A} \Rightarrow d : A \wedge B, c : \mathsf{t}}\,Rt}{Racd, c : \mathsf{t}, \mathcal{A} \Rightarrow d : A \wedge B, c : \mathsf{t}}\,L\mathsf{t}
\qquad
\begin{array}{c}\vdots\ \delta_1\\ d : A, Racd, Rbcd, c : \mathsf{t}, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B \Rightarrow d : A \wedge B\end{array}
}{Racd, Rbcd, c : \mathsf{t}, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B \Rightarrow d : A \wedge B}\,L\!\to
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Racd, Rbcd, c : \mathsf{t}, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B \Rightarrow d : A \wedge B}{Rbcd, c : \mathsf{t}, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B \Rightarrow d : A \wedge B}\,\mathrm{R3}}{R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B \Rightarrow b : \mathsf{t} \to (A \wedge B)}\,R\!\to\ (c, d \text{ fresh})}{R0ab, a : (\mathsf{t} \to A) \wedge (\mathsf{t} \to B) \Rightarrow b : \mathsf{t} \to (A \wedge B)}\,L\wedge}{\Rightarrow 0 : ((\mathsf{t} \to A) \wedge (\mathsf{t} \to B)) \to (\mathsf{t} \to (A \wedge B))}\,R\!\to\ (a, b \text{ fresh})}{\Rightarrow 0 : \blacksquare A \wedge \blacksquare B \to \blacksquare(A \wedge B)}\,\mathsf{df}\blacksquare
$$

and δ_1 is concluded by:

$$
\dfrac{
\dfrac{\dfrac{}{\mathsf{P}x, R0xc, Racd, c : \mathsf{t}, \mathcal{A} \Rightarrow d : A \wedge B, c : \mathsf{t}}\,R\mathsf{t}}{Racd, c : \mathsf{t}, \mathcal{A}' \Rightarrow d : A \wedge B, c : \mathsf{t}}\,L\mathsf{t}
\qquad
\begin{array}{c}\vdots\ \delta_2\\ d : B, d : A, Racd, \mathcal{A}'' \Rightarrow d : A \wedge B\end{array}
}{d : A, Racd, Rbcd, c : \mathsf{t}, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B \Rightarrow d : A \wedge B}\,L\!\to
$$

and δ_2 is obtained as follows:

$$
\dfrac{
\dfrac{R0dd, d : B, d : A, Racd, \mathcal{A}'' \Rightarrow d : A}{d : B, d : A, Racd, \mathcal{A}'' \Rightarrow d : A}\,\mathrm{R1}
\qquad
\dfrac{R0dd, d : B, d : A, Racd, \mathcal{A}'' \Rightarrow d : B}{d : B, d : A, Racd, \mathcal{A}'' \Rightarrow d : B}\,\mathrm{R1}
}{d : B, d : A, Racd, \mathcal{A}'' \Rightarrow d : A \wedge B}\,R\wedge
$$

where $\mathcal{A} = Rbcd, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B,$, $\mathcal{A}' = d : A, Rbcd, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B$ and $\mathcal{A}'' = Rbcd, c : \mathsf{t}, R0ab, a : \mathsf{t} \to A, a : \mathsf{t} \to B$. □

Extensions. The authors of [61, p. 422] mention some interesting extensions that can be obtained by adding axioms and their related conditions to Hilbert systems and frames for $\mathbf{E^t}$, respectively. They are given as follows:

(B3)	$B \to (A \to A)$	(q3)	$Rabc \implies R0bc$
(B4)	$A \to (B \to A)$	(q4)	$Rabc \implies R0ac$
(B5)	$B \to (A \vee {\sim}A)$[15]	(q5)	$R0a^*a$

The following extensions can be built:

S4:	$\mathbf{E^t}$ + (B3)	$\mathscr{F}_{\mathbf{S4}} = \mathscr{F}_{\mathbf{E^t}}$ + (q3)	(Lewis' modal logic **S4**)
CL:	$\mathbf{E^t}$ + (B4)	$\mathscr{F}_{\mathbf{CL}} = \mathscr{F}_{\mathbf{E^t}}$ + (q4)	(Classical logic)
EDS:	$\mathbf{E^t}$ + (B5)	$\mathscr{F}_{\mathbf{EDS}} = \mathscr{F}_{\mathbf{E^t}}$ + (q5)	

where **EDS** is a proper subsystem of Lewis' logic **S3** (and hence of **S4**), and it is "of importance in the debate on entailment since it includes Disjunctive Syllogism B5′ without conceding the higher degree of strict implication" [61, p. 422]. Finally, let

15
[15] Or, equivalently, (B5′), namely, $(A \wedge ({\sim}A \vee B)) \to B$.

us remark that by adding the rules obtained by converting the the frame conditions displayed above to **G3rE**$^{\mathsf{t}}$, namely:

$$\frac{R0bc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{RQ3} \qquad \frac{R0ac, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{RQ4} \qquad \frac{R0a^{*}a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{RQ5}$$

one obtains also labelled calculi for the logics just mentioned, i.e., **S4**, **CL** and **EDS**.

5 Proof analysis

The main aim of this section is to prove the CUT-admissibility theorem for our labelled sequent systems. The general proof presented here is similar to the proof for labelled systems for other non-classical logics (see, e.g., [41, 16, 24, 29, 13]). A list of the results proved in this section is as follows:

- Height-preserving admissibility of substitution of labels (Lem. 5.1).
- Derivability of generalized initial sequents (Prop. 5.2).
- Height-preserving admissibility of the weakening rules (Lem. 5.3).
- Height-preserving invertibility of logical and relational rules (Lem. 5.4).
- Height-preserving admissibility of the contraction rules (Lem. 5.5).
- CUT-admissibility (Thm. 5.6).
- Other useful admissibility results (Prop. 5.7).

As there are many cases to be analysed in these proofs, we only outline the important parts here.
Let's start by fixing some standard notions.

Definition 5.1. Let $\mathcal{A}$ be any labelled formula of the form $a : A$. We denote by $l(\mathcal{A})$ the label of a formula $\mathcal{A}$, and by $p(\mathcal{A})$ the pure part of the formula, that is, the unlabelled formula. The *weight* (or *complexity*) of a labelled formula is defined as a lexicographically ordered pair: $\langle \mathsf{w}(p(\mathcal{A})), \mathsf{w}(l(\mathcal{A}))\rangle$, where:

1. for all state labels $a \in W$, $\mathsf{w}(a) = 1$;
2. for all $p \in \mathsf{At}$, $\mathsf{w}(p) = 1$;
3. $\mathsf{w}(\mathsf{t}) = 1$;
4. $\mathsf{w}({\sim}A) = \mathsf{w}(A) + 1$;
5. $\mathsf{w}(A \heartsuit B) = \mathsf{w}(A) + \mathsf{w}(B) + 1$, for $\heartsuit \in \{\circ, \bullet, +, \rightarrow\}$.[16]

Definition 5.2. A rule **r** is *height-preserving admissible* just in case: if there is a derivation of the premise(s) of **r**, then there is a derivation of the conclusion of **r** that contains no application of **r** (with the height at most n, where n is the maximal height of the derivation of the premise(s)).

Definition 5.3. Let **q** be standing for either R or S. We define substitution as follows:

[16]As consequence of our definition, we obtain (i) $\mathsf{w}(A+B) = \mathsf{w}({\sim}A \rightarrow B) = \mathsf{w}({\sim}A)+\mathsf{w}(B)+1 = \mathsf{w}(A) + \mathsf{w}(B) + 2$, (ii) $\mathsf{w}(A \oplus B) = \mathsf{w}({\sim}({\sim}A \circ {\sim}B)) = \mathsf{w}({\sim}A \circ {\sim}B) + 1 = \mathsf{w}({\sim}A) + \mathsf{w}({\sim}B) + 2 = \mathsf{w}(A) + \mathsf{w}({\sim}B) + 3 = \mathsf{w}(A) + \mathsf{w}(B) + 4$, (iii) $\mathsf{w}(\mathsf{f}) = \mathsf{w}({\sim}\mathsf{t}) = \mathsf{w}(\mathsf{t}) + 1 = 2$ and (iv) $\mathsf{w}(\blacksquare A) = \mathsf{w}(\mathsf{t} \rightarrow A) = \mathsf{w}(\mathsf{t}) + \mathsf{w}(A) + 1 = \mathsf{w}(A) + 2$.

- $\mathbf{q}abc(d/e) \equiv \mathbf{q}abc$, if $e \neq a$, $e \neq b$ and $e \neq c$.
- $\mathbf{q}abc(d/a) \equiv \mathbf{q}dbc$, if $a \neq b$ and $a \neq c$.
- $\mathbf{q}abc(d/b) \equiv \mathbf{q}adc$, if $b \neq a$ and $b \neq c$.
- $\mathbf{q}abc(d/c) \equiv \mathbf{q}abd$, if $c \neq a$ and $c \neq b$.
- $\mathbf{q}aac(d/a) \equiv \mathbf{q}ddc$, if $a = b$ and $a \neq c$.
- $\mathbf{q}abb(d/b) \equiv \mathbf{q}add$, if $b = c$ and $b \neq a$.
- $\mathbf{q}cbc(d/c) \equiv \mathbf{q}dbd$, if $c = a$ and $c \neq b$.
- $\mathbf{q}aaa(d/a) \equiv \mathbf{q}ddd$, if $a = b$ and $a = c$.
- $A\mathsf{G}a(d/e) \equiv A\mathsf{G}a$, if $e \neq a$.
- $A\mathsf{G}a(d/a) \equiv A\mathsf{G}d$.
- $\mathsf{P}a(d/e) \equiv \mathsf{P}a$, if $e \neq a$.
- $\mathsf{P}a(d/a) \equiv \mathsf{P}d$.
- $a : A(d/e) \equiv a : A$, if $e \neq a$.
- $a : A(d/a) \equiv d : A$.

We are now in a position that allows us to extend this definition to multisets.

Lemma 5.1. *Let the variable e stand for either a, b or c. If $\vdash^n \Gamma \Rightarrow \Delta$ and, provided d is free for e in Γ, Δ, then $\vdash^n \Gamma(d/e) \Rightarrow \Delta(d/e)$ (allowing $*$-variables to be substituted to variables as well).*

Proof. By induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$.
Let $n = 0$. If $\Gamma \Rightarrow \Delta$ is an initial sequent and (d/e) is not a vacuous substitution, then the substitution $\Gamma(d/e) \Rightarrow \Delta(d/e)$ is also an initial sequent. Similarly for conclusions of the $\mathbf{G3rE}^{\mathsf{t}}$ rule $R\mathsf{t}$.
Let $n > 0$. We consider the last rule applied in the derivation.

1. If it is a rule for $\wedge$, $\vee$, or $\sim$, we apply the inductive hypothesis to the premise(s) of the rule, and then the rule.
2. We proceed similarly if the last rule is either $R\circ$, $L+^1$, $R\bullet$ or $L \rightarrow$.
3. If the last rule is either $R \rightarrow$, $L\circ$, $R+^1$ or $L\bullet$ and the substitution is vacuous ($e \neq a, b, c$), then there's nothing to do.
   Else, if d is not an eigenvariable, and the substitution d/e) is not vacuous, we apply the inductive hypothesis to the derivation of the premise and, finally, the rule.
   If d is the eigenvariable, we first apply the inductive hypothesis in order to replace the eigenvariable d with a fresh variable (by the variable condition the substitution does not affect neither Γ nor Δ). Finally, we apply the inductive hypothesis to the derivation of the premise(s) and then the rule.

A more detailed case analysis can be found and readapted to the systems of this paper from [13]. $\square$

Importantly, the heredity condition can be reflected at the calculus level by means of formal derivations:

Proposition 5.2. *Sequents of the following form: $R0ab, a : A, \Gamma \Rightarrow \Delta, b : A$ are derivable.*

Proof. By induction on the structure of A.
If $A = B \circ C$, then:

$$\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Rcdb, c : B, \mathcal{S}, \Gamma \Rightarrow \Delta, b : B \circ C, c : B}{Rcdb, c : B, \mathcal{S}, \Gamma \Rightarrow \Delta, b : B \circ C, c : B}\,\text{R1} \qquad \dfrac{R0dd, Rcdb, d : C, \mathcal{S}', \Gamma \Rightarrow \Delta, b : B \circ C, d : C}{Rcdb, d : C, \mathcal{S}', \Gamma \Rightarrow \Delta, b : B \circ C, d : C}\,\text{R1}}{Rcdb, c : B, d : C, R0ab, Rcda, \Gamma \Rightarrow \Delta, b : B \circ C}\,R\circ}{c : B, d : C, R0ab, Rcda, \Gamma \Rightarrow \Delta, b : B \circ C}\,\text{R3}'}{R0ab, a : B \circ C, \Gamma \Rightarrow \Delta, b : B \circ C}\,L\circ \; (c, d \text{ fresh})$$

where $\mathcal{S} = d : C, R0ab, Rcda$ and $\mathcal{S}' = c : B, R0ab, Rcda$.
If $A = B + C$, then:

$$\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Rcda, c : B, \mathcal{S}, \Gamma \Rightarrow \Delta, c : B, d : C}{Rcda, c : B, \mathcal{S}, \Gamma \Rightarrow \Delta, c : B, d : C}\,\text{R1} \qquad \dfrac{R0dd, Rcda, d : C, \mathcal{S}, \Gamma \Rightarrow \Delta, c : B, d : C}{Rcda, d : C, \mathcal{S}, \Gamma \Rightarrow \Delta, c : B, d : C}\,\text{R1}}{Rcda, Rcdb, R0ab, a : B + C, \Gamma \Rightarrow \Delta, c : B, d : C}\,L+^1}{Rcdb, R0ab, a : B + C, \Gamma \Rightarrow \Delta, c : B, d : C}\,\text{R3}''}{R0ab, a : B + C, \Gamma \Rightarrow \Delta, b : B + C}\,R+^1 \; (c, d \text{ fresh})$$

where $\mathcal{S} = Rcdb, R0ab, a : B + C$.
If $A = B \bullet C$, then:

$$\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Scdb, c : B, \mathcal{S}, \Gamma \Rightarrow \Delta, b : B \bullet C, c : B}{Scdb, c : B, \mathcal{S}, \Gamma \Rightarrow \Delta, b : B \bullet C, c : B}\,\text{R1} \qquad \dfrac{R0dd, Scdb, d : C, \mathcal{S}', \Gamma \Rightarrow \Delta, b : B \bullet C, d : C}{Scdb, d : C, \mathcal{S}', \Gamma \Rightarrow \Delta, b : B \bullet C, d : C}\,\text{R1}}{Scdb, c : B, d : C, R0ab, Scda, \Gamma \Rightarrow \Delta, b : B \bullet C}\,R\bullet}{c : B, d : C, R0ab, Scda, \Gamma \Rightarrow \Delta, b : B \bullet C}\,\text{R6}}{R0ab, a : B \bullet C, \Gamma \Rightarrow \Delta, b : B \bullet C}\,L\bullet \; (c, d \text{ fresh})$$

where $\mathcal{S}$ is $d : C, R0ab, Scda$, and $\mathcal{S}'$ abbreviates $c : B, R0ab, Scda$.
If $A = \mathsf{t}$, then:

$$\dfrac{\dfrac{\dfrac{}{R0cb, \mathsf{P}c, R0ca, R0ab, a : \mathsf{t}, \Gamma \Rightarrow \Delta, b : \mathsf{t}}\,R\mathsf{t}}{\mathsf{P}c, R0ca, R0ab, a : \mathsf{t}, \Gamma \Rightarrow \Delta, b : \mathsf{t}}\,\text{R5}}{R0ab, a : \mathsf{t}, \Gamma \Rightarrow \Delta, b : \mathsf{t}}\,L\mathsf{t}$$

In all cases the premises are derivable by the inductive hypothesis. Derivations for A being either $\sim B$ or $B \to C$ can be found in [13]. □

Let φ be standing for one of the relational atoms employed in some of the proof systems mentioned in this paper.

Lemma 5.3. *The rules of weakening:*

$$\frac{\Gamma \Rightarrow \Delta}{a : A, \Gamma \Rightarrow \Delta}\ \text{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : A}\ \text{RW} \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \text{LW}_L$$

are height-preserving admissible.

Proof. By induction on the height n of the derivation of the premise(s).
If $n = 0$, the cases are trivial.
Let $n > 0$. For derivations concluding with rules without variable condition, the result follows straightforwardly by applying the inductive hypothesis to the premise(s) of the rule.
In case the derivation terminates with one of the rules with the eigenvariable condition, we first apply the substitution lemma to the premise(s) of the rule in order to obtain a fresh eigenvariable not clashing with those in the weakening formula ($a : A$ or φ). The conclusion is then obtained by applying the inductive hypothesis and, finally, the rule. As an example, we consider $L\bullet$:

$$\frac{\vdash^n Sazx, a : B, z : C, \Gamma \Rightarrow \Delta}{\vdash^{n+1} x : B \bullet C, \Gamma \Rightarrow \Delta}\ L\bullet \quad \underset{Lem.5.1}{\overset{\text{SUB}(y/a)}{\rightsquigarrow}} \quad \frac{\dfrac{\vdash^n Syzx, y : B, z : C, \Gamma \Rightarrow \Delta}{\vdash^n a : A, Syzx, y : B, z : C, \Gamma \Rightarrow \Delta}\ i.h.}{\vdash^{n+1} a : A, x : B \bullet C, \Gamma \Rightarrow \Delta}\ L\bullet$$

where the clash of variables is avoided by substituting the clashing variable a with a fresh one, namely y.
The same procedure applies if the derivation terminates with an application of a relational rule, and $a : A$ or φ contain some of its eigenvariables. □

Definition 5.4. A rule $\mathbf{r}$ is *height-preserving invertible* just in case: if there is a derivation of the conclusion of $\mathbf{r}$, then there is a dedrivation of premise(s) of $\mathbf{r}$ (with the height at most n, where n is the maximal height of the derivation of the conclusion).

Lemma 5.4. *All logical and relational rules introduced in this article are height-preserving invertible.*

Proof. For each rule $\mathbf{r}$, we have to show that if there is a derivation δ of the conclusion, then there is a derivation δ' of the premise(s), of the same height. For all rules without eigenvariable condition, we use a standard induction on the height of the derivation. For rules subject to the eigenvariable condition as well, but we need to be sure that in the transformed derivation we make use of a fresh label by applying the substitution lemma inside δ', if needed. The same procedures apply to all relational rules as well.
As an interesting example, we show height-preserving invertibility of $L\bullet$. It is proved by induction on the height n of the derivation of $a : A \bullet B, \Gamma \Rightarrow \Delta$. We distinguish three main cases. (1) If $n = 0$, $a : A \bullet B, \Gamma \Rightarrow \Delta$ is an initial sequent, and then also $Sbca, b : A, c : B, \Gamma \Rightarrow \Delta$, is an initial sequent. Let $n > 0$. (2) If $\vdash^{n+1} a : A \bullet B, \Gamma \Rightarrow \Delta$ is concluded by any rule $\mathbf{r}$ other than $L\bullet$, we apply the inductive hypothesis to the premise(s) $a : A \bullet B, \Gamma' \Rightarrow \Delta'$ ($a : A \bullet B, \Gamma'' \Rightarrow \Delta''$) to obtain derivation(s) of height n of $Sbca, b : A, c : B, \Gamma' \Rightarrow \Delta'$ ($Sbca, b : A, c : B, \Gamma'' \Rightarrow \Delta''$). By applying $\mathbf{r}$ we obtain a derivation of height $n+1$ of

$Sbca, b : A, c : B, \Gamma \Rightarrow \Delta$, as desired. (3) If $\vdash^{n+1} a : A \bullet B, \Gamma \Rightarrow \Delta$ is concluded by $L\bullet$, then $Sbca, b : A, c : B, \Gamma \Rightarrow \Delta$ is the requested conclusion of height n, possibly with different eigenvariables, but the desired ones can be obtained by height-preserving substitutions (Lemma 5.1). An analogous reasoning can be applied to all logical and relational rules with variable condition. □

Lemma 5.5. *The rules of contraction:*

$$\frac{a : A, a : A, \Gamma \Rightarrow \Delta}{a : A, \Gamma \Rightarrow \Delta}\,\text{LC} \qquad \frac{\Gamma \Rightarrow \Delta, a : A, a : A}{\Gamma \Rightarrow \Delta, a : A}\,\text{RC} \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\text{LC}_L$$

are height-preserving admissible.

Proof. By induction on the height n of the derivation. If $n = 0$, then the premise is an initial sequent and, therefore, also the contracted sequent is an initial one. (Similarly for conclusions of $R\mathsf{t}$ in $\mathbf{G3rE^t}$).
Let $n > 0$. We consider the last rule applied to derive the premise of contraction.
If the contraction formula is not principal, both occurrences are to be found in the premise(s) of the rule application, which has (have) a decreased derivation height. By applying the inductive hypothesis, we contract those occurrences and apply the rule to obtain our desired sequent.
If the contraction formula is principal, we distinguish three cases:

1. **r** is a rule in which the principal formulas appear also in the premise(s);
2. **r** is a rule where the premises consist of proper subformulas of the conclusion;
3. **r** is a rule where both, labels and proper subformulas of the principal formula, are active formulas.

Ad 1. Contraction is applied by the inductive hypothesis to the premise(s) of the rule. Let $A = B \bullet C$ and $R\bullet$ be the last rule applied:

$$\frac{\vdash^n Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, a : B \bullet C, b : B \qquad \vdash^n Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, a : B \bullet C, c : C}{\vdash^{n+1} Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, a : B \bullet C}\,R\bullet$$

By applying the inductive hypothesis to the premises, we obtain the requested derivation:

$$\frac{\vdash^n Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, b : B \qquad \vdash^n Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, c : C}{\vdash^{n+1} Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C}\,R\bullet$$

Analogously if the last rule applied is $L \rightarrow$.
Let $A = B \circ C$ and suppose that the last rule applied is $R\circ$:

$$\frac{\vdash^n Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, a : B \circ C, b : B \qquad \vdash^n Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, a : B \circ C, c : C}{\vdash^{n+1} Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, a : B \circ C}\,R\circ$$

By applying the inductive hypothesis to the premises, we obtain the requested derivation:

$$\frac{\vdash^n Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, b : B \qquad \vdash^n Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, c : C}{\vdash^{n+1} Rbca, \Gamma \Rightarrow \Delta, a : B \circ C}\,R\circ$$

Finally, let $A = B + C$ and suppose that the last step is $L+^1$:

$$\frac{\vdash^n b : B, Rbca, a : B + C, a : B + C, \Gamma \Rightarrow \Delta \qquad \vdash^n c : C, Rbca, a : B + C, a : B + C, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rbca, a : B + C, a : B + C, \Gamma \Rightarrow \Delta}\ L+^1$$

By applying the inductive hypothesis to the premises, we obtain the desired derivation:

$$\frac{\vdash^n b : B, Rbca, a : B + C, \Gamma \Rightarrow \Delta \qquad \vdash^n c : C, Rbca, a : B + C, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rbca, a : B + C, \Gamma \Rightarrow \Delta}\ L+^1$$

Ad 2. The procedure is rather straightforward as applications of contraction are simply reduced to contractions on less complex formulas.
Ad 3. In this case, the inductive hypothesis is used on formulas of smaller complexity, previously obtained by applications of the inversion lemma. Let $A = B \bullet C$ and suppose that the derivation ends with an application of $L\bullet$:

$$\frac{\vdash^n Sbca, b : B, c : C, a : B \bullet C, \Gamma \Rightarrow \Delta}{\vdash^{n+1} a : B \bullet C, a : B \bullet C, \Gamma \Rightarrow \Delta}\ L\bullet \quad \overset{\text{INV.}}{\underset{Lem.5.4}{\rightsquigarrow}} \quad \text{(x3)}\ \frac{\dfrac{\vdash^n Sbca, b : B, c : C, Sbca, b : B, c : C, \Gamma \Rightarrow \Delta}{\vdash^n Sbca, b : B, c : C, \Gamma \Rightarrow \Delta}\ i.h.}{\vdash^{n+1} a : B \bullet C, \Gamma \Rightarrow \Delta}\ L\bullet$$

An identical reasoning can be carried out for $R \rightarrow$.
If $A = B \circ C$, and it is derived by $L\circ$, then:

$$\frac{\vdash^n Rbca, b : B, c : C, a : B \circ C, \Gamma \Rightarrow \Delta}{\vdash^{n+1} a : B \circ C, a : B \circ C, \Gamma \Rightarrow \Delta}\ L\circ \quad \overset{\text{INV.}}{\underset{Lem.5.4}{\rightsquigarrow}} \quad \text{(x3)}\ \frac{\dfrac{\vdash^n Rbca, b : B, c : C, Rbca, b : B, c : C, \Gamma \Rightarrow \Delta}{\vdash^n Rbca, b : B, c : C, \Gamma \Rightarrow \Delta}\ i.h.}{\vdash^{n+1} a : B \circ C, \Gamma \Rightarrow \Delta}\ L\circ$$

If $A = B + C$, and the derivation ends with an application of $R+^1$, then:

$$\frac{\vdash^n Rbca, \Gamma \Rightarrow \Delta, a : B + C, b : B, c : C}{\vdash^{n+1} \Gamma \Rightarrow \Delta, a : B + C, a : B + C}\ R+^1 \quad \overset{\text{INV.}}{\underset{Lem.5.4}{\rightsquigarrow}} \quad \text{(x3)}\ \frac{\dfrac{\vdash^n Rbca, Rbca, \Gamma \Rightarrow \Delta, b : B, c : C, b : B, c : C}{\vdash^n Rbca, \Gamma \Rightarrow \Delta, b : B, c : C}\ i.h.}{\vdash^{n+1} \Gamma \Rightarrow \Delta, a : B + C}\ R+^1$$

As a final example, let $A = \mathsf{t}$ and consider a derivation ending with $L\mathsf{t}$:

$$\frac{\vdash^n a : \mathsf{t}, \mathsf{P}b, R0ba, a : \mathsf{t}, \Gamma \Rightarrow \Delta}{\vdash^{n+1} a : \mathsf{t}, a : \mathsf{t}, \Gamma \Rightarrow \Delta}\ L\mathsf{t} \quad \overset{i.h.}{\rightsquigarrow} \quad \frac{\vdash^n \mathsf{P}b, R0ba, a : \mathsf{t}, \Gamma \Rightarrow \Delta}{\vdash^{n+1} a : \mathsf{t}, \Gamma \Rightarrow \Delta}\ L\mathsf{t}$$

Notice that when both contraction formulas are principal, we apply the closure condition. □

Theorem 5.6. *The rule of* CUT*:*

$$\frac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{CUT}$$

is admissible.

Proof. The proof is carried out by a lexicographic induction on the complexity of the CUT-formula $a : A$ and the sum of the heights $h(\delta_1) + h(\delta_2)$. We perform a case analysis on the last rule used in the derivation above the CUT, and consider whether it applies to the CUT-formula or not. We show that each application of CUT can either be eliminated, or be replaced by one or more applications of CUT of smaller complexity. The proof proceeds similarly to the CUT-elimination proofs for several labelled calculi, e.g., [41, 24, 29, 13]. Intuitively, we eliminate the left- and topmost CUT first, and proceed by repeating the procedure until we reach a CUT-free derivation. We start by showing that CUT can be eliminated if one of the CUT premises is an initial sequent (**case 1**). Then we show that the CUT-height can be reduced in all cases in which the CUT-formula is not principal in at least one of the CUT-premises (**case 2**). Finally, we show that if the CUT-formula is principal in both CUT-premises, then the CUT is reduced to one or more CUTs on less complex formulas or on shorter derivations (**case 3**). We present some interesting cases where the CUT-formula A is principal in both premises.

(**3.1**) We start by considering a derivation where the last rules applied to obtain the CUT-premises are $R\circ$ and $L\circ$, respectively. Let A be $B \circ C$:

$$\dfrac{\dfrac{Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, b : B \qquad Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, c : C}{Rbca, \Gamma \Rightarrow \Delta, a : B \circ C}\, R\circ \qquad {\scriptstyle (x, y \text{ fresh})}\,\dfrac{Rxya, x : B, y : C, \Gamma' \Rightarrow \Delta'}{a : B \circ C, \Gamma' \Rightarrow \Delta'}\, L\circ}{Rbca, \Gamma, \Gamma' \Rightarrow, \Delta, \Delta'}\, \textsc{cut}$$

It is transformed into the following one:

$$\dfrac{\dfrac{\dfrac{Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, c : C \qquad a : B \circ C, \Gamma' \Rightarrow \Delta'}{Rbca, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : C}\, \textsc{cut} \qquad \begin{array}{c} \vdots\, \delta_1 \\ Rbca, Rbca, c : C, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \end{array}}{Rbca, Rbca, Rbca, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\, \textsc{cut}}{Rbca, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, {\scriptstyle (\text{Lem. } 5.5)}\ \textsc{lc}{+}\ \textsc{rc}{+}\ \textsc{lc}_L$$

where δ_1 is concluded by:

$$\dfrac{\dfrac{Rbca, \Gamma \Rightarrow \Delta, a : B \circ C, b : B \qquad a : B \circ C, \Gamma' \Rightarrow \Delta'}{Rbca, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b : B}\, \textsc{cut} \qquad {\scriptstyle (\text{Lem. } 5.1)}\,\dfrac{{\scriptstyle (\text{Lem. } 5.1)}\,\dfrac{Rxya, x : B, y : C, \Gamma' \Rightarrow \Delta'}{Rbya, b : B, y : C, \Gamma' \Rightarrow \Delta'}\, \textsc{sub}(b/x)}{Rbca, b : B, c : C, \Gamma' \Rightarrow \Delta'}\, \textsc{sub}(c/y)}{Rbca, Rbca, c : C, \Gamma, \Gamma' \Rightarrow, \Delta, \Delta'}\, \textsc{cut}$$

(**3.2**) Assume that the premises of CUT are derived by $R+^1$ and $L+^1$, respectively. Let $A = B + C$:

$$\dfrac{{\scriptstyle (b, c \text{ fresh})}\,\dfrac{Rbca, \Gamma \Rightarrow \Delta, b : B, c : C}{\Gamma \Rightarrow \Delta, a : B + C}\, R+^1 \qquad \dfrac{d : B, Rdea, a : B + C, \Gamma' \Rightarrow \Delta' \qquad e : C, Rade, a : B + C, \Gamma' \Rightarrow \Delta'}{Rdea, a : B + C, \Gamma' \Rightarrow \Delta'}\, L+^1}{Rdea, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \textsc{cut}$$

It is transformed into the following derivation:

$$
\text{(Lem. 5.5)}\ \dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_1\\ Rdea, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', e : C\end{array} \qquad \dfrac{\Gamma \Rightarrow \Delta, a : B + C \qquad e : C, Rdea, a : B + C, \Gamma' \Rightarrow \Delta'}{e : C, Rdea, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}}{Rdea, Rdea, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\ \textsc{cut}}{Rdea, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{lc+ rc+ lc}_L
$$

where the conclusion of δ_1 is derived by:

$$
\text{(Lem. 5.5)}\ \dfrac{\dfrac{\text{(Lem. 5.1)}\ \dfrac{\text{(Lem. 5.1)}\ \dfrac{Rbca, \Gamma \Rightarrow \Delta, b : B, c : C}{Rbea, \Gamma \Rightarrow \Delta, b : B, e : C}\ \textsc{sub}(e/c)}{Rdea, \Gamma \Rightarrow \Delta, d : B, e : C}\ \textsc{sub}(d/b) \qquad \dfrac{\Gamma \Rightarrow \Delta, a : B + C \qquad d : B, Rdea, a : B + C, \Gamma' \Rightarrow \Delta'}{d : B, Rdea, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}}{Rdea, Rdea, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta', e : C}\ \textsc{cut}}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', e : C}\ \textsc{lc+ rc+lc}_L
$$

(**3.3**) Assume that the premises of CUT are derived by $R\bullet$ and $L\bullet$, respectively. Let $A = B \bullet C$:

$$
\dfrac{\dfrac{Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, b : B \qquad Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, c : C}{Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C}\ R\bullet \qquad {\scriptstyle (x,\, y \text{ fresh})}\ \dfrac{Sxya, x : B, y : C, \Gamma' \Rightarrow \Delta'}{a : B \bullet C, \Gamma' \Rightarrow \Delta'}\ L\bullet}{Sbca, \Gamma, \Gamma' \Rightarrow, \Delta, \Delta'}\ \textsc{cut}
$$

The desired transformed derivation is obtained as follows:

$$
\text{(Lem. 5.5)}\ \dfrac{\dfrac{\dfrac{Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, c : C \qquad a : B \bullet C, \Gamma' \Rightarrow \Delta'}{Sbca, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : C}\ \textsc{cut} \qquad \begin{array}{c}\vdots\ \delta_1\\ Sbca, Sbca, c : C, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\end{array}}{Sbca, Sbca, Sbca, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\ \textsc{cut}}{Sbca, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{lc+ rc+ lc}_L
$$

where δ_1 is concluded by:

$$
\dfrac{\dfrac{Sbca, \Gamma \Rightarrow \Delta, a : B \bullet C, b : B \qquad a : B \bullet C, \Gamma' \Rightarrow \Delta'}{Sbca, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b : B}\ \textsc{cut} \qquad \text{(Lem. 5.1)}\ \dfrac{\text{(Lem. 5.1)}\ \dfrac{Sxya, x : B, y : C, \Gamma' \Rightarrow \Delta'}{Sbya, b : B, y : C, \Gamma' \Rightarrow \Delta'}\ \textsc{sub}(b/x)}{Sbca, b : B, c : C, \Gamma' \Rightarrow \Delta'}\ \textsc{sub}(c/y)}{Sbca, Sbca, c : C, \Gamma, \Gamma' \Rightarrow, \Delta, \Delta'}\ \textsc{cut}
$$

(**3.4**) As a final example, assume that the premises of CUT are derived by $R\mathsf{t}$ and $L\mathsf{t}$, respectively. Let $A = \mathsf{t}$:

$$
\dfrac{\dfrac{}{\mathsf{P}b, R0ba, \Gamma \Rightarrow \Delta, a : \mathsf{t}}\ R\mathsf{t} \qquad {\scriptstyle (x \text{ fresh})}\ \dfrac{\mathsf{P}x, R0xa, a : \mathsf{t}, \Gamma' \Rightarrow \Delta'}{a : \mathsf{t}, \Gamma' \Rightarrow \Delta'}\ L\mathsf{t}}{\mathsf{P}b, R0ba, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}
$$

It is transformed into the following derivation:

$$
\text{(Lem. 5.5)}\ \dfrac{\dfrac{\mathsf{P}b, R0ba, \Gamma \Rightarrow \Delta, a : \mathsf{t} \qquad \text{(Lem. 5.1)}\ \dfrac{\mathsf{P}x, R0xa, a : \mathsf{t}, \Gamma' \Rightarrow \Delta'}{\mathsf{P}b, R0ba, a : \mathsf{t}, \Gamma' \Rightarrow \Delta'}\ \textsc{sub}(b/x)}{\mathsf{P}b, R0ba, \mathsf{P}b, R0ba, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}}{\mathsf{P}b, R0ba, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{lc+lc}_L
$$

The proofs for A being either $\sim B$ or $B \to C$ can be found in [13]. □

Proposition 5.7. *The following results can be shown:*

1. *The heredity rules for atomic formulas are height-preserving admissible:*

$$\frac{b : p, R0ab, a : p, \Gamma \Rightarrow \Delta}{R0ab, a : p, \Gamma \Rightarrow \Delta}\ \textsc{AtHer-l} \qquad \frac{R0ab, \Gamma \Rightarrow \Delta, b : p, a : p}{R0ab, \Gamma \Rightarrow \Delta, b : p}\ \textsc{AtHer-r}$$

2. *The heredity rules for compound formulas are admissible:*

$$\frac{b : A, R0ab, a : A, \Gamma \Rightarrow \Delta}{R0ab, a : A, \Gamma \Rightarrow \Delta}\ \textsc{GenHer-l} \qquad \frac{R0ab, \Gamma \Rightarrow \Delta, b : A, a : A}{R0ab, \Gamma \Rightarrow \Delta, b : A}\ \textsc{GenHer-r}$$

3. *The following rules, relating the concepts of entailment and implication (i.e., the sequent-style version of the Verification Lemma, p. 4), are admissible:*

$$\frac{\Rightarrow 0 : A \to B}{a : A \Rightarrow a : B}\ \textsc{ver1} \qquad \frac{a : A \Rightarrow a : B}{\Rightarrow 0 : A \to B}\ \textsc{ver2}$$

4. *The following rules for the two defined fission connectives (see Section 2, p. 8) are admissible:*

$$\frac{b^* : A, Rabc, a : A + B, \Gamma \Rightarrow \Delta \qquad c : B, Rabc, a : A + B, \Gamma \Rightarrow \Delta}{Rabc, a : A + B, \Gamma \Rightarrow \Delta}\ L{+}^2$$

$$\text{(b, c fresh)}\ \frac{Rabc, \Gamma \Rightarrow \Delta, b^* : A, c : B}{\Gamma \Rightarrow \Delta, a : A + B}\ R{+}^2$$

$$\frac{b^* : A, Rbca^*, a : A \oplus B, \Gamma \Rightarrow \Delta \qquad c^* : B, Rbca, a : A \oplus B, \Gamma \Rightarrow \Delta}{Rbca^*, a : A \oplus B, \Gamma \Rightarrow \Delta}\ L\oplus$$

$$\text{(b, c fresh)}\ \frac{Rbca^*, \Gamma \Rightarrow \Delta, b^* : A, c^* : B}{\Gamma \Rightarrow \Delta, a : A \oplus B}\ R\oplus$$

5. *The "generation rule" (see Section 3, p. 12) is height-preserving admissible.*

$$\frac{A\mathsf{G}a, a : A, \Gamma \Rightarrow \Delta}{A\mathsf{G}a, \Gamma \Rightarrow \Delta}\ L\mathsf{G}$$

6. *The "necessitation rule" (i.e.,* Rt2*, Section 4, p. 16) is admissible.*

$$\frac{\Rightarrow 0 : A}{\Rightarrow 0 : \mathsf{t} \to A}\ \textsc{nec}$$

7. *The rules for ■ (see Section 4, pp. 20 and ff.) are admissible.*

$$\text{(x fresh)}\ \frac{\mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta, b : A}{\Gamma \Rightarrow \Delta, a : \blacksquare A}\ R\blacksquare \qquad \frac{b : A, a : \blacksquare A, \mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta}{a : \blacksquare A, \mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta}\ L\blacksquare$$

Proof. The proofs of *1.*, *2.* and *3.* can all be found in [13].

Ad *4.* We proceed by root-first derivation. Let's start by considering the rules for $A + B =_{\mathsf{df}} \sim A \to B$. For $L{+}^2$:

$$\dfrac{\dfrac{\dfrac{b^* : A, Rabc, a : A + B, \Gamma \Rightarrow \Delta}{b^* : A, Rabc, a : {\sim}A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{df}+}{Rabc, a : {\sim}A \to B, \Gamma \Rightarrow \Delta, b : {\sim}A}\ R{\sim} \qquad \dfrac{c : B, Rabc, a : A + B, \Gamma \Rightarrow \Delta}{c : B, Rabc, a : {\sim}A \to B, \Gamma \Rightarrow \Delta}\ \mathsf{df}+}{\dfrac{Rabc, a : {\sim}A \to B, \Gamma \Rightarrow \Delta}{Rabc, a : A + B, \Gamma \Rightarrow \Delta}\ \mathsf{df}+}\ L\to$$

For $R+^2$:

$$(b, c \text{ fresh})\quad \dfrac{\dfrac{\dfrac{Rabc, \Gamma \Rightarrow \Delta, b^* : A, c : B}{Rabc, b : {\sim}A, \Gamma \Rightarrow \Delta, c : B}\ L{\sim}}{\Gamma \Rightarrow \Delta, a : {\sim}A \to B}\ R\to}{\Gamma \Rightarrow \Delta, a : A + B}\ \mathsf{df}+$$

Finally, we consider $A \oplus B =_{\mathsf{df}} {\sim}({\sim}A \circ {\sim}B)$. For $L\oplus$:

$$\dfrac{\dfrac{(\text{Lem. } 5.3)\ \dfrac{\dfrac{b^* : A, Rbca^*, a : A \oplus B, \Gamma \Rightarrow \Delta}{b^* : A, Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta}\ \mathsf{df}\oplus}{b^* : A, Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta, a^* : {\sim}A \circ {\sim}B}\ \text{RW}}{Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta, a^* : {\sim}A \circ {\sim}B, b : {\sim}A}\ R{\sim} \quad \dfrac{(\text{Lem. } 5.3)\ \dfrac{\dfrac{c^* : B, Rbca^*, a : A \oplus B, \Gamma \Rightarrow \Delta}{c^* : B, Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta}\ \mathsf{df}\oplus}{c^* : B, Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta, a^* : {\sim}A \circ {\sim}B}\ \text{RW}}{Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta, a^* : {\sim}A \circ {\sim}B, c : {\sim}B}\ R{\sim}}{Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta, a^* : {\sim}A \circ {\sim}B}\ R\circ$$

$$(\text{Lem. } 5.5)\ \dfrac{\dfrac{\dfrac{Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta, a^* : {\sim}A \circ {\sim}B}{Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta}\ L{\sim}}{Rbca^*, a : {\sim}({\sim}A \circ {\sim}B), \Gamma \Rightarrow \Delta}\ \text{LC}}{Rbca^*, a : A \oplus B, \Gamma \Rightarrow \Delta}\ \mathsf{df}\oplus$$

For $R\oplus$:

$$(b, c \text{ fresh})\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Rbca^*, \Gamma \Rightarrow \Delta, b^* : A, c^* : B}{Rbca^*, c : {\sim}B, \Gamma \Rightarrow \Delta, b^* : A}\ L{\sim}}{Rbca^*, b : {\sim}A, c : {\sim}B, \Gamma \Rightarrow \Delta}\ L{\sim}}{a^* : {\sim}A \circ {\sim}B, \Gamma \Rightarrow \Delta}\ L\circ}{\Gamma \Rightarrow \Delta, a : {\sim}({\sim}A \circ {\sim}B)}\ R{\sim}}{\Gamma \Rightarrow \Delta, a : A \oplus B}\ \mathsf{df}\oplus$$

Ad *5.* We consider **G3rB$^\bullet$** (and extensions thereof) and prove the claim by induction on the height n of the derivation. If $n = 0$, then the sequent is an initial one, and we obtain our desired result as follows:

$$A\mathsf{G}a, a : A, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p \ \overset{i.h.}{\rightsquigarrow}\ A\mathsf{G}a, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$$

where the rightmost sequent is obtained by applying the induction hypothesis to the leftmost one.

Let $n > 0$. If the derivation ends with an application of rules without eigenvariable condition, then we simply apply the inductive hypothesis to the premises and finally the rule. We display the cases for derivations ending with $L\bullet$ and $R\to$ (subject to the eigenvariable condition):

$$\dfrac{\vdash^n Sbca, b : B, c : C, A\mathsf{G}a, a : A, \Gamma \Rightarrow \Delta}{\vdash^{n+1} a : B \bullet C, A\mathsf{G}a, a : A, \Gamma \Rightarrow \Delta}\ L\bullet \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n Sbca, b : B, c : C, A\mathsf{G}a, \Gamma \Rightarrow \Delta}{\vdash^{n+1} a : B \bullet C, A\mathsf{G}a, \Gamma \Rightarrow \Delta}\ L\bullet$$

And, finally:

$$\dfrac{\vdash^n Rabc, b : B, A\mathsf{G}a, a : A, \Gamma \Rightarrow \Delta, c : C}{\vdash^{n+1} A\mathsf{G}a, a : A, \Gamma \Rightarrow \Delta, a : B \to C}\, R{\to} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n Rabc, b : B, A\mathsf{G}a, \Gamma \Rightarrow \Delta, c : C}{\vdash^{n+1} A\mathsf{G}a, \Gamma \Rightarrow \Delta, a : B \to C}\, R{\to}$$

If the last rule applied is a relational rule, the procedure is identical.

Ad *6.* Consider the system **G3rE**$^{\mathsf{t}}$. We show that, if $\Rightarrow 0 : A$, then $\Rightarrow 0 : \mathsf{t} \to A$.

$$\dfrac{(a, b \text{ fresh})\quad \dfrac{(\text{Lem. } 5.3)\quad \dfrac{(\text{Lem. } 5.1)\quad \dfrac{\Rightarrow 0 : A}{\Rightarrow b : A}\,\textsc{sub}(b/0)}{R0ab, a : \mathsf{t} \Rightarrow b : A}\,\textsc{lw}{+}\textsc{lw}_L}{\Rightarrow 0 : \mathsf{t} \to A}\, R \to$$

Ad *7.* Consider the rules of **G3rE**$^{\mathsf{t}}$. We proceed by root-first derivation. For $R\blacksquare$:

$$\dfrac{(y, b \text{ fresh})\quad \dfrac{(x \text{ fresh})\quad \dfrac{\dfrac{(\text{Lem } 5.3)\quad \dfrac{\mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta, b : A}{\mathsf{P}x, Raxb, R0xy, Rayb, y : \mathsf{t}, \Gamma \Rightarrow \Delta, b : A}\,\textsc{lw}{+}\textsc{lw}_L}{\mathsf{P}x, R0xy, Rayb, y : \mathsf{t}, \Gamma \Rightarrow \Delta, b : A}\,\text{T3}}{Rayb, y : \mathsf{t}, \Gamma \Rightarrow \Delta, b : A}\, L\mathsf{t}}{\dfrac{\Gamma \Rightarrow \Delta, a : \mathsf{t} \to A}{\Gamma \Rightarrow \Delta, a : \blacksquare A}\,\mathsf{df}\blacksquare}\, R \to$$

For $L\blacksquare$:

$$\dfrac{\dfrac{(\text{Lem } 5.3)\quad \dfrac{\dfrac{\dfrac{}{\mathsf{P}x, R0xx, \Gamma \Rightarrow \Delta, x : \mathsf{t}}\, R\mathsf{t}}{\mathsf{P}x, \Gamma \Rightarrow \Delta, x : \mathsf{t}}\,\text{R1}}{a : \mathsf{t} \to A, \mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta, x : \mathsf{t}}\,\textsc{lw}{+}\textsc{lw}_L \qquad \dfrac{b : A, a : \blacksquare A, \mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta}{b : A, a : \mathsf{t} \to A, \mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta}\,\mathsf{df}\blacksquare}{\dfrac{a : \mathsf{t} \to A, \mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta}{a : \blacksquare A, \mathsf{P}x, Raxb, \Gamma \Rightarrow \Delta}\,\mathsf{df}\blacksquare}\, L \to$$

□

6 Soundness and completeness

We conclude the paper by showing that our labelled calculi are sound and complete. Roughly, we prove that:

$$\Gamma \Rightarrow \Delta \text{ is provable in } \mathbf{G3rX} \Longleftrightarrow \Gamma \Rightarrow \Delta \text{ is valid in every frame for } \mathbf{X}$$

where **X** is any of the relevant logics discussed in this paper.

Soundness ($\Longrightarrow$). The proof consists in showing that the rules of each labelled calculus presented in this article preserve validity over Routley-Meyer frames. Let $\mathscr{F}$ and $\mathscr{M}$ be, respectively, any of the Routley-Meyer frames and models discussed in this paper. We start by extending semantic notions to sequents as follows:

Definition 6.1. Let $\mathbf{s}$ be any labelled sequent of the form $\Gamma \Rightarrow \Delta$. An $\mathbf{s}$-*interpretation in* $\mathscr{M}$ is a mapping $[\![\cdot]\!]$ from the labels in $\mathbf{s}$ to the set K of states in $\mathscr{M}$, such that (i) $0 = [\![0]\!]$ and (ii) if $Rabc$ (or $Sbca$, or $A\mathsf{G}a$, or $\mathsf{P}a$) is in Γ, then $R[\![a]\!][\![b]\!][\![c]\!]$ (or $S[\![b]\!][\![c]\!][\![a]\!]$, or $A\mathsf{G}[\![a]\!]$, or $P[\![a]\!]$) holds in $\mathscr{M}$. Now we can define:

$\mathscr{M}, [\![\cdot]\!] \Vdash \mathbf{s}$ iff if for all $a : A \in \Gamma$, we have $\mathscr{M}, [\![a]\!] \Vdash A$, then there exists $b : B \in \Delta$, such that $\mathscr{M}, [\![b]\!] \Vdash B$.

Definition 6.2. A sequent $\mathbf{s}$ is *satisfied* in $\mathscr{M}$ if for all $\mathbf{s}$-interpretations $[\![\cdot]\!]$ we have $\mathscr{M}, [\![\cdot]\!] \Vdash \mathbf{s}$. A sequent $\mathbf{s}$ is valid in a frame $\mathscr{F}$, if for all valuations v, the sequent $\mathbf{s}$ is satisfied in $\mathscr{M}$.

Finally, we can prove the requested soundness result:

Theorem 6.1. *If a sequent* $\mathbf{s}$ *is provable in* **G3rX***, then it is valid in every Routley-Meyer frame for* **X***.*

Proof. We proceed by induction on the height of the derivation of $\mathbf{s}$. We show that for each rule $\mathbf{r}$ of the form $\mathcal{P}_1, \ldots, \mathcal{P}_n/\mathcal{C}$, if the premises $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are valid in all Routley-Meyer frames, then so is $\mathcal{C}$. It follows from a case analysis on $\mathbf{r}$:

In. By way of contradiction, assume that $R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$ is not valid in all Routley-Meyer frames. This means that there is a model $\mathscr{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathscr{M}, [\![\cdot]\!] \nVdash R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$, i.e., $R[\![0]\!][\![a]\!][\![b]\!]$ and $\mathscr{M}, a \Vdash p$, but $\mathscr{M}, b \nVdash p$. However, this is not possible given heredity (see p. 4).

$R+^1$. By way of contradiction, assume that $Rbca, \Gamma \Rightarrow \Delta, b : A, c : B$ is valid in all Routley-Meyer frames, but $\Gamma \Rightarrow \Delta, a : A + B$ is not, where $b, c \notin \Gamma, \Delta$. The latter means that there is a model $\mathscr{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathscr{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, a : A + B$. In particular, we know that there are worlds b' and c' such that $Rb'c'[\![a]\!]$ and $\mathscr{M}, b' \nVdash A$, and $\mathscr{M}, c' \nVdash B$. Now we define an extension $[\![\cdot]\!]'$ of $[\![\cdot]\!]$ such that $[\![b]\!]' = b'$, $[\![c]\!]' = c'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$. Then, $\mathscr{M}, [\![\cdot]\!]' \nVdash Rbca, \Gamma \Rightarrow \Delta, b : A, c : B$. Contradiction.

$L+^1$. By way of contradiction, assume that $b : A, Rbca, \Gamma \Rightarrow \Delta$ and $c : B, Rbca, \Gamma \Rightarrow \Delta$ are valid in all Routley-Meyer frames, but $Rbca, a : A + B, \Gamma \Rightarrow \Delta$ is not. The latter means that there is a model $\mathscr{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathscr{M}, [\![\cdot]\!] \nVdash Rbca, a : A + B, \Gamma \Rightarrow \Delta$, i.e., $R[\![b]\!][\![c]\!][\![a]\!]$ and $\mathscr{M}, a \Vdash A + B$, but $\mathscr{M}, x \nVdash D$ for all $x : D \in \Delta$. However, by the clause for $+$ (p. 5) we also have $\mathscr{M}, b \Vdash A$ or $\mathscr{M}, c \Vdash B$. Consequently, $\mathscr{M}, [\![\cdot]\!] \nVdash b : A, Rbca, \Gamma \Rightarrow \Delta$ or $\mathscr{M}, [\![\cdot]\!] \nVdash c : B, Rbca, \Gamma \Rightarrow \Delta$. Contradiction.

$R\bullet$. By way of contradiction, assume that $Sbca, \Gamma \Rightarrow \Delta, b : A$ and $Sbca, \Gamma \Rightarrow \Delta, c : B$ are valid in all Routley-Meyer frames, but $Sbca, \Gamma \Rightarrow \Delta, a : A \bullet B$ is not. The latter means that there is a model $\mathscr{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathscr{M}, [\![\cdot]\!] \nVdash Sbca, \Gamma \Rightarrow \Delta, a : A \bullet B$, i.e., $S[\![b]\!][\![c]\!][\![a]\!]$ but $\mathscr{M}, a \nVdash A \bullet B$. By the forcing clause for $\bullet$ (Section 3), then we have $\mathscr{M}, b \nVdash A$ or $\mathscr{M}, c \nVdash B$. As a consequence, we obtain $\mathscr{M}, [\![\cdot]\!] \nVdash Sbca, \Gamma \Rightarrow \Delta, b : A$ or $\mathscr{M}, [\![\cdot]\!] \nVdash Sbca, \Gamma \Rightarrow \Delta, c : B$. Contradiction.

$L\bullet$. By way of contradiction, assume that $Sbca, b : A, c : B, \Gamma \Rightarrow \Delta$ is valid, but $a : A \bullet B, \Gamma \Rightarrow \Delta$ is not, where $b, c \notin \Gamma, \Delta$. The latter means that there is a model $\mathscr{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathscr{M}, [\![\cdot]\!] \nVdash a : A \bullet B, \Gamma \Rightarrow \Delta$, i.e., $\mathscr{M}, a \nVdash A \bullet B$, but $\mathscr{M}, x \nVdash D$, for $x : D \in \Delta$. By the forcing clause for $\bullet$, then we have that there are worlds b', c' such that $Sb'c'[\![a]\!]$, $\mathscr{M}, b' \Vdash A$ and $\mathscr{M}, c' \Vdash B$. Let $[\![\cdot]\!]'$ be an extension of $[\![\cdot]\!]$ such that $[\![b]\!]' = b'$, $[\![c]\!]' = c'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$. As a consequence, we obtain $\mathscr{M}, [\![\cdot]\!]' \nVdash Sbca, b : A, c : B, \Gamma \Rightarrow \Delta$. Contradiction.

An analogous reasoning can be carried out with rules $R\circ$ and $L\circ$ of Section 2. As a final example, we consider the rules for t of $\mathbf{G3rE^t}$ (Section 4):

$R\mathsf{t}$. By way of contradiction, assume that $\mathsf{P}b, R0ba, \Gamma \Rightarrow \Delta, a : \mathsf{t}$ is not valid in all Routley-Meyer frames. This means that there is a model $\mathscr{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathscr{M}, [\![\cdot]\!] \nVdash \mathsf{P}b, R0ba, \Gamma \Rightarrow \Delta, a : \mathsf{t}$, i.e., $\mathsf{P}[\![b]\!]$ and $R[\![0]\!][\![b]\!][\![a]\!]$, but $\mathscr{M}, a \nVdash \mathsf{t}$. However, by the forcing condition for t, from $\mathsf{P}[\![b]\!]$ and $R[\![0]\!][\![b]\!][\![a]\!]$, it follows that $\mathscr{M}, a \Vdash \mathsf{t}$. Contradiction.

$L\mathsf{t}$. By way of contradiction, assume that $\mathsf{P}b, R0ba, a : \mathsf{t}, \Gamma \Rightarrow \Delta$ is valid in all Routley-Meyer frames, but $a : \mathsf{t}, \Gamma \Rightarrow \Delta$ is not, where $b \notin \Gamma, \Delta$. The latter means that there is a model $\mathscr{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathscr{M}, [\![\cdot]\!] \nVdash a : \mathsf{t}, \Gamma \Rightarrow \Delta$. In particular, we know that there are worlds b' such that $\mathsf{P}b'$, $R[\![0]\!]b'[\![a]\!]$ and $\mathscr{M}, b' \Vdash \mathsf{t}$, but $\mathscr{M}, x \nVdash D$, for all $x : D \in \Delta$. Now we define an extension $[\![\cdot]\!]'$ of $[\![\cdot]\!]$ such that $[\![b]\!]' = b'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$. Then, $\mathscr{M}, [\![\cdot]\!]' \nVdash \mathsf{P}b, R0ba, a : \mathsf{t}, \Gamma \Rightarrow \Delta$. Contradiction.

The remaining cases, especially those concerning the rules for $\sim$ and $\rightarrow$, can be recovered from [13]. $\square$

Completeness ($\Longleftarrow$). In what follows, we prove that, for every sequent $\mathbf{s}$, the proof search either terminates in a proof or fails, and the failed proof tree is used to obtain a countermodel for $\mathbf{s}$. Intuitively, to see whether A is derivable, we check if it is valid at the actual world $0 \in W$, i.e., $0 \Vdash A$. This, indeed, will correspond to have the sequent $\Rightarrow 0 : A$ in our calculus.[17] Again, let $\mathbf{X}$ be any of the relevant logics dealt with in this article.

Theorem 6.2. *Let* $\mathbf{s}$ *be a sequent. Then either* $\mathbf{s}$ *is derivable in* $\mathbf{G3rX}$ *or* $\mathbf{s}$ *has a countermodel with the frame properties peculiar for* $\mathbf{X}$.

Proof. We proceed with the construction of a derivation tree for $\Gamma \Rightarrow \Delta$ by applying the rules of $\mathbf{G3rX}$ root-first. (For the complete development of the argument see [13, Appendix 1]). If the reduction tree is finite, i.e., all leaves are initial sequents, we have a proof in $\mathbf{G3rX}$. Assume that the derivation tree is infinite. By Königs lemma, it has an infinite branch that is used to build the needed counterexample. Suppose that $\Gamma \Rightarrow \Delta \equiv \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_i \Rightarrow \Delta_i \ldots$ is one of such branches. Consider the sets $\mathbf{\Gamma} \equiv \bigcup \Gamma_i$ and $\mathbf{\Delta} \equiv \bigcup \Delta_i$, for $i \geq 0$. We now construct a countermodel, i.e. a model that makes all labelled formulas and relational atoms in $\mathbf{\Gamma}$

[17] As remarked in [13], this correspond to reflect, at the calculus level, the *actualistic* notion of validity employed in reduced Routley-Meyer models. See also [46, p. 276].

true and all labelled formulas in $\mathbf{\Delta}$ false. We show that, A is forced in the model at 0 if $0 : A$ is in $\mathbf{\Gamma}$ and A is not forced at 0 if $0 : A$ is in $\mathbf{\Delta}$. We will end up with a countermodel to the endsequent. We display some relevant cases:

- If p is atomic, the claim holds by definition of the model.
- If $0 : A + B$ is in $\mathbf{\Delta}$, at the successive step of the reduction tree we find that $Rab0$ and that both $a : A$, $b : B$ are in $\mathbf{\Delta}$. By the inductive hypothesis we obtain $Rab0$ and $a \nVdash A$ and $b \nVdash B$, that is, $0 \nVdash A + B$ in the model.
- If $0 : A + B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms $Rab0$ that occur in $\mathbf{\Gamma}$. If there's no relational atom, the accessibility condition is vacuously satisfied and, therefore, $0 \Vdash A + B$ is in the model. For any occurrence of $Rba0$ in $\mathbf{\Gamma}$, by construction of the tree $a : A$ is in $\mathbf{\Gamma}$ or $b : A$ is in $\mathbf{\Gamma}$. By the inductive hypothesis $a \Vdash A$ or $b \Vdash B$, and since $Rab0$, we obtain $0 \Vdash A + B$ in the model.
- If $0 : A \bullet B$ is in $\mathbf{\Delta}$, we consider all the relational atoms $Sab0$ that occur in $\mathbf{\Gamma}$. If there's no relational atom, the accessibility condition is vacuously satisfied and, therefore, $0 \nVdash A \bullet B$ is in the model. For any occurrence of $Sab0$ in $\mathbf{\Gamma}$, by construction of the tree $a : A$ is in $\mathbf{\Delta}$ or $b : A$ is in $\mathbf{\Delta}$. By the inductive hypothesis $a \nVdash A$ or $b \nVdash B$, and since $Sab0$, we obtain $0 \nVdash A \bullet B$ in the model. (The case for $0 : A \circ B$ in $\mathbf{\Delta}$ is analogous.)
- If $0 : A \bullet B$ is in $\mathbf{\Gamma}$, at the successive step of the reduction tree we find that $Sab0$, $a : A$ and $b : B$ are in $\mathbf{\Gamma}$. By the inductive hypothesis we obtain $Sab0$, $a \Vdash A$ and $b \Vdash B$, that is, $0 \Vdash A \bullet B$ in the model. (The case for $0 : A \circ B$ in $\mathbf{\Gamma}$ is analogous.)
  As a final example, we consider the rules for $\mathtt{t}$ of Section 4:
- If $0 : \mathtt{t}$ is in $\mathbf{\Delta}$ and $\mathsf{P}b, R0b0$ is in $\mathbf{\Gamma}$, then by the inductive hypothesis we obtain $\mathsf{P}b, R0b0$, but $0 \nVdash \mathtt{t}$ in the model.
- If $0 : \mathtt{t}$ is in $\mathbf{\Gamma}$, then at the successive step of the reduction tree we find that $\mathsf{P}b, R0b0$ and $0 : \mathtt{t}$ are in $\mathbf{\Gamma}$. By the inductive hypothesis we obtain $\mathsf{P}b, R0b0$, and $0 \Vdash \mathtt{t}$ in the model.

The remaining cases can be checked in [13]. □

This proof directly implies our desired result:

Corollary 6.2.1. If a sequent **s** is valid in every Routley-Meyer frame for **X**, then **s** is derivable in the system **G3rX**.

7 Alternative proof systems and final remarks

Despite their complexities and subtleties, proof-theoretic studies of relevant logics have received extensive attention. In this paper, we have introduced labelled sequent calculi for a broad range of relevant logics including fusion, fission, and Ackermann's truth constant. Our approach involves incorporating semantic information from reduced Routley-Meyer models at the syntactic level. We have established soundness and completeness of these calculi. Additionally, we have demonstrated height-preserving invertibility of the rules, height-preserving admissibility of structural rules, and CUT-admissibility. In this final section, we will contextualize our current work within the broader landscape of proof systems for relevant logics by exploring various frameworks. Additionally, we will conclude this section by discussing both the advantages and limitations of the approach taken in this paper, while also suggesting potential directions for future research.

Firstly, by employing standard Gentzen sequents, as outlined, for instance, in [49, pp. 50–51], one can start with a sequent calculus for linear propositional logic and incorporate the left and right contraction rules to define the *distribution-free* version of **R**. Indeed, in such a sequent system one cannot deduce the distribution axiom $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$, meaning it lacks the strength required to establish completeness with respect to the full Hilbert-style presentation of **R**. Other systems, such as $\mathbf{R}^{\to}$ and $\mathbf{E}^{\to}$, which represent the implicational fragments of **R** and **E** respectively, can be characterized using Gentzen's sequent calculus for intuitionistic logic **LJ** as base framework. Specifically, these systems are defined by introducing specific constraints, particularly on the right rule for $\to$, if $\mathbf{E}^{\to}$ is considered. This additional constraint requires that in each root-first application of the right rule for $\to$, every formula in the antecedent must take the form of $A \to B$ (see [25, p. 264]). In addition to the previously mentioned fragments of both **R** and **E**, Bimbó [9, pp. 103–122] also explores ordinary sequent systems for the $\{\sim, \to\}$-fragments of **R** and **E**, their reformulation with the truth constant $\mathtt{t}$, as well as systems for some of **R**'s and **E**'s modal expansions. However, the success of these strategies is mostly limited to the mentioned fragments of **R** and **E**, and to capture a broader spectrum of relevant logics, more finely-grained proof-theoretic techniques are needed.

To begin with, Anderson and Belnap's early work [5] provides well-known examples of proof systems, encompassing both sequent and natural deduction calculi for relevant logics. Concerning natural deduction systems, they proposed rules with labels, specifically natural numbers, to track the *actual uses* of assumptions in derivations. The rules, presented in Jaśkowski-Fitch style, for introducing and eliminating implications take the following form:

$$(\to E) \quad \left|\begin{array}{l} A_a \\ A \to B_b \\ B_{a \cup B} \end{array}\right. \qquad\qquad (\to I) \quad \left|\begin{array}{l} \left|\begin{array}{l} \underline{A_{\{k\}}} \\ \vdots \\ B_a \end{array}\right. \\ A \to B_{a-\{k\}} \end{array}\right.$$

Alternatively, these rules can be presented in a Gentzen-Prawitz style as follows:

$$\frac{A_a \qquad A \to B_b}{B_{a \cup B}}\,(\to E) \qquad\qquad \frac{\begin{array}{c} \overline{A_{\{k\}}} \\ \vdots \\ B_a \end{array}}{A \to B_{a-\{k\}}}\,(\to I)$$

Importantly, in $(I \to)$, we require that $k \in a$. Intuitively, these rule represent an easy way of handling with numerals to express the relevance conditions by ensuring that for a formula B to follow from a formula A, the assumption A must be *actually used* in proving the conclusion B. More precisely, the rules for $\to$ displayed above allow for: (1) the inference of $B_{a\cup b}$ from A_a and $A \to B_b$; and (2) the deduction of $A \to B_{a-\{k\}}$ from a proof of B_a based on the hypothesis $A_{\{k\}}$, provided that $k \in a$ (with $a, b, \dots$ ranging over numerals). The resulting natural deduction systems, extensively studied in [5], have subsequently been adapted to encompass a wide range of quantified relevant logics in Brady's work [10]. Brady's research in [12] also focuses on normalization procedures, using the Fitch-style natural deduction system for propositional **DW** as a case study. Additionally, Standefer in [64] devised Fitch-style natural deduction systems for propositional **E**, while Anderson, in his earlier publication [3], introduced Fitch-style calculi also for quantified **E**. Interestingly, Standefer's work in [66] also explores translations between linear and tree natural deduction systems for relevant logics.

As for sequent-style frameworks, it's worth mentioning that besides the use of ordinary sequents, Anderson and Belnap in [5, pp. 57–69] introduced the systems they labelled *merge calculi* (see also [9, pp. 135–137]). The idea is that in the formulation of rules, $\Gamma, \Delta, \dots$ represent sequences of formulas subject to a special operation called *merge*, which combines sequences of formulas. Specifically, the elements of the sequence represented by Γ may be distributed in the "interstices" of Δ as long as the elements of Γ retain their internal ordering, formally denoted as $\mu(\Gamma, \Delta)$.[18] For example, if $\Gamma = A_1, A_2, A_3$ and $\Delta = B_1, B_2, B_3$, then the sequence $\Pi = A_1, B_1, B_2, A_2, A_3, B_3$ is one of the merges of Γ and Δ, that is $\Pi \in \mu(\Gamma, \Delta)$. These systems were used to capture relevant logics $\mathbf{T}^{\to}$, $\mathbf{E}^{\to}$, and $\mathbf{R}^{\to}$. The following rules for $\to$ are used:

$$\frac{\Gamma, \Delta \Rightarrow A \qquad \Pi, B, \Sigma \Rightarrow C}{\mu(\Gamma, \Pi, A \to B), \Delta, \Sigma \Rightarrow C}\ L \to_\mu \qquad\qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ R \to_\mu$$

In addition to the single-conclusion restriction on the right-hand side of sequents, the following restrictions must be imposed on $L \to_\mu$: for $\mathbf{T}^{\to}$, we impose that $\Delta \neq \varnothing$; for $\mathbf{E}^{\to}$, it is required that either $\Delta \neq \varnothing$ or, if $\Delta = \varnothing$, then A must be of the form $B \to C$, for some B, C. For $\mathbf{R}^{\to}$, no restriction on $L \to_\mu$ is needed.

To explore more complex languages, including those with conjunction and disjunction connectives, and hence to reintroduce the distribution axiom at the level of Hilbert-style systems, it becomes necessary to introduce those frameworks known as *Dunn-Mints* (or *consecution*) *calculi*. These systems were independently developed by Mints in his 1972 paper [39] and by Dunn in his 1973 contribution [14]. In this framework, there are two distinct operations for composing data on the left-hand side of sequents. In addition to the standard comma ",", which is used in Gentzen's sequents to represent extensional conjunction at the meta-level, we introduce the semicolon ";" to denote intensional conjunction (see [49, pp. 127–130], [9, pp. 123-129], [25, pp. 267–268]). Usually, the elements that in consecution calculi are grouped together by the separators "," and ";" are referred to as *structures* and are denoted by X, Y, etc. Formally, (X, Y) and $(X; Y)$ represent extensional and intensional structures, respectively, and the notation $X[Y]$ is used to denote that Y

[18]"Metaphorically speaking, a merge of two sequences is like closing a zipper with unevenly spaced juts: the juts on each strip retain their place with respect to each other, but a group of juts on one strip can jostle between a pair of juts on the other strip" [9, p. 135].

is a substructure of X. For example, the consecution calculus for the $\{\wedge, \vee, \rightarrow, \circ, \mathsf{t}\}$-fragment of **R** comprises the following rules for $\rightarrow$, $\circ$ and t employing ";" rather than ",":

$$\frac{X \Rightarrow A \qquad Y[B] \Rightarrow C}{Y[X; A \rightarrow B] \Rightarrow C}\ L\rightarrow; \qquad\qquad \frac{X; A \Rightarrow B}{X \Rightarrow A \rightarrow B}\ R\rightarrow;$$

$$\frac{X[A; B] \Rightarrow C}{X[A \circ B] \Rightarrow C}\ L\circ; \qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X; Y \Rightarrow A \circ B}\ R\circ; \qquad \frac{X[A] \Rightarrow C}{X[A; \mathsf{t}] \Rightarrow C}\ L\mathsf{t};$$

Furthermore, the precise distinction between ";" and "," enables one to preserve a version of the rule of weakening while simultaneously preventing the derivation of irrelevant formulas. Such a weakening rule takes the following form:

$$\frac{X[Y] \Rightarrow C}{X[Y, Z] \Rightarrow C}\ LW,$$

where $Y \neq \varnothing$, and any root-first application of the rule can be performed only on extensional structures.

Consecution calculi represent a cornerstone in the development of proof theory for relevant logics. However, their application becomes more complex when negation is among the connectives. A successful approach to dealing with negation is provided by the display calculus, a framework first discussed by Belnap in [8]. In this framework, not only are ";" and "," introduced, but a variety of *structural connectives* is also employed, making the structural reasoning used in derivations *visually* explicit. The fundamental feature of display calculi is that given a formula and a sequent containing it, it is always possible to transform that sequent into an equivalent one where the given formula is either the entire antecedent or the entire succedent (such a feature is called *display property*, hence the name display calculus). Structures denoted by X, Y, etc., are used, multiple-conclusion sequents are allowed, and as a consequence structures are combined not only on the left-hand side but also on the right-hand side of sequents. Belnap introduced display logic as a proof-theoretic tool for the investigation of a wide range of logics, not limited to relevant logics alone. His most significant contribution was the proof of a general CUT-elimination theorem applicable to all display calculi whose rules adhere to a small number of easily verified formal conditions (see [8, pp. 389–392], [9, pp. 199–208]). This result demonstrates that display logic is a powerful framework for the unified treatment of the proof theory of various logics. For example, we can accommodate $\sim$ by introducing an additional structural connective denoted by "$*$" and provide a calculus for the full language of **R**. The rules, taken from Restall's emendation of Belnap's original calculus, can be defined as follows (see [57], [49, pp. 130–135]):

$$\frac{X \Rightarrow \varnothing}{X \Rightarrow \mathsf{f}}\ R\mathsf{f}; \qquad \frac{\varnothing \Rightarrow X}{\mathsf{t} \Rightarrow X}\ L\mathsf{t}; \qquad \frac{X \Rightarrow *A}{X \Rightarrow \sim A}\ R{\sim}* \qquad \frac{*A \Rightarrow X}{\sim A \Rightarrow X}\ L{\sim}*$$

$$\frac{X; A \Rightarrow B}{X \Rightarrow A \rightarrow B}\ R\rightarrow; \qquad \frac{X \Rightarrow A \qquad B \Rightarrow Y}{A \rightarrow B \Rightarrow *X; Y}\ L\rightarrow;* \qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X; Y \Rightarrow A \circ B}\ R\circ;$$

$$\frac{A; B \Rightarrow X}{A \circ B \Rightarrow X}\ L\circ; \qquad \frac{X \Rightarrow A; B}{X \Rightarrow A + B}\ R+; \qquad \frac{A \Rightarrow X \qquad B \Rightarrow Y}{A + B \Rightarrow X; Y}\ L+;$$

Similar to consecution calculi, the rules governing $\wedge$ and $\vee$ rely on the use of the extensional separator "," instead of ";". Moreover, just as the comma on the left-(resp. right-)hand side of sequents denotes extensional conjunction (resp. disjunction), the semicolon on the left (resp. right) of $\Rightarrow$ denotes intensional conjunction (resp. disjunction). The empty structure, $\varnothing$, is employed to introduce $\mathtt{t}$ (resp. $\mathtt{f}$) on the left (resp. right) of $\Rightarrow$. As illustrated by the rule $L \rightarrow; *$, different structural connectives can be combined. For further information on display calculi one can refer to Wansing's book [71].

Another approach, aimed at preserving the distinctions between extensional and intensional combinations of premises, was introduced by Read [56]. This approach is sometimes referred to as the *Scottish Plan* in relevant logics (see [56, pp. 131–143]). The calculus consists of sequents of the following form: "$X : A$", where X is referred to as a *bunch* of premises, and A is a formula. In this setting, a bunch of premises X can be conjoined both extensionally and intensionally, and the formula A is the conclusion inferred based on those premises. Bunches are represented using two pairing operations symbolized by the comma and the semicolon: the comma denotes extensional conjunction, while the semicolon denotes intensional conjunction [56, p. 58]. Within this framework, Read introduces two general schemas, one for introduction and one for elimination rules. Based on these schemas, he defines several bunch proof systems for a variety of relevant logics, with particular attention to **DW**, **E**, and especially **R** [56, pp. 40-41, 51–77].

All the approaches we have discussed so far modify the sequent calculus in a manner that enhances its expressive power, enabling it to capture more nuanced distinctions in the intensional and extensional combination of data. These distinctions are not only important for the study of relevant logics but also extend to several non-classical systems. Another noteworthy approach involves the introduction of *hypersequent* calculi, as presented in Avron's work [7]. In essence, a hypersequent takes the form $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$, where each $\Gamma \Rightarrow \Delta$ is an ordinary sequent, with Γ and Δ representing sequences of formulas. The hypersequent is essentially a sequence of sequents, where the separator "$|$" is interpreted as a meta-level (extensional) disjunction. Avron initially introduced the hypersequent formulation for **RM**, and, for the sake of simplicity, we present it in a simplified manner, relying on the hypersequent calculus for linear logic as a foundational framework (see [25, p. 266]). The hypersequent structure is used in formulating the additional rules necessary for transitioning from the linear logic base framework to **RM**. These additional rules are presented as follows:

$$\frac{G \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma}{G \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}\ \textit{Split} \qquad \frac{G \mid \Gamma \Rightarrow \Delta \qquad H \mid \Pi \Rightarrow \Sigma}{G \mid H \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \textit{Mingle}$$

where, G and H represent side hypersequents. This hypersequent formulation of **RM** is deductively equivalent with respect to the axiomatization of **RM**, i.e., **R** with the mingle axiom, $A \rightarrow (A \rightarrow A)$. That is, the hypersequent calculus allows the derivation of all axioms of **RM** and, conversely, that each hypersequent, suitably translated into an object language formula, is derivable in the axiomatic system for **RM**. The translation, denoted by τ, can be displayed as follows: $\tau(\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n) = (\circ\Gamma_1 \rightarrow +\Delta_1) \vee \cdots \vee (\circ\Gamma_n \rightarrow +\Delta_n)$, where $\circ\Gamma$

and $+\Delta$ denote the intensional conjunction and disjunction of formulas in Γ and Δ, respectively.

On a different, albeit related, note, besides the natural deduction systems labelling formulas with numerals to express relevance between premises and conclusions in derivations, a variety of labelling techniques have been used in the development of the proof theory for relevant logics. For example, important instances of labelled calculi for relevant logics include the contribution made by E. Orlowska. In [48], she introduced a methodology aimed at constructing natural deduction-style systems for various propositional relevant logics. This method shares some resemblance with Negri's approach, as it involves characterizing relevant logics using ternary relational frames and subsequently formulating deduction rules based on these relational frames. Similarly, Priest [54, pp. 188-220, 535–563] introduced labelled tableaux systems for various propositional and quantified relevant logics, referring to the unreduced Routley-Meyer relational semantics.

An important aspect of the development of labelled proof theory has encompassed the study of semilattice relevant logics (referenced in works such as [69, 23, 27, 28, 75]). The semantics incorporated into the proof-theoretic structure does not involve three different states a, b, c related by a ternary relation of the form $Rabc$; instead, rather than introducing a third state c, the approach considers the union of a and b. Urquhart, intuitively, motivates his semantic choice by arguing that for a piece of information b to determine a formula $A \to B$ is to say that, whenever we can conclude A based on a piece of information a, we can also conclude B based on both pieces of information a and b taken together, formally $a \cup b$ [69, p. 160]. The resulting rules for $\to$ can be expressed as follows:

$$\frac{\Gamma, A_a \Rightarrow \Delta, A \to B_b, B_{b\cup a}}{\Gamma \Rightarrow \Delta, A \to B_b}\ R\to' \qquad \frac{\Gamma, A \to B_b \Rightarrow \Delta, A_a \qquad \Gamma, A \to B_b, B_{b\cup a} \Rightarrow \Delta}{\Gamma, A \to B_b \Rightarrow \Delta}\ L\to'$$

Another important example of labelled deductive systems for relevant logics was presented by L. Viganò in [70]. As noted in [42, p. 125], the main difference between our approach and Viganò's lies in the construction of sequent calculi with explicit structural rules, along with the use of single-conclusion rules for the accessibility relation (Horn clauses) confined within Harrop theories, i.e., theories lacking disjunctions in the right-hand side of axioms (see also [70, pp. 61–70]). In addition to sequent systems, Viganò also introduced labelled natural deduction calculi. It's worth noting that the proof systems presented in this paper can also be represented as natural deduction systems, with significant differences between Orlowska's and Viganò's approaches. To elaborate, Negri and von Plato [46, pp. 268–277] describe a methodology and its corresponding philosophical advantages for establishing a set of introduction and elimination rules, which mirror the left and right rules of a certain labelled sequent system. This method is based on starting with the semantic clause of the particular connective (such as $\sim$, $\to$, $\circ$, $+$, $\bullet$, t) and following the instructions outlined in Negri and von Plato's adaptation of the *inversion principle*, originally formulated by Gentzen and Prawitz. In essence, Negri and von Plato's strategy can be summarized as follows: guided by their inversion principle, which stipulates that *whatever can be deduced from the direct reasons for asserting a proposition must*

also be deducible from that proposition, one examines the introduction rule of a given connective and deduces the corresponding well-formed labelled elimination rule.[19]
In line with the mentioned labelled techniques, our approach has enabled us to *modularly capture* various systems within the same family of logics, alongside the already formal results, through the incorporation of simple additions to a base system. In our case, instead of altering the logical rules of a given labelled system, it amounts to making adjustments within the group of relational rules. Additionally, it is essential to emphasize that many of the techniques outlined in this section often enable us to characterize only small fragments of relatively strong relevant logics. However, labelled techniques broaden our horizons, enabling the consideration of more extensive sets of connectives. This expanded expressive power stems from the incorporation of labels into the syntax. Such an observation is of utmost importance, as labelled methods facilitate the development of calculi for weaker systems of relevant logics possibly including a variety of both intensional and extensional connectives. Achieving such results in non-labelled frameworks would be considerably more challenging. Secondly, as noted on several occasions, labelled calculi constructed using the methodology we employed enable the straightforward extrapolation of a countermodel for a sequent from a failed proof-search procedure (see, e.g., [43, 44]). However, in relation to specific limitations of labelled sequents, we present two important remarks.
Firstly, a common weakness associated with labelled calculi is the absence of a *formula interpretation*, i.e., a formal procedure for translating sequents into object-language formulas. One approach to address this challenge stems from the recognition that the proliferation of non-classical logics has spurred the exploration of various generalizations of sequent systems, prompting inquiries into their interrelationships. Therefore, an avenue to mitigate the necessity for independent proofs could involve establishing translations between labelled calculi, such as those introduced in this paper, and other characterizations. To obtain alternative versions of the formula interpretation, one might investigate translations into non-labelled frameworks, such as the previously mentioned hypersequents (e.g., [7]) and display sequents (e.g., [57]).
Secondly, certain concerns arise when investigating the *subformula property* and its implications within labelled frameworks. Without delving into the intricacies of (un)decidability issues, let us outline potential avenues for future research in this area.[20] CUT-elimination plays a crucial role, yielding the subformula property in sequent systems, ensuring that all formulas in a derivation are subformulas of those in the endsequent. However, labelled sequent calculi lack a full subformula property due to geometrical rules, wherein relational atoms vanish from premises to conclusions. Nevertheless, they exhibit a *weaker* version of the property, where all formulas in a derivation are either subformulas of formulas in the endsequent or relational atoms such as *Rabc* (see, e.g., [41, 16]). Yet, this property alone is insufficient to

[19] For the sake of brevity, I won't delve into the details of this approach. However, one can find the details in [46]. See also [45, pp. 17–23].

[20] It is important to acknowledge that relevant logics encounter challenges in establishing decidability, with many of them being undecidable.

prove syntactic decidability. To address this, one needs to avoid non-terminating proof-search procedures. Roughly, this involves putting bounds on the number of eigenvariables in derivations, along with additional bounds on the number of applications of rules with endformulas in the premises.

Lastly, a concluding observation. In our paper, we have extensively explored labelled rules that encompass various intensional operators. However, it is worth mentioning that relevant logics might include additional connectives, such as modal operators, or quantifiers as well.[21] Despite these possibilities, we have made a deliberate choice to focus solely on the specific set of connectives we considered, leaving the exploration of broader sets of connectives for future research. Nevertheless, it is essential to note that the methodology we have employed thus far can, in principle, be extended to accommodate the mentioned extensions.

[21]Recently, there has been notable progress in the field of quantified and modal relevant logics. One can explore e.g., [65, 68, 18, 67].

(Ar) $\Rightarrow 0 : {\sim}(A \to {\sim}A)$

$$
\dfrac{\dfrac{
\text{(Prop. 5.7)}\;\dfrac{\dfrac{R0yy, y:A, R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A \Rightarrow y:A}{y:A, R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A, \Rightarrow y:A}\,\text{R1}}{R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A \Rightarrow y:A}\,L\mathsf{G}
\qquad
\dfrac{\text{(Prop. 5.7)}\;\dfrac{\dfrac{R0yy^{**}, R0y^{**}y, y:A, R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A \Rightarrow y^{**}:A}{y:A, R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A, \Rightarrow y^{**}:A}\,\text{R2}}{R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A \Rightarrow y^{**}:A}\,L\mathsf{G}}{y^*:{\sim}A, R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A \Rightarrow}\,L{\sim}
}{(y \text{ fresh})\;\dfrac{R0^*yy^*, A\mathsf{G}y, 0^*:A\to{\sim}A \Rightarrow}{0^*:A\to{\sim}A \Rightarrow}\,\text{AR}}\,L\to}{\Rightarrow 0:{\sim}(A\to{\sim}A)}\,R{\sim}
$$

(Bo) $\Rightarrow 0 : (A \to B) \to {\sim}(A \to {\sim}B)$

$$
\dfrac{\text{(Prop. 5.7)}\;\dfrac{\dfrac{(s,t,u \text{ fresh})\;\dfrac{
\text{(Prop. 5.7)}\;\dfrac{\dfrac{R0ss, s:A, Ra^*su, A\mathsf{G}s, \mathcal{A} \Rightarrow s:A}{s:A, Ra^*su, A\mathsf{G}s, \mathcal{A} \Rightarrow s:A}\,\text{R1}}{Ra^*su, A\mathsf{G}s, \mathcal{A} \Rightarrow s:A}\,L\mathsf{G}
\qquad
\dfrac{\dfrac{\text{(Prop. 5.7)}\;\dfrac{\dfrac{R0ss, s:A, Rast, A\mathsf{G}s, \mathcal{A}' \Rightarrow u^*:B, s:A}{s:A, Rast, A\mathsf{G}s, \mathcal{A}' \Rightarrow u^*:B, s:A}\,\text{R1}}{Rast, A\mathsf{G}s, \mathcal{A}' \Rightarrow u^*:B, s:A}\,L\mathsf{G}
\qquad
t:B, Rast, R0tu^*, \mathcal{A}'' \Rightarrow u^*:B}{Rast, Ra^*su, A\mathsf{G}s, R0tu^*, a^*:A\to{\sim}B, a:A\to B \Rightarrow u^*:B}\,L\to}{u:{\sim}B, Rast, Ra^*su, A\mathsf{G}s, R0tu^*, a^*:A\to{\sim}B, a:A\to B \Rightarrow}\,L{\sim}
}{Rast, Ra^*su, A\mathsf{G}s, R0tu^*, a^*:A\to{\sim}B, a:A\to B \Rightarrow}\,L\to}{a^*:A\to{\sim}B, a:A\to B \Rightarrow}\,\text{BO}}{a:A\to B \Rightarrow a:{\sim}(A\to{\sim}B)}\,R{\sim}}{\Rightarrow 0:(A\to B)\to{\sim}(A\to{\sim}B)}\,\text{VER}_2
$$

where $\mathcal{A} = Rast, R0tu^*, a^*: A \to {\sim}B, a : A \to B$, $\mathcal{A}' = Ra^*su, R0tu^*, a^* : A \to {\sim}B, a : A \to B$, and $\mathcal{A}'' = Ra^*su, A\mathsf{G}s, a^* : A \to {\sim}B, a : A \to B$.

Table 3: Derivations of Ar and Bo in **G3rB•** (and extensions thereof)

$$(\mathsf{St}) \Rightarrow 0 : {\sim}((A \to B) \bullet (A \to {\sim}B))$$

$$\dfrac{R0yy, y : A, Rcyz^*, A\mathsf{G}y, Sbc0^*, \mathcal{A} \Rightarrow y : A}{y : A, Rcyz^*, A\mathsf{G}y, Sbc0^*, \mathcal{A} \Rightarrow y : A}\ \text{R1}$$

$$\text{(Prop. 5.7)}\ \dfrac{y : A, Rcyz^*, A\mathsf{G}y, Sbc0^*, \mathcal{A} \Rightarrow y : A}{Rcyz^*, A\mathsf{G}y, Sbc0^*, \mathcal{A} \Rightarrow y : A}\ L\mathsf{G}$$

$$\dfrac{R0yy, y : A, Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B, y : A}{y : A, Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B, y : A}\ \text{R1}$$

$$\text{(Prop. 5.7)}\ \dfrac{y : A, Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B, y : A}{Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B, y : A}\ L\mathsf{G}$$

$$\dfrac{R0zz^{**}, R0z^{**}z, z : B, Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B}{z : B, Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B}\ \text{R2}$$

$$\dfrac{Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B, y : A \qquad z : B, Rbyz, A\mathsf{G}y, \mathcal{A}' \Rightarrow z^{**} : B}{Rbyz, Rcyz^*, A\mathsf{G}y, Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow y^{**} : B}\ L\to$$

$$\dfrac{Rbyz, Rcyz^*, A\mathsf{G}y, Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow y^{**} : B}{y^* : {\sim}B, Rbyz, Rcyz^*, A\mathsf{G}y, Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow}\ L{\sim}$$

$$\dfrac{Rcyz^*, A\mathsf{G}y, Sbc0^*, \mathcal{A} \Rightarrow y : A \qquad y^* : {\sim}B, Rbyz, Rcyz^*, A\mathsf{G}y, Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow}{Rbyz, Rcyz^*, A\mathsf{G}y, Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow}\ L\to$$

$$(y, z \text{ fresh})\ \dfrac{Rbyz, Rcyz^*, A\mathsf{G}y, Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow}{Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow}\ \text{ST}$$

$$(b, c \text{ fresh})\ \dfrac{Sbc0^*, b : A \to B, c : A \to {\sim}B \Rightarrow}{0^* : (A \to B) \bullet (A \to {\sim}B) \Rightarrow}\ L\bullet$$

$$\dfrac{0^* : (A \to B) \bullet (A \to {\sim}B) \Rightarrow}{\Rightarrow 0 : {\sim}((A \to B) \bullet (A \to {\sim}B))}\ R{\sim}$$

where $\mathcal{A} = Rbyz, c : A \to {\sim}B, b : A \to B$ and $\mathcal{A}' = Sbc0^*, Rcyz^*, b : A \to B, c : A \to {\sim}B$.

Table 4: Derivation of St in $\mathbf{G3rB}^{\bullet}$ (and extensions thereof)

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