Topics in the proof theory of non-classical logics

Philosophy and Applications

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Summary (English)

CHAPTER 1 constitutes an introduction to Gentzen calculi from two perspectives, logical and philosophical. It introduces the notion of generalisations of Gentzen sequent calculus and the discussion on properties that characterize *good* inferential systems. Among the variety of Gentzen-style sequent calculi, I divide them in two groups: syntactic and semantic generalisations. In the context of such a discussion, the *inferentialist* philosophy of the meaning of logical constants is introduced, and some potential objections – mainly concerning the choice of working with semantic generalizations – are addressed. Finally, I'll introduce the case studies that I'll be dealing with in part II.

CHAPTER 2 is concerned with the origins and development of Jaśkowski's discussive logic. The main idea of this chapter is to systematize the various stages of the development of discussive logic related researches from two different angles, i.e., its connections to modal logics and its proof theory, by highlighting virtues and vices.

CHAPTER 3 focuses on the Gentzen-style proof theory of discussive logic, by providing a characterization of it in terms of labelled sequent calculi.

CHAPTER 4 deals with the Gentzen-style proof theory of relevant logics, again by employing the methodology of labelled sequent calculi. This time, instead of working with a single logic, I'll deal with a whole family of them. More precisely, I'll study in terms of proof systems those relevant logics that can be characterised, at the semantic level, by *reduced* Routley-Meyer models, i.e., relational structures with a ternary relation between states and a unique base element.

CHAPTER 5 investigates the proof theory of a modal expansion of intuitionistic propositional logic obtained by adding an 'actuality' operator to the connectives. This logic was introduced also using Gentzen sequents. Unfortunately, the original proof system is not CUT-free. This chapter shows how to solve this problem by moving to hypersequents.

CHAPTER 6 concludes the investigations and discusses the *future* of the research presented throughout the dissertation.

Zusammenfassung (Deutsch)

KAPITEL 1 stellt eine Einführung in die Gentzen-Kalküle aus logischer und philosophischer Sicht dar. Zudem führt es in den Begriff der Generalisierung des Gentzen'schen Sequenzkalküls und in die Diskussion über Eigenschaften ein, die gute Inferenzsysteme kennzeichnen. Unter der Vielzahl von Gentzen-artigen Sequenzialkalkülen unterteile ich diese in zwei Gruppen: die syntaktische und die semantische Generalisierung. Im Kontext dieser Erörterung wird die inferentialistische Philosophie der Bedeutung logischer Konstanten eingeführt und mögliche Widersprüche - vor allem die Entscheidung zur Arbeit mit semantischen Generalisierungen betreffend- werden aufgezeigt. Schließlich stelle ich die Fallstudien vor, mit denen ich mich in Teil II genauer auseinandersetzen werde.
KAPITEL 2 befasst sich mit den Ursprüngen und der Entwicklung von Jaśkowskis diskursiver Logik. Der Hauptgedanke dieses Kapitels ist es, die verschiedenen Entwicklungsstufen der diskursiven Logik aus zwei verschiedenen Blickwinkeln zu systematisieren, nämlich ihren Verbindungen zur Modallogik und ihrer Beweistheorie, wobei Vorteile und Schwächen aufgezeigt werden.
KAPITEL 3 konzentriert sich auf die Beweistheorie der diskursiven Logik im Gentzen-Stil, indem es eine Charakterisierung dieser Theorie in Form von etikettierten Sequentenkalkülen anbietet.
KAPITEL 4 befasst sich mit der Gentzen'schen Beweistheorie der relevanten Logiken unter Anwendung der Methodologie der gelabelten Sequenzkalküle. In diesem Fall arbeite ich jedoch nicht mit einer einzelnen Logik, sondern mit einer Familie logischer Systeme. Ich werde jene relevanten Logiken auf ihre Beweistheorie untersuchen, die auf semantischer Ebene durch reduzierte Routley-Meyer Modelle charakterisiert werden können, d.h. relationale Strukturen mit einer ternären Beziehung zwischen Zuständen und einem eindeutigen Basiselement aufweisen.
In KAPITEL 5 wird die Beweistheorie einer modalen Erweiterung der intuitionistischen Aussagenlogik erforscht, die durch das Hinzufügen eines "Aktualitäts"-Operators zu den Konnektiven entsteht. Diese Logik wurde ebenfalls unter Verwendung von Gentzen-Sequenzen eingeleitet. Leider ist ein solches Beweissystem nicht schnittfrei. Daher wird in diesem Kapitel aufgezeigt, wie dieses Problem gelöst werden kann.
KAPITEL 6 schließt die Untersuchungen ab und stellt Überlegungen über die Zukunft der in meiner Dissertation vorgestellten Forschung an.

List of published parts

The published parts of the dissertation are the following ones:

1. Chapters 2 and 3 are based on my paper: "Discussive Logic. A Short History of the First Paraconsistent Logic".[1]

2. Chapter 4 is based on my article: "Modular labelled calculi for relevant logics".[2]

3. Chapter 5 is based on my papers: "A Cut-free Hypersequent Calculus for Intuitionistic Modal Logic IS5"[3] and "A note on Cut-elimination for intuitionistic logic with 'actuality'".[4]

I am the only author of all aforementioned papers.

[1] In: Ingolf Max and Jens Lemanski (eds.), *Historia Logicae*, vol. 1, College Publications, London (Forthcoming), pp. 267–296. Accepted Nov. 2021.

[2] In: *The Australasian Journal of Logic* Vol. 20, No. 1 (2023): 47–87.

[3] In: Butler, A. (ed.), *Proceedings of the 18th International Workshop of Logic and Engineering of Natural Language Semantics 18 (LENLS18), JSAI-isAI2021* (2021) pp. 217–230.

[4] Currently under review for *The Logic Journal of the IGPL. Special issue on Non-classical Modal and Predicate Logics.*

Contents

Chapter 0

Foreword

In the last few decades, many generalizations of sequent calculi have been proposed in order to cope with the flourishing of non-classical logics. This tradition, which finds its roots in Gentzen's doctoral dissertation, has paved the way to a huge literature and various results. Nonetheless, there still are problems and issues concerning philosophical and logical aspects of generalizations of sequent calculi that need to be further developed and tackled:

> "There are many logical systems that logicians have studied through the years. Most of them were obtained by constructing a set of axioms for Hilbert-type systems, and even more, by tinkering with the axioms of such systems. It is our strong belief that those logical systems that might really be useful (in Artificial Intelligence or elsewhere) should meet, first of all, two intrinsically logical criteria. One of them is the existence of a simple semantics. *The other is the possibility of developing a corresponding good proof theory, in particular, Gentzen-type systems with an appropriate version of the* CUT*-elimination theorem*." [Avr91a, pp. 244–245, Emphasis mine]

In order to supportively discuss the idea expressed by A. Avron, I'll propose an investigation within the proof theory of some non-classical logics, i.e., discussive (Chapter 3), relevant (Chapter 4) and modal intuitionistic ones (Chapter 5). The conceptual and philosophical side of such applications are discussed at length in Chapter 1.
Before entering the discussion, it is good to recall the words stated by A. S. Troelstra and H. Schwichtenberg:

> "in dealing with Gentzen systems, no particular variant is to be preferred over all the others; one should choose a variant suited for the purpose at hand." [TS00, p. 51]

Taking these words as inspiration, throughout my investigation, I'll proceed according to the following two *methodological leitmotivs*:

1. *Practice-first view*. Any philosophical understanding of proof theory should be practice-oriented, i.e., the concrete applications of proof theoretic structures should be the primary source for philosophical reflections on proof theory.

2. *Non-absolutistic approach*. There's no such thing as *the* correct inferential system: different purposes and logics motivate the choice of different sequent systems.

Part I

Logical, philosophical and historical preliminaries

Chapter 1

Proof theory as logical and philosophical subject

Layout of the chapter. This chapter is meant to introduce and delimit the field of inquiry of the dissertation. After having introduced Gentzen's work, I'll specifically discuss the notion of generalisations of the sequent calculus. The discussion will be mainly focused on determining those properties that characterize *good* inferential systems. To be precise, I'll divide generalizations of Gentzen-style sequent calculi in two groups: syntactic and semantic generalisations. Moreover, the *inferentialist* philosophy of the meaning of logical constants will be introduced, and some potential objections will be addressed. In the last part of the chapter, I'll describe the case studies that I'll be dealing with in part II.

1.1 Gentzen's calculi in a nutshell

Proofs, especially mathematical and logical ones, have occupied the mind of philosophers since ever. Nonetheless, the creation of the subject nowadays known under the label *proof theory* is a rather modern achievement. Its foundation is usually tracked back to the works of the German mathematician D. Hilbert and his research program for the foundations of mathematics, i.e., *Hilbert's program*. Famously, he wrote:

> "[...] we must make the proofs as such the object of our investigation; we are thus compelled to a sort of proof theory which studies operations with the proofs themselves. [...] proof itself is something concrete and displayable; the contentual reflections follow the proofs themselves. Just as the physicist investigates his apparatus and the astronomer investigates his location; just as the philosopher practises the critique of reason; so, in my opinion, the mathematician has to secure his theorems by a critique of his proofs, and for this he needs proof theory." [Hil05, pp. 1127–1128]

Hilbert viewed the axiomatic method as the crucial tool to develop an adequate analysis of logical and mathematical proofs. The central idea of Hilbert's program was to ground all existing theories in a recursive, complete set of axioms,

and provide a *consistency* proof for all such sets. However, as it is well-known, the program failed and its failure is due to Kurt Gödel's incompleteness theorems (1931), which, roughly, showed that any consistent theory that is sufficiently strong to express some arithmetic truths, cannot prove its own consistency. However, the failure of Hilbert's program did not discourage logicians from engaging in *proof theory*: modified versions of Hilbert's program emerged and research on related topics has been carried out. Indeed, in parallel to the rise and fall of Hilbert's program, S. Jaśkowski[1] and G. Gentzen[2], independently, laid the basis of so-called *structural* and *ordinal proof theory*.[3] Their central idea was to use systems including specific *inference rules*, rather than relying on the axiomatic method advocated by Hilbert's program.
For the purposes of my work, it suffices to gently introduce Gentzen's logical work, its philosophical meaning, and move to the contemporary approaches towards proof theory, always keeping in mind both aspects, logical and philosophical.

Natural deduction. Gentzen, motivated by the aim of proving the consistency of arithmetic, elaborated a system in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning. Specifically, it was noticed that *reasoning from assumptions* in mathematical logic occupies an important place and that it, therefore, plays a central role in our understanding of the mechanism of proofs:

> "The inference rules were designed to model the patterns of ordinary reasoning that mathematicians carry out in proving results, such as hypothetical proofs or reasoning by cases." [MGZ21, p. 8]

Roughly, to formalize such an intuition, logical constants (propositional operators, quantifiers) are no longer characterised in terms of axioms, but in terms of inferential rules that permit to either *introduce* or *eliminate* them. For example, some of the introduction rules for classical (propositional) logic are the following ones[4]:

$$\begin{array}{c} A^1 \\ \vdots \\ \bot \\ \hline \neg A \end{array}\, {}^{1}\, I\neg \qquad \frac{A \quad B}{A \wedge B}\, I\wedge \qquad \frac{A}{A \vee B}\, I\vee_1 \qquad \frac{B}{A \vee B}\, I\vee_2 \qquad \begin{array}{c} A^1 \\ \vdots \\ B \\ \hline A \supset B \end{array}\, {}^{1}\, I\supset$$

whereas the elimination rules are:

[1] See Jaśkowski's paper [Jaś34], as well as the remarks in [PH23].

[2] See [Gen35a; Gen35b], as well as their English translation, namely, *Investigations into logical deduction* [Gen69b] (the version included the *Collected Papers of G. Gentzen* edited by M. E. Szabo [Gen69a]). A comparison between Jaśkowski's and Gentzen's approaches can be found in [PH14].

[3] "Gentzen's work contains the beginnings of what we call structural proof theory [...], as well as ordinal proof theory." [MGZ21, p. 10]

[4] If not stated differently, $\neg, \wedge, \vee, \supset$ represent negation, conjunction, disjunction and material implication, respectively.

$$\frac{\neg A \quad A}{\bot}\,E\neg \qquad \frac{A \wedge B}{A}\,E\wedge_1 \quad \frac{A \wedge B}{B}\,E\wedge_2 \qquad {}_{1,2}\frac{A \vee B \quad \begin{matrix}A^1\\ \vdots \\ C\end{matrix} \quad \begin{matrix}B^2\\ \vdots \\ C\end{matrix}}{C}\,E\vee \qquad \frac{A \supset B \quad A}{B}\,E\supset$$

Notice that $I\neg$, $I\supset$ and $E\vee$ deal with the notion of *hypothetical derivation*. In the case of $I\supset$, the intuition standing behind is: if there's a way of obtaining B from the presence of A, then a statement of the form "if A, then B" can be concluded. More formally, if there's a derivation that concludes B from an assumption A, then $A \supset B$ can be derived ($I\neg$ is just a special case of the rule $I\supset$, as $\neg A = A \supset \bot$). Similarly for $E\vee$: C is derivable from $A \vee B$, if there are derivations of C from A and C from B. Additionally, let me remark that in both rules, $I\supset$ and $E\vee$, we need to *discharge* the assumption(s) in order to apply the rule under scope.[5]
As a concrete example, to see how derivations look like in a natural deduction system consider the following proof of $A \supset (B \supset (A \wedge B))$:

$$ {}_{1}\dfrac{{}_{2}\dfrac{\dfrac{A^1 \quad B^2}{A \wedge B}\,I\wedge}{B \supset (A \wedge B)}\,I\supset}{A \supset (B \supset (A \wedge B))}\,I\supset $$

where the discharge of assumptions A and B is denoted by numbers 1 and 2, respectively.
Importantly, when constructing proofs one can easily make some inferences which are unnecessary to obtain the desired conclusion. Aware of this possibility, Gentzen was not only interested in elaborating an adequate system of inference rules, but also in showing that everything which may be proved in it, may be proved in the most straightforward and direct way. In Gentzen's own words:

> "No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of the result." [Gen69b, p. 69]

In particular, unnecessary moves, which are often called *detours*, occur in derivations when, both, an introduction rule for some logical constant is used, and when the conclusion of such an introduction is in turn used as a premise for the application of the corresponding elimination rule. Nevertheless, unnecessary moves are dispensable and it is possible to transform derivations including detours into derivations without detours. This follows by observing that, in derivations with detours, the final conclusion is either already somewhere in the proof or may be directly deduced from premises of the introduction rule. As an example, consider the following transformation:

[5] We point out that Gentzen's approach is not the only one and that the notion of discharge just mentioned was criticized, for example, in [NvP01, p. 11] and [TS00, pp. 43-44]. For surveys on other approaches one can see, e.g., [PH12; PH23].

$$
\dfrac{{}^{1}\dfrac{\begin{matrix} A^1 \\ \vdots \\ B \end{matrix}}{A \supset B}\, I\supset \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B}\, E\supset \qquad \text{converts into} \qquad \begin{matrix}\vdots \\ A \\ \vdots \\ B\end{matrix}
$$

The explanation can be given as follows: if one deduces $A \supset B$ on the basis of $I\supset$ and then, by $E\supset$, deduces B from $A \supset B$ and A, then it is simpler to derive B directly from A. Notice that the existence of such a derivation is guaranteed because it is a subproof of the proof of $A \supset B$. More precisely, such a proof without detours is called *normal*. As a consequence, a natural deduction systems is said to be *normalizable* just in case a procedure that transforms all derivations into normal form can be actually given.
Finally, on more philosophical notes, one might wonder, why the rules displayed above possess a certain shape and what's the idea that they're communicating. To answer this question, it's worth reading Gentzen's own opinion:

> "The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'." [Gen69b, p. 80]

These words greatly influenced an important part of the philosophical understanding of proof theory and, indeed, have paved the way to so-called *logical inferentialism*[6], that is, a philosophical position claiming that the *meaning* of logical constants is given, not by identifying some objects as their meaning, but by stating the rules for their *use in inferences*[7]. Accordingly, rules usually state:

1. the *grounds* for asserting propositions – the conditions under which such assertions can be inferred.

2. the *consequences* of the asserted propositions – what can be inferred from asserting them.

As remarked by Gentzen himself, indeed, the *grounds* for a constant to be derived are spelled out in its introduction rule, while the *consequences* thereof are given by the corresponding elimination rule. I'll come back to logical inferentialism in the next few pages.
Although the idea of a normal proof is rather simple to grasp it is not so simple to prove that all derivations can be converted into normal proof. In fact for many natural deduction systems such a result is not available yet. Therefore,

[6] To be precise, I'll discuss only *proof theoretic* logical inferentialism. For other approaches, one might consult [Gar13].

[7] The *Meaning is Use* perspective is sometimes referred to as an *anti-realistic* conception of the meaning of logical constants (as opposed to a *realistic* – usually, model-theoretic – conception of meaning): "Gentzen-style proof theory is usually associated with a certain 'anti-realistic' philosophy of meaning" [Wan98, p. 7]. For an interesting reconstruction consider [NvP15, pp. 261-268].

Gentzen, well aware of the limitations of natural deduction systems, always motivated by the desire to establish the consistency of arithmetic, introduced a second system, called sequent calculus, and proved for it the famous *Hauptsatz* (the cut-elimination theorem).

Sequent calculus. Before dealing with some philosophical aspects of sequent systems, let me introduce their formalism. A sequent is an object of the following form:

$$\Gamma \Rightarrow \Delta$$

where, $\Rightarrow$ is referred to as *sequent arrow* and left- and right-hand sides of sequents are usually referred to as antecedents and succedents (or consequents), respectively. Finally, Γ, Δ are used to denote collections of formulas – usually, sets, multisets or lists. The differences between such structures can be given as follows:

1. In lists both the *multiplicity* and the *order* of elements counts.
2. In multisets the *multiplicity* of elements matters, but not their *order*.
3. In sets neither the *multiplicity* nor the *order* of elements matters.

If Γ, Δ are treated as either multisets or sets, it was proposed to understand a sequent $\Gamma \Rightarrow \Delta$ as the following object-language formula:

$$\wedge\Gamma \supset \vee\Delta$$

where, $\wedge\Gamma$ and $\vee\Delta$ stand for conjunctions and disjunctions of formulas, respectively. In this case, writing A, Γ can be understood as shorthand for $\{A\} \cup \Gamma$. Instead, if Γ is treated as list, the comma in the notation A, Γ is interpreted as *concatenation* between a formula A and Γ. Accordingly, if Δ is a list as well, then writing Γ, Δ denotes a concatenation between two lists of formulas. Nevertheless, their significance can be understood also in terms of model-theoretic validity. Without disentangling all details, it might be said:

$$\vDash \Gamma \Rightarrow \Delta \quad \text{iff} \quad \text{either, for some } A \in \Gamma, \nvDash A, \text{ or, for some } A \in \Delta, \vDash A.$$

In other words, a sequent is valid just in case either some formula in the antecedent is false, or some formula in the succedent is true.

Working with sequent-style structures offers an alternative way of keeping track of assumptions and discharges thereof. To be clear, in a sequent $\Gamma \Rightarrow C$, the conclusion C is dependent from the assumptions contained in Γ, which are listed on the same line. Moreover, in sequent calculi, we no longer work with elimination rules, but instead we deal with two kinds of introduction rules, i.e., those introducing a constant on the *right* and those introducing it on the *left*. As an example, let's consider the right rules of Gentzen's system for classical propositional logic first:

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ R\neg \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee_1 \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee_2 \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ R\supset$$

And the left rules correspond to the following ones:

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ L\neg \qquad \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge_1 \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge_2$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A \quad B, \Sigma \Rightarrow \Pi}{A \supset B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ L\supset$$

where, $\Gamma, \Sigma, \Delta, \Pi$ stand for lists of formulas. $L\supset$ is stated in a *context-free* form, whereas all other rules are called *context-sharing*[8]. In sequent calculi, in addition to rules allowing one to introduce a connective, there's a group of rules, called *structural*, acting directly on the *structure* of derivations. For a complete presentation of Gentzen's calculus, in addition to the left and right rules mentioned above, one needs to include the following structural rules as well:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \textsc{rw} \quad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \textsc{lw} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \textsc{rc} \quad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \textsc{lc}$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, B, A}\ \textsc{rp} \quad \frac{A, B, \Gamma \Rightarrow \Delta}{B, A, \Gamma \Rightarrow \Delta}\ \textsc{lp} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \textsc{cut}$$

Plus, the following *axiom* or *initial sequent*: $A \Rightarrow A$. The resulting system was termed **LK**, where **K** identifies classical logic (from the German word *klassische*). Derivations in sequent calculi are constructed *backwards*. Such process is often referred to as *root-first proof search*. Intuitively, by starting from the conclusion, we *decompose* each sequent, through application of logical and structural rules, until we reach initial sequents. At this point, the derivation *terminates* and, the resulting construction corresponds to a proof of the decomposed sequent. Intuitively, this gives us back so-called *analytic proofs*, i.e., structures in which complex formulas are reduced to simpler ones in a logically significant way, no matter how long the inferential process may be. In more logical terms, the analytic decomposition process can be described by a rooted tree graph. The roots of such trees contain the sequents we wish to prove; leaves consist of initial sequents only. For example, a backwards derivation of $A \supset (B \supset (A \wedge B))$ is given as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B, A \Rightarrow A}\ \textsc{lw}}{A, B \Rightarrow A}\ \textsc{lp} \qquad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\ \textsc{lw}}{A, B \Rightarrow A \wedge B}\ R\wedge}{A \Rightarrow B \supset (A \wedge B)}\ R\supset}{\Rightarrow A \supset (B \supset (A \wedge B))}\ R\supset$$

where the application of the two-premise rule $R\wedge$ results in a branched tree, and the presence of two initial sequents, one in each branch, allows us to stop the proof search procedure. Hence, the resulting tree corresponds to an analytic proof of the endsequent.

[8]The former rules are sometimes also called *multiplicative*, and the latter ones *additive*.

Remark 1. To be precise, the choice of our sequent system is not arbitrary. Indeed, the choice of a set of structural rules is sensitive to both the logic under scope and the definition of sequents. Let's pick up as an example *intuitionistic logic*. The dispute between a classicist and an intuitionist can be put as follows. Consider the *law of exclude middle* (LEM), i.e.,:

$$A \vee \neg A$$

In classical logic, LEM is a theorem, and it, intuitively, says that every statement A is either true or false. In more philosophical terms, it could be said that a classicist endorses a position for which we know that $A \vee \neg A$ is true, no matter whether we are able to establish that either A or its negation, $\neg A$, is true. With a suggestive terminology, classical logic is said to allow for *verification-transcendent propositions*.

> "For the classicist, all propositions have truth values, including propositions whose truth values we are not in a position to ascertain. These so-called verification-transcendent propositions must be either true or false, even though there are no means of determining which." [AR09, p. 644]

Intuitionists, instead, led by the philosophical considerations of L. E. J. Brouwer[9], reject the idea that there are verification-transcendent propositions: to say that A, or $\neg A$, is true, one must exhibit a construction showing its truth:

> "This [conception] makes asserting the existence of mathematical objects illegitimate unless there are proofs of the existence of specific examples of each such object, that is to say a means of constructing the object in finitely many steps." [AR09, p. 642]

Anyway, by turning our attention back to proof theory, if we consider another time the sequent-style rules for classical logic displayed above, we derive LEM as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}\ R\neg}{\Rightarrow A \vee \neg A, \neg A}\ R\vee_1}{\Rightarrow A \vee \neg A, A \vee \neg A}\ R\vee_2}{\Rightarrow A \vee \neg A}\ \text{RC}$$

Gentzen [Gen69b] (see also [NvP01; Pao02; MGZ21]) noticed that a way to build a successful sequent calculus for intuitionistic logic, was to impose a restriction on the number of formulas that can appear in succedents. More precisely, by considering *single-conclusion* sequents, i.e., structures where no more than one formula is allowed to appear on the right, it is possible to get a calculus for intuitionistic logic. The idea is simple: take the rules displayed above for classical

[9]For more on Brouwer's intuitionism see [Att20, §§3-4]. For a more general orientation on the mathematical and logical developments of intuitionism after Brouwer, see [Iem20] and [Pos20]. Finally, an interesting philosophical introduction to intuitionistic logic, as opposed to classical logic, can be found in [AR09, §2.1].

logic, and delete any multiple occurrence of formulas from the right of sequents. Of course, the choice of handling with only single-succedent sequents affects the presence or absence of certain structural rules as well. Gentzen's calculus for intuitionistic logic, for example, included the following single-succedent structural rules:

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ \textsc{rw} \qquad \frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ \textsc{lw} \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ \textsc{lc}$$

$$\frac{A, B, \Gamma \Rightarrow C}{B, A, \Gamma \Rightarrow C}\ \textsc{lp} \qquad \frac{\Gamma \Rightarrow A \quad A, \Sigma \Rightarrow C}{\Gamma, \Sigma \Rightarrow C}\ \textsc{cut}$$

The resulting systems is usually acknowledged under the label **LJ**. To see that LEM is not a theorem in our single-succedent sequent calculus **LJ**, consider the following two failed derivations:

$$\cfrac{\vdots \atop \Rightarrow A}{\Rightarrow A \vee \neg A}\ R\vee_1 \qquad \cfrac{\cfrac{\vdots \atop A \Rightarrow}{\Rightarrow \neg A}\ R\neg}{\Rightarrow A \vee \neg A}\ R\vee_2$$

Notice that although further structural rules may be applied, we will never be able to reach initial sequents and conclude the derivations. The single-succedent restriction is exactly the tool that blocks the derivability of intuitionistically undesirable formulas, such as LEM.
An additional remark, concerning the choice of structural rules, is needed. At the beginning of this paragraph (see p. 13), I've specified that elements like $\Gamma, \Delta, \ldots$ are collections of formulas, such as lists, sets or multisets. Gentzen [Gen69b] defined calculi for both, classical and intuitionistic logic, by using finite lists of formulas. According to such a formulation, sequents such as $A, B, C \Rightarrow D$ and $C, A, B \Rightarrow D$, are not the same sequent. Indeed, Gentzen's choice of working with lists, and the consequent need to *permute* the order of formulas, is what motivates the presence of LP (and, for classical logic, also of RP) among the structural rules. Differently, if one wishes to work without considering the order of elements, but only taking into account their multiplicity, the natural choice is to opt for multisets. In such a context LP (and, for classical logic, also RP), can be excluded from the group of structural rules, exactly because sequents, such as $A, B, C \Rightarrow D$ and $C, A, B \Rightarrow D$, are the same.

Before moving ahead, let me spend some final words on a fundamental trait of sequent systems. Consider a derivation of the following form:

$$\frac{\vdots \atop B \Rightarrow C, A \qquad \vdots \atop A, D \Rightarrow E}{B, D \Rightarrow C, E}\ \textsc{cut}$$

Notice that, among all rules stated so far, CUT is the only one allowing us to *get rid* of formulas from derivations (A in our example above). Although its presence simplifies significantly the length and complexity of derivations, some considerations are needed. First of all, recall that we have to take care that "no concepts enter into the proof other than those contained in its final result" [Gen69b, p. 69]. According to what cut allows us to do, then it seems that, in sequent-style derivations, further formulas than those included in the conclusion, might enter and even play a central role in the construction of tree proofs. This is, for example, the case for the formula A displayed in the derivation above. To avoid the possibility of constructing trees by using more formulas than those included in the conclusion, Gentzen had the ingenious idea of showing that, roughly, applications of cut can be eliminated from derivations. The proof given by Gentzen himself in his dissertation is famously acknowledged under the label *Hauptsatz*. As a consequence of his proof, Gentzen was able to regain a system in which formulas, belonging to some endsequent, are already to be found in some previous step of the derivation. Usually, such feature is known as *subformula property*, and, along with cut-freeness, it tells us something fundamental – both philosophically and logically speaking – of sequent-style derivations: they're effectively *analytic*.

Philosophically speaking, let me highlight the epistemic side of eliminating cut. Usually, cut is interpreted as formally encoding reasoning *by lemmas* or *by subsidiary assumptions*:

> "Viewed as a transformation of mathematical proofs, cut- elimination corresponds to the removal of intermediate statements (lemmas) from a proof. The mathematical interest in this transformation lies in the fact that frequently these lemmas may contain mathematical concepts which do not occur in the theorem that is shown. Removing these lemmas also removes these concepts therefore allowing the computation of an elementary proof from a more abstract one. [...] Therefore, on the mathematical level, the abstract concepts – up to a certain degree – determine the form of the elementary argument." [Het10, pp. 1–2][10]

So, the elimination of *abstract concepts* not occurring in the conclusion of a proof, via cut, results in a more *elementary* construction where fewer elements are used within the proof. Epistemologically said, in deductions, there's no need to rely on *external information*, because we can make sure that whatever is to be proved may be derived only by relying on its *internal information*, i.e., by using the elements contained in it. So, in elementary derivations, nothing must be guessed, and, whatever is needed to conclude an argument, is already given. Notice that the terminology *elementary proofs* only refers to those derivations where applications of cut have been eliminated and it does not refer to their length. Indeed, what we might call the *epistemic gain*[11] of cut eliminability results, is not related to the length of proofs, (especially given that usually cut-free derivations increase in their size), but, once again, to their analyticity. Therefore, not

[10] To strengthen this idea, let's consider that, aside from mathematics, "[...] in our reasoning, we often employ proofs in which we use subsidiary conclusions that help us to shorten the process of demonstration. The cut-rule is nothing but the formal equivalent of this exploitation of subsidiary conclusions" [Pog09a, p. 24].

[11] Although I am borrowing the terminology from N. Tennant's book [Ten17, Ch. 7], I am using it within a rather different logical context.

only analytic derivations are desirable from a technical perspective, as already recognized by Gentzen himself, but also from an epistemological point of view: once we get rid of subsidiary assumptions, we're in a position to know that each conclusion can be deconstructed, and that its subelements are the only objects occurring in the proof search process, no matter how long the resulting tree is. Summing up: elementary proofs, for their being more direct than their abstract counterparts, are epistemologically informative, although they might be much longer.

Remark 2. I have introduced Gentzen's **LK** and **LJ**, however, they are *not* the only possible formulations of sequent calculi:

> "Gentzen systems [...] have many variants. There is no reason for the reader to get confused by this fact. Firstly, we wish to stress that in dealing with Gentzen systems, *no particular variant is to be preferred over all the others; one should choose a variant suited for the purpose at hand*. Secondly, there is some method in the apparent confusion." [TS00, p. 51, Emphasis mine]

Given the purposes of my work, let's consider a family of sequent systems, usually called *logical sequent calculi*. Such systems are characterised by having only logical rules as primitive. Differently, structural rules are shown to be *admissible*. As the notion of admissibility will be discussed in several parts of this work, let's introduce it properly. A sequent-style rule:

$$\frac{\mathcal{P}_1 \ldots \mathcal{P}_n}{C}\ \mathbf{r}$$

is admissible in a sequent calculus **L**, if $\vdash_{\mathbf{L}} \mathcal{P}_1, \ldots, \vdash_{\mathbf{L}} \mathcal{P}_n$, together imply $\vdash_{\mathbf{L}} C$. We remark that admissibility results are often proved by induction and admissible rules can be used in derivations like normal rules.
Among logical calculi one finds, for example, the famous systems known as **G3c** (for classical logic) and **G3i** (for intuitionistic logic). The former is formulated by including atomic initial sequents of the form $p, \Gamma \Rightarrow \Delta, p$ (which can be generalized to compound formulas A, as explained below). Moreover, $R\vee_i$, $L\wedge_i$ ($i = 1, 2$) and $L\supset$ are included in their additive forms:

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee \qquad \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$$

G3i is, similarly to **LJ**, obtained by restricting the consequent of sequents in **G3c** rules to contain at most one formula on the right. So, atomic initial sequents have the form: $p, \Gamma \Rightarrow p$ and $R\vee$ is splitted again into two rules. Moreover, notice that the single-succedent condition might cause some troubles in showing that the contraction rule is (height-preserving) admissible. For the formal definitions of *height* and *height-preserving admissibility*, consider Definitions 3.3.2 and 3.3.3, respectively. However, intuitively, we can say that a sequent-style rule **r** is height-preserving admissible in a sequent calculus **L**, if **r** is admissible (see above), along with the additional condition that the number of steps needed to derive C is at most n, where n is the maximal numbers of steps in the derivation(s) of the premise(s) of **r**. In order to avoid troubles with such a proof, the idea (due to Dragalin [Dra88][12]) is to include the following version of $L\supset$:

$$\frac{A \supset B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C}\ L\supset$$

where the left-premise is said to be *contraction-absorbing*. This terminology is used to indicate that $L\supset$ already contains the effects of applying contraction in its left-premise. Similarly, the presence of Γ, Δ in the formulation of initial sequents makes them *weakening-absorbing*. More generally said, the admissibility of structural rules tells us that their effects are completely absorbed by the logical rules.

For the purposes of my work, no other remarks on Gentzen's sequent calculi are needed, as in the next chapters we will mainly deal with *generalizations* thereof and the specific details will be given case by case.

1.2 Towards generalizations of sequent systems

The discovery of so-called *paradoxes of material implication* and the crisis in foundations of mathematics highlighted some technical and philosophical limitations of classical logic. This series of events brought scholars in trying to identify and formalize other systems of logic to escape the problematic features of classical logic. Among such alternative systems, one finds intuitionistic logic, which I briefly mentioned above. Besides intuitionistic logic, however, there is a variety of logics, which deviate, in some way or another, from classical logic, and we will work with some of them in the following chapters. For the time being, we should notice that, unfortunately, not all interesting non-classical logics can be given a CUT-free sequent calculus. Nonetheless, the lack of such a characterisation was shown to be far from being an insurmountable obstacle, and, indeed, logicians came up with many so-called *generalizations* of sequents, capable of dealing with the flourishing of non-classical logics. First of all, let's try to fix how to understand the word "generalization":[13]

GSC Extension of the standard sequent calculus, obtained by introducing a more abstract version of the notion of sequent, and flexible enough to generate calculi for at least the logic(s) under scope.

To understand what the words "more abstract version" are referring to, let's consider some concrete proposals of generalizations of sequent systems:

1. Firstly, one can modify the structure of the standard sequent calculus in a purely syntactic fashion. The systems constructed by modifying, in some way or another, the structure of Gentzen's sequents will be referred to as *Syntactic generalizations*. Among such calculi one finds, for example:

[12] R. Dyckhoff's papers [Dyc92; Dyc97] are also fundamental to the understanding of logical systems. Moreover, among others, one can see also: Troelstra and Schwichtenberg [TS00], Negri and von Plato [NvP01], as well as Indrzejczak [Ind21].

[13] See especially [Pog09a, pp. 51–52].

(a) multiple sequent calculi: allowing more than just one sequent arrow;
(b) higher-arity sequent calculi: we allow more than just one antecedent and one succedent;
(c) hypersequent calculi: treating n different sequents at the same time;
(d) display calculi: dealing with different ways of combining formulas;

2. Secondly, one can enrich a standard sequent calculus by adding semantic expressions in its language. At least two specific kinds of semantic elements were used to develop such generalizations: algebraic and model-theoretic. We will refer to such calculi as *Semantic based* – or, simply, *labelled – generalizations*.

All such different calculi have been proposed within specific and sometimes very different research programmes. Their importance is not only related to their specific usefulness as logical tools, but also to their philosophical value. As said above, *logical inferentialists* believe that the inferential use of logical constants determines their meaning, as encapsulated in introduction and elimination rules of natural deduction systems. Nonetheless, I believe sequent-style systems to be more *appropriate* reference calculi for inferentialism. In other terms, instead of the pair introduction-elimination rules, I consider the left and the right introduction rules of the sequent calculus as those rules providing us with the meaning of the constant they're concerned with. Indeed, in agreement with Paoli's idea:

> "[C]ut-free sequent calculi are even more apt than natural deduction systems for a molecularistic semantics of logical constants: not only do we have separate rules for each connective, but we are also guaranteed that larger fragments conservatively extend smaller fragments containing fewer connectives." [Pao03, p. 536]

The idea that rules, left and right, of a sequent system determine the meaning of the constant, by specifying its use, can (and should) be extended to the generalizations of sequent calculi as well. To this extent, in the last decades, several logical, methodological and philosophical desiderata that proof systems should be required to enjoy, have been put forward. In the literature, generalizations satisfying a certain amount of desiderata are sometimes said to provide *good* proof systems. A list of such desiderata is as follows:[14]

1. Separation: Each constant is introduced independently from any other constant. More formally, a rule for a logical constant $\bullet$ should not exhibit any other constants in antecedent and succedent than $\bullet$.

2a. Weak symmetry: each constant $\bullet$ has at least one pair of rules for introducing it into an antecedent and a succedent of a conclusion-sequent. 2b. Symmetry: each constant $\bullet$ has exactly one pair of rules for introducing it into an antecedent and a succedent of a conclusion-sequent.

[14] The idea of defining criteria to identify *good* generalizations of sequent systems has attracted some attention. See, for example, Avron [Avr91c; Avr96], Wansing [Wan92; Wan94; Wan98; Wan00; Wan02], Indrzejczak [Ind97; Ind21], Paoli [Pao02], Negri [Neg07; NvP15], Poggiolesi [Pog08b; Pog09a; Pog09b; Pog12], Parisi [Par22].

3a. Weak explicitness: rules for constants exhibit • only in conclusion sequents, but never in premises. 3b. Explicitness: rules are weakly explicit and exhibit only one occurrence of •.

4. Uniqueness: Each connective should be uniquely characterized by its rules in a given system.[15]

5. Invertibility: not only the conclusion of a rule follows from its premise(s), but also viceversa, i.e. rules are doubly sound.

6. Conservativeness: a calculus C', obtained by adding to the calculus C one or more constants, and rules concerning these newly introduced constants, should prove exactly those sequents (including only constants of C) which were already provable in C.

7. Modularity: adding or deleting one or more axioms from the Hilbert-style presentation of a logic L corresponds to add or delete one or more rules from a sequent calculus for L. Each combination of rules is meant to be sound and complete for the corresponding logic.

8a. Došen's principle[16]: in modular extensions of a calculus, rules for logical constants stay unaltered, and different systems can be obtained exclusively by modifying the structural rules. 8b. Poggiolesi's principle: in modular extensions of a calculus, different systems can be obtained by modifying both logical and structural rules.

Observation 1. One might wonder what's the difference between item 7. and items 8a, 8b. To answer this question, let's firstly consider Poggiolesi's analysis [Pog09a, p. 34]:

> "In the literature, Došen's principle is sometimes also referred to as the "modularity property". We find this second name a possible source of misunderstanding. Došen's principle describes the relationships between different sequent calculi, while the modularity property requires the link between Hilbert systems and Gentzen systems to be straightforward. Therefore the two properties are related but not the same."

Hence, modularity expresses the idea that, if a Hilbert system $\mathcal{H}'$ is obtained by adding, let's say, a new axiom to $\mathcal{H}$, then, according to modularity, the calculus should systematically reflect such an addition by adjoining rules. If modularity is accepted, then one might reasonably ask what kind of rules – logical and/or structural – are we supposed to work with to get modular extensions of a given calculus. This is exactly the point raised by both Došen's and Poggiolesi's principles:

[15] For discussions on uniqueness one can consult, among others, [Wan92; Wan94; Par22].

[16] For more on the origins of such a concept, and discussions thereof, see [Wan92; Wan94; Wan98], but also [Pog08b; Pog09a].

Došen's principle	All "rules for the logical operations are never changed: all changes are made in the structural rules" [Doš88, p. 353].[17]
Poggiolesi's principle	A "general variant of a sequent calculus $\mathcal{G}$ can be obtained from it by both varying its logical and structural parts" [Pog09a, p. 34].

So, item 7. is directly concerned with the description of a desirable property of proof systems, modularity, useful in laying down the core meaning of logical constants. The latter two principles, 8a.,8b., instead, describe two ways to achieve modular formulations of proof systems.[18]

9. Identity theorem: we show that from an assumption A, we can always prove the same statement A, formally $A \Rightarrow A$.

10. CUT-elimination/admissibility: the CUT rule is dispensable.

11. Subformula property: each formula displayed in the premise(s) is present as a subformula of the final formula in the conclusion.

Observation 2. There are some main points that should be raised in connection with the last three items listed above. First of all, as Belnap wrote:

> "I take the Identity theorem to constitute half of what is required to show that [...] formulas "mean the same" in both antecedent and consequent position. (The [CUT-]Elimination Theorem is the other half of what is required for this purpose.)". [Bel82, p. 383]

In other words, to claim that rules determine the meaning of constants, by specifying their use in inferences, one needs to show that there's no *asymmetry* between what one can prove in both sides of a sequent. However, dependently on the choice of the calculus, identity and CUT-elimination/admissibility results may be formulated differently. Generally speaking, let's summarize them as follows:

(a) Identity theorem. As remarked above, the identity theorem tells us that all sequents of the form $A \Rightarrow A$ can be shown to be admissible (A being either compound or atomic). To give you an example, suppose one opts for a *logical system*, i.e., a calculus with (height-preserving) admissible, instead of primitive, structural rules. This requires one to deal with atomic initial sequents. Let's consider the case for the system **G3i** mentioned above, which is usually formulated by including initial sequents of the following

[17] Although I am viewing Došen's principle as indicating a way to achieve modular formulations of sequent calculi, it must be pointed out that Došen's emphasis was rather on the role played by structural rules in determining the core meaning of logical constants. Roughly, following Došen's idea, the meaning of a logical operator is laid down by the logical rules, which stay unaltered across different sequent systems – these latter, obtained by addition or deletion of one or more structural rules. This allows one to think of, for example, $\wedge$ as conjunction in different logical systems.

[18] Poggiolesi's proposal was introduced as an emendation of the weaknesses surrounding Došen's principle. Her detailed examination can be found in [Pog08b], as well as in [Pog09a, pp. 31–34].

form: $p, \Gamma \Rightarrow p$. Notice that the presence of Γ is crucial, given that **G3i** contains no structural rule. To generalize $p, \Gamma \Rightarrow p$ to compound formulas A, one usually needs to perform an induction on the structure of A. For instance, if $A = B \supset C$, then the desired result can be obtained by means of formal derivations as follows:

$$\dfrac{\dfrac{B, \overbrace{B \supset C}^{\Gamma} \Rightarrow B \qquad C, \overbrace{B}^{\Gamma} \Rightarrow C}{B \supset C, B \Rightarrow C}\,L\supset}{B \supset C \Rightarrow B \supset C}\,R\supset$$

where the premises are derivable by the inductive hypothesis, and there's no need to apply any of the weakening rules, exactly because of the presence of Γ. These types of initial sequents are said to be *weakening-absorbing*, in the sense that the effect of weakening is already included within their formulation, and this is why in logical systems, such as **G3i**, one is allowed to work with primitive logical rules only (recall the notion of admissibility stated on p. 18).

(b) CUT-elimination/admissibility. Its pretty common, in the literature on proof theory, to consider at least two forms of CUT-rules and two ways of obtaining CUT-free calculi. First of all, CUT is either additive (context-sharing) or multiplicative (context-free):

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textsc{cut}_A \qquad \dfrac{\Gamma \Rightarrow \Delta, A \qquad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\,\textsc{cut}_M$$

For what concerns the *eliminability* strategies of CUT, it's common practice to distinguish the two following results. Let $\mathcal{C}$ + CUT be a sequent calculus with CUT and $\mathcal{C}$ be its CUT-free version:

- ★ CUT-elimination: If $\vdash_{\mathcal{C}+\textsc{cut}} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathcal{C}} \Gamma \Rightarrow \Delta$.
- ♡ CUT-admissibility: If $\vdash_{\mathcal{C}} \Gamma \Rightarrow \Delta, A$ and $\vdash_{\mathcal{C}} A, \Sigma \Rightarrow \Pi$, then $\vdash_{\mathcal{C}} \Gamma, \Sigma \Rightarrow \Delta, \Pi$.[19]

Again, the choices are not arbitrary and depend on both the formulation of the calculus as a whole and the peculiarities of the logic(s) under scope. Nonetheless, the two eliminability results are related: CUT is eliminable in $\mathcal{C}$ + CUT iff CUT is admissible in $\mathcal{C}$. Once an eliminability result is given, as remarked by Kremer [Kre88, pp. 62–66], and endorsed by Poggiolesi in [Pog08b, p. 138], we are granted that:

> "[...] logical rules do not prove anything except that which concerns the symbol they introduce (and, therefore, they do not give anything more than the meaning of the constant they introduce)."

[19] Similarly, CUT-admissibility may be formulated so to deal with additive CUTs.

(c) Subformula property. A decisive feature of CUT-free sequent calculi is the property for which all formulas appearing in a derivation, are subformulas of formulas in the conclusion. Such derivations are usually referred to as *analytic*.[20] It is important to remark that, analyticity is not only a desirable logical property of proof systems, but also one of their philosophically relevant features: "Support for the analytic method has a long and venerable history in philosophy" [Pog09a, p. 12][21], which can be tracked back to ancient Greece, as well as to modern philosophers (Descartes, Arnauld, Pascal), scientists and mathematicians (Galileo, Newton, Bolzano). In logic, analyticity follows from CUT-freeness and the subformula property, and it is usually strictly correlated with two other fundamental features of a calculus: consistency and decidability. For the former, the argument is fairly simple. Suppose that some calculus C allows to derive the empty sequent with applications of CUT, i.e., $\vdash \varnothing \Rightarrow \varnothing$, and that it enjoys an eliminability result. Accordingly, $\vdash \varnothing \Rightarrow \varnothing$ can be derived without applications of CUT and derivations enjoy the subformula property. However, from a closer inspection, we see that no rule allows us to obtain a derivation concluding with the empty sequent, and, therefore, $\nvdash \varnothing \Rightarrow \varnothing$. For decidability, instead, we can reason as follows: given a CUT-free system, enjoying the subformula property, for each sequent $\mathcal{S}$, it can be established (in a finite number of steps) whether $\mathcal{S}$ is derivable or not. Notice, that we are guaranteed that the proof search procedure will effectively discriminate between derivable and underivable sequents, exactly because:

> "we know that it is not possible to lose any formula during the derivation process, and that we can only pass from logically more complex formulas to logically simpler ones (reading derivation process bottom-up)." [Pog09a, p. 14]

I have discussed some philosophical features that can serve as heuristics to produce and evaluate the *goodness* of generalizations of sequent calculi in a rather general way. Nevertheless, a specific case analysis will be developed in the second part of this work, by considering how to deal with two generalizations, namely labelled calculi and hypersequent systems, when applied to certain non-classical logics. Before moving ahead, let me, finally, engage in a discussion concerning some potential worries and objections that might be raised in connection with my choice of working with systems taken from the group of semantic-based generalizations.

1.3 Addressing some potential objections

Semantic-based generalizations are sometimes seen as *improper* Gentzen sequent systems as their formalism is polluted[22] with semantic information. Usually,

[20] See, for example, [Ind21, pp. 16–17] for an alternative definition.

[21] One should also consider the investigation on analyticity performed in [Pog12].

[22] Semantic pollution is an "epithet attributed [...] to Rajeev Goré in conversation" [Rea15, p. 650]. The debate on semantic pollution was briefly reconstructed also by L. Humberstone in

arguments against semantic based systems include the following criticism:

> "Despite their impact, labelled proof systems have been criticized as impure, in contrast to the more traditional proof systems, and difficult to use in practice" [Neg07, p. 109].

> "The labelled method [...] is a semantic method [in that] it imports in its language the whole structure of Kripke semantics in an explicit and significant way" [PR12, p. 49].

Accordingly, the literature seems to suggest that an another desideratum might be added to the list stated above. Let's call the requirement, for which in the formulation of generalizations of sequent systems, we should "not make any use of semantic parameters beyond the language of formulas" [PR12, p. 49], *Syntactic purity principle*.
At this point, naturally, one can ask *whether inferentialism can make use of labelled rules to characterise the meaning of logical constants*. I'll try to argue that negative answers to that question are misguided and, ultimately, that in semantic based generalizations there's no such thing as semantic pollution.[23]
First of all, inferentialism seeks for an anti-realistic conception of meaning of logical constants by capturing, via rules, their inferential behaviour. On the other hand:

> "Model-theoretic semantics is often described as being *realistic*; it establishes a kind of correspondence between linguistic expressions and elements of formal structures which are either thought of as a part reality or as representing part of reality." [Wan00, p. 3, Emphasis mine]

It is clear that proof-theoretic and model-theoretic philosophies of meaning are radically different one from the other. According to the former, the meaning of logical constants is encoded in their actual use, while, in the latter, the meaning of each constant is spelled out, usually, through a function from the logical vocabulary to a domain of objects. Among the most notorious model-theoretic structures, and modifications thereof, used to characterize non-classical logics, one can certainly think of so-called *Kripke relational semantics* (or *possible worlds semantics*). Such structures are also the starting point for the construction of a variety of labelled calculi. These type of sequent systems raise a first possible worry, that is, to loose the anti-realistic flavour of an inferentialist characterization of logical constants in virtue of expressions usually interpreted in a *reified* manner. The word *reification* has a very long tradition in the history of philosophy, and, in this context, I'll use it in a rather common and general way. More specifically, *reified* indicates all formal expressions that are interpreted as referring to some extra-logical object, no matter whether abstract or concrete. Transferring the reified interpretation of model-theoretic expressions (as some

Remark 1.21.8, [Hum11, pp. 111–112].

[23] My reflections are, especially, inspired by the different works on semantic pollution and syntactic purity by Poggiolesi (and Restall) [Pog09a; PR12], Negri and von Plato [NvP11; NvP15], Read [Rea15] and Martinot [Mar2x].

kind of extra-logical objects), to their proof-theoretic counterparts is one of the moves that motivates the rejection of labelled systems as good tools for inferentialism. The point that I would like to raise is that the presence of auxiliary expressions is *not* a threat to an anti-realistic conception of meaning. Indeed, we're considering the value of labelled deductive systems as proper proof theoretic frameworks, allowing us to grasp the correct inferential use of constants, and not the ontology surrounding their formalism:

> "Whether they serve to denote something is a separate matter, [...] a matter of metaphysics, not of semantics." [Rea15, p. 656]

So, the questions, *do labels and relational atoms serve to the scope of finding meaning-defining rules?* and *do labels and relational atoms exist, denote,. . . ?*, should be kept separate.[24] As the overlap between the two issues engenders confusion, to see the value of labelled generalizations, indeed, it might be a better choice to refrain from our *reification attitudes*:

> "The claim that signs and expressions of a [labelled proof system] depict (and thus refer to) something may quite rapidly lead to a reification of entities one may not be prepared to reify, resulting in various forms of realism or Platonism." [Dut12, p. 92]

More importantly, we should inspect whether the transfer of reified interpretations – typical of model-theoretic expressions – really fits the objects we find in a labelled proof system. To answer this question, consider the following example due to Read:

> "Leibniz [believed that:] 'There is no need to let mathematical analysis depend on metaphysical controversies.' But the infinitesimal calculus (to which Leibniz was referring) is still meaningful whether or not infinitesimal quantities exist." [Rea15, p. 656]

Accordingly, for labelled calculi, there's no need to let their proof theoretic legitimacy depend on the metaphysical interpretation of the auxiliary expressions.
Indeed, by getting closer to the practice, labels and relational atoms are no more and no less than technical devices supporting proof-construction (as it will be shown at length in the first, as well as in second, case study of part II). As the transfer of reified interpretations, usually ascribed to model-theoretic expressions, to proof theoretic elements is rejected, semantic based generalizations *satisfy* the syntactic purity requirement. The question of whether logical expressions (such as labels and relational atoms) denote, and of what they denote, is therefore a separate and independent concern. Moreover, if these reflections against the *transfer of metaphysical interpretations* (usually of model-theoretic expressions in terms of reified entities), in considerations on the *proof theoretic legitimacy* of semantic based generalizations, is or seems convincing, then the anti-realistic spirit of inferentialism is preserved:

[24] Accordingly, it seems to me that such notion of pollution can be reasonably referred to as *metaphysical pollution*.

> "The charge [of semantic pollution] is shown to be mistaken. It is argued on inferentialist grounds that labelled deductive systems are as syntactically pure as any formal system in which the rules define the meanings of the logical constants." [Rea15, p. 649]

> "The lesson from the [...] correspondence between syntax and semantics is that one direction of a semantical clause corresponds to an introduction rule, the other direction to an elimination rule. In perfect analogy to the proof terms of typed lambda-calculus [...], we can make the semantics of possible worlds [...] formal, by including these worlds and the forcing relation as parts of a system of rules." [NvP15, p. 270]

It is suggested that, in analogy to typed systems (see our example below), also in labelled calculi, rules governing the additional expressions are *genuine* proof-formation rules (albeit obtained by conversion of semantic clauses and frame conditions), allowing one to construct more "detailed" derivations. Following this suggestion, "$a : A$, [can be read as] a is a *proof-object* for A" [NvP01, p. 13, Emphasis mine], and elements like $a \leq b$ or aRb (often included in labelled calculi), can be seen as expressing interactions between such proof-objects. Indeed, in labelled generalizations, for each logic, the derivability of formulas is subject, not only to rules for logical constants, but also to rules stating which labels can occur and how they interact with each other. Informally speaking, rules encoding interactions between labels are no more, no less than those "instructions" we need to construct bottom-top proof-searches.

Example 1. Let $\to$ be denoting intuitionistic implication and take again the sequent $\Rightarrow A \to (B \to (A \wedge B))$. Let's consider a term-annotated system (see, e.g., [TS00]), where initial sequents are as follows $x : p, \Gamma \Rightarrow x : p$[25] and right rules for $\to$ and $\wedge$ have the following shape:

$$\frac{x : A, \Gamma \Rightarrow y : B}{\Gamma \Rightarrow \lambda x.y : A \to B}\ R\to^{\lambda} \qquad \frac{\Gamma \Rightarrow x : A \qquad \Gamma \Rightarrow y : B}{\Gamma \Rightarrow \langle x, y\rangle : A \wedge B}\ R\wedge^{\lambda}$$

Accordingly, the derivation of $A \to (B \to (A \wedge B))$ can be displayed as follows:

$$\dfrac{\dfrac{\dfrac{x : A, y : B \Rightarrow x : A \qquad x : A, y : B \Rightarrow y : B}{x : A, y : B \Rightarrow \langle x, y\rangle : A \wedge B}\ R\wedge^{\lambda}}{x : A \Rightarrow (\lambda y.\langle x, y\rangle) : B \to (A \wedge B)}\ R\to^{\lambda}}{\Rightarrow (\lambda x.\lambda y.\langle x, y\rangle) : A \to (B \to (A \wedge B))}\ R\to^{\lambda}$$

The idea behind the formulation of $R\wedge^{\lambda}$ and $R\to^{\lambda}$ can be understood by relying on the so-called BHK-interpretation[26], that:

[25] We specify that Γ is a multiset (and not a list). Initial sequents are weakening-absorbing, and their generalized version for arbitrary formulas is admissible.

[26] See [TS00, p. 23 & p. 55] and [Bim14, pp. 7–8]

> "[...] explains what it means to prove a logically compound statement in terms of what is means to prove its components; the explanations use the notions of *construction* and *constructive proof* as unexplained primitive notions." [TS00, p. 55]

For example, the clauses for $\wedge$ and $\rightarrow$ are stated as follows:

($\wedge$) z is a proof of $A \wedge B$ iff z is a pair $\langle x, y \rangle$ and x is a proof of A and y is a proof of B.

($\rightarrow$) x is a proof of $A \rightarrow B$ iff x is a construction transforming any proof y of A into a proof $x(y)$ of B.

More closely, the explanation of $\wedge$ and $\rightarrow$ furnished by the BHK-clauses, is made explicit also within in the rules $R\wedge^{\lambda}$ and $R \rightarrow^{\lambda}$. For example, the latter one tells us that, if, starting from a proof x of A, we can find a proof, say y, of B, then we have in fact given a proof of $A \rightarrow B$, which we denote as $\lambda x.y$. Similarly, for $R\wedge^{\lambda}$.
However, as remarked, in [NvP15, p. 269]:

> "Thirty years after Gentzen, and well before the computational semantics was understood in detail, Saul Kripke gave another semantics for intuitionistic logic in terms of *possible worlds*."

Intuitively, Kripke's idea was to define truth relative to the discovery process of an idealized mathematician (or, community of mathematicians). At each point in such development, a body of mathematical results has been provided. Knowledge of past results is presumed, and the body of known results grows as time proceeds. Formally, we have a set of *worlds*, *states* or *situations* (denoted as $x, y, v, z, \dots$), connected to each other by a reflexive and transitive relation (denoted $\leq$). Truth conditions for $\wedge$ and $\rightarrow$ are formulated as follows:

$A \wedge B$ is true at x	iff	A is true at x and B is true at x.
$A \rightarrow B$ is true at x	iff	for all y, if $x \leq y$ and A is true at y, then B is true at y.

Also in this case, the clauses just displayed can be used to lay down sequent-style rules. Initial sequents $x \leq y, x : p, \Gamma \Rightarrow \Delta, y : p$, whereas rules for $\rightarrow$ and $\wedge$ are as follows:

$$\text{(y fresh)}\ \frac{x \leq y, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A \rightarrow B}\ R{\rightarrow}^{L} \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ R\wedge^{L}$$

Moreover, notice that we add rules for relational atoms $x \leq y$ which reflect, at the calculus level, the conditions of reflexivity and transitivity previously mentioned:

$$\frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref} \qquad \frac{x \leq z, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}\ \mathit{Trs}$$

Accordingly, $\Rightarrow A \rightarrow (B \rightarrow (A \wedge B))$ is derived as follows.

$$
\dfrac{y \leq z, x \leq y, y : A, y : B \Rightarrow z : A \qquad \dfrac{z \leq z, y \leq z, x \leq y, y : A, z : B \Rightarrow z : B}{y \leq z, x \leq y, y : A, z : B \Rightarrow z : B}\ Ref}{\dfrac{y \leq z, x \leq y, y : A, z : B \Rightarrow z : A \wedge B}{\dfrac{x \leq y, y : A \Rightarrow y : B \to (A \wedge B)}{\Rightarrow x : A \to (B \to (A \wedge B))}\ R \to^L \ (y \text{ fresh})}\ R \to^L \ (z \text{ fresh})}\ R\wedge^L
$$

This example is meant to simply illustrate that there's (at least) a certain *conceptual vicinity* between labelled and term-annotated proof systems. In the cases briefly analysed, we always started from semantic explanations to provide inference rules. More precisely, we enriched the syntax of sequents by adding specific symbols to represent either proofs or possible worlds. Nevertheless, conceptually speaking, terms interpreted as denoting proofs or constructions, differently from labels and relational atoms, are not seen as polluting the syntax of term-annotated sequents. However, suppose that one relies on a reified interpretation of the syntax of term-annotated calculi, for example, by claiming that terms denote *real proofs*. A question naturally arises: an (intuitive, pre-theoretic) interpretation, maybe fully denotational, of the elements enriching the syntax of term-annotated sequents, should encourage us in leaving aside such frameworks while doing proof theory, despite their usefulness? Can an answer to the previous question be negative for term-annotated systems and be positive if asked with respect to labelled calculi? Also in this case, I believe, the point is whether denotational questions should enter considerations concerning the legitimacy of proof systems. Additionally, I believe that the questions stated above can be further understood and addressed when some (formal) result, concerning a possible correspondence between labelled and term-annotated calculi, will be properly given.

The practical use of labelled generalizations seems to suggest that there's neither room, nor any need to rely on our reification attitudes in the understanding of the value of labelled deductive systems, and, ultimately, that, by leaving denotational questions aside, we can perfectly make sense of their enriched formalism in purely proof theoretic terms.
These few philosophical considerations, along with our technical case studies (see Chapters 3–4), (should) allow us to glimpse that *there's more to semantic based generalizations, than semantic pollution!*

1.4 What's included in this thesis?

As I have remarked several times, instead of keeping the discussion only at the theoretical level, I'll attempt to characterise some non classical logics through two generalizations of sequent systems, namely labelled sequent calculi and hypersequent systems. More specifically, benefits of such characterisations will be explicitly mentioned and discussed case by case.

1.4.1 Case study 1. Discussive logic

Stanisław Jaśkowski is acknowledged as one of the founders of paraconsistent logics inasmuch as he proposed one of the first inconsistency-tolerant and non-trivial systems known under the label of *discussive* (or, sometimes, *discursive*) logic (abbreviated, $\mathbf{D_2}$). In order to present his idea, Jaśkowski questioned so-called *explosion laws*, i.e, those logical principles according to which any statement can be proven from a contradiction. To motivate his criticism, Jaśkowski suggested to consider discussions as contexts where inconsistent theses may be expressed, without leading discussants to infer every meaningful expression from them. More precisely, to formalize his paraconsistent representation of discussions, Jaśkowski's suggested to replace, firstly, material implication $\supset$ and, secondly, also classical conjunction $\wedge$ in favour of more fine-grained connectives defined through the modal operator of possibility $\Diamond$. Roughly:

$$p \rightarrow_{\mathsf{d}} q = \Diamond p \supset q$$

$$p \wedge_{\mathsf{d}} q = p \wedge \Diamond q$$

We remark that $\rightarrow_{\mathsf{d}}$ denotes *discussive implication*, whereas $\wedge_{\mathsf{d}}$ is *discussive conjunction*. Intuitively, the former one can be read as "if someone states p, then q", and the latter can be understood as "p and someone states q". Now, let $p_1, \ldots, p_n$ be representing some opinions uttered in a discussion and take q as a possible conclusion. The idea is that q *discussively follows* from $p_1, \ldots, p_n$ just in case $\Diamond q$ follows from $\Diamond p_1, \ldots, \Diamond p_n$ in **S5**. This characterization, according to Jaśkowski, "[...] is how an impartial arbiter might understand the theses of the various participants in the discussion." Indeed, according to the perspective of an external observer (i.e. someone who is not involved in a discussion) all that is uttered in a discussion is *only possible*. It seems a reasonable observation, as people not involved in discussions can disbelief, disagree or dissociate from discussants' statements. For the same reason, also conclusions following from discussants' statements in a discussion are *only possible*. In this setting, discussions consist of both statements uttered by some discussant and the conclusions that can be inferred from them.

Example 2. For instance, informally, one might think of certain *legal situations* as those settings in which discussants can put forward contradictory statements without making the whole situation "meaningless". For example, during trials it is common for lawyers involved to either support a certain perspective on certain facts or to refute it. One might think of those trials in which there's a prosecutor, usually trying to argue in favour of the guilt of someone, and defence attorneys, usually trying to argue in favour of the innocence of their defendant(s). All statements uttered in order to advocate for the validity of their respective positions, and the consequences thereof, are understood as self-consistent and coherent with the role each expert is called to fulfil within the courtroom. Nonetheless, it often happens that lawyers argue by putting forward statements in contradiction with other lawyers' argumentations. Importantly, the presence of some inconsistencies, does not make the situation, i.e., the trial, trivial, insignificant. Rather, it seems part of the internal structure of trials to allow each of the involved parts to argue in favour of or against a certain these, even at the cost of reaching two completely opposite points of view. Moreover, according to the court – the *impartial arbiter* in Jaśkowski's words – each of the positions expressed by the parts involved in the trial is merely *possible*, exactly in the sense that all of them are understood as equally possible descriptions of the facts under consideration during the trial. The final decision, then, ascribes different degrees of plausibility to the thesis advocated by the parts, by considering other data, such as material evidences, legal procedures, judiciary precedents, and so on.

Jaśkowski provided neither a deductive system nor a semantic characterization of $\mathbf{D_2}$. Nevertheless, discussive systems have attracted some attention amongst logicians and several formal structures have been employed to offer detailed and systematic assessments of Jaśkowski's logic. In what follows, we wish to address two distinct, but strictly related, issues concerning discussive logic.
First of all, we will present an overview, with historical and critical remarks, of two articles by S. Jaśkowski, which contain the formulation of his paraconsistent logic. Roughly, after having introduced Jaśkowski's methodology of building $\mathbf{D_2}$ and his main philosophical motivations for providing such a system, we will explore some of the main contributions to the development of $\mathbf{D_2}$.
As it will be examined and highlighted in part of the historical survey on discussive logic, the task of finding axiomatic systems for $\mathbf{D_2}$ has not only occupied a privileged place, but has also marked a troublesome path in the field of discussive logic-related researches. As an alternative to the Hilbert-style proof theory, in Chapter 3, I wish to propose another proof theoretic characterization of discussive logic in terms of labelled sequents. These latter structures, as sketched above, will be used to provide a rule-based calculus for $\mathbf{D_2}$. Labelled calculi are well-known sequent systems that internalize, at the syntactic level, semantic informations taken, in our specific case, from relational models for $\mathbf{D_2}$. The plan is to introduce the semantics, present our intended labelled calculus, and show that it enjoys a variety of proof-theoretic properties. We will conclude with some methodological observations, by highlighting virtues and benefits of our approach, and by pointing out some topics for future research.

1.4.2 Case Study 2. Relevant logics

Relevant logics are a well-known family of non-classical logics introduced to cope with so-called paradoxes of material and strict implication. According to relevantists, a connective standing for implication is intended to express a more fine-grained and philosophically motivated notion of conditional. Part of the philosophical intuition of relevant logics, at least in the early development by Anderson and Belnap [AB75], was that the antecedent and consequent of a valid conditional must be relevant to each other, in the sense that, in expressions of the form "*if A, then B*" there must be a strong connection between antecedent A and consequent B.
Relevant logics have attracted a lot of attention among logicians and many formal structures were applied to offer detailed and systematic characterizations. Proof theoretic studies on relevant logics have a long and troubled history.

Gentzen-style sequents were proposed, among others, in [AB75; Pao02]. For what concerns generalizations of sequents, instead, there are different trends in the literature. To cite a few of them: *cognate* sequents ([Kri59]), *hypersequents* ([Avr87; Avr91b]), Dunn-Mints calculi ([Dun73]), *consecution* calculi ([Bim14]), *display* sequents [Res98; Res00; Bim14]. From the perspective of labelled proof systems, instead, there is a variety of approaches. Among the early significant contributions, one finds A. Urquhart and S. Giambrone's U- and G-systems for some positive fragments of a family of relevant logics (called *semilattice logics*) in [GU87]. Urquhart and Giambrone's systems correspond to a *weakly labelled calculus* in the sense of [Ind21, p. 204], that is, labels are limited technical devices supporting proof construction. Indeed, no special rules operating on labels are introduced. More precisely, the behaviour of labels in derivations is subject only to some specific restrictions, established directly on the application of rules. Moreover, the labelling of formulas in the rules for $\rightarrow$ refers to a different treatment of the ternary relation $Rabc$ at the semantic level, that is, by putting $c = a \cup b$. An analogous work was conducted by R. Kashima in [Kas01; Kas03] always in the context of semilattice relevant logics. L. Viganò [Vig00] pursued a characterization of some relevant logics by using a calculus enriched with rules acting on labels and which restates the presence of the third element c, rather than $a \cup b$. Similarly in [KN20], H. Kurokawa and S. Negri introduced a wide range of labelled calculi constructed with reference to the *original* (or *non reduced*) ternary relational semantics proposed by Routley and Meyer.
For the purposes of this chapter, however, we will introduce relevant logics in terms of *reduced* Routley-Meyer models, i.e., by means of relational structures employing a ternary relation between states (see, e.g., [RM73; Rou+82]), along with a distinct element interpreted as the *real* (or *actual*) world. Intuitively, in Routley-Meyer models, a relevant implication "*if A, then B*" is true at world a just in case, for all worlds b, c, related to a, if A is true at b, then B is true at c. The aim of this chapter is to define modular proof systems for a variety of relevant logics on the basis of these models. More specifically, we will introduce a family of modular labelled sequent calculi for relevant logic **B** and its extensions, namely, **DW**, **DJ**, **TW**, **T**, **RW**, **R** and **RM**. The calculi are based on Routley-Meyer semantics, in the sense that, by following the well-established methodology proposed by [Neg05], sequents internalize, by means of syntactic tools, semantic information taken exactly from reduced Routley-Meyer models. I'll present the rules of the labelled calculi and some related preliminary results, along with a comparison with other related works. Central results include a proof of soundness, as well as proofs of completeness (both, semantic and syntactic). Finally, we will proceed towards the proof analysis of the systems and conclude with a proof of cut-admissibility. In the conclusions, I'll highlight the benefit of such an approach by addressing how the methodology adopted could be expanded to cover further topics within the proof theory of relevant logics.

1.4.3 Case Study 3. Modal expansions of intuitionistic logic

In this chapter, the idea is to pick up a syntactic generalization of Gentzen sequents, i.e., hypersequents, and deal with a modal expansion of intuitionistic logic. To be more specific, a hypserquent is a structure of the following form:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_{n-1} \Rightarrow \Delta_{n-1} \mid \Gamma_n \Rightarrow \Delta_n$$

where each $\Gamma_i \Rightarrow \Delta_i$ is a Gentzen sequent, interpreted in the usual manner, and the symbol $\mid$ is interpreted as an object language disjunction, i.e.:

$$(\wedge\Gamma_1 \to \vee\Delta_1) \vee (\wedge\Gamma_2 \to \vee\Delta_2) \vee \cdots \vee (\wedge\Gamma_{n-1} \to \vee\Delta_{n-1}) \vee (\wedge\Gamma_n \to \vee\Delta_n)$$

The investigation in this chapter continues the research initiated in [DMP21], where the modal intuitionistic logic **IS5** was under scope. It was noticed that by elaborating a single-conclusion version of the hypersequent calculus in [Pog08a], along with all necessary additions and modifications, one can build a successful cut-free system for **IS5**. The positive results in [DMP21] seem to be in contrast with the following negative remark:

> "the hypersequent structure [...] is a multiset of sequents, called components, separated by a symbol denoting disjunction, in the sense that it is a multi-contextual structure. [It] does not really enrich the sequent structure in this case and it appears [as not] appropriate to deal with intuitionistic and modal operators." [GS10, §3.1]

To strengthen our idea that hypersequent systems are appropriate to deal also with intuitionistic and modal operators, I'll propose an additional case study. More closely, I'll study intuitionistic logic expanded via the addition of a so-called *actuality operator*, denoted '@' (see, [NO20]). Roughly, formulas like $@A$ can be read as 'A is actual'. I'll prove a cut-elimination result and discuss the consequences thereof. This provides more positive evidence of the appropriateness of using hypersequent structures to deal with modal intuitionistic logics.

Chapter 2

A little prelude. Origins and development of discussive logic

Layout of the chapter. In this chapter, I'll propose a philosophical and historical systematization of the various stages of the development of Jaśkowski's discussive logic. Such an investigation will be carried out by considering two different – albeit related – perspectives: the connections between $\mathbf{D_2}$ and modal logics, as well as the proof theoretic approaches to $\mathbf{D_2}$.

2.1 The Origins of Discussive Logic

Throughout this chapter we will consider the following classical connectives, $\neg$ *(negation),* $\wedge$ *(conjunction),* $\vee$ *(disjunction),* $\supset$ *(material implication), plus the modal operators,* $\Box$ *(necessary) and* $\Diamond$ *(possible). All additions and changes will be explicitly stated and explained.* $\Gamma, \Delta, \Sigma, \ldots$ *and* $A, B, C, \ldots$ *denote sets of formulas and formulas, respectively.* $p, q, r \ldots$ *stand for propositional variables..*

2.1.1 The first discussive system

S. Jaśkowski (1906-1965)[1] is the author of several important logical and mathematical studies. To cite some of them, Jaśkowski is usually acknowledged as one of the inventors of the natural deduction calculus (accomplishing this work almost at the same time of G. Gentzen) and as the proponent of the first paraconsistent logic known as "discussive" (or "discursive") logic[2]. In [Jaś99a] (which corresponds to the English translation of Jaśkowski's original article [Jaś48], published in 1948), the logician proposed a logic which should capture situations where discussants are in conflict. Jaśkowski's main idea was to consider a discussant's statement, p, as inherently consistent, but potentially incoherent with some other discussant's proposition. With this in mind, Jaśkowski focused his attention on a classically valid law, namely *ex contradictione quodlibet [sequitur]*

[1] For biographical informations one can consider [KP67; Dub75; Ind18]. For synthetic introductions to Jaśkowski's discussive logic, see, for example, [Pri84; PTW22].

[2] Jaśkowski denoted this logic by $\mathbf{D}_2$, where the label '2' indicates that we are dealing with the 'two-valued discussive sentential calculus'.

((ECQ), "from a contradiction everything [follows]") – $p \supset (\neg p \supset q)$ – claiming that it should not be generally valid. His strategy, in order to invalidate (ECQ), has been that of getting rid of the classical connective of material implication, i.e., $\supset$, in favour of so-called "discussive implication", i.e., $\rightarrow_\mathsf{d}$. Lewis' modal logic **S5** has played a fundamental role in the formulation of such discussive systems, so, let's recall the definition of **S5**:

Definition 2.1.1. **S5** is axiomatized is follows:

If A is a theorem of **PC**, then A is a theorem of **S5**.

$$\Box(A \supset B) \supset (\Box A \supset \Box B) \tag{K}$$

$$\Box A \supset A \tag{T}$$

$$\Diamond A \supset \Box\Diamond A \tag{5}$$

and the following rules:

$$\frac{A \qquad A \supset B}{B}\ \text{MP} \qquad\qquad \frac{A}{\Box A}\ \text{Nec}$$

Finally, we say that a modal logic **L** is of **S5**-type iff $\mathbf{L} \subseteq \mathbf{S5}$[3].

Thanks to Lewis' modal system, Jaśkowski established the definition of discussive implication in the following way: $p \rightarrow_\mathsf{d} q = \Diamond p \supset q$, validating thus the discussive version of *modus ponens*:

$$\frac{A \qquad A \rightarrow_\mathsf{d} B}{B}\ \text{MP}_d$$

Additionally, we can get also the definition of "discussive bi-implication", $p \leftrightarrow_\mathsf{d} q = (\Diamond p \supset q) \wedge (\Diamond q \supset \Diamond p)$. Notice that, so defined, both, $\rightarrow_\mathsf{d}$ and $\leftrightarrow_\mathsf{d}$, are asymmetric connectives. One might wonder what the $\Diamond$ operator is meant to represent in a discussive framework. According to Jaśkowski's own perspective:

> "To bring out the nature of the theses of such a system it would be proper to precede each thesis by the reservation: "in accordance with the opinion of one of the participants in the discussion" or "for a certain admissible meaning of the terms used". Hence the joining of a thesis to a discussive system has a different intuitive meaning than has assertion in an ordinary system. *Discussive assertion* includes an implicit reservation of the kind specified above, which [...] has its equivalent in $\Diamond$ [Jaś99a, p. 43]."

In a latest note, [Jaś99b] (the English translation of the 1949 paper [Jaś49]), Jaśkowski proposed to substitute from the set of connectives also classical conjunction in favour of "discussive conjunction" and chose the following definition: $p \wedge_\mathsf{d} q = p \wedge \Diamond q$. With this additional connective, then Jaśkowski defined again discussive bi-implication in the following manner: $p \leftrightarrow_\mathsf{d} q = (p \rightarrow_\mathsf{d} q) \wedge_\mathsf{d} (q \rightarrow_\mathsf{d} p)$. So, in sum, to prove discussive formulas, i.e., formulas including discussive connectives, Jaśkowski suggested to transform such formulas accordingly to their modal definitions and to prove the resulting modal formula in **S5**. In more rigorous terms:

[3]As known, **S5** has several equivalent axiomatization; for instance, one can employ (4) ($\Box A \supset \Box\Box A$) and (B) ($A \supset \Box\Diamond A$) instead of axiom (5).

Definition 2.1.2. $\mathbf{D}_2$ is the system whose language $\mathcal{L}$ includes the following set of connectives $\mathsf{S} = \{\neg, \vee, \wedge_{\mathsf{d}}, \rightarrow_{\mathsf{d}}, \leftrightarrow_{\mathsf{d}}\}$. Take a function $\tau : \mathsf{Form}_{\mathbf{D}_2} \mapsto \mathsf{Form}_{\mathbf{S5}}$ such that, for any $A, B \in \mathsf{Form}_{\mathbf{D}_2}$:

$$\begin{aligned}
&\tau(p) = p\\
&\tau(\neg A) = \neg\tau(A)\\
&\tau(A \vee B) = \tau(A) \vee \tau(B)\\
&\tau(A \wedge_{\mathsf{d}} B) = \tau(A) \wedge \Diamond\tau(B)\\
&\tau(A \rightarrow_{\mathsf{d}} B) = \Diamond\tau(A) \supset \tau(B)\\
&\tau(A \leftrightarrow_{\mathsf{d}} B) = (\Diamond\tau(A) \supset \tau(B)) \wedge \Diamond(\Diamond\tau(B) \supset \tau(A))
\end{aligned}$$

Let $\Diamond\Gamma = \{\Diamond\tau(A_1), \ldots, \Diamond\tau(A_n) \mid A_1, \ldots, A_n \in \Gamma\}$, then for all $\Gamma \subseteq \mathsf{Form}_{\mathbf{D}_2}$ and $B \in \mathsf{Form}_{\mathbf{D}_2}$, we set:

$$\Gamma \models_{\mathbf{D}_2} B \text{ iff } \Diamond\Gamma \models_{\mathbf{S5}} \Diamond\tau(B).$$

In other words, a formula B is said to be a *discussive consequence* of a set of premises $\{A_1, \ldots, A_n\}$ just in case $\Diamond\tau(B)$ follows from the set $\{\Diamond\tau(A_1), \ldots, \Diamond\tau(A_n)\}$ in **S5**. Following Jaśkowski:

> "[...] if a thesis A is recorded in a discussive system, its intuitive sense ought to be interpreted so as if it were preceded by the symbol $\Diamond$, that is, the sense: "it is possible that A". This is how an impartial arbiter might understand the theses of the various participants in the discussion." [Jaś99a, p. 43]

The motivation behind this quote and Definition 2.1.2 can be intuitively explained with the following example. If we take formulas including $\rightarrow_{\mathsf{d}}$ and replace it simply accordingly to τ we will obtain a great number of **S5** invalid formulas. In this case, even the identity, $A \rightarrow_{\mathsf{d}} A$, if transformed in $\Diamond A \supset A$, turns out to be **S5**-invalid. However, many of this negative results can be avoided, if we prefix $\Diamond$ to every modally translated formula. For example, $A \rightarrow_{\mathsf{d}} A$, if translated as follows $\Diamond(\Diamond A \supset A)$, turns out to be **S5**-valid.

Observation 3. To see the paraconsistent character of $\mathbf{D}_2$ consider that already in [Jaś99a], the discussive version of (ECQ), $A \rightarrow_{\mathsf{d}} (\neg A \rightarrow_{\mathsf{d}} B)$, was not included as a theorem of $\mathbf{D}_2$. To see this, consider always the modal translation of (ECQ), i.e., $\Diamond(\Diamond A \supset (\Diamond\neg A \supset B))$, which is not valid in **S5**. Consequently to the rejection of (ECQ), the existence of contradictory statements, $\Diamond A$ and $\Diamond\neg A$, is possible without that their presence entails the 'overfilling' (triviality) of the system. However, the logic is not paraconsistent with respect to conjuncted contradictions, indeed, $\Diamond(\Diamond(A \wedge \neg A) \supset B)$ is still a theorem of **S5** . Moreover, notice that in this framework $\wedge$ adjunction fails (i.e., $A \wedge B$ cannot be inferred from A and B) and, for this specific reason, the $\{\neg, \vee, \wedge, \rightarrow_{\mathsf{d}}\}$-fragment of $\mathbf{D}_2$ is usually classified among the *non-adjunctive* approaches to paraconsistent logics:

> "...] discussive logic represents an ideology that is, to my mind, the most appropriate one for paraconsistency. To put it informally: at the very core of paraconsistency lies not negation, but conjunction. [...] With respect to inconsistency tolerating calculi, this connective seems to be the most important one." [Urc02, p. 487]

Nonetheless, in [Jaś99b], thanks to the presence of discussive conjunction, adjunction can be successfully restated in the system. The discussive version of the law of non contradiction (LNC), $\neg(A \wedge_{\mathsf{d}} \neg A)$, remains a valid law. To see this consider always the **S5** invalid formula $\diamond(A \wedge \diamond\neg A)$. Finally, the discussive version of conjunctive (ECQ), $(A \wedge_{\mathsf{d}} \neg A) \rightarrow_{\mathsf{d}} B$, is no longer valid, making, thus, $\mathbf{D}_2$ paraconsistent also with respect to conjuncted contradictions.

Observation 4. Jaśkowski's definition of $\wedge_{\mathsf{d}}$ and $\rightarrow_{\mathsf{d}}$ are not the only ones available and, indeed, experts considered different variants, such as:

$$A \wedge^{l}_{\mathsf{d}} B = \diamond A \wedge B$$
$$A \wedge^{s}_{\mathsf{d}} B = \diamond A \wedge \diamond B$$
$$A \rightarrow^{s}_{\mathsf{d}} B = \diamond A \supset \diamond B$$

As one can easily see, the introduction of these new connectives tries to recover the asymmetry present in Jaśkowski's original proposal. Anyway, notice that the formulas $\diamond(A \wedge \diamond B)$, $\diamond(\diamond A \wedge B)$ and $\diamond(\diamond A \wedge \diamond B)$ are all equivalent in **S5**, while $\diamond(\diamond A \supset B)$ and $\diamond(\diamond A \supset \diamond B)$ are already equivalent in **S4** (a subset of **S5**). Moreover, as known since [Jaś99a], $\mathbf{D}_2$ is a paraconsistent extension of the $\{\vee, \wedge, \supset\}$-fragment of classical logic. In other words, the discussive operators in $\mathbf{D}_2$ behave just like their classical counterparts. Interestingly, however, if we consider also an enriched language which includes a negation connective, the discussive logics generated by these new operators will no longer coincide with the $\{\neg, \vee, \wedge, \supset\}$-fragment of classical logic.

> "It is not true thus that different translation clauses 'would have just the same consequences' [...]. Different choices of discussive conjunction and discussive implication would in fact define logics distinct from $\mathbf{D}_2$." [Joa05, p. 215]

This is a struggling point. Indeed, as we will see in section 2.2.3, some notable problems arise in the formulation and comparison of axiomatic systems including different discussive connectives and negation.

2.1.2 Jaśkowski's Philosophical Motivations

In his celebrated *Metaphysics*, Aristotle claimed that "the most indisputable of all beliefs is that contradictory statements are not at the same time true" ([Ari01, Γ, 1011b13–14]), establishing, thus, – in a crystal clear way for the first time in the history of philosophy – one of the most celebrated and debated logical, psychological and ontological laws, i.e., the so-called law of non-contradiction (LNC). Roughly, Aristotle was convinced that the principle for which two opposite propositions, usually, one the negation of the other, cannot both be true at the same time had a very special status. Indeed, (LNC) corresponds, according to the Greek philosopher, to the most certain principle, which has a triple valence: it is a law of human rationality and reasoning (logic), it is a law governing reality (ontology) and, finally, it is a law concerning human beliefs (psychology). The discussions continued and, finally, during the middle ages, the debates on contradictions reached another fundamental turning point. An unknown author, usually acknowledged under the pseudonymous of Pseudo-Scotus, defined for the first time the principle of *ex contradictione quodlibet [sequitur]* in a commentary to Aristotle's *Analytica Priora* [Pse01]. Importantly, William of Soissons, during the XII century, proposed the first known proof of the aforementioned principle and it is documented that already during the XIV century logicians knew about its existence and accepted (ECQ) as true[4]. However, the birth and the growing interest towards formal logical systems, strictly matched to philosophical considerations and objectives, has led some philosophers and logicians to reconsider also the validity and the truth of (LNC) and (ECQ). Jaśkowski has been among them. Indeed, in the first paragraphs of his celebrated 1949 article he develops a brief survey concerning the most important philosophical positions which, according to his reading, have provided some motivations to accept the presence of contradictory sentences (especially, Hegel and Marx)[5]. For instance, with respect to empirical sciences, Jaśkowski wrote:

> "[...] it is known that the evolution of the empirical disciplines is marked by periods in which the theorists are unable to explain the results of experiments by a homogenous and consistent theory, but use different hypotheses, which are not always consistent with one another, to explain the various groups of phenomena. This applies, for instance, to physics in its present-day stage. Some hypotheses are even termed "working" hypotheses when they result in certain correct predictions, but have no chance to be accepted for good, since they fail in some other cases." [Jaś99a, p. 37]

The theoretical solution, according to Jaśkowski, is the following:

> "we have to take into account the fact that in some cases we have to do with a system of hypotheses which, if subjected to a too consistent analysis, would result in a contradiction between themselves or with a certain accepted law, but which we use in a way that is restricted so as not to yield a self evident falsehood." [Jaś99a, p. 37]

Indeed, in the paragraphs were he begins to elaborate more formally his ideas, Jaśkowski distinguishes very strictly between "inconsistent" and "trivial" system. The first notion is linked to the presence, within the logical system under consideration, of two theses, one the negation of the other (p and $\neg p$); the second concept, instead, asserts that in a system it is possible to derive any formula if there is a couple of contradictory statements. So, as obvious, systems in which every proposition is derivable have no practical significance, since everything can be asserted. So, finally:

> "[...] the task is to find a system of the sentential calculus which: (1) when applied to the inconsistent systems would not always entail their overfilling, (2) would be rich enough to enable practical inference, (3) would have an intuitive justification." [Jaś99a, p. 38]

[4] Importantly, the works by William of Soissons have not been preserved, however a witness of his work is contained in John of Salisbury's *Metalogicon*.

[5] For more philosophical details on the consequences of adopting a paraconsistent point of view, one might consider [Pri08].

Jaśkowski did not further elaborate his philosophical considerations, but, nowadays, scholars provided – by taking inspiration directly from Jaśkowski's brief suggestions – some interesting philosophical applications of $\mathbf{D}_2$ (for example, to the foundations of physical theories, to the notion of pragmatic (or partial) truth [CD95; CKB07], to the formal study of belief structures and argumentation schemes [DPS18]).

2.2 The Development of Discussive Logic

Discussive systems have attracted discrete attention and various experts contributed to their development[6]. Our aim, in what follows, is to systematize and explain some of the main works concerning Jaśkowski's discussive logic. To keep the presentation as much as possible self-contained, we will restrict our attention to three distinct, even if connected, paths. More precisely, we will focus our attention on:

§2.1 the connections between discussive logic and modal systems;
§2.2 a family of logics, called "**J**" systems;
§2.3 the "direct" axiomatizations of $\mathbf{D}_2$, i.e., those systems which include axioms for discussive connectives.

2.2.1 Connections to Modal Logics

Early developments

The tradition of modal studies connected to $\mathbf{D}_2$ started already in 1968 thanks to a paper by N. da Costa [CD68] and continued uninterrupted throughout the years. Roughly said:

> "Besides non-adjunctiveness, another common obsession of discussivists concerns the alleged 'modal character' of $\mathbf{D}_2$." [Joa05, p. 217]

Early remarkable results have been provided by J. Kotas in [Kot74] from 1974. First of all, let's fix the next definition:

Definition 2.2.1. Let $\heartsuit \in \{\Box, \Diamond\}$. A $\heartsuit$-counterpart of a modal system $\mathbf{M}$ is defined as follows: $\heartsuit^n(\mathbf{M}) = \{A \mid \heartsuit^n A \in \mathbf{M}\}$, for $n \geq 1$.

With respect to Jaśkowski's $\mathbf{D}_2$, Kotas elaborated an axiomatization having as primitive connectives only $\neg$, $\supset$, $\Box$. We will denote this system $\mathbf{D}_2^K$, where

[6] At the best of our knowledge, one previous attempt in that direction was made by Ciuciura in [Ciu99] from 1999. Nonetheless, in what follows, we wish to consider also alternative approaches towards discussive systems and enrich our considerations by commenting more recent works.

'K' stands for Kotas. The axioms of $\mathbf{D}_2^K$ are:

$$\Box(A \supset (\neg A \supset B)) \tag{K1}$$
$$\Box((A \supset B) \supset ((B \supset C) \supset (A \supset C))) \tag{K2}$$
$$\Box((\neg A \supset A) \supset A) \tag{K3}$$
$$\Box(\Box(A \supset B) \supset (\Box A \supset \Box B)) \tag{K4}$$
$$\Box(\Box A \supset A) \tag{K5}$$
$$\Box(\neg\Box A \supset \Box\neg\Box A) \tag{K6}$$
$$\text{Substitution} \tag{Sub}$$

$$\frac{\Box A \qquad \Box(A \supset B)}{\Box B}\,\Box\text{MP} \qquad \frac{\Box A}{\Box\Box A}\,\text{R4} \qquad \frac{\Box A}{A}\,\text{Den} \qquad \frac{\neg\Box\neg A}{A}\,\text{Dep}\Box$$

As usual, if we want to add the possibility operator, we can define it: $\Diamond A = \neg\Box\neg A$. Notice that by having $\Diamond$ as a defined connective, Dep$\Box$ may be substituted by:

$$\frac{\Diamond A}{A}\,\text{Dep}$$

An important achievement of [Kot74] is the presentation of the following equivalences between **S5**-type systems and various combinations of axioms and rules of $\mathbf{D}_2^K$:

K1-K6	(Sub)	($\Box$MP)	(R4)	(Den)	(Dep$\Box$)/(Dep)	Equivalent System
✓	✓	✓	✓	-	-	$\Box$**S5**
✓	✓	✓	✓	✓	-	**S5**
✓	✓	✓	✓	✓	✓	$\Diamond$**S5**

Notice that, according to the table above, Kotas proved that $\mathbf{D}_2^K$ is equivalent to $\Diamond$**S5**. This result allowed him, finally, to prove that $\mathbf{D}_2^K$ is finitely axiomatizable. To obtain his results, Kotas relied on two different Jaśkowski- style translation functions. Take τ of Definition 2.1.2 and substitute the clauses for $\wedge_\mathsf{d}$ and $\rightarrow_\mathsf{d}$ with the following ones:

$$\tau^*(A \wedge_\mathsf{d} B) = \neg(\neg\tau^*(A) \vee \Box\neg\tau^*(B))$$
$$\tau^*(A \rightarrow_\mathsf{d} B) = (\neg\Box\neg\tau^*(A) \supset \tau^*(B)$$

In addition, consider a map τ_1 such that $\mathsf{Form}_{\Diamond\mathbf{S5}} \mapsto \mathsf{Form}_{\mathbf{D}_2^K}$. For any $A, B \in \Diamond$**S5**:

$$\tau_1(p) = p$$
$$\tau_1(\neg A) = \neg\tau_1(A)$$
$$\tau_1(A \supset B) = \neg\tau_1(A) \vee \tau_1(B)$$
$$\tau_1(\Box A) = \neg((\neg p \vee p) \wedge_\mathsf{d} \tau_1(A))$$

First of all, the equivalence between $\mathbf{D}_2^K$ and $\Diamond$**S5** follows also thanks to the introduction of two additional connectives [Kot74, p. 197], [Vas01, p. 37], namely:

$$A \multimap B = \Box(A \supset B) \tag{$\multimap$}$$
$$A \rightharpoonup B = \neg((\neg p \vee p) \wedge_\mathsf{d} \neg(\neg A \vee B)) \tag{$\rightharpoonup$}$$

In particular, Kotas showed that the interpretation τ turns the implication $\rightharpoonup$ in the strict implication $\dashv$, and the interpretation τ_1 turns the implication $\dashv$ in $\rightharpoonup$. Collecting all this together, Kotas proved that:

1. The translations maps τ and τ_1 establish that $\mathbf{D}_2^K$ and $\Diamond\mathbf{S5}$ are equivalent. In other words, if $\models_{\mathbf{D}_2^K} A$ then $\models_{\Diamond\mathbf{S5}} \tau(A)$ and if $\models_{\Diamond\mathbf{S5}} B$ then $\models_{\mathbf{D}_2^K} \tau_1(B)$, [Kot74, pp. 198–199].

2. $\mathbf{D}_2^K$ is a finitely axiomatizable system [Kot74, p. 199].

Along these lines of studies, the polish logician T. Furmanowski [Fur75] published a paper concerning the smallest modal system whose $\Diamond$- counterpart coincides with discussive logic. So, by starting from Kotas' axiomatization K1-K5, Furmanowski defined $\Diamond\mathbf{S4}$, i.e., the $\Diamond$- counterpart of $\mathbf{S4}$. As usual, by adding axiom K6 to the axiomatization, we get $\Diamond\mathbf{S5}$. In particular, in [Fur75], what's interesting, with respect to these systems, is the equality between $\Diamond\mathbf{S4}$ and $\Diamond\mathbf{S5}$. This result is obtained by showing that both inclusions, (i) $\Diamond\mathbf{S4} \supseteq \Diamond\mathbf{S5}$ and (ii) $\Diamond\mathbf{S5} \supseteq \Diamond\mathbf{S4}$, are satisfied. The latter inclusion is trivial since it is well-known that $\mathbf{S5} \supseteq \mathbf{S4}$. For (i), instead, we need to show that the axioms K1-K5 and the rules of inferences of [Kot74] constitute a complete axiomatization of $\Diamond\mathbf{S4}$ ([Fur75, p. 39]) and, secondly, to prove that the characteristic axiom of $\Diamond\mathbf{S5}$ K6 is also a formula of $\Diamond\mathbf{S4}$ ([Fur75, p. 41]). This equality states that, for any A, $\models_{\Diamond\mathbf{S4}} A$ just in case $\models_{\Diamond\mathbf{S5}} A$. So, roughly, the quality of modality in $\Diamond\mathbf{S4}$ is the same as in $\Diamond\mathbf{S5}$. From this result and the axiomatizations of $\Diamond\mathbf{S4}$ and $\Diamond\mathbf{S5}$, Furmanowski proved that, for any system $\mathbf{S}$ such that, $\mathbf{S4} \subseteq \mathbf{S} \subseteq \mathbf{S5}$: $\models_{\mathbf{S}} \Diamond A$ if and only if $\models_{\Diamond\mathbf{S5}} \Diamond A$. At this point, with this background, Furmanowski defined Jaśkowski's discussive logic by starting from such a system $\mathbf{S}$:

Definition 2.2.2. Let $\mathbf{D}(\mathbf{S})$ be a discussive system as based on a modal system $\mathbf{S}$, such that $\mathbf{S4} \subseteq \mathbf{S} \subseteq \mathbf{S5}$:

$$\mathbf{D}(\mathbf{S}) = \{A \in \mathsf{Form}_{\mathbf{D}(\mathbf{S})} \mid \Diamond\tau(A) \in \mathbf{S}\}$$

Take Jaśkowski's translation map τ. Then: $\models_{\mathbf{D}(\mathbf{S})} A$ iff $\models_{\mathbf{S}} \Diamond\tau(A)$.

Notice that, if $\mathbf{S} = \mathbf{S5}$, then $\mathbf{D}(\mathbf{S5}) = \mathbf{D}_2$. From this fact, and by the previous result for which, for any system $\mathbf{S4} \subseteq \mathbf{S} \subseteq \mathbf{S5}$, it holds that $\models_{\mathbf{S}} \Diamond A$ if and only if $\models_{\Diamond\mathbf{S5}} \Diamond A$, we may conclude that, for any such modal system $\mathbf{S}$: $\mathbf{D}(\mathbf{S}) = \mathbf{D}_2$.

Recent developments

The tradition of modal studies connected to Jaśkowski's logic continued and largely increased. Recently, the gigantic work of M. Nasieniewski and A. Pietruszczak in [NP08; NP09a; NP09b] contributed to the development of the weakest regular modal logic[7] (denoted by $\mathbf{rS5}^{\mathrm{M}}$) that defines $\mathbf{D}_2$. In [NP08],

[7] As usual, we define a regular modal logic $\mathbf{L}$ as a set of modal formulas satisfying the following conditions: (i) $\mathbf{PC} \subseteq \mathbf{L}$, (ii) $\Diamond p \leftrightarrow \neg\Box\neg p \in \mathbf{L}$ and (iii) $\mathbf{L}$ is closed under *modus pones* for $\supset$, under the regularity rule $(A \wedge B) \supset C/(\Box A \wedge \Box B) \supset \Box C$, and under uniform substitution A/A', where A' is the result of uniform substitution of propositional variables in A. Moreover, $\mathbf{L}$ is said to be normal if $\mathrm{K} \in \mathbf{L}$ and $\mathrm{Nec} \in \mathbf{L}$.

the authors analyse $\mathbf{S5}^{\mathrm{M}}$, i.e., a normal modal logic presented previously by J. Perzanowski. Let $\mathbf{L}$ be any modal logic such that $\mathbf{L}$ defines $\mathbf{D}_2$ iff $\mathbf{D}_2 = \{A \in \mathsf{Form}_{\mathbf{D}_2} \mid \Diamond\tau(A) \in \mathbf{L}\}$. We denote with $\Diamond\mathbf{NS5}$ the set of all normal logics from $\Diamond\mathbf{S5}$. By having this in mind and by following the authors of [NP08], let's introduce the system $\mathbf{S5}^{\mathrm{M}}$ with the following axioms:[8]

$$\Box p \supset \Diamond p \tag{D}$$

$$\Diamond\Box(\Diamond\Box p \supset \Box p) \tag{ML5}$$

$$\Diamond\Box(\Box p \supset p) \tag{MLT}$$

and the rule:

$$\frac{\Diamond\Diamond A}{\Diamond A}\ \mathrm{RM}^2_1$$

A preliminary result is that $\mathbf{S5}^{\mathrm{M}}$ is the smallest logic in $\Diamond\mathbf{NS5}$ [NP08, p. 199] but, also, that $\mathbf{S5}^{\mathrm{M}}$ is the smallest normal logic defining $\mathbf{D}_2$.
Starting from $\mathbf{S5}^{\mathrm{M}}$, the authors consider $\mathbf{rS5}^{\mathrm{M}}$, which is the smallest regular logic which contains (MLT) and (RM^2_1). As expected, $\mathbf{rS5}^{\mathrm{M}} \in \Diamond\mathbf{RS5}$ and, moreover, it constitutes the smallest logic belonging to $\Diamond\mathbf{RS5}$. With respect to discussive logic, Nasieniewski and Pietruszczak aimed at showing that $\mathbf{rS5}^{\mathrm{M}}$ is the smallest regular (non-normal) modal logic defining Jaśkowski's $\mathbf{D}_2$. To do this, the author of [NP09a] consider again the function τ of Definition 2.1.2 together with the following map, labelled τ_2. Let τ_2 be a map such that $\mathsf{Form}_{\mathbf{rS5}^{\mathrm{M}}} \mapsto \mathsf{Form}_{\mathbf{D}_2}$. For any formula $A, B \in \mathbf{rS5}^{\mathrm{M}}$:

$$\begin{aligned}
&\tau_2(p) = p\\
&\tau_2(\neg A) = \neg\tau_2(A)\\
&\tau_2(A \vee B) = \tau_2(A) \vee \tau_2(B)\\
&\tau_2(A \wedge B) = \neg(\neg\tau_2(A) \vee \neg\tau_2(B))\\
&\tau_2(A \supset B) = \neg\tau_2(A) \vee \neg\tau_2(B)\\
&\tau_2(A \leftrightarrow B) = \neg(\neg(\neg\tau_2(A) \vee \tau_2(B)) \vee \neg(\neg\tau_2(B) \vee \tau_2(A)))\\
&\tau_2(\Diamond A) = (p \vee \neg p) \wedge_{\mathsf{d}} \tau_2(A)\\
&\tau_2(\Box A) = \neg\tau_2(A) \rightarrow_{\mathsf{d}} \neg(p \vee \neg p)
\end{aligned}$$

With this in mind, we are able to introduce $\mathbf{D}_2$ as follows:

Definition 2.2.3. Let $\mathbf{L}$ be any modal logic such that:

$$\mathbf{D}(\mathbf{L}) = \{A \in \mathsf{Form}_{\mathbf{D}_2} \mid \Diamond\tau(A) \in \mathbf{L}\}$$

Then: $\mathbf{L}$ defines $\mathbf{D}_2$ iff $\mathbf{D}(\mathbf{L}) = \mathbf{D}_2$.

So, for any modal logic $\mathbf{L}$ such that, if $\mathbf{L} \in \Diamond\mathbf{S5}$ then $\mathbf{L}$ defines $\mathbf{D}_2$. Additionally, $\mathbf{rS5}^{\mathrm{M}} \in \Diamond\mathbf{RS5}$ and $\mathbf{S5}^{\mathrm{M}} \in \Diamond\mathbf{NS5}$. For $\Diamond\mathbf{RS5}$ and $\Diamond\mathbf{NS5}$ being subsets of $\Diamond\mathbf{S5}$, we get that $\mathbf{rS5}^{\mathrm{M}} \in \Diamond\mathbf{S5}$ and $\mathbf{S5}^{\mathrm{M}} \in \Diamond\mathbf{S5}$. So, $\mathbf{rS5}^{\mathrm{M}}$ and $\mathbf{S5}^{\mathrm{M}}$ both define

[8]Notice that in all normal and regular modal logics axiom (D) can be equivalently formulated as $\Diamond(p \supset p)$.

$\mathbf{D}_2$ and, hence, $\mathbf{D}_2 = \mathbf{D}(\mathbf{rS5}^{\mathrm{M}}) = \mathbf{D}(\mathbf{S5}^{\mathrm{M}})$. In other words, $\mathbf{rS5}^{\mathrm{M}}$ is the regular version of the smallest normal modal logic $\mathbf{S5}^{\mathrm{M}}$ such that (i) $\mathbf{rS5}^{\mathrm{M}} \subsetneq \mathbf{S5}^{\mathrm{M}}$ and (ii) every theorem beginning with $\Diamond$ of $\mathbf{rS5}^{\mathrm{M}}$ is also a theorem of $\mathbf{S5}^{\mathrm{M}}$ [NP08, p. 204]. So, finally, collecting together all these results, we get the main *desiderata* of [NP08]: $\mathbf{rS5}^{\mathrm{M}}$ is the smallest regular non-normal modal logic defining $\mathbf{D}_2$.
Additionally, in [NP09a], the authors showed that $\mathbf{rS5}^{\mathrm{M}}$ can be axiomatized without the rule of inference (RM^2_1) and that it is the smallest regular logic which contains the following theorems:

$$\Box p \supset \Diamond \Box\Box p \tag{4_s}$$
$$\Box p \supset \Diamond \Box p \tag{5_c}$$

In other terms, $\mathbf{rS5}^{\mathrm{M}} = \mathbf{C4_s5_c}$. Moreover, (i) if $\mathbf{rS5}^{\mathrm{M}}$ contains (4_s) and (MLT), we get that $\mathbf{rS5}^{\mathrm{M}} = \mathbf{C4_s(MLT)}$. Finally, (ii) $\mathbf{rS5}^{\mathrm{M}} = \mathbf{C5_c(RM^2_1)}$ iff it contains (5_c) and is closed under the rule (RM^2_1) [NP09a, p. 49].
In [NP09b], Nasieniewski and Pietruszczak gave a Kripke semantics for the smallest regular modal logic $\mathbf{rS5}^{\mathrm{M}}(= \mathbf{C4_s5_c})$. The paper contains specific frame conditions for $\mathbf{rS5}^{\mathrm{M}}$ and completeness results. Let's begin with the next definition:

Definition 2.2.4. A frame for regular modal logic $\mathbf{rS5}^{\mathrm{M}}(= \mathbf{C4_s5_c})$ is a triple $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}} = \langle W, \mathcal{R}, N\rangle$, where W is the set of *worlds*, $N \subseteq W$ consists of *regular worlds* and $\mathcal{R}$ is the accessibility relation[9]. Furthermore, $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}} = \langle W, \mathcal{R}, N\rangle$ satisfies the following conditions:

$$\forall w \in N, \exists u \in N(w\mathcal{R}u \wedge \forall x \in W(u\mathcal{R}x \Rightarrow w\mathcal{R}x)) \tag{Fr1}$$
$$\forall w \in N, \exists u \in N(w\mathcal{R}u \wedge \forall x \in W(\exists y \in N(u\mathcal{R}y \wedge y\mathcal{R}x) \Rightarrow w\mathcal{R}x)) \tag{Fr2}$$

(5_c) is valid in frames satisfying (Fr1) [NP09b, p. 177] and (4_s) is valid if the frame satisfies (Fr2) [NP09b, p. 178]. Notice that both frame conditions constitute strengthening of seriality [NP09b, p. 179]. Finally, as usual:

Definition 2.2.5. A model $\mathcal{M}_{\mathbf{rS5}^{\mathrm{M}}} = \langle W, \mathcal{R}, N, v\rangle$ for $\mathbf{rS5}^{\mathrm{M}}(= \mathbf{C4_s5_c})$ is based on a frame $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}}$ and on a valuation $v : \mathsf{Form}_{\mathbf{rS5}^{\mathrm{M}}} \times W \longrightarrow \{0, 1\}$ such that for any $A \in \mathsf{Form}_{\mathbf{rS5}^{\mathrm{M}}}$ and $w \in W$:

$$\begin{aligned} v(\Box A) = 1 \quad &\text{iff} \quad w \in N \text{ and } \forall x \in \mathcal{R}(w),\ v(A, x) = 1 \\ v(\Diamond A) = 0 \quad &\text{iff} \quad w \notin N \text{ or } \exists x \in \mathcal{R}(w),\ v(A, x) = 1 \end{aligned}$$

where $\mathcal{R}(w) = \{x \in W \mid w\mathcal{R}x\}$.
A formula A is true in a model $\mathcal{M}_{\mathbf{rS5}^{\mathrm{M}}}$ iff $v(A, w = 1)$ for any $w \in W$.
A formula A is valid in a given frame $\mathcal{F}_{\mathbf{rS5}^{\mathrm{M}}}$ iff it is true in all models $\mathcal{M}_{\mathbf{rS5}^{\mathrm{M}}}$ based on the aforementioned frame.

In sum, the authors of [NP08; NP09a; NP09b] provided both an axiomatic system and a possible worlds semantics for the regular version of **S5** and, consequently, defined discussive logic on that formal basis[10]. From the perspective of

[9] If we let $W = N$, then we get the pair $\langle W, \mathcal{R}\rangle$, which corresponds to a frame for normal modal logics.

[10] Notice, finally, that we have restricted our attention just to some of the papers that Nasieniewski, Pietruszczak and collaborators devoted to $\mathbf{D}_2$. For more on their work see our conclusive remarks.

Jaśkowski's $\mathbf{D}_2$, the work of Nasieniewski and Pietruszczak is interesting since it shows, not only that there other normal modal logics different from **S5** defining discussive logic, but that there are also non-normal regular versions of **S5** which define $\mathbf{D}_2$.

2.2.2 The 'J' Systems

Remarkably,

> "[t]he year 1967 was a turning point in the development of the discussive logic. Newton C.A. da Costa and Lech Dubikajtis met in Paris and gradually commenced the development of the logic." [Ciu99, p. 10]

Indeed, as said above, in a paper from 1968 [CD68], da Costa and Dubikajtis presented the first modal-type axiomatization of $\mathbf{D}_2$. The **S5**-type system they proposed, known as **J**, has become famous in the context of discussive systems. As remarked by the authors, **J** has several axiomatizations[11] and, in what follows, we will refer to the axiomatic system presented in [CD95] from 1995. Interestingly, **J**, and in particular some of its extensions, have been applied to philosophical problems, such as to the debate on the underlying logic of scientific theories. However, before turning to the philosophical applications of **J** and related systems, let's introduce them. **J** is the system composed by the following axioms and rules [CD95, p. 45]:

If A is a theorem of **S5**, then $\Box A$ is a theorem of **J**.

$$\frac{\Box A \qquad \Box(A \supset B)}{\Box B}\ \Box\text{MP} \qquad \frac{\Box A}{A}\ \text{Den} \qquad \frac{\Diamond A}{A}\ \text{Dep} \qquad \frac{\Box A}{\Box\Box A}\ \text{R4}$$

J has been introduced in the literature as another $\Diamond$-counterpart of **S5** and, indeed, $\models_{\mathbf{J}} A$ iff $\models_{\mathbf{S5}} \Diamond A$. Starting from **J**, da Costa and Doria presented a first-order variant of it, denoted $\mathbf{J}^*$, by adding the universal quantifier $\forall$ among the connectives. As usual, the existential quantifier can be defined $\exists x A = \neg\forall x \neg A$. Before, defining $\mathbf{J}^*$, it is useful to recall the axiomatic system for $\mathbf{S5Q}^=$ (quantified **S5** with identity):

If A is a theorem of **PC**, then A is an theorem of $\mathbf{S5Q}^=$.

All axioms of **S5** (Definition 2.1.1), plus :

$$x = x \qquad \text{(Id1)}$$

$$x = y \supset (A(x) \leftrightarrow B(x)) \qquad \text{(Id2)}$$

$$\forall x A(x) \supset A(t), \qquad (\forall 1)$$

where t is either a variable free for x in $A(x)$ or an individual constant. And the following rule:

[11] For other synthetic reconstructions one can also consider [Ciu99; Vas01].

$$\frac{A \supset B(x)}{A \supset \forall x B(x)}\ \mathrm{R}\forall$$

Now, the language of $\mathbf{J}^*$ coincides the language of $\mathbf{S5Q}^=$ and, indeed, $\mathbf{J}^*$ is introduced as follows:

> If A is an theorem of $\mathbf{S5Q}^=$, then $\Box A$ is a theorem of $\mathbf{J}^*$
>
> The axioms and rules of $\mathbf{J}$, plus:

$$\frac{\Box(A \supset B(x))}{\Box(A \supset \forall x B(x))}\ \mathrm{R}\Box\forall$$

where, in the rule (R$\Box\forall$), x is not free in A.
Notice that, differently from Jaśkowski's papers, da Costa and Doria considered left-discussive conjunction. Roughly, by adding both, $\wedge^l_{\mathsf{d}}$ and $\rightarrow_{\mathsf{d}}$, to $\mathbf{J}$ and $\mathbf{J}^*$, the paraconsistent character of such systems. Indeed, the following formulas, governing the 'explosion' of logical systems, are not valid neither in $\mathbf{J}$ nor in $\mathbf{J}^*$. Let $\wedge$ be classical conjunction:

$$A \rightarrow_{\mathsf{d}} (B \rightarrow_{\mathsf{d}} A \wedge B)$$
$$((A \wedge B) \rightarrow_{\mathsf{d}} C) \rightarrow_{\mathsf{d}} (A \rightarrow_{\mathsf{d}} (B \rightarrow_{\mathsf{d}} C))$$
$$A \rightarrow_{\mathsf{d}} (\neg A \rightarrow_{\mathsf{d}} B)$$
$$(A \rightarrow_{\mathsf{d}} \neg A) \rightarrow_{\mathsf{d}} B$$

Furthermore, let $\Gamma = \{A \mid \Gamma \vdash_{\mathbf{J}^*} A\}$. As usual, if Γ is the set of all formulas, then Γ is trivial. If not, Γ is non-trivial; if we have a formula A such that both $\Gamma \vdash_{\mathbf{J}^*} A$ and $\Gamma \vdash_{\mathbf{J}^*} \neg A$, then Γ is inconsistent. If not, Γ is consistent. With respect to these definitions, the two authors – who where interested in modelling situations in which scientists may reason through inconsistent sets of sentences, considered as "working hypothesis" [CD95, p. 46] – showed that their $\mathbf{J}$-systems allow to deal with inconsistent and non-trivial sets of premises. In other words, da Costa and Doria proved that $\mathbf{J}$ and $\mathbf{J}^*$ are paraconsistent logics.

$\mathbf{D_2}$, $\mathbf{J}^*$ & the foundations of physics

Recall that Jaśkowski believed that "the evolution of the empirical disciplines is marked by periods in which [...] the results of experiments [...] are not always consistent with one another" [Jaś99a]. Accordingly, the inconsistent results are to be considered as 'working hypothesis', i.e., as sentences that are taken *as if they were true* to inspect their respective consequences and establish which one describes more accurately scientific phenomena. da Costa and Doria tried to make sense of Jaśkowski's idea by elaborating a variant of $\mathbf{J}^*$ which can be used as underlying logic for physical theories. The starting point has been represented by the (formal) conceptions of physical structure and theory, due to M.L. Dalla Chiara and G. Toraldo di Francia (see [DT81; CD95; CKB07]). First of all, a 'physical structure' $\mathcal{A}$ is a set-theoretic structure of the following form:

$$\mathcal{A} = \langle M, S, < Q_0, Q_1, \ldots, Q_n >, \rho\rangle$$

where, M represents a set of mathematical structures. Notice, the authors of [DT81] aimed at modelling physical concepts, such as vector spaces, as set-theoretic structures, by taking the axioms of **ZF** set theory. Secondly, S is a set of "physical situations", i.e., a set of physical states assumed by a physical system in a certain time interval. In other words, S is the element of the physical structure that 'mirrors' the physical theory that $\mathcal{A}$ is trying to capture. Each Q_k ($0 \leq k \leq n$) is an "operationally defined quantity" whose domain of definition is some $S_1 \subseteq S$. As a convention, let Q_0 denote time. To be clear, if we wish to measure a quantity Q_k of a physical system in a state $s \in S$ at a time t_k, with $1 \leq k \leq n$, we get an interval $I(k, t_k)$ of the real number line $\mathbb{R}$. So, if we measure time, i.e., Q_0, the result we obtain is a "time interval". t and t_k represent time instants and we express, in $\mathcal{L}$, the "acceptable values" of Q_k at t_i as $q_k(t_i)$. So, in a certain sense, *all values* in a interval $I(k, t_k)$ are "appropriate values" for the measurement of the quantity Q_k of the physical system in a state $s \in S$. Finally, ρ associates mathematical structures of M to their physically meaningful quantity. To see how this framework is supposed to work, as usual, let $A(t, q_k(t_k))$ be a formula whose only free variables are those one expressing time instants, (t and t_k). Formally, $\models_s A(t, q_k(t_k))$ means that a formula A, in a certain interval of time, is true for a physical state s if there are values t^0 and q_k^0 (of Q_k) in the interval $I(t, t_k)$, with $1 \leq k \leq n$, such that $A(t^0, q_k^0)$ is true in s. Now, let $\models_{\mathcal{A}} A(t, q_k(t_k))$ denote that A is true in the physical structure $\mathcal{A}$. If we obtain t in I_t and q_k in $I(t, t_k)$, so that $\neg A(t, q_k(t_k))$ is also true in $\mathcal{A}$, then the physical theory captured by $\mathcal{A}$ is paraconsistent. In other words, as one might have expected, with respect to $\mathcal{A}$, we get a paraconsistent physical theory whenever $\models_{\mathcal{A}} A(t, q_k(t_k))$ and $\models_{\mathcal{A}} \neg A(t, q_k(t_k))$.[12]. At this point, da Costa and Doria aimed at demonstrating that:

> "[...] the underlying logic of a physical theory in Dalla Chiara and di Francia approach is most adequately represented by Jaśkowski's discussive logic." [CD95, p. 57]

and, more precisely, by $\mathbf{J}^{**}$. This system is similar to $\mathbf{J}^{*}$, but imposes some more restrictions on bound variables [CD95; CKB07]. Take again $\mathbf{S5Q}^{=}$ and let $\uplus A = \forall x_n\, A(x_n)$ be denoting a formula A preceded by a sequences of universal quantifiers so that all variables in A are bound. $\mathbf{J}^{**}$ is constituted by the following axioms and rules:

If A is an instance of a theorem of $\mathbf{S5Q}^{=}$, then $\Box \uplus A$ is a theorem of $\mathbf{J}^{*}$.

[12] To be clear, consider the following example due to [CKB07, pp. 849–850]. Take Newton's second law: $F = m \cdot a$. The variables appearing in the equation corresponds to the physical quantities to be measured: "force" (F), "mass" (m) and "acceleration" (a). If we take a state $s \in S$, their values stand in the following three intervals $I(F_1, F_2) \subseteq \mathbb{R}$, $I(m_1, m_2) \subseteq \mathbb{R}$ and $I(a_1, a_2) \subseteq \mathbb{R}$. When we are able to find three real numbers $p_1 \in I(F_1, F_2)$, $q_1 \in I(m_1, m_2)$ and $r_1 \in I(a_1, a_2)$, such that $p_1 = q_1 \cdot r_1$, then it holds that $\models_s F = m \cdot a$. Likewise, if we encounter the opposite situation, namely we find three real numbers, in their respective intervals, such that $p_2 \neq q_2 \cdot r_2$, also these three real numbers can be considered as acceptable values for solving the equation. So, $\models_s \neg(F = m \cdot a)$ and, hence, Newton's second law, in the very same physical situation s, is both, true and false. In this case, for the same situation s, Newton's law is a proposition C, such that $\models_s C$ and $\models_s \neg C$. However, $\models_s C \wedge \neg C$ does not hold, since it would mean to find three real numbers p, q, r, in their respective intervals, for which the conjunction $p = q \cdot r \wedge p \neq q \cdot r$ holds.

$$\Box \uplus (\Box(A \supset B) \supset (\Box A \supset \Box B)) \tag{J_1^{**}}$$

$$\Box \uplus (\Box A \supset A) \tag{J_2^{**}}$$

$$\Box \uplus (\Diamond A \supset \Box\Diamond A) \tag{J_3^{**}}$$

$$\Box \uplus (\forall x A(x) \supset A(t)) \tag{J_4^{**}}$$

$$\Box \uplus (x = x) \tag{J_5^{**}}$$

$$\Box \uplus (x = y \supset (A(x) \leftrightarrow A(y))) \tag{J_6^{**}}$$

$$\frac{\Box \uplus A \qquad \Box \uplus (A \supset B)}{\Box \uplus B}\ {\scriptstyle \uplus\Box\mathrm{MP}} \qquad \frac{\Box \uplus A}{A}\ {\scriptstyle \uplus\mathrm{Den}} \qquad \frac{\Box \uplus A}{\Box \uplus \Box A}\ {\scriptstyle \uplus\mathrm{R4}}$$

$$\frac{\Diamond \uplus A}{A}\ {\scriptstyle \uplus\mathrm{Dep}} \qquad \frac{\Box \uplus (A \supset B(x))}{\Box \uplus (A \supset \forall x B(x))}\ {\scriptstyle \mathrm{R}\uplus\Box\forall}$$

So: $\models_{\mathbf{J}^{**}} A$ iff $\models_{\mathbf{S5Q}^{=}} \Diamond \uplus A$. Notice that vacuous quantification can be introduced/eliminated in any formula.
The only difference between $\mathbf{J}^{**}$ and $\mathbf{J}^{*}$ concerns the applications: the first one is more suitable than the second one to handle with, since there's no problem on the discussive interpretation of the free variables. Accordingly, a physical theory, denoted, $\mathcal{T}$, extends the notion of physical structure and, in sum, it is composed by the following elements:

1. A formal language $\mathcal{L}$.
2. A set of axioms $\mathcal{A}$ expressed in $\mathcal{L}$ such that $\mathcal{A} = \mathcal{A}_L \cup \mathcal{A}_M \cup \mathcal{A}_P$, where $\mathcal{A}_L$, $\mathcal{A}_M$ and $\mathcal{A}_P$ are, respectively, the set of logical, mathematical and physical axioms.
3. A language $\mathcal{L}_0 \subset \mathcal{L}$. The logic $\mathcal{L}_0$, used to deal with the mathematical structures of $\mathcal{T}$, is classical and, hence, $\mathcal{A}_M$ includes all classically valid formulas.
4. The axioms of $\mathbf{J}^{**}$ are included in $\mathcal{A}_L$ to deal with inconsistent sets of premises.
5. $\mathcal{A}_M$ must contain all axioms for the structures of M.
6. $\mathcal{A}_P$ contains all "physically motivated sentences".

So, finally, for A being a theorem of $\mathcal{T}$, then it holds that: if A is formulated in $\mathcal{L}_0$, then A is closed under classical consequence relation. Furthermore, from the perspective of inconsistent theories: for all $A \in \mathcal{T}$, A is closed under $\mathbf{J}^{**}$-consequence relation.
Notice that terms of $\mathcal{L}_0$ cannot refer to the quantities Q_k, but exclusively to mathematical structures of M, which are totally classical. More precisely, exactly the quantities Q_k induce the language to be paraconsistent. Indeed, if we are given a formula B such that its terms refer to some of the Q_k, generally, it can result that both, B and $\neg B$ are true in $\mathcal{T}$. Consequently, both sentences should be included in $\mathcal{A}_P$. Here's exactly the paraconsistent character of the definition of truth, i.e., in a physical theory $\mathcal{T}$, for some state $s \in S$ and a formula B, we can reach both, $\models_s B$ and $\models_s \neg B$. As said above, the acceptance of pairs of contradictory statements, such as B and $\neg B$, is meant to mirror those situations in which two inconsistent sentences are taken to be true with the aim to inspect

their respective consequences and chose which one provides a more accurate description of the scientific phenomena under consideration. Of course, this does not mean that: $\models_s B \wedge \neg B$.

2.2.3 Introducing Discussive Connectives

In the previous discussion we have left apart the centrality of discussive connectives in formulating Jaśkowski's discussive logic in favour of an analysis principally focused on the development of the connections between $\mathbf{D}_2$ and modal systems. In what follows, we reverse the perspective by analysing some of the major attempts to give axioms to Jaśkowski's $\mathbf{D}_2$, without relying on translations and by considering directly a language including $\wedge_\mathsf{d}$, $\rightarrow_\mathsf{d}$ instead of $\wedge$, $\supset$. The challenge of providing such an axiomatization, usually known as 'Jaśkowski's problem' [Vas01, p. 42], has been stated by N. da Costa already in 1975 [Cos75, p. 14]:

> "Is it possible to formulate a **natural** and **simple** axiomatization for $\mathbf{D}_2$ employing $\rightarrow_\mathsf{d}$, $\wedge_\mathsf{d}$, $\vee$ and $\neg$ as the only primitive connectives?"

According to [Kot75], the first non modal axiomatization of $\mathbf{D}_2$ has been proposed by Furmanowski but has never been published before Kotas' paper from 1975 [Vas01]. It is worth having a look at Furmanowski's work not only for its historical importance, but also for the originality of the proposed axioms. Let $A, B, C, \ldots$ be formulas and let $\perp = \neg(A \vee \neg A)$. The discussive logic $\mathbf{D}_2^F$ is axiomatized by the following axioms:

$$\neg(A \supset (\neg A \supset B)) \rightarrow_\mathsf{d} \perp \tag{F1}$$
$$(A \supset B) \supset ((B \supset C) \supset (A \supset C)) \rightarrow_\mathsf{d} \perp \tag{F2}$$
$$\neg((\neg A \supset B) \supset A) \rightarrow_\mathsf{d} \perp \tag{F3}$$
$$\neg((\neg A \supset B) \supset A) \rightarrow_\mathsf{d} B \tag{F4}$$
$$\neg((\neg(A \supset B) \rightarrow_\mathsf{d} \perp) \rightarrow_\mathsf{d} ((\neg A \rightarrow_\mathsf{d} \perp) \supset (\neg B \rightarrow_\mathsf{d} \perp))) \rightarrow_\mathsf{d} \perp \tag{F5}$$
$$\neg(\neg\neg(\neg A \supset \perp) \vee \neg\neg(\neg A \rightarrow_\mathsf{d} \perp)) \rightarrow_\mathsf{d} \perp \tag{F6}$$
$$(\neg(A \supset B) \rightarrow_\mathsf{d} C) \rightarrow_\mathsf{d} ((\neg A \rightarrow_\mathsf{d} C) \rightarrow_\mathsf{d} (\neg B \rightarrow_\mathsf{d} C)) \tag{F7}$$
$$(\neg A \rightarrow_\mathsf{d} \perp) \rightarrow_\mathsf{d} A \tag{F8}$$
$$(A \rightarrow_\mathsf{d} B) \rightarrow_\mathsf{d} (\neg(A \rightarrow_\mathsf{d} B) \rightarrow_\mathsf{d} B) \tag{F9}$$
$$\neg(\neg\neg A \rightarrow_\mathsf{d} B) \rightarrow_\mathsf{d} A \tag{F10}$$

Notice that, $\mathbf{D}_2^F$ is still 'impure' in the sense that, even though, Furmanowski did not include the modal operators, he still kept the presence of two conditionals, including the material one. So, strictly speaking, accordingly to [Cos75], $\mathbf{D}_2^F$ cannot be regarded as a solution to 'Jaśkowski's problem'. In 1977 [CD77; CK77] da Costa and Dubikajtis presented the first complete axiomatization of discussive logic including directly discussive connectives in the axiom schemata. In particular, da Costa and Dubikajtis [CD77] presented some axioms including $\rightarrow_\mathsf{d}$ and $\wedge_\mathsf{d}^l$. From now on, we will denote the discussive logic so formalized by $\mathbf{D}_2^l$, where 'l' indicates the presence of $\wedge_\mathsf{d}^l$, instead of Jaśkowski's $\wedge_\mathsf{d}$. So, the discussive logic $\mathbf{D}_2^l$ is axiomatized as follows

$$A \to_{\mathsf{d}} (B \to_{\mathsf{d}} A) \tag{Ax1}$$
$$(A \to_{\mathsf{d}} (B \to_{\mathsf{d}} C)) \to_{\mathsf{d}} ((A \to_{\mathsf{d}} B) \to_{\mathsf{d}} (A \to_{\mathsf{d}} C)) \tag{Ax2}$$
$$(A \wedge_{\mathsf{d}}^l B) \to_{\mathsf{d}} A \tag{Ax3}$$
$$(A \wedge_{\mathsf{d}}^l B) \to_{\mathsf{d}} B \tag{Ax4}$$
$$A \to_{\mathsf{d}} (B \to_{\mathsf{d}} (A \wedge_{\mathsf{d}}^l B)) \tag{Ax5}$$
$$A \to_{\mathsf{d}} (A \vee B) \tag{Ax6}$$
$$B \to_{\mathsf{d}} (A \vee B) \tag{Ax7}$$
$$(A \to_{\mathsf{d}} C) \to_{\mathsf{d}} ((B \to_{\mathsf{d}} C) \to_{\mathsf{d}} (A \vee B) \to_{\mathsf{d}} C) \tag{Ax8}$$
$$A \to_{\mathsf{d}} \neg\neg A \tag{Ax9}$$
$$\neg\neg A \to_{\mathsf{d}} A \tag{Ax10}$$
$$((A \to_{\mathsf{d}} B) \to_{\mathsf{d}} A) \to_{\mathsf{d}} A \tag{Ax11}$$
$$\neg(A \vee \neg A) \to_{\mathsf{d}} B \tag{Ax12}$$
$$\neg(A \vee B) \to_{\mathsf{d}} \neg(B \vee A) \tag{Ax13}$$
$$\neg(A \vee B) \to_{\mathsf{d}} (\neg B \wedge_{\mathsf{d}}^l \neg A) \tag{Ax14}$$
$$\neg(\neg\neg A \vee B) \to_{\mathsf{d}} \neg(A \vee B) \tag{Ax15}$$
$$(\neg(A \vee B) \to_{\mathsf{d}} C) \to_{\mathsf{d}} ((\neg A \to_{\mathsf{d}} B) \vee C) \tag{Ax16}$$
$$\neg((A \vee B) \vee C) \to_{\mathsf{d}} \neg(A \vee (B \vee C)) \tag{Ax17}$$
$$\neg((A \to_{\mathsf{d}} B) \vee C) \to_{\mathsf{d}} (A \wedge_{\mathsf{d}}^l \neg(B \vee C)) \tag{Ax18}$$
$$\neg((A \wedge_{\mathsf{d}}^l B) \vee C) \to_{\mathsf{d}} (A \to_{\mathsf{d}} \neg(B \vee C)) \tag{Ax19}$$
$$\neg(\neg(A \vee B) \vee C) \to_{\mathsf{d}} (\neg(\neg A \vee C) \vee \neg(\neg B \vee C)) \tag{Ax20}$$
$$\neg(\neg(A \to_{\mathsf{d}} B) \vee C) \to_{\mathsf{d}} (A \to_{\mathsf{d}} \neg(\neg B \vee C)) \tag{Ax21}$$
$$\neg(\neg(A \wedge_{\mathsf{d}}^l B) \vee C) \to_{\mathsf{d}} (A \wedge_{\mathsf{d}}^l \neg(\neg B \vee C)) \tag{Ax22}$$

$$\frac{A \qquad A \to_{\mathsf{d}} B}{B}\ \mathrm{MP}_d$$

Remark 3. $\mathbf{D}_2^l$ includes the following set of connectives into its language $\{\neg, \vee, \wedge_{\mathsf{d}}^l, \to_{\mathsf{d}}\}$, where the only difference, as said, with $\mathbf{D_2}$ is the presence of left-discussive conjunction. Notice that, even though $\mathbf{D}_2^l$ constitutes a complete axiomatization [CD77, p. 54], from the perspective of [Cos75], it might be still considered only as a 'partial' solution to 'Jaśkowski's problem'. Indeed, this time the 'impurity' of the axioms is not linked to the inclusion of other connectives than the discussive ones, plus $\neg$ and $\vee$, but to the presence of $\wedge_{\mathsf{d}}^l$. As remarked above (Observation 4), the interaction of $\neg$ with different discussive operators defines logics distinct from Jaśkowki's $\mathbf{D_2}$. Indeed, strictly speaking, since Jaśkowski's $\mathbf{D_2}$ included right-discussive conjunction, $\mathbf{D}_2^l$ can be considered only as a variation of $\mathbf{D_2}$.

More recently, J. Alama and H. Omori [AO18] presented a complete and sound axiomatization for discussive logic, including Jaśkowski's right-discussive conjunction (denoted $\mathbf{D}_2^r$). The starting point of [AO18] are the axioms of $\mathbf{D}_2^l$. The only necessary change to get $\mathbf{D}_2^r$, is to drop Ax19 and Ax22 in favour of:

$$\neg((A \wedge_{\mathsf{d}} B) \vee C) \rightarrow_{\mathsf{d}} (B \rightarrow_{\mathsf{d}} \neg(A \vee C)) \tag{Ax19$'$}$$
$$\neg(\neg(A \wedge_{\mathsf{d}} B) \vee C) \rightarrow_{\mathsf{d}} (\neg(\neg A \vee C) \wedge_{\mathsf{d}} B) \tag{Ax22$'$}$$

Moreover, in the axioms involving conjunction, one simply needs to replace $\wedge_{\mathsf{d}}^l$ with $\wedge_{\mathsf{d}}$. Of course, Ax19 and Ax22 of $\mathbf{D}_2^l$ mirrored the behaviour of negated left-discussive conjunction. Ax19′ and Ax22′ absolve the same job, but with respect to right-discussive conjunction. Both axioms are $\mathbf{D}_2$-valid if and only if their modally translated versions are **S5**-valid, i.e., just in case the following formulas are valid in **S5**, accordingly to τ (see 2.1.2):

$$\Diamond(\Diamond\neg((A \wedge \Diamond B) \vee C) \supset (\Diamond B \supset \neg(A \vee C)))$$
$$\Diamond(\Diamond\neg(\neg(A \wedge \Diamond B) \vee C) \supset (\neg(\neg A \vee C) \wedge \Diamond B))$$

Following the changes proposed in [AO18], it seems that $\mathbf{D}_2^r$ accomplishes, at least, the task of finding a correct and complete axioms system for Jaśkowski's discussive logic. At this point, it might be naturally asked if $\mathbf{D}_2^r$ goes even further and gives a positive and definitive answer to 'Jaśkowski's problem'. Up to now it seems to be the best candidate.

We wish to strengthen this idea by considering briefly two other axiomatizations for $\mathbf{D}_2$, both elaborated by J. Ciuciura in [Ciu05; Ciu08]. First of all, consider again a set of connectives including lef-discussive conjunction and the axiomatic system proposed in [Ciu05] (denoted $\mathbf{D}_2^C$). Take Ax1-Ax8, plus MP$_d$, of $\mathbf{D}_2^l$, and add the following axioms:

$$A \vee (A \rightarrow_{\mathsf{d}} B) \tag{C1}$$
$$A \rightarrow_{\mathsf{d}} \neg(\neg(A \vee B) \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg A) \tag{C2}$$
$$\neg(\neg(A \vee B) \wedge_{\mathsf{d}} \neg B \wedge_{\mathsf{d}}^l \neg A) \rightarrow_{\mathsf{d}} \\ \rightarrow_{\mathsf{d}} \neg(\neg(A \vee B \vee C) \wedge_{\mathsf{d}}^l \neg C \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg A) \tag{C3}$$
$$\neg(\neg(A \vee B \vee C) \wedge_{\mathsf{d}}^l \neg C \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg A) \rightarrow_{\mathsf{d}} \\ \rightarrow_{\mathsf{d}} \neg(\neg(A \vee B \vee C) \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg C \wedge_{\mathsf{d}}^l \neg A) \tag{C4}$$
$$\neg(\neg(A \vee B) \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg A) \rightarrow_{\mathsf{d}} ((A \vee \neg B) \rightarrow_{\mathsf{d}} A) \tag{C5}$$
$$\neg(\neg(A \vee B \vee C) \wedge_{\mathsf{d}}^l \neg C \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg A) \rightarrow_{\mathsf{d}} \\ \rightarrow_{\mathsf{d}} ((A \vee B \vee \neg C) \rightarrow_{\mathsf{d}} (A \vee B)) \tag{C6}$$
$$\neg(\neg(A \vee B \vee C) \wedge_{\mathsf{d}}^l \neg C \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg A) \rightarrow_{\mathsf{d}} \\ \rightarrow_{\mathsf{d}} (\neg(\neg(A \vee B \vee \neg C) \wedge_{\mathsf{d}}^l \neg\neg C \wedge_{\mathsf{d}}^l \neg B \wedge_{\mathsf{d}}^l \neg A) \rightarrow_{\mathsf{d}} \neg(\neg B \wedge_{\mathsf{d}}^l \neg A)) \tag{C7}$$
$$\neg(\neg A \wedge_{\mathsf{d}}^l \neg B) \rightarrow_{\mathsf{d}} (A \vee B) \tag{C8}$$
$$(A \vee \neg\neg B) \rightarrow_{\mathsf{d}} (A \vee B) \tag{C9}$$
$$(A \vee B) \rightarrow_{\mathsf{d}} (A \vee \neg\neg B) \tag{C10}$$

As usual, the consequence relation $\vdash_{\mathbf{D}_2^C}$ is determined by the axioms Ax1-Ax8, C1-C10 and by the rule MP$_d$. Additionally, to prove soundness and completeness results, Ciuciura proposed a possible world semantics for $\mathbf{D}_2^C$, in which all elements are identical to those of Definition 2.2.5, except that $W = N$ and that we include the following clauses:

$$v(A \wedge^l_{\mathsf{d}} B, w) = 1 \quad \text{iff} \quad \exists x \in \mathcal{R}(w),\ v(A, x) = 1 \text{ and } v(B, w) = 1$$
$$v(A \to_{\mathsf{d}} B, w) = 1 \quad \text{iff} \quad \forall x \in \mathcal{R}(w),\ v(A, x) = 0 \text{ or } v(B, w) = 1$$

Since $\mathbf{D}^C_2$ relies on an equivalence relation between worlds, the accessibility relation may be not explicitly stated in the clauses. In any case, these changes will not affect their meaning, [Ciu05, pp. 239–240.]. Importantly, Ciuciura aimed at proving soundness and completeness of $\mathbf{D}^C_2$, but, unfortunately, in [AO18, p. 1171], it was proved that in $\mathbf{D}^C_2$ there is (at least) one unprovable formula. The point is struggling, since the formula in question, i.e., $\neg(A \vee \neg A) \to_{\mathsf{d}} B$, is valid according to the Jaśkowski-style translation τ of Definition 2.1.2. Consequently, one might naturally doubt whether $\mathbf{D}^C_2$ *is*, in some sense, an axiomatization of Jaśkowski's discussive logic in the sense of [Cos75], given also the presence of $\wedge^l_{\mathsf{d}}$ instead of $\wedge_{\mathsf{d}}$. However, in an another paper [Ciu08], Ciuciura restated the presence of right-discussive conjunction among the connectives and provided an axiomatic system for it. We denote Ciuciura's second axiomatization by $\mathbf{D}^{C*}_2$. Take again Ax1-Ax8 (replacing $\wedge^l_{\mathsf{d}}$ with $\wedge_{\mathsf{d}}$ in Ax3, Ax4, Ax5) and MP$_d$ of $\mathbf{D}^l_2$, plus the following axioms:

$$A \vee (A \to_{\mathsf{d}} B) \tag{C1*}$$
$$\neg(\neg A \wedge_{\mathsf{d}} \neg\neg A \wedge_{\mathsf{d}} \neg(A \vee \neg A)) \tag{C2*}$$
$$\neg(\neg A \wedge_{\mathsf{d}} \neg B \wedge_{\mathsf{d}} \neg(A \vee B)) \to_{\mathsf{d}} \\ \to_{\mathsf{d}} \neg(\neg A \wedge_{\mathsf{d}} \neg B \wedge_{\mathsf{d}} \neg C \wedge_{\mathsf{d}} \neg(A \vee B \vee C)) \tag{C3*}$$
$$\neg(\neg A \wedge_{\mathsf{d}} \neg B \wedge_{\mathsf{d}} \neg C \wedge_{\mathsf{d}} \neg(A \vee B \vee C)) \to_{\mathsf{d}} \\ \to_{\mathsf{d}} \neg(\neg A \wedge_{\mathsf{d}} \neg C \wedge_{\mathsf{d}} \neg B \wedge_{\mathsf{d}} \neg(A \vee C \vee B)) \tag{C4*}$$
$$\neg(\neg A \wedge_{\mathsf{d}} \neg B \wedge_{\mathsf{d}} \neg C \wedge_{\mathsf{d}} \neg(A \vee B \vee C)) \to_{\mathsf{d}} \\ \to_{\mathsf{d}} ((A \vee B \vee \neg C) \to_{\mathsf{d}} (A \vee B)) \tag{C5*}$$
$$\neg(\neg A \wedge_{\mathsf{d}} \neg B) \to_{\mathsf{d}} (A \vee B) \tag{C6*}$$
$$(A \vee (B \vee \neg B)) \to_{\mathsf{d}} \neg(\neg A \wedge_{\mathsf{d}} \neg(B \vee \neg B)) \tag{C7*}$$

As in the case of $\mathbf{D}^C_2$, to prove soundness and completeness, Ciuciura proposed a possible worlds semantics, but dropping out the clause for $\wedge^l_{\mathsf{d}}$ in favour of the following one for $\wedge_{\mathsf{d}}$:

$$v(A \wedge_{\mathsf{d}} B, w) = 1 \quad \text{iff} \quad \exists x \in \mathcal{R}(w),\ v(A, w) = 1 \text{ and } v(B, x) = 1$$

Some criticism has been moved against Ciuciura's $\mathbf{D}^{C*}_2$. J. Alama [Ala06] noticed that if we take the axioms Ax1-Ax22 of da Costa's and Dubikajtis' $\mathbf{D}^l_2$, in comparison to the ones of Ciuciura, we will get a troublesome situation: the two axiomatizations share some theses (Ax1-Ax8), while some others are respectively unprovable. Technically, if we encounter this situation, the two logics under considerations are said to be "orthogonal". In this specific case [Ala06, pp. 4–8]:

Proposition 2.2.1. $\mathbf{D}_2^{C*} \nvdash$ Ax9, Ax12, Ax13, Ax14, Ax15, Ax16, Ax17, Ax18, Ax19, Ax20, Ax21, Ax22.

At this point, consequently, it might be naturally asked whether $\mathbf{D}_2^{C*}$ corresponds to a restriction of $\mathbf{D}_2^{l}$. The answer is no, since there is (at least) one axiom of $\mathbf{D}_2^{C*}$ which is $\mathbf{D}_2^{l}$-unprovable [Ala06, p. 11]:

Proposition 2.2.2. $\mathbf{D}_2^{l} \nvdash$C5*

In sum, $\mathbf{D}_2^{C*}$. and $\mathbf{D}_2^{l}$, one with respect to the other, are not complete axiomatizations and, moreover, they ought to be called as orthogonal, i.e., they overlap and each one has theorems which are not formulas of the other. Finally, also the addition of new axioms still confirms that Ciuciura's axiomatization $\mathbf{D}_2^{C*}$ is an incomplete system of axioms [AO18, p. 1168.].
Notice , finally, that $\mathbf{D}_2^{C*}$ also fails to be an axiomatization Jaśkowski's $\mathbf{D}_2$, in the sense that there are $\mathbf{D}_2$-valid formulas, that are unprovable in $\mathbf{D}_2^{C*}$ [AO18, pp. 1167–1170], namely[13]:

$$\begin{aligned}
&A \to_{\mathsf{d}} \neg\neg A \\
&\neg(A \vee \neg A) \to_{\mathsf{d}} B \\
&\neg(A \vee B) \to_{\mathsf{d}} \neg(B \vee A) \\
&\neg(\neg\neg A \vee B) \to_{\mathsf{d}} \neg(A \vee B)
\end{aligned}$$

Remark 4. In conclusion, all these considerations lead us in doubting that $\mathbf{D}_2^{C}$ and $\mathbf{D}_2^{C*}$ did provide a solution to 'Jaśkowksi's problem'. Furthermore, given the presence of both, Observation 4 and of Proposition 2.2.2, also $\mathbf{D}_2^{l}$ seems to be far from providing a solution. Nonetheless, $\mathbf{D}_2^{r}$, as elaborated in [AO18], seems to be an adequate candidate to settle positively the problem raised in [Cos75].

2.3 Conclusive remarks

We have selected some of the perspectives under which discussive systems can be considered and, for the sake of brevity, we have chosen to explain and discuss just some of the main contributions present in the literature. For example, we have analysed how 'Jaśkowski's problem' might be solved, given the axiomatic systems we discussed. Nonetheless, many other works could have been considered (to cite a few of them, see [Kot75; Cos75; Vas01; MN19]). J. Perzanowski, in the critical notes to [Jaś99b, p. 59], showed how to define 'discussive negation', i.e., $\neg_d A = \Diamond\neg A$. Interestingly, the equivalence between $\Diamond\neg A$ and $\neg\Box A$, makes, in fact, $\neg_d$ equal to 'un-necessity'. However, there are only few articles considering these kind of extensions of the set of discussive connectives. Remarkably, in [Ciu06], there's an axiomatization of discussive logic including also $\neg_d$ among the connectives, but, unfortunately, this attempt has some problems (see, [AO18, pp. 1178–1179]). Hence, the challenge of developing an axiomatization for $\mathbf{D}_2$,

[13]Notice that those $\mathbf{D}_2^{C*}$ unprovable formulas correspond to Ax9, Ax12, Ax13 and Ax15 of both, $\mathbf{D}_2^{r}$ and $\mathbf{D}_2^{l}$.

including also $\neg_d$, is still open.
As remarked several times, Jaśkowski's logic has attracted discrete attention and many other research paths have been inaugurated. For instance, there has been some interest in developing discussive logic by getting rid of classical **S5**, in favour of other non-classical systems (see, among others [KC79; Ciu00; AAN11]). Additionally, the work of connecting $\mathbf{D}_2$ to modal logics (especially, the articles by M. Nasieniewski and colleagues) increased (for example, [NP11; MNP19; NP14]). Among their gigantic work, it's worth mentioning the proposal of an 'adaptive' (inconsistency-tolerant) version of discussive logic (see [Nas01; Nas03; Nas04] and [Meh06]).
From a more philosophical perspective, instead, one might find another interesting application of discussive logic to the philosophy of sciences in [CD95], where, in addition to the applications of $\mathbf{J}^{**}$ to the foundations of physical theories, the authors propose also a theory of 'pragmatic' (or 'partial') truth. The intuition underlying their idea, roughly, is that, with respect to inconsistent informations, scientists work with such informations *as if they were true*, and do not take them to be true *simpliciter*. Also in this case, $\mathbf{J}^{*}$ and $\mathbf{J}^{**}$ show their usefulness in modelling reasoning with inconsistent sets of premises. Importantly, in [DPS18], the authors – by taking inspiration from Jaśkowski's main motivation to build $\mathbf{D}_2$ – propose a four-valued discussive logic ($\mathbf{D}_4$) with the aim of capturing situations in which discussants put forward inconsistent opinions. Roughly, this work includes a 'doxastic' variant of discussive logic, allows to distinguish among different agents, each one with its respective set of beliefs, and models (through a function) the agents' capabilities (e.g., perception, expert-supplied knowledge, communication, discussion). The idea is that a reasoner, that starts from a lack of informations, can – in the process of acquiring more data – reach either support or refutation of such data. However, if there's an overload of informations, the reasoner may reach both, truth and falsity, i.e., inconsistent data.

In conclusion, as said, this overview is not exhaustive and, indeed, our aim was to indicate just some of the most interesting directions that discussive logic oriented researches have taken, by starting from Jaśkowski's papers. We think that thanks to its historical importance as the first known formulation of a paraconsistent logic and to its subsequent developments, discussive logic is still an interesting and vital field of investigation.

Part II

Applications

Chapter 3

Jaśkowski's discussive logic meets Gentzen-style calculi

Layout of the chapter. In this chapter, we investigate the proof theory of Jaśkowski's $\mathbf{D_2}$ by using labelled sequent calculi. Central topics include proofs of soundness and completeness as well as a complete proof analysis of the systems under scope. I'll conclude the chapter with several remarks highlighting the virtues of my investigation by pointing out some topics for further research on discussive logic related issues.

3.1 Semantic preliminaries

In this section, we introduce the formal language and Kripke's relational semantics for $\mathbf{D_2}$.

Definition 3.1.1. Let $\mathcal{L}$ be the language of $\mathbf{D_2}$. We denote by At a set of atoms $p, q, \ldots$. The set of $\mathbf{D_2}$ formulas, denoted Form, is defined recursively for all A as follows:

$$A ::= p \mid \neg A \mid A \vee A \mid A \wedge_{\mathsf{d}} A \mid A \rightarrow_{\mathsf{d}} A$$

Definition 3.1.2. A *relational frame* for $\mathbf{D_2}$, denoted $\mathcal{F}$, is a structure of the following shape $\langle W, R\rangle$, where W is a set of points (worlds, states) and $R \subseteq W^2$, satisfying the following conditions: (1) wRw, (2) $wRv \implies vRw$ and (3) $vRw \wedge wRz \implies vRz$.

Definition 3.1.3. A *relational model* for $\mathbf{D_2}$, denoted $\mathcal{M}$, is a pair $\langle \mathcal{F}, \mathsf{v}\rangle$, where $\mathcal{F}$ is a relational frame and $\mathsf{v} : \mathsf{At} \mapsto \mathcal{P}(W)$ is a valuation function on atomic formulas. The valuation is then extended to the whole language in the following way:

$$\mathcal{M}, w \Vdash p \text{ iff } w \in \mathsf{v}(p) \tag{1}$$

$$\mathcal{M}, w \Vdash \neg A \text{ iff } \mathcal{M}, w \nVdash A \tag{2}$$

$$\mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B \tag{3}$$

$$\mathcal{M}, w \Vdash A \wedge_{\mathsf{d}} B \text{ iff } \mathcal{M}, w \Vdash A \text{ and } \exists v \in W : \mathcal{M}, v \Vdash B \tag{4}$$

$$\mathcal{M}, w \Vdash A \rightarrow_{\mathsf{d}} B \text{ iff } \exists v \in W : wRv \text{ and } \mathcal{M}, v \Vdash A, \text{ imply } \mathcal{M}, w \Vdash B \tag{5}$$

Finally, A is *satisfied* in a model $\mathcal{M} = \langle \mathcal{F}, \mathsf{v} \rangle$, that is, $w \Vdash A$ iff there is a $v \in W$ such that wRv and $v \vDash A$. A formula A is *valid* in a frame $\mathcal{F}$ iff, for all valuations v, the formula A is satisfied in $\mathcal{M}$.

3.2 Labelled Sequent System for $\mathbf{D_2}$

Over the years, labelled proof systems have been widely studied and applied to the vast realm of non-classical logics. Specifically with respect to discussive logic, for example, J. Ciuciura [Ciu04] has developed a labelled tableau system. Inspired by this work and by starting from the semantics that we have just sketched, in what follows, we introduce a labelled sequent calculus for $\mathbf{D_2}$ and start considering its major proof-theoretic properties.

Definition 3.2.1. The notation $w : A$ is used to denote *labelled formulas*. Objects of the form wRv are called *relational atoms*. Given two multisets Γ, Δ of both, labelled formulas and relational atoms, a *labelled sequent* is an object of the following form: $\Gamma \Rightarrow \Delta$.

The labelled rules of our sequent system are subject to the following closure condition. Consider a rule $\mathcal{R}$ of the following form:

$$\frac{A, B_1, \dots, B_n, B_{n+1}, B_{n+1}, \Gamma \Rightarrow \Delta}{B_1, \dots, B_n, \Gamma \Rightarrow \Delta}\ (\mathcal{R})$$

Applying the *closure condition* on $\mathcal{R}$ means to substitute the multiple occurrence B_{n+1}, B_{n+1} with a single one to obtain a rule $\mathcal{R}^*$ of the following shape:

$$\frac{A, B_1, \dots, B_n, B_{n+1}, \Gamma \Rightarrow \Delta}{B_1, \dots, B_n, \Gamma \Rightarrow \Delta}\ (\mathcal{R}^*)$$

Axiomatic sequents

$$w : p, \Gamma \Rightarrow \Delta, w : p$$

where p is an atomic formula.

Propositional rules

$$\frac{\Gamma \Rightarrow \Delta, w : A}{w : \neg A, \Gamma \Rightarrow \Delta}\ (L\neg) \qquad \frac{w : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg A}\ (R\neg)$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta}\ (L\vee) \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B}\ (R\vee)$$

Discussive rules

$$\frac{wRv, w : A, v : B, \Gamma \Rightarrow \Delta}{w : A \wedge_{\mathsf{d}} B, \Gamma \Rightarrow \Delta}\ (L\wedge_{\mathsf{d}}^{+})$$

$$\frac{wRv, \Gamma \Rightarrow \Delta, w : A \wedge_{\mathsf{d}} B, w : A \qquad wRv, \Gamma \Rightarrow \Delta, w : A \wedge_{\mathsf{d}} B, v : B}{wRv, \Gamma \Rightarrow \Delta, w : A \wedge_{\mathsf{d}} B}\ (R\wedge_{\mathsf{d}})$$

$$\frac{wRv, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta, v : A \qquad wRv, w : B, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta}{wRv, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta}\ (L \rightarrow_{\mathsf{d}})$$

$$\frac{wRv, v : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}^{+})$$

where the symbol $^+$ denotes that v is an *eigenvariable*.

Relational rules

$$\frac{wRw,\ \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathit{Ref}) \qquad \frac{vRw, wRv, \Gamma \Rightarrow \Delta}{wRv, \Gamma \Rightarrow \Delta}\ (\mathit{Sym})$$

$$\frac{zRv, zRw, wRv, \Gamma \Rightarrow \Delta}{zRw, wRv\Gamma \Rightarrow \Delta}\ (\mathit{Trs})$$

The propositional rules displayed above are the usual ones for $\neg$ and $\vee$; the rules for $\wedge_{\mathsf{d}}$ and $\rightarrow_{\mathsf{d}}$ have been constructed out of the semantic clauses of Definition 3.1.3 by following the methodology presented, among other, in [Neg05; NvP11; Pog09a]. As relational atoms are never active in the right-hand side of sequents, we do not include axioms of the form $wRv, \Gamma \Rightarrow \Delta, wRv$.

3.3 Proof Analysis and Cut-admissibility

Definition 3.3.1. Let $\mathcal{A}$ be any labelled formula of the form $v : A$. We denote by $l(\mathcal{A})$ the label of a formula $\mathcal{A}$, and by $p(\mathcal{A})$ the pure part of the formula, that is, the part of the formula without the label. The *weight* (or *complexity*) of a labelled formula is defined as a lexicographically ordered pair: $\langle \mathsf{w}(p(\mathcal{A})), \mathsf{w}(l(\mathcal{A}))\rangle$, where:

1. for all state labels $v \in W$, $\mathsf{w}(v) = 1$;
2. for all $p \in \mathsf{At}$, $\mathsf{w}(p) = 1$;
3. $\mathsf{w}(\neg A) = \mathsf{w}(A) + 1$;
4. $\mathsf{w}(A \wedge B) = \mathsf{w}(A) + \mathsf{w}(B) + 1$, for $\wedge \in \{\vee, \wedge_{\mathsf{d}}, \rightarrow_{\mathsf{d}}\}$.

Definition 3.3.2. We denote by $h(\delta)$ the natural number indicating the height of a derivation. We associate the height with the longest branch in a proof-tree $\delta - 1$. The height of a derivation $h(\delta)$ is defined by induction on the construction of δ:

$$\delta \equiv \{\ \Gamma \Rightarrow \Delta \qquad\qquad h(\delta) = 0$$

$$\delta \equiv \left\{ \begin{array}{c} \vdots\ \}\delta_1 \\ \dfrac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}\ \mathcal{R} \end{array} \right. \qquad\qquad h(\delta) = h(\delta_1) + 1$$

$$\delta \equiv \left\{ \dfrac{\dfrac{\vdots\,\}\delta_1}{\Gamma' \Rightarrow \Delta'}\mathcal{R}' \qquad \dfrac{\vdots\,\}\delta_2}{\Gamma'' \Rightarrow \Delta''}\mathcal{R}''}{\Gamma \Rightarrow \Delta}\mathcal{R} \right. \qquad h(\delta) = h(\delta_1) + 1,\ h(\delta_2) + 1$$

Finally, let $\mathcal{S}$ be a sequent. The notation '$\delta \vdash \mathcal{S}$' stands for '$\delta$ is a proof/derivation of $\mathcal{S}$' with $h(\delta) \leq n$ and '$\vdash^n \mathcal{S}$' stands for '$\mathcal{S}$ has a derivation δ of height n'.

Proposition 3.3.1. $\vdash_{\mathbf{G3D_2}} w : A, \Gamma \Rightarrow \Delta, w : A$.

Proof. By induction on the complexity of A. Let $A = B \wedge_{\mathsf{d}} C$:

$$\dfrac{\dfrac{\vdots \atop wRv, w : B, v : C, \Gamma \Rightarrow \Delta, w : B \qquad \vdots \atop wRv, w : B, v : C, \Gamma \Rightarrow \Delta, v : C}{wRv, w : B, v : C, \Gamma \Rightarrow \Delta, w : B \wedge_{\mathsf{d}} C}(R\wedge_{\mathsf{d}})}{w : B \wedge_{\mathsf{d}} C, \Gamma \Rightarrow \Delta, w : B \wedge_{\mathsf{d}} C}(L\wedge_{\mathsf{d}})$$

Now, let $A = B \rightarrow_{\mathsf{d}} C$:

$$\dfrac{\dfrac{\vdots \atop wRv, v : B, w : B \rightarrow_{\mathsf{d}} C, \Gamma \Rightarrow \Delta, v : B, w : C \qquad \vdots \atop wRv, w : C, v : B, w : B \rightarrow_{\mathsf{d}} C, \Gamma \Rightarrow \Delta, w : C}{wRv, w : B \rightarrow_{\mathsf{d}} C, v : B, \Gamma \Rightarrow \Delta, w : C}(L \rightarrow_{\mathsf{d}})}{w : B \rightarrow_{\mathsf{d}} C, \Gamma \Rightarrow \Delta, w : B \rightarrow_{\mathsf{d}} C}(R \rightarrow_{\mathsf{d}})$$

In both cases, the premises are derivable by the induction hypothesis. □

Definition 3.3.3. A rule $\mathcal{R}$ of $\mathbf{G3D_2}$ is height-preserving admissible just in case: if there is a derivation of the premise(s) of $\mathcal{R}$, then there is a derivation of the conclusion of $\mathcal{R}$ that contains no application of $\mathcal{R}$ (with the height at most n, where n is the maximal height of the derivation of the premise(s)).

Definition 3.3.4. We define substitution as follows:

- $wRv(z/x) \equiv wRv$, if $x \neq w$ and $x \neq v$.
- $wRv(z/w) \equiv zRv$, if $w \neq v$.
- $wRv(z/v) \equiv wRz$, if $w \neq v$.
- $wRw(z/w) \equiv zRz$.
- $w : A(z/v) \equiv w : A$, if $v \neq w$.
- $w : A(z/w) \equiv z : A$.

Next we extend this definition to multisets:

Lemma 3.3.2 (Substitution)**.** If $\vdash^n_{\mathbf{G3D_2}} \Gamma \Rightarrow \Delta$ and, provided v is free for w in Γ, Δ, then $\vdash^n_{\mathbf{G3D_2}} \Gamma(v/w) \Rightarrow \Delta(v/w)$.

Proof. If $n = 0$ and (z/w) is not a vacuous substitution, then it can be an axiomatic sequent of the form $w : p, \Gamma \Rightarrow \Delta, w : p$ or of the form $wRv, \Gamma \Rightarrow \Delta, wRv$. In both cases, also the substitution $\Gamma(z/w) \Rightarrow \Delta(z/w)$ is an axiomatic sequent. Let $n > 0$. If we are considering a propositional rule, we apply the inductive hypothesis to the premise(s) of the rule, and then the rule again. We proceed similarly if the last rule is a discussive rule without a variable condition, namely $R\wedge_{\mathsf{d}}$ and $L \rightarrow_{\mathsf{d}}$. Finally, let's consider a discussive rule with the *eigenvariable* condition, such as $R \rightarrow_{\mathsf{d}}^{+}$. (i) Let $w : A \rightarrow_{\mathsf{d}} B$ be principal and let z be a fresh variable. Let $\Delta = w : A \rightarrow_{\mathsf{d}} B, \Delta'$:

$$\dfrac{\vdots \atop \vdash^{n} wRz, z : A, \Gamma \Rightarrow \Delta', w : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', w : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}^{+})$$

Notice that $z \neq w$ and that $z \notin \Gamma, \Delta'$. By the application of the inductive hypothesis we obtain the following application of $R \rightarrow_{\mathsf{d}}^{+}$ with the same derivation height:

$$\dfrac{\vdots \atop \vdash^{n} vRz, z : A, \Gamma(v/w) \Rightarrow \Delta'(v/w), v : B}{\vdash^{n+1} \Gamma(v/w) \Rightarrow \Delta'(v/w), v : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}^{+})$$

(ii) Assume again $w : A \rightarrow_{\mathsf{d}} B$ as the principal formula, but with v not being a fresh variable. So, our derivation ends as follows:

$$\dfrac{\vdots \atop \vdash^{n} wRv, v : A, \Gamma \Rightarrow \Delta', w : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', w : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}^{+})$$

First, we replace v by a fresh variable z. By the variable condition the substitution does not affect Γ, Δ'. Indeed, we get the following premise of height n: $wRz, z : A, \Gamma \Rightarrow \Delta', w : B$. So, by applying inductive hypothesis, we substitute the label w with v to conclude:

$$\dfrac{\vdots \atop \vdash^{n} vRz, z : A, \Gamma(v/w) \Rightarrow \Delta'(v/w), v : B}{\vdash^{n+1} \Gamma(v/w) \Rightarrow \Delta'(v/w), v : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}^{+})$$

(iii) If w is not the label of the principal formula $A \rightarrow_{\mathsf{d}} B$ in the derivation, then the proof proceeds analogously. A similar reasoning applies if the derivation ends with an application of $L\wedge_{\mathsf{d}}^{+}$. ☐

Lemma 3.3.3. The rules of weakening:

$$\dfrac{\Gamma \Rightarrow \Delta}{w : A, \Gamma \Rightarrow \Delta}\ (Lw) \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : A}\ (Rw) \qquad \dfrac{\Gamma \Rightarrow \Delta}{wRv, \Gamma \Rightarrow \Delta}\ (Lw_R)$$

are height-preserving admissible in $\mathbf{G3D_2}$.

Proof. By induction on the height of the derivation. For $n = 0$, the case is trivial. For $n > 0$, we simultaneously display the transformed derivations for Lw and Rw on the left and on the right, respectively. As an example we will deal with $R \rightarrow_{\mathsf{d}}{}^{+}$. (i) Let $\Delta = \Delta', w : A \rightarrow_{\mathsf{d}} B$. Suppose $w : A \rightarrow_{\mathsf{d}} B$ is the principal formula and the label v in a fresh variable:

$$\dfrac{\begin{matrix}\vdots\\ \vdash^n wRv, v : A, \Gamma \Rightarrow \Delta', w : B\end{matrix}}{\vdash^{n+1} \Gamma \Rightarrow \Delta', w : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}{}^{+})$$

By applying the inductive hypothesis (on the left and on the right) to the premise and, finally, also the rule, we obtain the requested derivations:

$$\dfrac{\begin{matrix}\vdots\\ \vdash^n w : A, wRv, v : A, \Gamma \Rightarrow \Delta', w : B\end{matrix}}{\vdash^{n+1} w : A, \Gamma \Rightarrow \Delta', w : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}^{+}) \qquad \dfrac{\begin{matrix}\vdots\\ \vdash^n wRv, v : A, \Gamma \Rightarrow \Delta', w : B, w : A\end{matrix}}{\vdash^{n+1} \Gamma \Rightarrow \Delta', w : A \rightarrow_{\mathsf{d}} B, w : A}\ (R \rightarrow_{\mathsf{d}}^{+})$$

(ii) If the label v in the premise is not a fresh label, we need to avoid clashes of variables. So, we apply the substitution lemma to the premise of the rule to replace v with a fresh variable, say z, and obtain the following premise: $wRz, z : A, \Gamma \Rightarrow \Delta', w : B$. Finally, we apply the inductive hypothesis and the rule to get:

$$\dfrac{\begin{matrix}\vdots\\ \vdash^n w : A, wRz, z : A, \Gamma \Rightarrow \Delta', w : B\end{matrix}}{\vdash^{n+1} w : A, \Gamma \Rightarrow \Delta',\ w : A \rightarrow_{\mathsf{d}} B}\ (R \rightarrow_{\mathsf{d}}^{+}) \qquad \dfrac{\begin{matrix}\vdots\\ \vdash^n wRz, z : A, \Gamma \Rightarrow \Delta', w : B, w : A\end{matrix}}{\vdash^{n+1} \Gamma \Rightarrow \Delta', w : A \rightarrow_{\mathsf{d}} B, w : A}\ (R \rightarrow_{\mathsf{d}}^{+})$$

as desired. The same reasoning applies if the last rule applied is $L\wedge_{\mathsf{d}}^{+}$. □

Definition 3.3.5. A rule $\mathcal{R}$ of $\mathbf{G3D_2}$ is height-preserving invertible just in case: if there is a derivation of the conclusion of $\mathcal{R}$, then there is a dedrivation of premise(s) of $\mathcal{R}$ (with the height at most n, where n is the maximal height of the derivation of the conclusion).

Lemma 3.3.4 (Inversion)**.** All rules of $\mathbf{G3D_2}$ are height-preserving invertible.

Proof. For each rule $\mathcal{R}$, we have to show that if there is a derivation δ of the conclusion, then there is a derivation δ' of the premise(s), of the same height. For $L\neg$, $R\neg$, $L\vee$, $R\vee$, $R\wedge_{\mathsf{d}}$ and $L \rightarrow_{\mathsf{d}}$ we use a standard induction on the height of δ. For $L\wedge_{\mathsf{d}}$ and $R \rightarrow_{\mathsf{d}}$ as well, but we need to be sure that in the transformed derivation we make use of a fresh label by applying the substitution lemma inside δ', if needed. □

Lemma 3.3.5. The rules of contraction:

$$\dfrac{w : A, w : A, \Gamma \Rightarrow \Delta}{w : A, \Gamma \Rightarrow \Delta}\ (Lc) \qquad \dfrac{\Gamma \Rightarrow \Delta, w : A, w : A}{\Gamma \Rightarrow \Delta, w : A}\ (Rc) \qquad \dfrac{wRv, wRv, \Gamma \Rightarrow \Delta}{wRv, \Gamma \Rightarrow \Delta}\ (Lc_R)$$

are height-preserving admissible in $\mathbf{G3D_2}$.

Proof. By induction on the height of derivation. As usual, if $n = 0$, then the premise is an axiomatic sequent and so also the contracted sequent is an axiomatic one. If $n > 0$, we consider the last rule applied to the premise of contraction. If the contraction formula is not principal in the premise of some $\mathcal{R}$, then both occurrences are found in the premises of the rule and they have a smaller derivation height. By applying the induction hypothesis, we contract them and apply $\mathcal{R}$ to obtain a derivation of the conclusion with the same derivation height. As an example, consider a rule where the principal formula and the relational atoms are both active, for instance:

$$\dfrac{\vdash^n wRv, v : A, \Gamma \Rightarrow \Delta, w : A \rightarrow_{\mathsf{d}} B, w : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta, w : A \rightarrow_{\mathsf{d}} B, w : A \rightarrow_{\mathsf{d}} B} \; (R \rightarrow_{\mathsf{d}}^{+})$$

By height-preserving invertibility applied to the premise, we obtain the following derivation:

$$\dfrac{\dfrac{\vdash^n wRv, v : A, wRv, v : A, \Gamma \Rightarrow \Delta, w : B, w : B}{\vdash^n wRv, v : A, \Gamma \Rightarrow \Delta, w : B} \; i.h.}{\vdash^{n+1} \Gamma \Rightarrow \Delta, w : A \rightarrow_{\mathsf{d}} B} \; (R \rightarrow_{\mathsf{d}}^{+})$$

as requested. Notice that if both contraction formulas are principal in $(R \rightarrow_{\mathsf{d}}^{+})$, we apply the closure condition.
Finally, consider a rule in which only the labelled formula is principal:

$$\dfrac{\vdash^n wRv, w : A \rightarrow_{\mathsf{d}} B, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta, v : A \qquad \vdash^n wRv, w : B, w : A \rightarrow_{\mathsf{d}} B, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta}{\vdash^{n+1} wRv, w : A \rightarrow_{\mathsf{d}} B, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta} \; (L \rightarrow_{\mathsf{d}})$$

By inductive hypothesis, we obtain the following derivation:

$$\dfrac{\vdash^n wRv, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta, v : A \qquad \vdash^n wRv, w : B, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta}{\vdash^{n+1} wRv, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta} \; (L \rightarrow_{\mathsf{d}})$$

The cases for $L\wedge_{\mathsf{d}}$ and $R\wedge_{\mathsf{d}}$ can be treated analogously. □

So, finally, we can show that CUT is an admissible rules of $\mathbf{G3D_2}$:

Theorem 3.3.6. The rule of CUT:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \; (\text{CUT})$$

is admissible in $\mathbf{G3D_2}$.

Proof. The proof is by a lexicographic induction on the complexity of the CUT-formula $w : A$ and the sum of the heights $h(\delta_1) + h(\delta_2)$. We perform a case analysis on the last rule used in the derivation above the CUT and whether it applies to the CUT-formula or not. We show that each application of CUT can either be eliminated, or be replaced by one or more applications of CUT of smaller complexity. The proof proceeds similarly to the CUT-elimination proof for modal and intermediate logics, see [Neg05; NvP11]. Intuitively, we eliminate the topmost CUT first, and proceed by repeating the procedure until we reach a CUT-free derivation. We start by showing that CUT can be eliminated if one of the CUT premises is an axiom (**case 1**). Then we show that the CUT-height can be reduced in all cases in which the CUT-formula is not principal in at least one of the CUT-premises (**case 2**). Finally, we show that if the CUT-formula is principal in both CUT-premises, then the CUT is reduced to one or more *cuts* on less complex formulas or on shorter derivations (**case 3**). The complete case analysis is performed in Appendix A1.
Here, we present two interesting cases where the CUT-formula A is principal in both premises. Let $w : A = w : B \wedge_{\mathsf{d}} C$ and consider the following derivation:

$$\dfrac{\dfrac{wRv,\Gamma' \Rightarrow \Delta', w:B\wedge_{\mathsf{d}} C, w:B \qquad wRv,\Gamma'\Rightarrow\Delta',w:B\wedge_{\mathsf{d}} C, v:C}{wRv,\Gamma'\Rightarrow\Delta',w:B\wedge_{\mathsf{d}} C}\,(R\wedge_{\mathsf{d}}) \qquad \dfrac{wRz,w:B,z:C,\Gamma\Rightarrow\Delta}{w:B\wedge_{\mathsf{d}} C,\Gamma\Rightarrow\Delta}\,(L\wedge_{\mathsf{d}})}{wRv,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(\textsc{cut})$$

It is transformed into:

$$\text{(Lem. 3.3.5)}\ \dfrac{\dfrac{\begin{matrix}\vdots\,\delta_1\\ wRv,\Gamma,\Gamma'\Rightarrow\Delta,\Delta',v:C\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2\\ wRv,v:C,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'\end{matrix}}{wRv,wRv,\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow\Delta,\Delta,\Delta',\Delta'}\,(\textsc{cut})}{wRv,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Lc+Rc+Lc_R)$$

where δ_1 is concluded by:

$$\dfrac{wRv,\Gamma'\Rightarrow\Delta',w:B\wedge_{\mathsf{d}} C,v:C \qquad w:B\wedge_{\mathsf{d}} C,\Gamma\Rightarrow\Delta}{wRv,\Gamma,\Gamma'\Rightarrow\Delta,\Delta',v:C}\,(\textsc{cut})$$

and δ_2 is obtained by:

$$\text{(Lem. 3.3.5)}\ \dfrac{\dfrac{\dfrac{wRv,\Gamma'\Rightarrow\Delta',w:B\wedge_{\mathsf{d}} C,w:B \qquad w:B\wedge_{\mathsf{d}} C,\Gamma\Rightarrow\Delta}{wRv,\Gamma,\Gamma'\Rightarrow\Delta,\Delta',w:B}\,(\textsc{cut}) \qquad \dfrac{wRz,w:B,z:C,\Gamma\Rightarrow\Delta}{wRv,w:B,v:C,\Gamma\Rightarrow\Delta}\,[v/z]}{wRv,wRv,v:C,\Gamma,\Gamma,\Gamma'\Rightarrow\Delta,\Delta,\Delta'}\,(\textsc{cut})}{wRv,v:C,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Lc+Rc+Lc_R)$$

Notice that the two topmost CUTs, those on $w : B \wedge_{\mathsf{d}} C$, are derived with a shorter derivation height, while the other two are applied on formulas of smaller complexity, i.e., $w : B$ and $v : C$..
Assume that the premises of CUT are derived by $R \rightarrow_{\mathsf{d}}$ and $L \rightarrow_{\mathsf{d}}$, respectively. Let $A = B \rightarrow_{\mathsf{d}} C$:

$$\dfrac{\dfrac{wRv,v:B,\Gamma\Rightarrow\Delta,w:C}{\Gamma\Rightarrow\Delta,w:B\rightarrow_{\mathsf{d}} C}\,(R\rightarrow_{\mathsf{d}}) \qquad \dfrac{wRz,w:B\rightarrow_{\mathsf{d}} C,\Gamma'\Rightarrow\Delta',z:B \qquad wRz,w:C,w:B\rightarrow_{\mathsf{d}} C,\Gamma'\Rightarrow\Delta'}{wRz,w:B\rightarrow_{\mathsf{d}} C,\Gamma'\Rightarrow\Delta'}\,(L\rightarrow_{\mathsf{d}})}{wRz,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(\textsc{cut})$$

It is transformed into the following derivation:

$$\text{(Lem. 3.3.5)}\ \dfrac{\dfrac{\begin{matrix}\vdots\,\delta_1\\ wRz,\Gamma,\Gamma'\Rightarrow\Delta,\Delta',w:C\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2\\ wRz,w:C,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'\end{matrix}}{wRz,wRz,\Gamma,\Gamma,\Gamma',\Gamma'\Rightarrow\Delta,\Delta,\Delta',\Delta'}\,(\textsc{cut})}{wRz,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(Lc+Rc+Lc_R)$$

where the conclusion of δ_1 is derived by:

$$\text{(Lem.3.3.5)}\ \dfrac{\dfrac{\dfrac{\Gamma\Rightarrow\Delta,w:B\rightarrow_{\mathsf{d}} C \qquad wRzw:B\rightarrow_{\mathsf{d}} C,\Gamma'\Rightarrow\Delta',z:B}{wRz,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'z:B}\,\textsc{cut} \qquad \text{(Lem. 3.3.2)}\ \dfrac{wRv,v:B,\Gamma\Rightarrow\Delta,w:C}{wRz,z:B,\Gamma\Rightarrow\Delta,w:C}\,[z/v]}{wRz,wRz,\Gamma,\Gamma,\Gamma'\Rightarrow\Delta,\Delta,\Delta',w:C}\,(\textsc{cut})}{wRz,\Gamma,\Gamma'\Rightarrow\Delta,\Delta',w:C}\,Lc{+}Rc{+}Lc_R$$

while the conclusion of δ_2 is derived by:

$$\dfrac{\Gamma\Rightarrow\Delta,w:B\rightarrow_{\mathsf{d}} C \qquad wRz,w:C,w:B\rightarrow_{\mathsf{d}} C,\Gamma'\Rightarrow\Delta'}{wRz,w:C,\Gamma,\Gamma'\Rightarrow\Delta,\Delta'}\,(\textsc{cut})$$

Notice that the two topmost CUTS, those on $w : B \rightarrow_{\mathsf{d}} C$, are derived with a shorter derivation height, while the other two are applied on formulas of smaller complexity, i.e., $z : B$ and $w : C$. □

Remark 5. Jaśkowski argued that a fundamental feature of discussive systems is the possibility of having a detachable $\rightarrow_{\mathsf{d}}$:

> "In every discussive system two theses, one of the form: $\mathfrak{P} \rightarrow_{\mathsf{d}} \mathfrak{Q}$, and the other of the form: $\mathfrak{P}$, entail the thesis $\mathfrak{Q}$ [...]. Thus the rule of *modus ponens* may be applied to discussive theses if discussive implication is used instead of ordinary implication."

In light of the CUT-admissibility result just presented, we can show that in our newly introduced $\mathbf{G3D_2}$, it is possible to simulate the detachability of $\rightarrow_{\mathsf{d}}$ by the following result.

Proposition 3.3.7. The rule of *modus ponens*:

$$\frac{A \rightarrow_{\mathsf{d}} B \qquad A}{B}$$

is admissible in $\mathbf{G3D_2}$.

Proof. We show that, in $\mathbf{G3D_2}$, given $\Rightarrow w : A \rightarrow_{\mathsf{d}} B$ and $\Rightarrow w : A$, we derive $\Rightarrow w : B$:

$$\dfrac{\dfrac{\Rightarrow w : A \qquad \dfrac{\Rightarrow w : A \rightarrow_{\mathsf{d}} B \qquad \dfrac{wRw, w : A, w : A \rightarrow_{\mathsf{d}} B \Rightarrow w : A \qquad wRw, w : B, w : A \rightarrow_{\mathsf{d}} B \Rightarrow w : B}{wRw, w : A \rightarrow_{\mathsf{d}} B, w : A \Rightarrow w : B}\,(L \rightarrow_{\mathsf{d}})}{wRw, w : A \Rightarrow w : B}\,(\textsc{cut})}{wRw \Rightarrow w : B}\,(\textsc{cut})}{\Rightarrow w : B}\,(Ref)$$

where the rightmost premises are derivable by Proposition 3.3.1 and the applications of CUT are admissible by Theorem 3.3.6. □

3.3.1 Derivations, syntactic completeness and paraconsistency

Before delving into the proof of syntactic completeness, it is essential to highlight the consideration of additional structural-like rules.
As per the relational semantics previously introduced, the modal aspect of the consequence relation of $\mathbf{D_2}$ is expressed by including the following proviso: $w \Vdash A$ holds if and only if there exists a state $v \in W$ such that wRv and $v \Vdash A$. Given that our calculus fully internalizes the relational semantics for $\mathbf{D_2}$, we need to introduce additional machinery to correctly express the discussive consequence relation within $\mathbf{G3D_2}$. In his paper on labelled tableau system for discussive logic, Ciuciura remarks that:

> "Jaśkowski suggested treating a discussion as a set of opinions given by participants. There follows an idea to precede each opinion by the provision: for a certain admissible meaning of the terms used. [...] The idea is reflected in an additional rule.

> **Special rule**:
> $$\frac{\varrho :: FP}{\tau :: FP}\ (\mathrm{S})$$
> where ϱ is a root label and τ is a label that has been already used in the branch. The application of the rule is always limited to root labels." [Ciu04, p. 228]

Similarly, we incorporate the following two rules to mirror Ciuciura's special rule at the calculus level:

$$(w \text{ root label})\ \frac{wRv, \Gamma \Rightarrow \Delta, w : p, v : p}{wRv, \Gamma \Rightarrow \Delta, w : p}\ \textsc{AtJ} \qquad (w \text{ root label})\ \frac{wRv, \Gamma \Rightarrow \Delta, w : A, v : A}{wRv, \Gamma \Rightarrow \Delta, w : A}\ \textsc{GenJ}$$

Since these rules correspond to forms of contraction, it is preferable to have a system in which these rules are height-preserving admissible rather than primitive. There's no eigenvariable requirement; however, it is necessary to ensure that whenever we encounter a sequent of the following form $wRv \Rightarrow w : A$, before applying bottom-up the GenJ rule, we need to verify that w is a root label to obtain $wRv \Rightarrow w : A, v : A$. By *root label*, we mean a label that labels a formula in the most bottom sequent of a derivation.

Lemma 3.3.8. The following rules:

$$(w \text{ root label})\ \frac{wRv, \Gamma \Rightarrow \Delta, w : p, v : p}{wRv, \Gamma \Rightarrow \Delta, w : p}\ \textsc{AtJ} \qquad (w \text{ root label})\ \frac{wRv, \Gamma \Rightarrow \Delta, w : A, v : A}{wRv, \Gamma \Rightarrow \Delta, w : A}\ \textsc{GenJ}$$

are height-preserving admissible.

Proof. By induction on the height n of the derivation.
(a) We start by considering AtJ. If $n = 0$, then:

$$\vdash^0 wRv, x : p, \Gamma \Rightarrow \Delta, x : p, w : p, v : p \quad \overset{i.h.}{\rightsquigarrow} \quad \vdash^0 wRv, x : p, \Gamma \Rightarrow \Delta, x : p, w : p$$

The remaining base cases consist of relational rules. As an example consider *Ref*:

$$\frac{\vdash^n xRx, wRv, \Gamma \Rightarrow \Delta, w : p, v : p}{\vdash^{n+1} wRv, \Gamma \Rightarrow \Delta, w : p, v : p}\ Ref \quad \overset{i.h.}{\rightsquigarrow} \quad \frac{\vdash^n xRx, wRv, \Gamma \Rightarrow \Delta, w : p}{\vdash^{n+1} wRv, \Gamma \Rightarrow \Delta, w : p}\ Ref$$

The other relational rules are dealt with analogously. The inductive step is performed by permutation of the rules.
(b) For GenJ. Let $n = 0$ and take initial sequents:

$$\vdash^0 wRv, x : p, \Gamma \Rightarrow \Delta, x : p, w : A, v : A \quad \overset{i.h.}{\rightsquigarrow} \quad \vdash^0 wRv, x : p, \Gamma \Rightarrow \Delta, x : p, w : A$$

For $n > 0$, we proceed as follows. Suppose the last rule applied are $L\wedge_{\mathsf{d}}$ and $R\wedge_{\mathsf{d}}$, respectively. We have the following transformations. For $L\wedge_{\mathsf{d}}$:

$$\frac{\vdash^n xRy, x : B, y : C, wRv, \Gamma \Rightarrow \Delta, w : A, v : A}{\vdash^{n+1} x : B \wedge_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A, v : A}\ L\wedge_{\mathsf{d}} \quad \overset{i.h.}{\rightsquigarrow} \quad \frac{\vdash^n xRy, x : B, y : C, wRv, \Gamma \Rightarrow \Delta, w : A}{\vdash^{n+1} x : B \wedge_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A}\ L\wedge_{\mathsf{d}}$$

For $R\wedge_{\mathsf{d}}$:

$$\dfrac{\vdash^{n} xRy, wRv, \Gamma \Rightarrow \Delta, w : A, x : B \wedge_{\mathsf{d}} C, x : B, v : A \qquad \vdash^{n} xRy, wRv, \Gamma \Rightarrow \Delta, w : A, x : B \wedge_{\mathsf{d}} C, y : C, v : A}{\vdash^{n+1} xRy, wRv, \Gamma \Rightarrow \Delta, w : A, x : B \wedge_{\mathsf{d}} C, v : A}\, R\wedge_{\mathsf{d}}$$

$$i.h.\ \rightsquigarrow$$

$$\dfrac{\vdash^{n} xRy, wRv, \Gamma \Rightarrow \Delta, w : A, x : B \wedge_{\mathsf{d}} C, x : B \qquad \vdash^{n} xRy, wRv, \Gamma \Rightarrow \Delta, w : A, x : B \wedge_{\mathsf{d}} C, y : C}{\vdash^{n+1} xRy, wRv, \Gamma \Rightarrow \Delta, w : A, x : B \wedge_{\mathsf{d}} C}\, R\wedge_{\mathsf{d}}$$

We now display the transformations for $L \rightarrow_{\mathsf{d}}$ and $R \rightarrow_{\mathsf{d}}$, respectively. For $L \rightarrow_{\mathsf{d}}$:

$$\dfrac{\vdash^{n} xRy, x : B \rightarrow_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A, v : A, y : B \qquad \vdash^{n} x : C, xRy, x : B \rightarrow_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A, v : A}{\vdash^{n+1} xRy, x : B \rightarrow_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A, v : A}\, L\rightarrow_{\mathsf{d}}$$

$$i.h.\ \rightsquigarrow$$

$$\dfrac{\vdash^{n} xRy, x : B \rightarrow_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A, y : B \qquad \vdash^{n} x : C, xRy, x : B \rightarrow_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A}{\vdash^{n+1} xRy, x : B \rightarrow_{\mathsf{d}} C, wRv, \Gamma \Rightarrow \Delta, w : A}\, L\rightarrow_{\mathsf{d}}$$

For $R \rightarrow_{\mathsf{d}}$:

$$\dfrac{\vdash^{n} xRy, y : B, wRv, \Gamma \Rightarrow \Delta, w : A, v : A, x : C}{\vdash^{n+1} wRv, \Gamma \Rightarrow \Delta, w : A, v : A, x : B \rightarrow_{\mathsf{d}} C}\, R\rightarrow_{\mathsf{d}} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^{n} xRy, y : B, wRv, \Gamma \Rightarrow \Delta, w : A, x : C}{\vdash^{n+1} wRv, \Gamma \Rightarrow \Delta, w : A, v : A, x : B \rightarrow_{\mathsf{d}} C}\, R\rightarrow_{\mathsf{d}}$$

The cases for the other connectives are dealt with analogously. □

To illustrate the practical usefulness of the AT/GENJ rules, we will now proceed with the derivation of the axioms and rules for discussive logic, as stated in [AO18, p. 1172] (denoted $\mathbf{D_2}^*$). More precisely, we relate the notion of derivability in $\mathbf{D_2}^*$ and derivability in $\mathbf{G3D_2}$, that is we will prove a syntactic completeness result:

Theorem 3.3.9. If $\vdash_{\mathbf{D_2}^*} A$, then $\vdash_{\mathbf{G3D_2}+\textsc{cut}} \Rightarrow w : A$.

Proof. By root first search. I showcase the derivations of some salient examples. $\vdash_{\mathbf{G3D_2}} \Rightarrow w : A \rightarrow_{\mathsf{d}} (B \rightarrow_{\mathsf{d}} A)$.

$$\dfrac{\dfrac{\dfrac{wRx, wRv, v : A, x : B, \Rightarrow w : A, v : A}{wRx, wRv, v : A, x : B, \Rightarrow w : A}\,\textsc{GenJ}\ (\text{Lem. 3.3.8})}{wRv, v : A \Rightarrow w : B \rightarrow_{\mathsf{d}} A}\, R\rightarrow_{\mathsf{d}}\ (x \text{ fresh})}{\Rightarrow w : A \rightarrow_{\mathsf{d}} (B \rightarrow_{\mathsf{d}} A)}\, R\rightarrow_{\mathsf{d}}\ (v \text{ fresh})$$

$\vdash_{\mathbf{G3D_2}} \Rightarrow w : ((A \rightarrow_{\mathsf{d}} B) \rightarrow_{\mathsf{d}} A) \rightarrow_{\mathsf{d}} A$.

$$
\begin{prooftree}
\AxiomC{$wRx, x : A, vRw, wRv, \mathcal{S} \Rightarrow w : A, w : B, x : A$}
\RightLabel{\textsc{GenJ} (Lem. 3.3.8)}
\dashedLine
\UnaryInfC{$wRx, x : A, vRw, wRv, \mathcal{S} \Rightarrow w : A, w : B$}
\LeftLabel{(x fresh)}
\RightLabel{$R \to_{\mathsf{d}}$}
\UnaryInfC{$vRw, wRv, \mathcal{S} \Rightarrow w : A, w : A \to_{\mathsf{d}} B$}
\AxiomC{$v : A, vRw, wRv, \mathcal{S} \Rightarrow w : A, v : A$}
\RightLabel{\textsc{GenJ} (Lem. 3.3.8)}
\dashedLine
\UnaryInfC{$v : A, vRw, wRv, \mathcal{S} \Rightarrow w : A$}
\RightLabel{$L \to_{\mathsf{d}}$}
\BinaryInfC{$vRw, wRv, v : (A \to_{\mathsf{d}} B) \to_{\mathsf{d}} A \Rightarrow w : A$}
\RightLabel{Sym}
\UnaryInfC{$wRv, v : (A \to_{\mathsf{d}} B) \to_{\mathsf{d}} A \Rightarrow w : A$}
\LeftLabel{(v fresh)}
\RightLabel{$R \to_{\mathsf{d}}$}
\UnaryInfC{$\Rightarrow w : ((A \to_{\mathsf{d}} B) \to_{\mathsf{d}} A) \to_{\mathsf{d}} A$}
\end{prooftree}
$$

where $\mathcal{S}$ abbreviates $v : (A \to_{\mathsf{d}} B) \to_{\mathsf{d}} A$.
$\vdash_{\mathbf{G3D_2}} \Rightarrow w : \neg\neg A \to_{\mathsf{d}} A$ and $\vdash_{\mathbf{G3D_2}} \Rightarrow w : A \to_{\mathsf{d}} \neg\neg A$.

$$
\begin{prooftree}
\AxiomC{$wRv, v : A \Rightarrow w : A, v : A$}
\LeftLabel{(Lem. 3.3.8)}
\RightLabel{\textsc{GenJ}}
\dashedLine
\UnaryInfC{$wRv, v : A \Rightarrow w : A$}
\RightLabel{$R\neg$}
\UnaryInfC{$wRv \Rightarrow w : A, v : \neg A$}
\RightLabel{$L\neg$}
\UnaryInfC{$wRv, v : \neg\neg A \Rightarrow w : A$}
\LeftLabel{(v fresh)}
\RightLabel{$R \to_{\mathsf{d}}$}
\UnaryInfC{$\Rightarrow w : \neg\neg A \to_{\mathsf{d}} A$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$wRv, v : A \Rightarrow w : A, v : A$}
\LeftLabel{(Lem. 3.3.8)}
\RightLabel{\textsc{GenJ}}
\dashedLine
\UnaryInfC{$wRv, v : A \Rightarrow w : A$}
\RightLabel{$L\neg$}
\UnaryInfC{$wRv, w : \neg A, v : A \Rightarrow$}
\RightLabel{$R\neg$}
\UnaryInfC{$wRv, v : A \Rightarrow w : \neg\neg A$}
\LeftLabel{(v fresh)}
\RightLabel{$R \to_{\mathsf{d}}$}
\UnaryInfC{$\Rightarrow w : A \to_{\mathsf{d}} \neg\neg A$}
\end{prooftree}
$$

$\vdash_{\mathbf{G3D_2}} \Rightarrow w : \neg(A \vee \neg A) \to_{\mathsf{d}} B$.

$$
\begin{prooftree}
\AxiomC{$wRv, v : A \Rightarrow w : B, v : A$}
\RightLabel{$R\neg$}
\UnaryInfC{$wRv \Rightarrow w : B, v : A, v : \neg A$}
\RightLabel{$R\vee$}
\UnaryInfC{$wRv \Rightarrow w : B, v : A \vee \neg A$}
\RightLabel{$L\neg$}
\UnaryInfC{$wRv, v : \neg(A \vee \neg A) \Rightarrow w : B$}
\LeftLabel{(v fresh)}
\RightLabel{$R \to_{\mathsf{d}}$}
\UnaryInfC{$\Rightarrow w : \neg(A \vee \neg A) \to_{\mathsf{d}} B$}
\end{prooftree}
$$

$\vdash_{\mathbf{G3D_2}} \Rightarrow w : \neg(A \vee B) \to_{\mathsf{d}} (\neg A \wedge_{\mathsf{d}} \neg B)$.

$$
\begin{prooftree}
\AxiomC{$vRv, wRv, v : A \Rightarrow \mathcal{S}, v : A, v : B$}
\RightLabel{$R\neg$}
\UnaryInfC{$vRv, wRv \Rightarrow \mathcal{S}, v : A, v : B, v : \neg A$}
\AxiomC{$vRv, wRv, v : B \Rightarrow \mathcal{S}, v : A, v : B$}
\RightLabel{$R\neg$}
\UnaryInfC{$vRv, wRv \Rightarrow \mathcal{S}, v : A, v : B, v : \neg B$}
\RightLabel{$R\wedge_{\mathsf{d}}$}
\BinaryInfC{$vRv, wRv \Rightarrow w : \neg A \wedge_{\mathsf{d}} \neg B, v : A, v : B, v : \neg A \wedge_{\mathsf{d}} \neg B$}
\RightLabel{Ref}
\UnaryInfC{$wRv \Rightarrow w : \neg A \wedge_{\mathsf{d}} \neg B, v : A, v : B, v : \neg A \wedge_{\mathsf{d}} \neg B$}
\LeftLabel{(Lem. 3.3.8)}
\RightLabel{\textsc{GenJ}}
\dashedLine
\UnaryInfC{$wRv \Rightarrow w : \neg A \wedge_{\mathsf{d}} \neg B, v : A, v : B$}
\RightLabel{$R\vee$}
\UnaryInfC{$wRv \Rightarrow w : \neg A \wedge_{\mathsf{d}} \neg B, v : A \vee B$}
\RightLabel{$L\neg$}
\UnaryInfC{$wRv, v : \neg(A \vee B) \Rightarrow w : \neg A \wedge_{\mathsf{d}} \neg B$}
\LeftLabel{(v fresh)}
\RightLabel{$R \to_{\mathsf{d}}$}
\UnaryInfC{$\Rightarrow w : \neg(A \vee B) \to_{\mathsf{d}} (\neg A \wedge_{\mathsf{d}} \neg B)$}
\end{prooftree}
$$

where $\mathcal{S} = w : \neg A \wedge_{\mathsf{d}} \neg B, v : \neg A \wedge_{\mathsf{d}} \neg B$.
The admissibility of the *modus ponens* rule for $\to_{\mathsf{d}}$ is shown in Prop. 3.3.7 above. □

Let me conclude with an observation. As discussed in Chapter 2, Jaśkowski made a deliberate choice to define the $\mathbf{D_2}$ consequence relation by relying on the **S5** possibility operator $\Diamond$. He recognized that this approach was necessary even for proving very simple theses, which can be considered as not counter-intuitive in discussive logic. Indeed, to establish $A \to_{\mathsf{d}} A$ as a theorem of $\mathbf{D_2}$, a straightforward translation like $\Diamond A \supset A$ in **S5** is insufficient. Instead, an **S5** valid translation requires the prefixing of an additional $\Diamond$ before the translated formula, i.e., $\Diamond(\Diamond A \supset A)$:

> "The system defined in this way is discussive, i.e., its theses are provided with discussive assertion which implicitly includes the functor $\Diamond$. This is an essential fact, since even such a simple law as $p \supset p$, on the replacement of $\supset$ by $\rightarrow_{\mathsf{d}}$, becomes $p \rightarrow_{\mathsf{d}} p$ ($D_2 1$), which is not a theorem in **S5**, and becomes such only when preceded by the symbol $\Diamond$: $\Diamond(p \rightarrow_{\mathsf{d}} p)$ ($M_2 4$)." [Jaś99a, 45, Notation adapted]

In relational semantics, this idea is expressed – as mentioned earlier – by the following proviso: $w \Vdash A$ holds if and only if there exists a state $v \in W$ such that wRv and $v \Vdash A$. This is precisely what AtJ and GenJ are designed to express in $\mathbf{G3D_2}$. Indeed, Jaśkowski's suggestion of having $A \rightarrow_{\mathsf{d}} A$ as a discussive theorem is preserved in $\mathbf{G3D_2}$:

$$\dfrac{\dfrac{wRv, v : A \Rightarrow w : A, v : A}{wRv, v : A \Rightarrow w : A}\ \textsc{GenJ}\ \text{(Lem. 3.3.8)}}{\Rightarrow w : A \rightarrow_{\mathsf{d}} A}\ R\rightarrow_{\mathsf{d}}\ (v \text{ fresh})$$

Moreover, to further demonstrate that $\mathbf{G3D_2}$ adequately represents Jaśkowski's discussive logic, let's examine the paraconsistent nature of $\mathbf{D_2}$ and how it is incorporated into our calculus. Specifically, while $\mathbf{D_2}$ is inconsistency-tolerant, it is not explosive. As previously mentioned (see Chapter 2), Jaśkowski argued against explosion principles in relation to discussive connectives, and we can observe that $\mathbf{G3D_2}$ also fulfils this requirement.
First of all, $\nvdash_{\mathbf{G3D_2}} \Rightarrow w : A \rightarrow_{\mathsf{d}} (\neg A \rightarrow_{\mathsf{d}} B)$ [Jaś99a, p. 49].

$$\dfrac{\dfrac{\dfrac{\vdots}{wRx, wRv, v : A \Rightarrow w : B, x : A}}{wRx, wRv, v : A, x : \neg A \Rightarrow w : B}\ L\neg}{\dfrac{wRv, v : A \Rightarrow w : \neg A \rightarrow_{\mathsf{d}} B}{\Rightarrow w : A \rightarrow_{\mathsf{d}} (\neg A \rightarrow_{\mathsf{d}} B)}\ R\rightarrow_{\mathsf{d}}\ (v \text{ fresh})}\ R\rightarrow_{\mathsf{d}}\ (x \text{ fresh})$$

In the displayed proof search, there is no way to reach an initial sequent and terminate the procedure. This is because there is no possibility to obtain a formula of the form $w : A$ on the right-hand side of the sequent, which would be labeled by the root label w, and would enable us to apply GenJ and derive $v : A$ on the right-hand side of the sequent, thus terminating the proof search. The only possible application of GenJ is restricted to $w : B$, and this would lead to $x : B$ (resp. $v : B$) on the right-hand side, given wRx (resp. wRv), and this is unhelpful for reaching an initial sequent as well.
Likewise, $\nvdash_{\mathbf{G3D_2}} \Rightarrow w : (A \wedge_{\mathsf{d}} \neg A) \rightarrow_{\mathsf{d}} B$ [Jaś99b, p. 58].

$$\dfrac{\dfrac{\dfrac{\vdots}{vRx, wRv, v : A \Rightarrow w : B, x : A}}{vRx, wRv, v : A, x : \neg A \Rightarrow w : B}\ L\neg}{\dfrac{wRv, v : A \wedge_{\mathsf{d}} \neg A \Rightarrow w : B}{\Rightarrow w : (A \wedge_{\mathsf{d}} \neg A) \rightarrow_{\mathsf{d}} B}\ R\rightarrow_{\mathsf{d}}\ (v \text{ fresh})}\ L\wedge_{\mathsf{d}}\ (x \text{ fresh})$$

3.4 Soundness and Semantic Completeness

This section is devoted to the proofs of the soundness and completeness for our systems. We will show that rules of $\mathbf{G3D_2}$ preserve validity over Kripke frames obeying the conditions appropriate for $\mathbf{D_2}$. In order to do that, we start by extending semantic notions to sequents as follows:

Definition 3.4.1. Let $\mathcal{M} = \langle W, R, \mathsf{v}\rangle$ be a model and let $\mathcal{S}$ be the sequent $\Gamma \Rightarrow \Delta$. We define a $\mathcal{S}$-*interpretation in* $\mathcal{M}$ is a mapping $[\![\cdot]\!]$ from the labels in $\mathcal{S}$ to the set W of states in $\mathcal{M}$, such that, if wRv is in Γ, then $[\![w]\!]R[\![v]\!]$ in $\mathcal{M}$. Now we can define:

$\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{S}$ iff if for all $w : A \in \Gamma$, we have $\mathcal{M}, [\![w]\!] \Vdash A$, then there exists $v : B \in \Delta$, such that $\mathcal{M}, [\![v]\!] \Vdash B$.

Definition 3.4.2. A sequent $\mathcal{S}$ is *satisfied* in $\mathcal{M} = \langle W, R, \mathsf{v}\rangle$ if for all $\mathcal{S}$-interpretations $[\![\cdot]\!]$ we have $\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{S}$. A sequent $\mathcal{S}$ is valid in a frame $\mathcal{F} = \langle W, R\rangle$, if for all valuations v, the sequent $\mathcal{S}$ is satisfied in $\mathcal{M} = \langle W, R, \mathsf{v}\rangle$.

Finally, we can prove the soundness theorem:

Theorem 3.4.1. If a sequent $\mathcal{S}$ is provable in $\mathbf{G3D_2}$, then it is valid in frame $\mathcal{F}$.

Proof. We proceed by induction on the height of the derivation of $\mathcal{S}$. We show that for each rule $\mathcal{R}$ of the form $\mathcal{P}_1, \dots, \mathcal{P}_n/C$, if the premises $\mathcal{P}_1, \dots, \mathcal{P}_n$ are valid in all frames $\mathcal{F}$, then so is C. It follows from a case analysis on $\mathcal{R}$:

Ax. By way of contradiction, assume that $w : p, \Gamma \Rightarrow \Delta, w : p$ is not valid in all frames $\mathcal{F}$. This means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash w : p, \Gamma \Rightarrow \Delta, w : p$, i.e., $\mathcal{M}, w \Vdash p$, but $\mathcal{M}, w \nVdash p$. Contradiction.

$L\wedge_{\mathsf{d}}$. By way of contradiction, assume that $wRv, w : A, v : B, \Gamma \Rightarrow \Delta$ is valid in all frames $\mathcal{F}$, but $w : A \wedge_{\mathsf{d}} B, \Gamma \Rightarrow \Delta$ is not, where v is a fresh variable. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash w : A \wedge_{\mathsf{d}} B, \Gamma \Rightarrow \Delta$. In particular, there is a world v' such that $[\![w]\!]Rv'$ and $\mathcal{M}, w \Vdash A \wedge_{\mathsf{d}} B$, but $\mathcal{M}, z \nVdash C$, for all $z : C \in \Delta$. It follows that $\mathcal{M}, w \Vdash A$ and $\mathcal{M}, v' \Vdash B$. By defining an extension $[\![\cdot]\!]'$ of $[\![\cdot]\!]$ such that $[\![v]\!]' = v'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$, we obtain $\mathcal{M}, [\![\cdot]\!]' \nVdash wRv, w : A, v : B, \Gamma \Rightarrow \Delta$. Contradiction.

$R\wedge_{\mathsf{d}}$. By way of contradiction, assume that $wRv, \Gamma \Rightarrow \Delta, w : A$ and $wRv, \Gamma \Rightarrow \Delta, v : B$ are valid in frames $\mathcal{F}$, but $wRv, \Gamma \Rightarrow \Delta, w : A \wedge_{\mathsf{d}} B$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash wRv, \Gamma \Rightarrow \Delta, w : A \wedge_{\mathsf{d}} B$, i.e., $[\![w]\!]R[\![v]\!]$ and $\mathcal{M}, w \nVdash A \wedge_{\mathsf{d}} B$. Then, $\mathcal{M}, w \nVdash A$ or $\mathcal{M}, v \nVdash B$. Consequently, $\mathcal{M}, [\![\cdot]\!] \nVdash wRv, \Gamma \Rightarrow \Delta, w : A$ or $\mathcal{M}, [\![\cdot]\!] \nVdash wRv, \Gamma \Rightarrow \Delta, v : B$. Contradiction.

$L \rightarrow_{\mathsf{d}}$. By way of contradiction, assume that $wRv, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta, v : A$ and $wRv, w : A \rightarrow_{\mathsf{d}} B, w : B, \Gamma \Rightarrow \Delta$ are valid in frames $\mathcal{F}$, but $wRv, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash wRv, w : A \rightarrow_{\mathsf{d}} B, \Gamma \Rightarrow \Delta$, i.e., $[\![w]\!] R [\![v]\!]$ and $\mathcal{M}, w \Vdash A \rightarrow_{\mathsf{d}} B$, but $\mathcal{M}, z \nVdash C$, for all $z : C \in \Delta$. It follows that $\mathcal{M}, v \nVdash A$ or $\mathcal{M}, w \Vdash B$. Consequently, $\mathcal{M}, [\![\cdot]\!] \nVdash wRv, \Gamma \Rightarrow \Delta, v : A$ or $\mathcal{M}, [\![\cdot]\!] \nVdash wRv, w : B, \Gamma \Rightarrow \Delta$. Contradiction.

$R \rightarrow_{\mathsf{d}}$. By way of contradiction, assume that $wRv, v : A, \Gamma \Rightarrow \Delta, w : B$ is valid in all frames $\mathcal{F}$, but $\Gamma \Rightarrow \Delta, w : A \rightarrow_{\mathsf{d}} B$ is not, where $v \notin \Gamma, \Delta, w : A \rightarrow_{\mathsf{d}} B$. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, w : A \rightarrow_{\mathsf{d}} B$. In particular, we know that there is a world v' such that $[\![w]\!] Rv'$ and $\mathcal{M}, v' \Vdash A$, but $\mathcal{M}, w \nVdash B$. Let $[\![\cdot]\!]'$ be an extension of $[\![\cdot]\!]$, such that $[\![v]\!]' = v'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$. It follows that $\mathcal{M}, [\![\cdot]\!]' \nVdash wRv, v : A, \Gamma \Rightarrow \Delta, w : B$. Contradiction.

The other cases are similar and simpler. In particular, note that the cases for relational rules are trivial, as all frames $\mathcal{F}$ have to obey the corresponding conditions. □

Theorem 3.4.2. Let $\Gamma \Rightarrow \Delta$ be a sequent of $\mathbf{G3D_2}$. Then either the sequent is derivable in $\mathbf{G3D_2}$ or it has a countermodel with the frame properties of reflexivity, symmetry and transitivity.

Proof. We follow the pattern of the completeness proof in [**ne**; Neg09; NvP11]. We proceed with the construction of a derivation tree for $\Gamma \Rightarrow \Delta$ by applying the rules of $\mathbf{G3D_2}$ root-first (see Appendix A2). If the reduction tree is finite, i.e., all leaves are axiomatic sequents, we have a proof in $\mathbf{G3D_2}$. Assume that the derivation tree is infinite. By König's lemma, it has an infinite branch that is used to build the needed counterexample. Suppose that $\Gamma \Rightarrow \Delta \equiv \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_i \Rightarrow \Delta_i \ldots$ is one of such branches. Consider the sets $\mathbf{\Gamma} \equiv \bigcup \Gamma_i$ and $\mathbf{\Delta} \equiv \bigcup \Delta_i$, for $i \geq 0$. We now construct a countermodel, i.e. a model that makes all labelled formulas and relational atoms in $\mathbf{\Gamma}$ true and all labelled formulas in $\mathbf{\Delta}$ false. Let $\mathcal{F}$ be a frame, whose elements are all the labels occurring in $\mathbf{\Gamma}$. $\mathcal{F}$ is defined as follows:

- for all $w : p$ in $\mathbf{\Gamma}$ it holds that $w \Vdash p$ in $\mathcal{F}$.
- for all wRv in $\mathbf{\Gamma}$ it holds that wRv in $\mathcal{F}$.
- for all $w : p$ in $\mathbf{\Delta}$ it holds that $w \nVdash p$ in $\mathcal{F}$.

We show that for any formula A, $w \Vdash A$ if $w : A$ is in $\mathbf{\Gamma}$ and $w \nVdash A$ if $w : A$ is in $\mathbf{\Delta}$, where w is any arbitrary label.

- If p is atomic, the claim holds by definition of the model.
- The cases for $\neg$ and $\vee$ do not pose special difficulties.
- If $w : A \wedge_{\mathsf{d}} B$ is in $\mathbf{\Gamma}$, then $w : A \wedge_{\mathsf{d}} B$ appears in some Γ_i and, therefore, at some successive step of the reduction tree, for some $n > 0$, one finds that wRv, $w : A$ and $v : B$ are in Γ_{m+n}. By the inductive hypothesis we have that, wRv, $w \Vdash A$ and $v \Vdash B$. It follows that $w \Vdash A \wedge_{\mathsf{d}} B$ in the model.

- If $w : A \wedge_{\mathsf{d}} B$ is in $\mathbf{\Delta}$, then we consider all relational atoms of the form wRv that are in $\mathbf{\Gamma}$. If there's no relational atoms, the condition is vacuously satisfied and $w \nVdash A \wedge_{\mathsf{d}} B$ in the model. For any occurrence of wRv in $\mathbf{\Gamma}$, by construction of the tree, either $w : A$ or $v : B$ in $\mathbf{\Delta}$. So, by inductive hypothesis either $w \nVdash A$ or $v \nVdash B$ and, therefore, $w \nVdash A \wedge_{\mathsf{d}} B$.
- If $w : A \rightarrow_{\mathsf{d}} B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms wRv that occur in $\mathbf{\Gamma}$. If there's no relational atom, the accessibility condition is vacuously satisfied and, therefore, $w \Vdash A \rightarrow_{\mathsf{d}} B$ is in the model. For any occurrence of wRv in $\mathbf{\Gamma}$, by construction of the tree $v : A$ is in $\mathbf{\Delta}$ or $w : B$ is in $\mathbf{\Gamma}$. By the inductive hypothesis $v \nVdash A$ or $w \Vdash B$, and, given wRv, it follows that $w \Vdash A \rightarrow_{\mathsf{d}} B$.
- If $w : A \rightarrow_{\mathsf{d}} B$ is in $\mathbf{\Delta}$, at the successive step of the reduction tree we find that wRv and $v : A$ in $\mathbf{\Gamma}$, whereas $w : B$ is in $\mathbf{\Delta}$. By the inductive hypothesis we obtain wRv and $v \Vdash A$ but $w \nVdash B$, that is, $w \nVdash A \rightarrow_{\mathsf{d}} B$ in the model.

□

Finally, as a consequence, we obtain our desired result:

Corollary 3.4.2.1. If a sequent $\Gamma \Rightarrow \Delta$ is valid in every $\mathbf{D}_2$ frame $\mathcal{F}$, then it is derivable in the system $\mathbf{G3D_2}$.

3.5 Final remarks

- Throughout this chapter, we relied on the same set of connectives that Jaśkowski originally considered in his 1948 and 1949 articles. However, over the years, variants of Jaśkowski's discussive connectives have been introduced. For example, some of them are defined as follows:

  1. *discussive negation*, i.e., $\neg_{\mathsf{d}} A = \Diamond\neg A$.
  2. *left*-discussive conjunction, i.e., $A \wedge^{l}_{\mathsf{d}} B = \Diamond A \wedge B$.
  3. *symmetric*-discussive conjunction, i.e., $A \wedge^{s}_{\mathsf{d}} B = \Diamond A \wedge \Diamond B$.
  4. *symmetric*-discussive implication, i.e., $A \rightarrow^{s}_{\mathsf{d}} B = \Diamond A \supset \Diamond B$.

  First of all, notice that, as pointed out by J. Marcos, "different choices of discussive conjunction and discussive implication would in fact define logics distinct from $\mathbf{D_2}$". Here, we haven't considered *alternative* formulations of discussive logic, but rather we have furnished a proof system for Jaśkowski's discussive logic including *right*-discussive conjunction and *left*-discussive implication. However, we observe that, if equipped with appropriate semantic clauses defined in terms of relational models, all other discussive connectives listed above can be, in line of principle, treated according to the methodology we have adopted so far.

- In $\mathbf{G3D_2}$, we do not translate discussive formulas into modal ones, but we reflect the *diamond effect* of the definitions of $\wedge_{\mathsf{d}}$ and $\rightarrow_{\mathsf{d}}$ by using different related worlds. More precisely, by following Jaśkowski's suggestion, we

have defined $\mathbf{D_2}$ with the appeal to relational models whose accessibility relation is defined as in the models for **S5**. This choice is reflected, at the calculus level, by the presence of specific relational rules.
However, despite Jaśkowski's preference, a wide research program on discussive logic aims at the formulation of modal systems, different from **S5**, allowing to define $\mathbf{D_2}$. The literature on this specific topic is vast and, for the sake of our purpose, we will only sketch two examples of how the strategy used in this chapter can be adapted to define labelled calculi for discussive logic defined through alternative modal logics.

Example 3. In [NP13], it was shown that another possibility to define $\mathbf{D_2}$ is given by the normal modal logic **KD45** (See Corollary 9.2 and Theorem 9.1). Relational models for discussive logic, in this case, will have the accessibility relation defined as in models for **KD45**: serial ($\forall w \exists v(wRv)$), transitive ($vRw \wedge wRz \implies vRz$) and euclidean ($wRv \wedge wRz \implies vRz$). The semantic conditions for all operators, including those for $\rightarrow_{\mathsf{d}}$ and $\wedge_{\mathsf{d}}$, are exactly as in Def. By moving to the construction of the calculus, we observe that the rules for propositional and discussive operators will preserve the shape of those belonging to $\mathbf{G3D_2}$. At the level of the relational rules, instead, we have to perform a little change. We keep (*Trs*), but replace (*Ref*) and (*Sym*), with the following two rules:

$$(v \text{ fresh})\ \frac{wRv, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (Ser) \qquad \frac{vRz, wRv, wRz, \Gamma \Rightarrow \Delta}{wRv, wRz, \Gamma \Rightarrow \Delta}\ (Euc)$$

Example 4. As discussed in the previous chapter, one can employ also certain non-normal modal logics to define discussive logic. For example, the logic called $\mathbf{rS5}^{\mathrm{M}}$ was proved to be the smallest regular non-normal modal logics[1] defining $\mathbf{D_2}$. At the semantic level, is characterized by so-called *neighbourhood frames*, namely relational structures of the form $\langle W, N, R\rangle$, where N is a subset of W, whose elements are referred to as *normal worlds*. For R, the accessibility relation, one has the following two frame conditions:

$$\forall w \in N, \exists u \in N(wRu \wedge \forall x \in W(uRx \implies wRx)) \qquad (\times)$$
$$\forall w \in N, \exists u \in \mathcal{N}(wRu \wedge \forall x \in W(\exists y \in N(uRy \wedge yRx) \implies wRx)) \qquad (\star)$$

where $\mathcal{N} = \{v \in N \mid R(v) \subseteq N\}$ and $R(v) = \{z \in N \mid vRz\}$. A *neighbourhood model* is the following structure $\langle W, N, R, \mathsf{v}\rangle$, where the truth conditions for classical operators are preserved, whereas those for the modalities have to be modified. Consider, as an example, the clause for $\Diamond$:

$$\mathcal{M}, w \vDash \Diamond A \text{ iff } w \notin N \text{ or } \exists v \in W\,(wRv \text{ and } v \vDash A)$$

In, e.g., [NS16; GNS19], it was shown that is possible to deal with labelled calculi also in the case of non-normal modal logics characterised by neighbourhood structures. Nevertheless, given the complications in the semantics, also the development of the calculus might involve some subtleties. Roughly, in order to develop a proof system for $\mathbf{D_2}$ by employing $\mathbf{rS5}^{\mathrm{M}}$ models as starting point,

[1]See [NP09a; NP09b], as well as the previous chapter.

one needs firstly to define $\rightarrow_{\mathsf{d}}$ and $\wedge_{\mathsf{d}}$ in terms of neighbourhood models and, secondly, convert the resulting clauses into schematic labelled rules. Relational rules, obtained by converting $(\times)$, $(\star)$ and possibly other fundamental features of the semantics, shall be added as well.

Chapter 4

Modular labelled calculi for relevant logics

Layout of the chapter. In Section 4.1, relevant logics are introduced in terms of both, reduced Routley-Meyer models and axiomatic systems. Sections 4.2 and 4.3 present the rules of the labelled calculi for a variety of relevant logics and some related preliminary results, as well as a comparison with Kurokawa and Negri's approach [KN20]. Section 4.4 includes a proof of soundness, while Section 4.5 contains proofs of completeness. Finally, in Section 4.6, we will proceed towards the proof of CUT-admissibility.

4.1 Preliminaries

4.1.1 Semantics and axioms for relevant logic B

In this section, we will introduce Routley-Meyer relational semantics and an axiomatic system for relevant logic **B** (standing for *basic*). The former structures, employing a ternary relation between states, can be considered as generalizations of Kripke models for intuitionistic and modal logics. Notice that the interpretation of ternary relations is a controversial topic and there are different orientations in the literature.[1] Some possible readings are (notations adapted):

> "Well, to say that x determines $A \rightarrow B$ is to say that whenever we can conclude A on the basis of a piece of information y, we can conclude B on the basis of x and y jointly, that is, on the basis of $x \cup y$." [Urq72, p. 160]

> "Consider a natural English rendering of Kripke's binary R. xRy 'says' that 'world' y is possible relative to world x. An interesting ternary generalization is to read $xRyz$ to say that 'worlds' y and z are compossible (better, maybe, compatible) relative to x. (The reading is suggested by Dunn.)" [RM73, p. 200]

[1] A detailed overview can be found in [Bea+12].

> "$Rabc$ iff b and c are pairwise accessible from a, or, to take a more revealing modal analogue, iff a and b are compatible relative to c, or conversely iff c is compatible with a and b." [Rou+82, pp. 299–300]

> "[...] we may read $Rxyz$ as meaning that z contains all the information obtainable by pooling the information x and y. [Alternatively,] $Rxyz$ is [...] interpreted as saying that the information in y is carried to z by x." [Pri08, p. 207]

Let's turn to the formal details.

Definition 4.1.1. Let $\mathcal{L}$ be the language of **B**. We denote by At a set of atomic formulas $p, q, \dots$. The set of **B** formulas, denoted Form, is defined recursively for all A as follows:

$$A ::= p \mid {\sim}A \mid A \wedge A \mid A \vee A \mid A \rightarrow A$$

Definition 4.1.2. A *reduced Routley-Meyer frame* for relevant logic **B**, denoted $\mathcal{F}$, is a quadruple $\langle W, 0, *, R\rangle$, where W is a set of points, with 0 denoting its base element, * is a unary function $W \mapsto W$. Finally, $R \subseteq W^3$ and satisfies the following conditions:

$$a^{**} = a \tag{F1}$$
$$R0aa \tag{F2}$$
$$R0ab \wedge R0bc \implies R0ac \tag{F3}$$
$$R0da \wedge Rabc \implies Rdbc \tag{F4}$$
$$R0ab \implies R0b^*a^* \tag{F5}$$

Notice that relations of the form $R0ab$ and $R0ab \wedge R0ba$ can be abbreviated by writing $a \leq b$ and $a = b$, respectively. However, given that both symbols, $\leq$ and $=$, are precisely defined in terms of the ternary accessibility relation, we can employ only R to characterize relevant logics.

Definition 4.1.3. A *reduced Routley-Meyer model* for **B**, denoted $\mathcal{M}$, is a pair $\langle \mathcal{F}, v\rangle$, where $\mathcal{F}$ is a reduced Routley-Meyer frame and $v : \mathsf{At} \mapsto \wp(W)$ is a valuation function on atomic formulas, such that, if $R0ab$ and $a \in v(p)$, then $b \in v(p)$, for all $p \in \mathsf{At}$. The valuation is then extended to the whole language in the following way:

$$\mathcal{M}, a \Vdash p \text{ iff } a \in v(p) \tag{1}$$
$$\mathcal{M}, a \Vdash {\sim}A \text{ iff } \mathcal{M}, a^* \nVdash A \tag{2}$$
$$\mathcal{M}, a \Vdash A \wedge B \text{ iff } \mathcal{M}, a \Vdash A \text{ and } \mathcal{M}, a \Vdash B \tag{3}$$
$$\mathcal{M}, a \Vdash A \vee B \text{ iff } \mathcal{M}, a \Vdash A \text{ or } \mathcal{M}, a \Vdash B \tag{4}$$
$$\mathcal{M}, a \Vdash A \rightarrow B \text{ iff } \forall b, c \in W, \text{ if } Rabc \text{ and } \mathcal{M}, b \Vdash A, \text{ then } \mathcal{M}, c \Vdash B \tag{5}$$

Finally, we say that a formula A is *satisfied* in a model $\mathcal{M} = \langle \mathcal{F}, v\rangle$ iff $\mathcal{M}, 0 \Vdash A$ and that 'A *entails* B in $\mathcal{M}$' iff, for all $a \in W$, if $a \Vdash A$, then $a \Vdash B$. A formula A is *valid* in a frame $\mathcal{F} = \langle W, 0, *, R\rangle$ iff, for all valuations v, the formula A is satisfied in $\mathcal{M}$.

Observation 5. In the previous definitions we have introduced a so-called *reduced* model for relevant logics (see e.g., [Sla87; Gia92]). These models were introduced as alternative structures to what might be called *non reduced* models, see e.g., [RM73; Rou+82].[2] There are some main differences to consider. Let $\mathcal{F}'$ and $\mathcal{M}'$ be denoting non reduced frames and models, respectively. $\mathcal{F}'$ is the following structure $\langle W, 0, T, *, R\rangle$, where, 0 is taken to be a subset of W, rather than a singleton, and T is a distinct element $T \in 0$, called *designated situation*. The members of 0 are referred to as *regular situations*. A model $\mathcal{M}'$ is the structure $\langle \mathcal{F}', v\rangle$. Finally, satisfaction in a model is defined with respect to regular situations, i.e., A is *satisfied* in a model $\mathcal{M}'$ iff $\mathcal{M}', x \Vdash A$, for all $x \in 0$. Validity on $\mathcal{F}'$ is defined as before.

An important, standard lemma is that preservation of truth along the heredity ordering holds for arbitrary formulas:

Lemma 4.1.1 ([Rea88; Res00; DR02])**.** If $R0ab$ and $\mathcal{M}, a \Vdash A$, then $\mathcal{M}, b \Vdash A$.

Furthermore, we state a result showing the equivalence between the *satisfaction* of an implication in a model and the notion of *entailment* in that model. This results is often referred to as *verification lemma* (see [DR02]).

Lemma 4.1.2 ([Rea88; Res00; DR02])**.** A entails B in a given model $\mathcal{M}$ iff $A \rightarrow B$ is satisfied in that model, i.e., for all $a \in W$, $(\mathcal{M}, a \Vdash A \implies \mathcal{M}, a \Vdash B)$ iff $\mathcal{M}, 0 \Vdash A \rightarrow B$.

From the perspective of axiomatic systems, **B** is the least set of formulas containing all instances of the following axioms and closed under the following rules. (We employ $\Rrightarrow$ as a rule-forming operator, distinct from both, the meta-level symbol $\implies$ and the sequent arrow $\Rightarrow$.)

(A1) $A \rightarrow A$
(A2) $A_1 \wedge A_2 \rightarrow A_i$
(A3) $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C))$
(A4) $A_i \rightarrow (A_1 \vee A_2)$
(A5) $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)$
(A6) $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
(A7) $\sim\sim A \rightarrow A$

(R1) $A, A \rightarrow B \Rrightarrow B$
(R2) $A, B \Rrightarrow A \wedge B$
(R3) $A \rightarrow B \Rrightarrow (C \rightarrow A) \rightarrow (C \rightarrow B)$
(R4) $A \rightarrow B \Rrightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$
(R5) $A \rightarrow B \Rrightarrow \sim B \rightarrow \sim A$

[2] According to [Gia92, p. 442], "reduced models are technically and practically important for the practicing logician. They are simpler and hence easier to use".

4.1.2 Stronger relevant logics

In this subsection, we will present some Hilbert systems for some common stronger relevant logics, which can be obtained by the addition of axioms to the system for **B**. Likewise, frames for **B**, $\mathcal{F}_{\mathbf{B}}$, can be extended to capture stronger relevant logics by adding some further constraints on R. In what follows, we display a list of axioms and the frame conditions imposed on Routley-Meyer frames to validate them. Some of these conditions appeal to the standard definitions, $Rabcd ::= \exists x(Rabx \wedge Rxcd)$ and $Ra(bc)d ::= \exists x(Raxd \wedge Rbcx)$:

(A8)	$(A \to B) \to ({\sim}B \to {\sim}A)$	(F6)	$Rabc \implies Rac^*b^*$
(A9)	$(A \to B) \wedge (B \to C) \to (A \to C)$	(F7)	$Rabc \implies Ra(ab)c$
(A10)	$(A \to B) \to ((B \to C) \to (A \to C))$	(F8)	$Rabcd \implies Rb(ac)d$
(A11)	$(A \to B) \to ((C \to A) \to (C \to B))$	(F9)	$Rabcd \implies Ra(bc)d$
(A12)	$(A \to (A \to B)) \to (A \to B)$	(F10)	$Rabc \implies Rabbc$
(A13)	$(A \wedge (A \to B)) \to B$	(F11)	$Raaa$
(A14)	$(A \to {\sim}A) \to {\sim}A$	(F12)	Raa^*a
(A15)	$(A \to (B \to C)) \to (B \to (A \to C))$	(F13)	$Rabcd \implies Racbd$
(A16)	$A \to ((A \to B) \to B)$	(F14)	$Rabc \implies Rbac$
(A17)	$A \vee {\sim}A$	(F15)	$R00^*0$
(A18)	$((A \to A) \to B) \to B$	(F16)	$Ra0a$
(A19)	$A \to (A \to A)$	(F17)	$Rabc \implies (R0ac \vee R0bc)$

The following well known relevant logics can be obtained by combinations of the indicated axioms and frame conditions.

B = (A1) + ... + (A7) + (R1) + ... + (R5)	$\mathcal{F}_{\mathbf{B}}$ = (F1) + ... + (F5)
DW = **B** + (A8)	$\mathcal{F}_{\mathbf{DW}} = \mathcal{F}_{\mathbf{B}}$ + (F6)
DJ = **DW** + (A9)	$\mathcal{F}_{\mathbf{DJ}} = \mathcal{F}_{\mathbf{DW}}$ + (F7)
TW = **DJ** + (A10) + (A11)	$\mathcal{F}_{\mathbf{TW}} = \mathcal{F}_{\mathbf{DJ}}$ + (F8) + (F9)
T = **TW** + (A12) + (A13) + (A14) + (A17)	$\mathcal{F}_{\mathbf{T}} = \mathcal{F}_{\mathbf{TW}}$ + (F10) + (F11) + (F12) + (F15)
RW = **TW** + (A15) + (A16)	$\mathcal{F}_{\mathbf{RW}} = \mathcal{F}_{\mathbf{TW}}$ + (F13) + (F14)
R = **B** + (A8) + ... + (A18)	$\mathcal{F}_{\mathbf{R}} = \mathcal{F}_{\mathbf{B}}$ + (F6) + ... + (F16)
RM = **R** + (A19)	$\mathcal{F}_{\mathbf{RM}} = \mathcal{F}_{\mathbf{R}}$ + (F17)

Let $\mathbf{X} = \{\mathbf{B}, \mathbf{DW}, \mathbf{DJ}, \mathbf{TW}, \mathbf{T}, \mathbf{RW}, \mathbf{R}, \mathbf{RM}\}$.

Theorem 4.1.3 ([Rea88; Res00; DR02])**.** A formula A is a theorem of **X** if and only if A is valid in all Routley-Meyer frames, $\mathcal{F}_{\mathbf{X}}$.

Let us now proceed towards the construction of our intended labelled calculi.

4.2 Proof System

In this section, we shall define a family of modular calculi for relevant logics. First of all, we enrich our language with labels ($a, b, c, \ldots, x, y, z, \ldots$) denoting states in Routley-Meyer models and an expression to formalize the forcing relation. Formally:

Axioms For p atomic:

$$R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$$

(possibly, a^*, b^*)
Logical rules

$$\frac{\Gamma \Rightarrow \Delta, a^* : A}{a : {\sim}A, \Gamma \Rightarrow \Delta}\ L{\sim} \qquad \frac{a^* : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : {\sim}A}\ R{\sim}$$

$$\frac{a : A, a : B, \Gamma \Rightarrow \Delta}{a : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, a : A \quad \Gamma \Rightarrow \Delta, a : B}{\Gamma \Rightarrow \Delta, a : A \wedge B}\ R\wedge$$

$$\frac{a : A, \Gamma \Rightarrow \Delta \quad a : B, \Gamma \Rightarrow \Delta}{a : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, a : A, a : B}{\Gamma \Rightarrow \Delta, a : A \vee B}\ R\vee$$

$$\frac{Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \quad Rabc, a : A \to B, c : B, \Gamma \Rightarrow \Delta}{Rabc, a : A \to B, \Gamma \Rightarrow \Delta}\ L\to$$

$$(b, c \text{ fresh})\ \frac{Rabc, b : A, \Gamma \Rightarrow \Delta, c : B}{\Gamma \Rightarrow \Delta, a : A \to B}\ R\to$$

Relational rules for R

$$\frac{R0a^{**}a, R0aa^{**}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R1}$$

$$\frac{R0aa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R2} \qquad \frac{R0ac, R0ab, R0bc, \Gamma \Rightarrow \Delta}{R0ab, R0bc, \Gamma \Rightarrow \Delta}\ \text{R3}$$

$$\frac{Rdbc, R0da, Rabc, \Gamma \Rightarrow \Delta}{R0da, Rabc, \Gamma \Rightarrow \Delta}\ \text{R4} \qquad \frac{R0b^*a^*, R0ab, \Gamma \Rightarrow \Delta}{R0ab, \Gamma \Rightarrow \Delta}\ \text{R5}$$

Figure 4.1: **G3rB**

Definition 4.2.1. Let W be a set of labels, including a distinguished label denoted 0, and $\mathcal{L}$ be the language of **B**. To express the forcing relation $a \Vdash A$ via sequents we use the notation $a : A$, for $A \in \mathsf{Form}$ and $a \in W$. The set of well-formed formulas consists of (1) labelled formulas $a : A$ and (2) relational atoms $Rabc$, for all $A \in \mathsf{Form}$ and $a, b, c \in W$. Finally, given two multisets Γ, Δ of labelled formulas and relational atoms, a *labelled sequent* is an object of the following form: $\Gamma \Rightarrow \Delta$.

Furthermore, the labelled rules of our sequent system are subject to the following closure condition. Consider a rule $\mathcal{R}$ of the following form:

$$\frac{A, B_1, \ldots, B_n, B_{n1}, B_{n1}, \Gamma \Rightarrow \Delta}{B_1, \ldots, B_n, \Gamma \Rightarrow \Delta}\ \mathcal{R}$$

$$\frac{Rac^*b^*, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{R6} \qquad (x \text{ fresh})\ \frac{Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{R7}$$

$$(y \text{ fresh})\ \frac{Rbyd, Racy, Rabx, Rxcd, \Gamma \Rightarrow \Delta}{Rabx, Rxcd, \Gamma \Rightarrow \Delta}\text{R8}$$

$$(y \text{ fresh})\ \frac{Rayd, Rbcy, Rabx, Rxcd, \Gamma \Rightarrow \Delta}{Rabx, Rxcd, \Gamma \Rightarrow \Delta}\text{R9}$$

$$(x \text{ fresh})\ \frac{Rabx, Rxbc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{R10}$$

$$\frac{Raaa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{R11} \qquad \frac{Raa^*a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{R12}$$

$$(y \text{ fresh})\ \frac{Racy, Rybd, Rabx, Rxcd, \Gamma \Rightarrow \Delta}{Rabx, Rxcd, \Gamma \Rightarrow \Delta}\text{R13} \qquad \frac{Rbac, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{R14}$$

$$\frac{R00^*0, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{R15} \qquad \frac{Ra0a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\text{R16}$$

$$\frac{R0ac, Rabc, \Gamma \Rightarrow \Delta \qquad R0bc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{R17}$$

Figure 4.2: Further mathematical rules for R

Applying the *closure condition* on $\mathcal{R}$ means to substitute the multiple occurrences B_{n1}, B_{n1} with a single one to obtain a rule $\mathcal{R}^*$ of the following shape:

$$\frac{A, B_1, \ldots, B_n, B_{n1}, \Gamma \Rightarrow \Delta}{B_1, \ldots, B_n, \Gamma \Rightarrow \Delta}\mathcal{R}^*$$

We remark that the rules of **G3rB** are defined by analysing the semantic conditions of Definition 4.1.3 of the corresponding operators. More precisely, the sequent system is obtained by formulating the rules according to the methodology introduced for modal and intermediate logics in [Neg05; Neg07]. We remark that axiomatic sequents are stated in their weakening-absorbing version, while the premises of $L \rightarrow$ are contraction-absorbing. Importantly, $R \rightarrow$ has the eigenvariable condition, that is, each root-first application of the rule requires the introduction of *fresh* (i.e., not previously used) labels.

In addition to the newly introduced rules for $\rightarrow$, there are also rules for R constructed through the method of conversion of frame conditions into sequent calculus rules. More precisely, we first have observed that all frame conditions are formulated either as *universal axioms* or *geometric implications* and, then, by following the methodology described in [Neg05] (but previously also in [Neg03; Neg14; NvP19]), we have transformed them into well-constructed sequent-style rules. Universal axioms are first turned into conjunctive normal form, namely, $P_1 \wedge \cdots \wedge P_i \to Q_1 \vee \cdots \vee Q_j$ and, then, into suitably formulated rules. Geometric implications, instead, are formulas of the following shape $\forall \overline{z}(A \to B)$, where A and B are *geometric formulas*, i.e., they do not contain neither $\forall$ nor $\to$. As before, we first turn them into conjunctive normal form, namely, $\forall \overline{x}(P_1 \wedge \cdots \wedge P_i \to \exists y_1 M_1 \vee \cdots \vee \exists y_j M_j)$ and, then, convert them into the corresponding rule-schemes. Notice that according to this strategy, we are allowed to obtain modular extensions of **G3rB** (Figure 4.1) by transforming further frame conditions (see list on p. 76) into sequent-style rules (Figure 4.2). Such extensions can be characterised as follows:

G3rDW = **G3rB** + R6	**G3rDJ** = **G3rDW** + R7
G3rTW = **G3rDJ** + R8 + R9	**G3rT** = **G3rTW** + R10 + R11 + R12+ +R15
G3rRW = **G3rTW** + R14 + R13	**G3rR** = **G3rRW** + **G3rT** + R16
G3rRM = **G3rR** + R17	

Observation 6. Kurokawa and Negri [KN20] developed a family of labelled calculi for a wide range of relevant logics by using *non reduced* Routley-Meyer models as starting point. We recall that in these latter (i) 0 is taken to be a subset of W, rather than a singleton, and (ii) there is an element $T \in 0$. Although we followed the same methodology to obtain our intended systems, there are some substantial differences.

1. The notion of validity is not defined at the base element T, but it refers to all regular situations (see Observation 5 and [KN20, §3.2]) and this is reflected at the calculus level as follows (see [KN20, §6]). For all $x \in 0$, if $x \Vdash A$, then $0x \Rightarrow x : A$.
2. The formulations of the rules for relevant implication involves an auxiliary unary operator, i.e., the indexed modality $\Box_a$. The index a gives a ternary relation, denoted $bR_a c$, which is taken as an assignment of a binary relation to an index, rather than expressing a compossibility relation between situations. However, as the authors themselves remark, this "choice is not mandatory, i.e., the ternary relation for implication could be directly handled without using the indexed modality. But via the indexed modality we can obtain a uniformity with [...] works on conditional logics" [KN20, §1], i.e., with labelled systems proposed for conditional logics, for example, in [NS16; GNS19].
3. The semantic condition for indexed modalities is:

$$b \Vdash \Box_a A \text{ iff } \forall c(bR_a c \implies a \Vdash A) \tag{$\Box_a$}$$

It is, in turn, used to formulate the clause for $\to$ as follows:

$$a \Vdash A \to B \text{ iff } \forall b(b \Vdash A \implies b \Vdash \Box_a B) \tag{$\to'$}$$

Accordingly, the rules for both operators, $\Box_a$ and $\to$, are of the following shape:

$$\frac{a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad b : \Box_a B, a : A \to B, \Gamma \Rightarrow \Delta}{a : A \to B, \Gamma \Rightarrow \Delta} L'{\to} \qquad (b \text{ fresh}) \frac{b : A, \Gamma \Rightarrow \Delta, b : \Box_a B}{\Gamma \Rightarrow \Delta, a : A \to B} R'{\to}$$

$$\frac{c : A, bR_a c, b : \Box_a A, \Gamma \Rightarrow \Delta, b : \Box_a B}{bR_a c, b : \Box_a A, \Gamma \Rightarrow \Delta} L\Box_a \qquad (c \text{ fresh}) \frac{bR_a c, \Gamma \Rightarrow \Delta, c : A}{\Gamma \Rightarrow \Delta, b : \Box_a A} R\Box_a$$

4. Axiomatic sequents are only of the form $a : p, \Gamma \Rightarrow \Delta, a : p$ and, in order to preserve the heredity property at the calculus level, the following rule is included:

$$\frac{b : p, a \leq b, a : p, \Gamma \Rightarrow \Delta}{a \leq b, a : p, \Gamma \Rightarrow \Delta} \text{AtHer}$$

   Since this rule is a form of contraction, it is preferable to have a system in which this rule is height-preserving admissible (proved in Lemma 4.3.3). This is the reason why we have heredity incorporated in axioms. Moreover, in the presence of Proposition 4.3.2 (below) the generalized version of AtHer can be derived using (admissible) cut and contraction (see Proposition 4.3.4).

Although the non reduced Routley-Meyer semantics allows for a characterization of a wider range of relevant logics, the labelled systems constructed out of it can be shown to be semantically complete only indirectly (at least for the moment), and this is mainly due to the definition of validity on regular situations (elements of 0), see [KN20, §6]. Nonetheless, Kurokawa and Negri observe that the lack of a direct proof seems to be far from being an insurmountable problem and argue that such "a proof of completeness by proof-search must be possible, since labelled sequent calculi are in general suitable for proof-search and invertible rules preserve countermodels" [KN20, §8]. Instead, notice that if validity is defined w.r.t. the distinct element $0 \in W$ (considered as a singleton), we can lay out a direct completeness proof without encountering the difficulties connected to the presence of regular situations. Indeed, in [NvP15, p. 276], it was noticed:

> "The labelled approach allows for a fine distinction between various notions of logical consequence that can be adopted: *actualistic* logical consequence is logical consequence relative to the actual world, whereas *universal* (or *strong*) consequence is relative to an arbitrary world."

By keeping this distinction in mind, we will provide an *actualistic* completeness proof, i.e., we will show that if a formula A is valid at the actual world 0, then the sequent $\Rightarrow 0 : A$ is derivable (see Section 4.5).
Before going ahead, let us summarize the central results contained in the following sections.

1. A is a theorem of **X**.
2. A is provable in **G3rX** + cut, and cut has the following shape:

$$\frac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{cut}$$

3. A is provable in **G3rX**.
4. A is valid in every Routley-Meyer frame for **X**.

The equivalence between 1 and 4 is stated in Theorem 4.1.3. 1 $\Longrightarrow$ 2 and 4 $\Longrightarrow$ 3 are both proved in Section 4.5 (Theorems 4.5.2 and 4.5.3); 2 $\Longrightarrow$ 3 is proved in Section 4.6 (Theorem 4.6.4), 3 $\Longrightarrow$ 4 is proved in Section 4.4 (Theorem 4.4.1).

4.3 Preliminary results

In this section, we show some preliminary results. Let us start by introducing the notions of *weight* of formulas and *height* of derivations in the standard way. (Let $\mathbf{X} = \{\mathbf{B}, \mathbf{DW}, \mathbf{DJ}, \mathbf{TW}, \mathbf{T}, \mathbf{RW}, \mathbf{R}, \mathbf{RM}\}$.)

Definition 4.3.1. Let $\mathcal{A}$ be any labelled formula of the form $a : A$. We denote by $l(\mathcal{A})$ the label of a formula $\mathcal{A}$, and by $p(\mathcal{A})$ the pure part of the formula, that is, the part of the formula without the label. The *weight* (or *complexity*) of a labelled formula is defined as a lexicographically ordered pair: $\langle \mathsf{w}(p(\mathcal{A})), \mathsf{w}(l(\mathcal{A}))\rangle$, where:

1. for all state labels $a \in W$, $\mathsf{w}(a) = 1$;
2. for all $p \in \mathsf{At}$, $\mathsf{w}(p) = 1$;
3. $\mathsf{w}(\sim A) = \mathsf{w}(A) + 1$;
4. $\mathsf{w}(A \wedge B) = \mathsf{w}(A) + \mathsf{w}(B) + 1$, for $\wedge \in \{\wedge, \vee, \rightarrow\}$.

The height of derivations is measured as stated in Definition 3.3.2.

Definition 4.3.2. A rule $\mathcal{R}$ is *height-preserving admissible* just in case: if there is a derivation of the premise(s) of $\mathcal{R}$, then there is a derivation of the conclusion of $\mathcal{R}$ that contains no application of $\mathcal{R}$ (with the height at most n, where n is the maximal height of the derivation of the premise(s)).

Definition 4.3.3. We define substitution as follows:

- $Rabc(d/e) \equiv Rabc$, if $e \neq a$, $e \neq b$ and $e \neq c$.
- $Rabc(d/a) \equiv Rdbc$, if $a \neq b$ and $a \neq c$.
- $Rabc(d/b) \equiv Radc$, if $b \neq a$ and $b \neq c$.
- $Rabc(d/c) \equiv Rabd$, if $c \neq a$ and $c \neq b$.
- $Raac(d/a) \equiv Rddc$, if $a = b$ and $a \neq c$.
- $Rabb(d/b) \equiv Radd$, if $b = c$ and $b \neq a$.
- $Rcbc(d/c) \equiv Rdbd$, if $c = a$ and $c \neq b$.
- $Raaa(d/a) \equiv Rddd$, if $a = b$ and $a = c$.
- $a : A(d/b) \equiv a : A$, if $b \neq a$.
- $a : A(d/a) \equiv d : A$.

Next we extend this definition to multisets. Similar proofs for labelled calculi for logics characterised by ternary relations are included, e.g., in [NS16; HGT18; KN20].

Lemma 4.3.1. Let the variable e stand for either a, b or c. If $\mathbf{G3rX} \vdash^n \Gamma \Rightarrow \Delta$ and, provided d is free for e in Γ, Δ, then $\mathbf{G3rX} \vdash^n \Gamma(d/e) \Rightarrow \Delta(d/e)$ (allowing $*$-variables to be substituted to variables as well).

Proof. Let $n = 0$. If $\Gamma \Rightarrow \Delta$ is an axiom and (d/e) is not a vacuous substitution, then the substitution $\Gamma(d/e) \Rightarrow \Delta(d/e)$ is also an axiom. Let $n > 0$. If we are considering a propositional rule, we apply the inductive hypothesis to the premise(s) of the rule, and then the rule again. For example, let $\Gamma = e : {\sim}A, \Gamma'$ and $e = a, b, c$:

$$\frac{\vdash^{n} \Gamma' \Rightarrow \Delta, e^* : A}{\vdash^{n1} e : {\sim}A, \Gamma' \Rightarrow \Delta}\ L{\sim}$$

In this case, in order to apply $L{\sim}$, we substitute d^*/e^* by the inductive hypothesis, and get the following derivation of the same height:

$$\frac{\vdash^{n} \Gamma'(d^*/e^*) \Rightarrow \Delta(d^*/e^*), d^* : A}{\vdash^{n1} d : {\sim}A, \Gamma'(d^*/e^*) \Rightarrow \Delta(d^*/e^*)}\ L{\sim}$$

We proceed similarly if the last rule is $L \rightarrow$ (without the variable condition). Finally, let's consider the only rule with the eigenvariable condition, namely $R \rightarrow$.
(1) If the substitution is vacuous ($e \neq a, b, c$), then there's nothing to do.
(2) Assume the substitution d/e is not vacuous and d is not a fresh variable. We have to consider the case where d is substituted for a. Let $\Delta = a : A \rightarrow B, \Delta'$:

$$\frac{\vdash^{n} Razx, z : A, \Gamma \Rightarrow \Delta', x : B}{\vdash^{n1} \Gamma \Rightarrow \Delta', a : A \rightarrow B}\ R \rightarrow$$

By the application of the inductive hypothesis (d/a) we obtain the following application of $R \rightarrow$ with the same derivation height:

$$\frac{\vdash^{n} Rdzx, z : A, \Gamma(d/a) \Rightarrow \Delta'(d/a), x : B}{\vdash^{n1} \Gamma(d/a) \Rightarrow \Delta'(d/a), d : A \rightarrow B}\ R \rightarrow$$

(3) The substitution is non-vacuous, and d is an eigenvariable. So, our derivation ends as follows:

$$\frac{\vdash^{n} Radc, d : A, \Gamma \Rightarrow \Delta', c : B}{\vdash^{n1} \Gamma \Rightarrow \Delta', a : A \rightarrow B}\ R \rightarrow \qquad \frac{\vdash^{n} Rabd, b : A, \Gamma \Rightarrow \Delta', d : B}{\vdash^{n1} \Gamma \Rightarrow \Delta', a : A \rightarrow B}\ R \rightarrow$$

First, we rename the fresh variables d, c and b, d with z, x and x, z, respectively. By the variable condition the substitution does not affect Γ, Δ'. Indeed, we get the following premise of height n:

$$Raxz, x : A, \Gamma \Rightarrow \Delta', z : B \quad \text{and} \quad Razx, z : A, \Gamma \Rightarrow \Delta', x : B$$

. So, by applying inductive hypothesis, we substitute the labels d/b and d/c, respectively, to conclude:

$$\dfrac{\vdash^{n} Razx, z : A, \Gamma(d/b) \Rightarrow \Delta'(d/b), x : B}{\vdash^{n1} \Gamma(d/b) \Rightarrow \Delta'(d/b), a : A \to B}\, R\to \qquad \dfrac{\vdash^{n} Raxz, x : A, \Gamma(d/c) \Rightarrow \Delta'(d/c), z : B}{\vdash^{n1} \Gamma(d/c) \Rightarrow \Delta'(d/c), a : A \to B}\, R\to$$

Analogous results follow also for relational rules. Some of them subject to the eigenvariable condition and, as usual, more care is needed. Roughly, the cases for such relational rules follow the pattern of case 3 above: to avoid clashes of variables, we apply height-preserving substitution before the inductive hypothesis and conclude the argument by finally applying the rule. □

As in the case of other labelled calculi for intermediate logics (e.g., [DN12; MMS21]), the heredity property of the forcing relation (Lemma 4.1.1) can be expressed by means of formal derivations in the calculus:

Proposition 4.3.2. Sequents of the following form are derivable in **G3rX**: $R0ab, a : A, \Gamma \Rightarrow \Delta, b : A$.

Proof. By induction on A. Let $A = {\sim}B$ and consider the following derivation:

$$\dfrac{\dfrac{\dfrac{R0b^*a^*, R0ab, b^* : B, \Gamma \Rightarrow \Delta, a^* : B}{R0ab, b^* : B, \Gamma \Rightarrow \Delta, a^* : B}\, \text{R5}}{R0ab, b^* : B, a : {\sim}B, \Gamma \Rightarrow \Delta}\, L{\sim}}{R0ab, a : {\sim}B, \Gamma \Rightarrow \Delta, b : {\sim}B}\, R{\sim}$$

where the premises are derivable by inductive hypothesis.
If $A = B \to C$, then we obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Racd, \mathcal{S}, c : B, \Gamma \Rightarrow \Delta, d : C, c : B}{Racd, \mathcal{S}, c : B, \Gamma \Rightarrow \Delta, d : C, c : B}\, \text{R2} \qquad \dfrac{R0dd, Racd, \mathcal{S}, d : C, c : B, \Gamma \Rightarrow \Delta, d : C}{Racd, \mathcal{S}, d : C, c : B, \Gamma \Rightarrow \Delta, d : C}\, \text{R2}}{Racd, R0ab, Rbcd, a : B \to C, c : B, \Gamma \Rightarrow \Delta, d : C}\, L\to}{R0ab, Rbcd, a : B \to C, c : B, \Gamma \Rightarrow \Delta, d : C}\, \text{R4}}{R0ab, a : B \to C, \Gamma \Rightarrow \Delta, b : B \to C}\, R\to\ (c, d \text{ fresh})$$

where $\mathcal{S}$ abbreviates $R0ab, Rbcd, a : B \to C$. The cases for A being $B \wedge C$ or $B \vee C$ are straightforward. □

Lemma 4.3.3. The following rules:

$$\dfrac{b : p, R0ab, a : p, \Gamma \Rightarrow \Delta}{R0ab, a : p, \Gamma \Rightarrow \Delta}\, \textsc{AtHer-l} \qquad \dfrac{R0ab, \Gamma \Rightarrow \Delta, b : p, a : p}{R0ab, \Gamma \Rightarrow \Delta, b : p}\, \textsc{AtHer-r}$$

are height-preserving admissible.

Proof. We display the details for AtHer-l, but the argument is the same for AtHer-r. By induction on the height of δ, we prove that for any proof of $b : p, R0ab, a : p, \Gamma \Rightarrow \Delta$, there exists a proof of $R0ab, a : p, \Gamma \Rightarrow \Delta$ of the same (or smaller) height. The base cases are obtained as follows:

$$\vdash^{n} b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p \overset{i.h.}{\rightsquigarrow} \vdash^{n} R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$$

$$\dfrac{\vdash^{n} R0aa^{**}, R0a^{**}a, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\, \text{R1} \overset{i.h.}{\rightsquigarrow} \dfrac{\vdash^{n} R0aa^{**}, R0a^{**}a, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\, \text{R1}$$

$$\dfrac{\vdash^n R0aa, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R2} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n R0aa, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R2}$$

$$\dfrac{\vdash^n R0ac, R0bc, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} R0bc, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R3} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n R0ac, R0bc, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} R0bc, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R3}$$

$$\dfrac{\vdash^n Racd, Rbcd, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} Rbcd, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R4} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n Racd, Rbcd, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} Rbcd, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R4}$$

$$\dfrac{\vdash^n R0b^*a^*, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R5} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n R0b^*a^*, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n1} R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\text{R5}$$

The remaining cases are dealt with analogously. The inductive step is completed by permutation of the rules. □

Proposition 4.3.4. The following rules:

$$\dfrac{b : A, R0ab, a : A, \Gamma \Rightarrow \Delta}{R0ab, a : A, \Gamma \Rightarrow \Delta}\text{GenHer-l} \qquad \dfrac{R0ab, \Gamma \Rightarrow \Delta, b : A, a : A}{R0ab, \Gamma \Rightarrow \Delta, b : A}\text{GenHer-r}$$

corresponding to the heredity rules for compound formulas, are admissible.

Proof. GenHer-l can be derived as follows:

$$\dfrac{R0ab, a : A, \Gamma \Rightarrow \Delta, b : A \qquad b : A, R0ab, a : A, \Gamma \Rightarrow \Delta}{R0ab, a : A, \Gamma \Rightarrow \Delta}\text{cut+lc+rc+lc}_L$$

For GenHer-r we have the following derivation:

$$\dfrac{R0ab, \Gamma \Rightarrow \Delta, b : A, a : A \qquad R0ab, a : A, \Gamma \Rightarrow \Delta, b : A}{R0ab, \Gamma \Rightarrow \Delta, b : A}\text{cut+lc+ rc+lc}_L$$

where the leftmost (resp., rightmost) premise is derivable by Proposition 4.3.2, while the applications of contraction and cut are admissible by Lemma 4.6.3 and Theorem 4.6.4.[3] □

4.4 Soundness

This section is devoted to the proof of the soundness theorem for our systems ($3 \implies 4$, p. 80). We will show that the rules of each labelled calculus **G3rX** preserve validity over Routley-Meyer frames obeying the conditions appropriate for each relevant logic **X**. In order to do that, we start by extending semantic notions to sequents as follows:

[3]We observe that this proposition can be proved in the same way as we proved admissibility of AtHer-l and AtHer-r, i.e., by induction on the height of the derivation. This, in fact would provide us with a stronger result, namely that GenHer-l and GenHer-r are height-preserving admissible in **G3rX**. However, here we omit the details of such a proof as we do not need this result throughout the chapter.

Definition 4.4.1. Let $\mathcal{M} = \langle W, 0, *, R_{\mathcal{M}}, v\rangle$ be a model and let $\mathcal{S}$ be the sequent $\Gamma \Rightarrow \Delta$. We define a $\mathcal{S}$*-interpretation in* $\mathcal{M}$ is a mapping $[\![\cdot]\!]$ from the labels in $\mathcal{S}$ to the set W of states in $\mathcal{M}$, such that (i) $0 = [\![0]\!]$ and (ii) if $Rabc$ is in Γ, then $R_{\mathcal{M}}[\![a]\!][\![b]\!][\![c]\!]$. Now we can define:

$\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{S}$ iff if for all $a : A \in \Gamma$, we have $\mathcal{M}, [\![a]\!] \Vdash A$, then there exists $b : B \in \Delta$, such that $\mathcal{M}, [\![b]\!] \Vdash B$.

Definition 4.4.2. A sequent $\mathcal{S}$ is *satisfied* in $\mathcal{M} = \langle W, 0, *, R, v\rangle$ if for all $\mathcal{S}$-interpretations $[\![\cdot]\!]$ we have $\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{S}$. A sequent $\mathcal{S}$ is valid in a frame $\mathcal{F} = \langle W, 0, *, R\rangle$, if for all valuations v, the sequent $\mathcal{S}$ is satisfied in $\mathcal{M} = \langle W, 0, *, R, v\rangle$.

Finally, we can prove the soundness theorem:

Theorem 4.4.1. If a sequent $\mathcal{S}$ is provable in **G3rX**, then it is valid in every Routley-Meyer frame for **X**.

Proof. We proceed by induction on the height of the derivation of $\mathcal{S}$. We show that for each rule $\mathcal{R}$ of the form $\mathcal{P}_1, \ldots, \mathcal{P}_n/C$, if the premises $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are valid in all Routley-Meyer frames, then so is C. It follows from a case analysis on $\mathcal{R}$:

Ax. By way of contradiction, assume that $R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$ is not valid in all Routley-Meyer frames. This means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$, i.e., $R_{\mathcal{M}}[\![0]\!][\![a]\!][\![b]\!]$ and $\mathcal{M}, a \Vdash p$, but $\mathcal{M}, b \nVdash p$. However, this is not possible given heredity (lemma 4.1.1).

$L{\sim}$. By way of contradiction, assume that $\Gamma \Rightarrow \Delta, a^* : A$ is valid in all Routley-Meyer frames, but $a : {\sim}A, \Gamma \Rightarrow \Delta$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash a : {\sim}A, \Gamma \Rightarrow \Delta$, i.e., $\mathcal{M}, a \Vdash {\sim}A$, but $\mathcal{M}, d \nVdash C$ for all $d : C \in \Delta$. However, by the forcing clause (2), we also have $\mathcal{M}, a^* \nVdash A$. Consequently, $\mathcal{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, a^* : A$. Contradiction.

$R{\sim}$. By way of contradiction, assume that $a^* : A, \Gamma \Rightarrow \Delta$ is valid in all Routley-Meyer frames, but $\Gamma \Rightarrow \Delta, a : {\sim}A$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, a : {\sim}A$, i.e., $\mathcal{M}, d \Vdash C$, for all $d : C \in \Gamma$ but $\mathcal{M}, a \nVdash {\sim}A$. However, by the forcing clause (2), we also have $\mathcal{M}, a^* \Vdash A$. Then, $\mathcal{M}, [\![\cdot]\!] \nVdash a^* : A, \Gamma \Rightarrow \Delta$. Contradiction.

$L \rightarrow$. By way of contradiction, assume that $Rabc, \Gamma \Rightarrow \Delta, b : A$ and $Rabc, c : B, \Gamma \Rightarrow \Delta$ are valid in all Routley-Meyer frames, but $Rabc, a : A \rightarrow B, \Gamma \Rightarrow \Delta$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash Rabc, a : A \rightarrow B, \Gamma \Rightarrow \Delta$, i.e., $R_{\mathcal{M}}[\![a]\!][\![b]\!][\![c]\!]$ and $\mathcal{M}, a \Vdash A \rightarrow B$, but $\mathcal{M}, d \nVdash C$ for all $d : C \in \Delta$. However, by the forcing clause (5), we also have $\mathcal{M}, b \nVdash A$ or $\mathcal{M}, c \Vdash B$. Consequently, $\mathcal{M}, [\![\cdot]\!] \nVdash Rabc, \Gamma \Rightarrow \Delta, b : A$ or $\mathcal{M}, [\![\cdot]\!] \nVdash Rabc, c : B, \Gamma \Rightarrow \Delta$. Contradiction.

$R \rightarrow$. By way of contradiction, assume that $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$ is valid in all Routley-Meyer frames, but $\Gamma \Rightarrow \Delta, a : A \rightarrow B$ is not, where $b, c \notin \Gamma, \Delta$. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, a : A \rightarrow B$. In particular, we know that there are worlds b' and c' such that $R_{\mathcal{M}}[\![a]\!]b'c'$ and $\mathcal{M}, b' \Vdash A$, but $\mathcal{M}, c' \nVdash B$. Now we define an extension $[\![\cdot]\!]'$ of $[\![\cdot]\!]$ such that $[\![b]\!]' = b'$, $[\![c]\!]' = c'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$. Then, $\mathcal{M}, [\![\cdot]\!]' \nVdash Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$. Contradiction.

The other cases are similar and simpler. In particular, note that the cases for the mathematical rules are trivial, as all Routley-Meyer frames have to obey the corresponding conditions. □

4.5 Completeness

In this section, we will show the completeness of **G3rB**, and its extensions, by deriving the axioms of the corresponding logics ($1 \implies 2$, p. 80).
Let $\mathbf{X} = \{\mathbf{B}, \mathbf{DW}, \mathbf{DJ}, \mathbf{TW}, \mathbf{T}, \mathbf{RW}, \mathbf{R}, \mathbf{RM}\}$.
Before turning to the proof the theorem, we show a syntactic version of Lemma 4.1.2 within our labelled calculi:

Lemma 4.5.1. **G3rX** + CUT $\vdash a : A \Rightarrow a : B$ iff **G3rX** + CUT $\vdash \Rightarrow 0 : A \rightarrow B$.

Proof. ($\Longrightarrow$)

$$\dfrac{\dfrac{a : A \Rightarrow a : B \qquad R0ab, a : B \Rightarrow b : B}{R0ab, a : A \Rightarrow b : B}\,\text{CUT}}{\Rightarrow 0 : A \rightarrow B}\,R\rightarrow$$

($\Longleftarrow$)

$$\dfrac{\dfrac{\Rightarrow 0 : A \rightarrow B \qquad \dfrac{R0aa, 0 : A \rightarrow B, a : A \Rightarrow a : A, a : B \qquad R0aa, 0 : A \rightarrow B, a : A, a : B \Rightarrow a : B}{R0aa, 0 : A \rightarrow B, a : A \Rightarrow a : B}\,L\rightarrow}{R0aa, a : A \Rightarrow a : B}\,\text{CUT}}{a : A \Rightarrow a : B}\,\text{R2}$$

where, in both derivations, the rightmost premise(s) is (are) derivable by Proposition 4.3.2. □

Theorem 4.5.2. If a formula A is provable in an axiomatic system **X**, then the sequent $\Rightarrow 0 : A$ is derivable in the corresponding labelled system **G3rX** + CUT.

Proof. The proof proceeds by deriving root-first the axioms of each relevant logic **X** in the corresponding labelled system **G3rX** + CUT. As the derivations occupy much space, we display them in Appendix B1. □

Alternatively, one might be interested in proving a theorem of semantic completeness, that is, for every sequent $\mathcal{S}$, the proof search either terminates in a proof or fails, and the failed proof tree is used to obtain a countermodel for $\mathcal{S}$. Intuitively, to see whether A is derivable, we check if it is valid at the actual world $0 \in W$, i.e., $0 \Vdash A$. This, indeed, will amount to have the sequent

$\Rightarrow 0 : A$ in our calculus. As said above, this correspond to reflect, at the calculus level, the *actualistic* notion of validity employed in reduced Routley-Meyer models. Finally, notice that the countermodel construction argument, allows us to show completeness directly (although non-constructively, as the proof relies on König's lemma), for any labelled sequent and not only specifically for formulas.

Theorem 4.5.3. Let $\Gamma \Rightarrow \Delta$ be a sequent of **G3rX**. Then either the sequent is derivable in **G3rX** or it has a countermodel with the frame properties peculiar for **X**.

Proof. We follow the pattern of the completeness proof in [Neg09; NvP11]. We proceed with the construction of a derivation tree for $\Gamma \Rightarrow \Delta$ by applying the rules of **G3rX** root-first (see Appendix B2). If the reduction tree is finite, i.e., all leaves are axiomatic sequents, we have a proof in **G3rX**. Assume that the derivation tree is infinite. By König's lemma, it has an infinite branch that is used to build the needed counterexample. Suppose that $\Gamma \Rightarrow \Delta \equiv \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_i \Rightarrow \Delta_i \ldots$ is one of such branches. Consider the sets $\mathbf{\Gamma} \equiv \bigcup \Gamma_i$ and $\mathbf{\Delta} \equiv \bigcup \Delta_i$, for $i \geq 0$. We now construct a countermodel, i.e. a model that makes all labelled formulas and relational atoms in $\mathbf{\Gamma}$ true and all labelled formulas in $\mathbf{\Delta}$ false. Let $\mathcal{F}_{\mathbf{X}}$ be a frame, whose elements are all the labels occurring in $\mathbf{\Gamma}$. $\mathcal{F}_{\mathbf{X}}$ is defined as follows:

- for all $a : p$ in $\mathbf{\Gamma}$ it holds that $a \Vdash p$ in $\mathcal{F}_{\mathbf{X}}$.
- for all $Rabc$ in $\mathbf{\Gamma}$ it holds that $R_{\mathcal{M}}abc$ in $\mathcal{F}_{\mathbf{X}}$.
- for all $a : p$ in $\mathbf{\Delta}$ it holds that $a \nVdash p$ in $\mathcal{F}_{\mathbf{X}}$.

It can then be shown that A is forced in the model at 0 if $0 : A$ is in $\mathbf{\Gamma}$ and A is not forced at 0 if $0 : A$ is in $\mathbf{\Delta}$. We will end up with a countermodel to the endsequent.

1. If p is atomic, the claim holds by definition of the model.
2. If $0 : {\sim}A$ is in $\mathbf{\Gamma}$, then $0^* : A$ is in $\mathbf{\Delta}$. By the inductive hypothesis $0^* \nVdash A$, i.e., $0 \Vdash {\sim}A$.
3. If $0 : {\sim}A$ is in $\mathbf{\Delta}$, then $0^* : A$ is in $\mathbf{\Gamma}$. By the inductive hypothesis $0^* \Vdash A$, i.e., $0 \nVdash {\sim}A$.
4. If $0 : A \wedge B$ is in $\mathbf{\Gamma}$, then there exists i such that $0 : A \wedge B$ appears first in Γ_i, and, therefore, for some $j \geq 0$, we have $0 : A$ and $0 : B$ in Γ_{i+j}. By the inductive hypothesis $0 \Vdash A$ and $0 \Vdash B$ and, consequently, $0 \Vdash A \wedge B$. (The case for $0 : A \vee B$ in $\mathbf{\Delta}$ is analogous.)
5. If $0 : A \wedge B$ is in $\mathbf{\Delta}$, then either $0 : A$ or $0 : B$ in $\mathbf{\Delta}$. By the inductive hypothesis either $0 \nVdash A$ or $0 \nVdash B$ and, therefore, $0 \nVdash A \wedge B$. (The case for $0 : A \vee B$ in $\mathbf{\Gamma}$ is analogous.)
6. If $0 : A \rightarrow B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms $R0ab$ that occur in $\mathbf{\Gamma}$. If there's no relational atom, the accessibility condition is vacuously satisfied and, therefore, $0 \Vdash A \rightarrow B$ is in the model. For any occurrence of $R0ab$ in $\mathbf{\Gamma}$, by construction of the tree $a : A$ is in $\mathbf{\Delta}$ or $b : A$ is in $\mathbf{\Gamma}$.

By the inductive hypothesis $a \nVdash A$ or $b \Vdash B$, and since $R_{\mathcal{M}}0ab$, we obtain $0 \Vdash A \to B$ in the model.

7. If $0 : A \to B$ is in $\mathbf{\Delta}$, at the successive step of the reduction tree we find that $R0ab$ and $a : A$ in $\mathbf{\Gamma}$, whereas $b : B$ is in $\mathbf{\Delta}$. By the inductive hypothesis we obtain $R_{\mathcal{M}}0ab$ and $a \Vdash A$ but $b \nVdash B$, that is, $0 \nVdash A \to B$ in the model.

□

This result directly implies the implication $4 \implies 3$ stated on p. 80.

Corollary 4.5.3.1. If a sequent $\Gamma \Rightarrow \Delta$ is valid in every Routley-Meyer frame for **X**, then it is derivable in the system **G3rX**.

4.6 Proof analysis and Cut-admissibility

In this section we prove the CUT-admissibility theorem for our labelled sequent calculi. The general proof presented here is similar to the proof for labelled systems for modal and intermediate logics (see, e.g., [Neg05; NvP11; DN12; HGT18; KN20; MMS21]). More precisely, we proceed with the proofs of weakening and contraction admissibility. In conclusion, we show the central theorem of the section, i.e., CUT-admissibility. As there are many cases to be analysed in these proofs, we only outline the important parts here.

Lemma 4.6.1. The rules of weakening:

$$\frac{\Gamma \Rightarrow \Delta}{d : C, \Gamma \Rightarrow \Delta}\text{ LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, d : C}\text{ RW} \qquad \frac{\Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\text{ LW}_L$$

are height-preserving admissible in **G3rX**.

Proof. By induction on the height of the derivation. (1) For $n = 0$, the case is trivial. For $n > 0$, we simultaneously display the transformed derivations for LW and RW. (Analogous results hold for LW_L)
(2) For rules without variable condition, the lower sequent of the transformed derivation is the same as the lower one of the original derivation, obtained by applying several times weakening. This is also the case for $L \to$.
(3) Consider the rules with the variable condition, e.g., $R \to$. The derivations end as follows:

$$\frac{\vdash^n Radc, d : A, \Gamma \Rightarrow \Delta', c : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \to B}\, R \to \qquad \frac{\vdash^n Rabd, b : A, \Gamma \Rightarrow \Delta', d : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \to B}\, R \to$$

To avoid clashes of variables we apply height-preserving substitution (x/d) to obtain:

$$Raxc, x : A, \Gamma(x/d) \Rightarrow \Delta'(x/d), c : B \text{ and } Rabx, b : A, \Gamma(x/d) \Rightarrow \Delta'(x/d), x : B$$

Finally, by applying the inductive hypothesis (on the left and on the right) to the premise and, finally, also the rule, we obtain the requested derivations:

$$\frac{\vdash^{n} d : C, Raxc, x : A, \Gamma \Rightarrow \Delta', c : B}{\vdash^{n+1} d : C, \Gamma \Rightarrow \Delta', a : A \to B}\, R\to \qquad \frac{\vdash^{n} Raxc, x : A, \Gamma \Rightarrow \Delta', c : B, d : C}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \to B, d : C}\, R\to$$

and

$$\frac{\vdash^{n} d : C, Rabx, b : A, \Gamma \Rightarrow \Delta', x : B}{\vdash^{n+1} d : C, \Gamma \Rightarrow \Delta', a : A \to B}\, R\to \qquad \frac{\vdash^{n} Rabx, b : A, \Gamma \Rightarrow \Delta', x : B, d : C}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \to B, d : C}\, R\to$$

where, in all cases, the lower derivations are the result of applying weakening (on the left and on the right) to the premises of the derivations displayed above. If we consider relational rules without variable condition, the proof follows straightforwardly by applications of the inductive hypothesis.
For relational rules with eigenvariable conditions, we always are in need to consider possible clashes of variables. As an example, suppose that the rule applied is R7:

$$\frac{\vdash^{n} Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta}\, \text{R7}$$

If $d \neq x$, that is, the variable condition is not violated, then desired derivations follow by the inductive hypothesis and an application of the rule:

$$\frac{\vdash^{n} d : C, Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} d : C, Rabc, \Gamma \Rightarrow \Delta}\, \text{R7} \qquad \frac{\vdash^{n} Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta, d : C}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta, d : C}\, \text{R7}$$

If the fresh variable condition is violated, we substitute the clashing variable with a fresh one, apply the inductive hypothesis and then the rule. If the application of the rule looks like:

$$\frac{\vdash^{n} Rabd, Radc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta}\, \text{R7}$$

we substitute d with a fresh one, say y, to obtain the following premise

$$\vdash^{n} Raby, Rayc, Rabc, \Gamma(y/d) \Rightarrow \Delta(y/d)$$

By applying the inductive hypothesis and the rule, we obtain the desired derivations:

$$\frac{\vdash^{n} d : C, Raby, Rayc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} d : C, Rabc, \Gamma \Rightarrow \Delta}\, \text{R7} \qquad \frac{\vdash^{n} Raby, Rayc, Rabc, \Gamma \Rightarrow \Delta, d : C}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta, d : C}\, \text{R7}$$

where, as before, the lower derivations are the results of applying weakening (on the left and on the right) to the premise of the rule displayed above. □

Definition 4.6.1. A rule $\mathcal{R}$ is *height-preserving invertible* just in case: if there is a derivation of the conclusion of $\mathcal{R}$, then there is a dedrivation of premise(s) of $\mathcal{R}$ (with the height at most n, where n is the maximal height of the derivation of the conclusion).

Lemma 4.6.2. All rules of **G3rX** are height-preserving invertible.

Proof. For each rule $\mathcal{R}$, we have to show that if there is a derivation δ of the conclusion, then there is a derivation δ' of the premise(s), of the same height. For $L\sim$, $R\sim$, $L\vee$, $R\vee$, $R\wedge$, $L\wedge$ and $L\rightarrow$ we use a standard induction on the height of δ. For $R\rightarrow$ as well, but we need to be sure that in the transformed derivation we make use of a fresh label by applying the substitution lemma inside δ', if needed. The same procedures apply to all relational rules (R1-R17).
As an interesting example, we show height-preserving invertibility of $R\rightarrow$. It is proved by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta, a : A \rightarrow B$. We distinguish three main cases. (1) If $n = 0$, $\Gamma \Rightarrow \Delta, a : A \rightarrow B$ is an axiom, and then also $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$ is an axiom. Let $n > 0$. (2) If $\vdash^{n+1} \Gamma \Rightarrow \Delta, a : A \rightarrow B$ is concluded by any rule $\mathcal{R}$ other than $R\rightarrow$, we apply the inductive hypothesis to the premise(s) $\Gamma' \Rightarrow \Delta', a : A \rightarrow B$ ($\Gamma'' \Rightarrow \Delta'', a : A \rightarrow B$) to obtain derivation(s) of height n of $Rabc, b : A, \Gamma' \Rightarrow \Delta', c : B$ ($Rabc, b : A, \Gamma'' \Rightarrow \Delta'', c : B$). By applying $\mathcal{R}$ we obtain a derivation of height $n+1$ of $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$, as desired. (3) If $\vdash^{n+1} \Gamma \Rightarrow \Delta, a : A \rightarrow B$ is concluded by $R\rightarrow$, then $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$ is the requested conclusion of height n, possibly with different eigenvariables, but the desired ones can be obtained by height-preserving substitutions (Lemma 4.3.1). As an example for relational rules, we only deal with R7, i.e., a rule with eigenvariable. (1) If $n = 0$, $Rabc, \Gamma \Rightarrow \Delta$ is an axiom, and then also $Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta$ is an axiom. If $\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta$ is concluded by any rule $\mathcal{R}$ other than R7, we apply the inductive hypothesis to the premise(s) $Rabc, \Gamma' \Rightarrow \Delta'$ ($Rabc, \Gamma'' \Rightarrow \Delta''$) to obtain derivation(s) of height n of $Rabx, Raxc, Rabc, \Gamma' \Rightarrow \Delta'$ ($Rabx, Raxc, Rabc, \Gamma'' \Rightarrow \Delta''$). By applying $\mathcal{R}$ we obtain a derivation of height $n+1$ of $Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta$, as desired. (3) If $\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta$ is concluded by R7, then $Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta$ is the requested conclusion of height n (possibly by applying Lemma 4.3.1). ◻

Lemma 4.6.3. The rules of contraction:

$$\frac{a : C, a : C, \Gamma \Rightarrow \Delta}{a : C, \Gamma \Rightarrow \Delta}\,\text{LC} \qquad \frac{\Gamma \Rightarrow \Delta, a : C, a : C}{\Gamma \Rightarrow \Delta, a : C}\,\text{RC} \qquad \frac{Rabc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\,\text{LC}_L$$

are height-preserving admissible in **G3rX**.

Proof. By induction on the height of derivation. As usual, if $n = 0$, then the premise is an axiomatic sequent and so also the contracted sequent is an axiomatic one. If $n > 0$, we consider the last rule applied to the premise of contraction. If the contraction formula is not principal in the premise of some $\mathcal{R}$, then both occurrences are found in the premises of the rule and they have a smaller derivation height. By applying the induction hypothesis, we contract them and apply $\mathcal{R}$ to obtain a derivation of the conclusion with the same derivation height. If the contraction formula is principal, we distinguish three cases: (1) $\mathcal{R}$ is a rule where active formulas are proper subformulas of the principal formula (all rules for $\sim, \wedge, \vee$); (2) $\mathcal{R}$ is a rule where both, labels $Rabc$ and proper subformulas of the principal formula, are active formulas ($R\rightarrow$); (3) $\mathcal{R}$ is a rule in which the principal formula is repeated also in the premises of the rule ($L\rightarrow$).
(1) In the cases for $\sim, \wedge, \vee$ the contraction is reduced to contraction on formulas of smaller complexity (as in the cases for modal and intermediate logics, see,

e.g., [Neg05; Neg07; DN12]).
(2) We consider a rule where the principal formula and relational atoms are both active, for instance:

$$\dfrac{\vdash^{n} Rabc, b : A, \Gamma \Rightarrow \Delta, a : A \to B, c : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta, a : A \to B, a : A \to B} \; R\to$$

By height-preserving invertibility (Lemma 4.6.2) applied to the premise, we obtain the following derivation:

$$\dfrac{\dfrac{\vdash^{n} Rabc, b : A, Rabc, b : A, \Gamma \Rightarrow \Delta, c : B, c : B}{\vdash^{n} Rabc, b : A, \Gamma \Rightarrow \Delta, c : B} \; i.h.}{\vdash^{n+1} \Gamma \Rightarrow \Delta, a : A \to B} \; R\to$$

as requested. Notice that if both contraction formulas are principal in $R \to$, we apply the closure condition.
(3) Finally, we consider a rule in which only the labelled formula is principal, namely $L \to$:

$$\dfrac{\vdash^{n} Rabc, a : A \to B, a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad \vdash^{n} Rabc, c : B, a : A \to B, a : A \to B, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, a : A \to B, a : A \to B, \Gamma \Rightarrow \Delta} \; L\to$$

Again, by applying the inductive hypothesis to the premises, we obtain the desired derivation:

$$\dfrac{\vdash^{n} Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad \vdash^{n} Rabc, c : B, a : A \to B, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, a : A \to B, \Gamma \Rightarrow \Delta} \; L\to$$

□

Finally, we can prove that CUT is an admissible rule. This theorem directly entails the implication 2 $\implies$ 3 stated on p. 80:

Theorem 4.6.4. The rule of CUT:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \; \text{CUT}$$

is admissible in **G3rX**.

Proof. The proof is by a lexicographic induction on the complexity of the CUT-formula $a : A$ and the sum of the heights $h(\delta_1) + h(\delta_2)$. We perform a case analysis on the last rule used in the derivation above the CUT and whether it applies to the CUT-formula or not. We show that each application of CUT can either be eliminated, or be replaced by one or more applications of CUT of smaller complexity. The proof proceeds similarly to the CUT-elimination proofs for several logics, e.g., [Neg05; NvP11; HGT18; MMS21]. Intuitively, we eliminate the left- and topmost CUT first, and proceed by repeating the procedure until we reach a CUT-free derivation. We start by showing that CUT can be eliminated if one of the CUT premises is an axiom (**case 1**). Then we show that the CUT-height can be reduced in all cases in which the CUT-formula is not principal in at least one of the CUT-premises (**case 2**). Finally, we show that if the CUT-formula is

principal in both cut-premises, then the cut is reduced to one or more cuts on less complex formulas or on shorter derivations (**case 3**). The complete case analysis is performed in Appendix B3.
Here, we present two interesting cases where the cut-formula A is principal in both premises. We start by considering a derivation where the last rules applied to obtain the cut-premises are $R\sim$ and $L\sim$, respectively. Let $A = \sim B$:

$$\dfrac{\dfrac{a^* : B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : \sim B}\, R\sim \qquad \dfrac{\Gamma' \Rightarrow \Delta', a^* : B}{a : \sim B, \Gamma' \Rightarrow \Delta'}\, L\sim}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}$$

It is transformed into the following derivation:

$$\dfrac{\Gamma' \Rightarrow \Delta', a^* : B \qquad a^* : B, \Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}$$

where cut is applied on a formula of smaller complexity.
Assume that the premises of cut are derived by $R \to$ and $L \to$, respectively. Let $A = B \to C$:

$$\dfrac{(b, c \text{ fresh})\,\dfrac{Rabc, b : B, \Gamma \Rightarrow \Delta, c : C}{\Gamma \Rightarrow \Delta, a : B \to C}\, R\to \qquad \dfrac{Rade, a : B \to C, \Gamma' \Rightarrow \Delta', d : B \qquad Rade, e : C, a : B \to C, \Gamma' \Rightarrow \Delta'}{Rade, a : B \to C, \Gamma' \Rightarrow \Delta'}\, L\to}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}$$

It is transformed into the following derivation:

$$\text{(Lemma 4.6.3)}\;\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_1\\ Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', e : C\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2\\ Rade, e : C, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'\end{matrix}}{Rade, Rade, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\,\text{cut}}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{lc+ rc+ lc}_L$$

where the conclusion of δ_1 is derived by:

$$\text{(Lemma 4.6.3)}\;\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : B \to C \qquad Rade, a : B \to C, \Gamma' \Rightarrow \Delta', d : B}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' d : B}\,\text{cut} \qquad \text{(Lemma 4.3.1)}\;\dfrac{\text{(Lemma 4.3.1)}\;\dfrac{Rabc, b : B, \Gamma \Rightarrow \Delta, c : C}{Rabe, b : B, \Gamma \Rightarrow \Delta, e : C}\,\text{sub}(e/c)}{Rade, d : B, \Gamma \Rightarrow \Delta, e : C}\,\text{sub}(d/b)}{Rade, Rade, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta', e : C}\,\text{cut}}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', e : C}\,\text{lc+ rc+lc}_L$$

while the conclusion of δ_2 is derived by:

$$\dfrac{\Gamma \Rightarrow \Delta, a : B \to C \qquad Rade, e : C, a : B \to C, \Gamma' \Rightarrow \Delta'}{Rade, e : C, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\text{cut}$$

Notice that the two topmost cuts, those on $a : B \to C$, are derived with a shorter derivation height, while the other two are applied on formulas of smaller complexity, i.e., $d : B$ and $e : C$. □

4.7 Conclusions

In this chapter, we have presented labelled sequent calculi for a wide range of relevant logics by reflecting at the syntactic level semantic informations taken from reduced Routley-Meyer models, and have proved soundness and (syntactic and semantic) completeness. Least but not last, we have shown height-preserving invertibility of the rules, height-preserving admissibility of structural rules, and CUT-admissibility.
To conclude, we would like to point out some further topics of research, directly connected to the work developed so far:

- Along with labelled calculi, many generalizations of sequent systems have been proposed over the years. This flourishing of systems has also paved the way to investigations concerning the relations between them. In this context, an interesting task for future work is represented by establishing correspondences between the calculi presented in this work with other characterizations obtained by application of different proof-theoretic structures, e.g., hypersequents and display sequents.

- Notice that relevant logics face some troubles when it comes to establish decidability results and, indeed, many of them are undecidable. Given the subtleties that such a discussion might involve, we leave (un)decidability issues out from this investigation and we limit ourselves to some observations. One of the main consequences that can be drawn from CUT-elimination proofs is a fundamental trait of sequent systems, namely the so-called *subformula property*. This ensures that all formulas in a derivation are subformulas of formulas in the endsequent. Unfortunately, labelled sequent calculi, given the presence of geometrical rules in which relational atoms disappear from premise to conclusion, do not have a *full* subformula property. Nonetheless, by following the considerations expressed in [Neg05], we observe that all of our calculi enjoy a *weak* version of the property, namely: *All formulas in a derivation are either subformulas of formulas in the endsequent or formulas of the form $Rabc$.* This property alone, however, is not enough to prove syntactic decidability. Firstly, in order to provide such a proof, one needs to find a bound on the number of eigenvariables (fresh labels) in a derivation of a given sequent. Secondly, since the repetition of the principal formula in the premises of $L \rightarrow$ is another source of potentially non-terminating proof search, there's also the need of finding a bound on applications of $L \rightarrow$. This amounts to binding the number of applications of $L \rightarrow$ with principal formula $a : A \rightarrow B$ to the number of relational atoms of the form $Rabc$ that appear on the left-hand side of sequents in the derivation. This number, in turn, will be bounded by the number of existing relational atoms of that form and relational atoms that can be introduced by applications of $R \rightarrow$ with principal formula $a : A \rightarrow B$.

- Throughout our chapter, we have considered labelled rules for the following connectives $\sim$, $\wedge$, $\vee$ and $\rightarrow$. However, occasionally (see, amongst others, [Rou+82, Ch. 5]), relevant logics are presented also with further connectives, such as for example, 'fusion' (also known as 'intensional conjunction') and 'fission' (also known as 'intensional disjunction'). Some other relevantists would also welcome the addition of the so-called 'Ackerman truth constant' (often denote as **t**). Nonetheless, given our intentions in this chapter, we have preferred to omit the consideration of wider sets of connectives and have decided to leave this topic for further research. We only notice that all connectives mentioned above can be, in line of principle, treated according to the methodology we have adopted so far.

Chapter 5

A proof theoretic investigation of actuality in intuitionistic logic

Layout of the chapter. After having introduced both, the context and the aim of this chapter, in Section 5.1.1, we'll focus on the formal details of $\mathbf{IPC}^{@}$ following the presentation proposed by S. Niki and H. Omori. In Sections 5.2 and 5.3, we'll define an analytic hypersequent calculus, called $\mathbf{HI}^{@}$, and provide proofs of soundness and completeness, respectively. Finally, Section 5.4 contains a proof of the CUT-elimination theorem for $\mathbf{HI}^{@}$ and of some related consequences.

5.1 Introduction

S. Niki and H. Omori [NO20] – motivated by both, a philosophical project whose roots can be found in the work of M. Dummett (e.g., [Dum75]) and by some recent papers by M. De (e.g., [De13]) – took up the challenge of extending intuitionism from mathematical discourse to empirical discourse. In their joint contribution, Niki and Omori proposed a system of intuitionistic propositional logic modally expanded via the addition of a so-called *actuality* operator, denoted '$@A$'.[1] Their logic, namely $\mathbf{IPC}^{@}$, is firstly introduced in terms of possible words semantics and axiomatic system. In addition, the authors introduced another proof system, namely a sequent calculus, called LGJ@, by modifying the characteristic rules of the sequent calculus of Titani [Tit97] and Aoyama [Aoy98] for the so-called global intuitionistic logic (GIPC). Unfortunately, as noticed in the final part of [NO20] (but previously also in [Cia05]), LGJ@ is not CUT-free. Therefore, the authors proposed an open question, that is, whether there is a CUT-free *hypersequent* calculus for $\mathbf{IPC}^{@}$ [NO20, p. 477]. This chapter aims at solving this problem.

5.1.1 Preliminaries

After defining the language, we first introduce the semantics, and then present the proof systems of [NO20].

[1] We point out that this logic was independently considered also by L. Humberstone in [Hum06].

Definition 5.1.1. The language of **IPC**$^{@}$, (denoted $\mathcal{L}^{@}_{\perp}$) includes the following set of connectives $\{\perp, \wedge, \vee, \rightarrow, @\}$. Let At be a set of atoms, $p, q, \ldots$, and Form be the set of formulas, $A, B, \ldots$, these latter being defined inductively as follows:

$$A ::= p \mid \perp \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid @A$$

Definition 5.1.2. A *frame* for **IPC**$^{@}$ is a triple $\mathcal{F} = \langle W, g, \leq \rangle$, where W is a non-empty set of states (or points), g is the least element of W (called the *base state*), $\leq$ is a pre-order (reflexive and transitive) on W.

Definition 5.1.3. A *model* for **IPC**$^{@}$ is a structure of the form $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a frame for **IPC**$^{@}$ and $V : \mathsf{At} \mapsto \wp(W)$, such that, for each $p \in \mathsf{At}$ and all $w_1, w_2 \in W$, if $w_1 \in V(p)$ and $w_1 \leq w_2$, then $w_2 \in V(p)$. We define the relation $\vDash$ recursively, as follows:

$$\mathcal{M}, w \vDash p \text{ iff } w \in V(p); \tag{5.1}$$
$$\mathcal{M}, w \nvDash \perp; \tag{5.2}$$
$$\mathcal{M}, w \vDash @A \text{ iff } \mathcal{M}, g \vDash A; \tag{5.3}$$
$$\mathcal{M}, w \vDash A \wedge B \text{ iff } \mathcal{M}, w \vDash A \text{ and } \mathcal{M}, w \vDash B; \tag{5.4}$$
$$\mathcal{M}, w \vDash A \vee B \text{ iff } \mathcal{M}, w \vDash A \text{ or } \mathcal{M}, w \vDash B; \tag{5.5}$$
$$\mathcal{M}, w \vDash A \rightarrow B \text{ iff } \forall x \in W : w \leq x \text{ and } \mathcal{M}, x \vDash A \text{ imply } \mathcal{M}, x \vDash B. \tag{5.6}$$

Finally, we say that a formula A is *valid* in a model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ iff $\mathcal{M}, g \vDash A$. A formula A is *satisfied* in a frame $\mathcal{F} = \langle W, g, \leq \rangle$ iff, for all valuations V, the formula A is valid in $\mathcal{M}$.

From the perspective of Hilbert systems, **IPC**$^{@}$ is the least set of formulas containing all instances of the following axioms and closed under the following rules:

(Ax1)	$\perp \rightarrow A$
(Ax2)	$A \rightarrow (B \rightarrow A)$
(Ax3)	$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(Ax4)	$(C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow (A \wedge B)))$
(Ax5)	$(A_1 \wedge A_2) \rightarrow A_i$
(Ax6)	$A_i \rightarrow (A_1 \vee A_2)$
(Ax7)	$(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
(Ax8)	$@(A \rightarrow B) \rightarrow (@A \rightarrow @B)$
(Ax9)	$@A \rightarrow A$
(Ax10)	$@A \rightarrow @@A$
(Ax11)	$@A \vee (@A \rightarrow B)$
(Ax12)	$@(A \vee B) \rightarrow (@A \vee @B)$

$$\frac{A \quad A \rightarrow B}{B}\,\mathsf{MP} \qquad\qquad \frac{A}{@A}\,\mathsf{RN}$$

Furthermore, Niki and Omori [NO20, p. 470] introduced also a sequent calculus, called **LGJ@**, based on a modification of the sequent calculus proposed by Titani [Tit97] and Aoyama [Aoy98] for the so-called global intuitionistic logic (abbreviated as **GIPC**), where, instead of @, one has the *globalization* operator (denoted $\Box$).[2] Roughly, in **LGJ@** a sequent is an object of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are lists (or sequences) of formulas. The specific rules of **LGJ@** are the following ones:

$$\frac{A, \Gamma \Rightarrow \bar{\Delta}, B}{\Gamma \Rightarrow \bar{\Delta}, A \rightarrow B}\ R\rightarrow' \qquad \frac{A, \Gamma \Rightarrow \Delta}{@A, \Gamma \Rightarrow \Delta}\ L@' \qquad \frac{\bar{\Gamma} \Rightarrow \bar{\Delta}, A}{\bar{\Gamma} \Rightarrow \bar{\Delta}, @A}\ R@'$$

where $\bar{\Gamma}, \bar{\Delta}$ represent finite lists of @-closed formulas, i.e., of formulas built from $\bot$ and formulas of the form $@A$, by the connectives $\wedge, \vee, \rightarrow$. However, as remarked above, **LGJ@** is not CUT-free ([NO20; Cia05]).

5.2 Proof System

In this section, we introduce an analytic calculus for **IPC**$^{@}$ in terms of hypersequents, i.e., a simple generalization of Gentzen's sequents (see, among others, [Avr91a; Avr96; AL11] and [Ind21, pp. 209–230]). Let's start by fixing some conventions. $\Gamma, \Delta, \Sigma, \Pi, \Phi, \ldots$ denote multisets[3] of formulas and $\uplus$ is used to indicate multiset union; the notation $[\,]$ is used for multisets as follows: $\Gamma^0 = [\,]$, $\Gamma^{n+1} = \Gamma \uplus \Gamma^n$. Finally, $i, j, k, l, m, n, \lambda, \mu, \ldots$ (possibly subscripted) stand for natural numbers.

Definition 5.2.1. A *hypersequent* is a structure of the form:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

where each $\Gamma_i \Rightarrow \Delta_i$ (for $i = 1, \ldots, n$) is a sequent. Γ_i, Δ_i are multisets of formulas and we call them *internal contexts*. 'G','H', ..., denote side hypersequents and we refer to them as *external contexts*.
If Δ_i contains at most one formula then the sequent is single-succedent, written $\Gamma_i \Rightarrow A$. A hypersequent $S_1 \mid \ldots \mid S_n$ is single-succedent, if all sequents $S_1, \ldots, S_n$ are single-succedent.

HI$^{@}$ is constructed as follows:

Initial sequents:

$$\frac{}{G \mid A \Rightarrow A}\ \textsc{id} \qquad \frac{}{G \mid \bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Logical rules:

$$\frac{G \mid \Gamma \Rightarrow \Delta, A \qquad G \mid \Gamma \Rightarrow \Delta, B}{G \mid \Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge \qquad {\scriptstyle (i=1,2)}\ \frac{G \mid A_i, \Gamma \Rightarrow \Delta}{G \mid A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\ L\wedge_i$$

[2] To be precise, Niki and Omori considered only the rules for the propositional and modal fragments of **GIPC**. Notice, however, that Titani and Aoyama originally formalized **GIPC** by relying on a set of connectives including also quantifiers.

[3] We recall that a multiset is like a set except that the multiplicity of the elements counts, and it is like a list (or sequence) except that the order of the elements doesn't count.

$$\frac{G \mid \Gamma \Rightarrow \Delta, A_i}{G \mid \Gamma \Rightarrow \Delta, A_1 \vee A_2}\ R\vee_i \ (i=1,2) \qquad \frac{G \mid A, \Gamma \Rightarrow \Delta \quad G \mid B, \Gamma \Rightarrow \Delta}{G \mid A \vee B, \Gamma \Rightarrow \Delta}\ L\vee$$

$$\frac{G \mid A, \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \rightarrow B}\ R\rightarrow \qquad \frac{G \mid \Gamma_1 \Rightarrow \Delta_1, A \quad G \mid B, \Gamma_2 \Rightarrow \Delta_2}{G \mid A \rightarrow B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ L\rightarrow$$

Modal rules:

$$\frac{G \mid @\Gamma \Rightarrow A}{G \mid @\Gamma \Rightarrow @A}\ R@ \qquad \frac{G \mid A, \Gamma \Rightarrow \Delta}{G \mid @A, \Gamma \Rightarrow \Delta}\ L@$$

Internal stuctural rules:

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta, A}\ \textsc{iw, r} \qquad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ \textsc{iw, l}$$

$$\frac{G \mid \Gamma \Rightarrow \Delta, A, A}{G \mid \Gamma \Rightarrow \Delta, A}\ \textsc{ic, r} \qquad \frac{G \mid A, A, \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ \textsc{ic, l}$$

External stuctural rules:

$$\frac{G}{G \mid \Gamma \Rightarrow \Delta}\ \textsc{ew} \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ \textsc{ec}$$

Modal structural rule:

$$\frac{G \mid @\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \mid @\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ \textsc{split}@$$

Cut rule:

$$\frac{G \mid \Gamma_1 \Rightarrow \Delta_1, A \quad H \mid A, \Gamma_2 \Rightarrow \Delta_2}{G \mid H \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ \textsc{cut}$$

Remark 6. We have employed a Maehara-style formulation of **IPC$^@$** (see, [Mae54]). Indeed, not all rules are single-succedent (as, for example, in Gentzen's **LJ**), but we impose such a restriction only on $R \rightarrow$ and $R@$. The notation $@\Gamma$ denotes any set of @-formulas, namely, formulas prefixed by @ and the relationship between @-formulas and the @-closed formulas of **LGJ@** can be characterized as follows: if one replaces $@\Gamma$ with $\bar{\Gamma}$ in the rules $R@$ and SPLIT@ of **HI$^@$**, the resulting rules can be shown equivalent to the original ones. Indeed, all @-formulas are @-closed formulas and $@\bar{\Gamma} \dashv\vdash \bar{\Gamma}$.

We remark that SPLIT@ is needed to ensure completeness. More specifically, it guarantees that @-formulas behave as boolean formulas, as established by (Ax11), namely $@A \vee (@A \rightarrow B)$.

Finally, some standard notions are as follows:

Definition 5.2.2. A *derivation* d is defined as a finite tree of hypersequents such that (i) leaves are instances of ID or $L\bot$ and (ii) all hypersequents, except the lowest one, are upper hypersequents of a certain rule instance. The *length* of d, formally $|d|$, is the (maximal number of applications of inference rules)+1 occurring in in all branches of d. We write $\vdash^{\mathbf{HI}^@} H$ if there exists a derivation of H in **HI$^@$**. The *complexity* of a formula A, denoted $|A|$, is the number of occurrences of its connectives.

5.3 Soundness & Completeness

In this section, we prove soundness and completeness for **HI**$^{@}$. We start with the former and show that, whenever the premises of a rule-application are valid in **IPC**$^{@}$ models, then so is the conclusion (see, e.g., [AL11; Ind15]). To get the intended proof we first extend semantic notions to sequents and hypersequents as follows:

Definition 5.3.1. Let $\mathcal{M}$ be an **IPC**$^{@}$ model:

1. $\mathcal{M}, w \vDash \Gamma \Rightarrow \Delta$ iff $\mathcal{M}, w \nvDash A$, for some $A \in \Gamma$, or $\mathcal{M}, w \vDash A$, for some $A \in \Delta$.
2. $\mathcal{M}$ is a model of a hypersequent H iff there exists a component $S \in H$ such that $\mathcal{M}, g \vDash S$.

Notice that semantic validity of a hypersequent is now defined in terms of truth preservation at g of its components. Finally:

Definition 5.3.2. Let $\mathcal{M}$ be an **IPC**$^{@}$ model and H be a hypersequent. $\vdash^{\mathcal{M}} H$ iff every $\mathcal{M}$ is a model of H.

Theorem 5.3.1 (Soundness)**.** If $\vdash^{\mathbf{HI}^{@}} H$, then $\vdash^{\mathcal{M}} H$.

Proof. Let **r** be a rule of **HI**$^{@}$ and $\mathcal{M} = \langle W, g, \leq, V \rangle$ be an **IPC**$^{@}$ model. We show that each $\mathcal{M}$, which is a model of the premise(s) of **r**, is also a model of its conclusion. It follows from a case analysis on **r**. We start by considering the rules for initial sequents of **HI**$^{@}$:

ID Let $H = G \mid A \Rightarrow A$ and assume for contradiction that $\mathcal{M}$ is not a model of H. Thus, we have that for each $S \in H$ and $S \in G$, $\mathcal{M}, g \nvDash S$. In particular, it follows that $\mathcal{M}, g \vDash A$ and $\mathcal{M}, g \nvDash A$. Contradiction.

$L\bot$ Let $H = G \mid \bot, \Gamma \Rightarrow \Delta$ and assume for contradiction that $\mathcal{M}$ is not a model of H. Thus, we have that for each $S \in H$ and $S \in G$, $\mathcal{M}, g \nvDash S$. In particular, we obtain $\mathcal{M}, g \vDash \bot$ and $\mathcal{M}, g \vDash B$, for all $B \in \Gamma$, but $\mathcal{M}, g \nvDash B$, for all $B \in \Delta$. However, $\bot$ cannot be forced at any point in any **IPC**$^{@}$ model.

We outline the cases for $R@$ and $L@$:

$R@$ Let $H = G \mid @\Gamma \Rightarrow @A$ and suppose that it is derived from the premise $G \mid @\Gamma \Rightarrow A$ by an application of $R@$. Assume for contradiction that $\mathcal{M}$ is not a model of H. So, for all $S \in H$ and $S \in G$, we have that $\mathcal{M}, g \nvDash S$. We obtain in particular that $\mathcal{M}, g \nvDash @\Gamma \Rightarrow @A$, i.e., $\mathcal{M}, g \vDash @B$, for all $@B \in @\Gamma$, but $\mathcal{M}, g \nvDash @A$. The latter means $\mathcal{M}, g \nvDash A$. It follows that $\mathcal{M}, g \nvDash @\Gamma \Rightarrow A$, which implies that $\mathcal{M}$ is not a model of $G \mid @\Gamma \Rightarrow A$.

$L@$ Let $H = G \mid @A, \Gamma \Rightarrow \Delta$ and suppose that it is derived from the premise $G \mid A, \Gamma \Rightarrow \Delta$ by an application of $L@$. Assume for contradiction that $\mathcal{M}$ is not a model of H. So, for all $S \in H$ and $S \in G$, we have that $\mathcal{M}, g \nvDash S$. We obtain in particular that $\mathcal{M}, g \nvDash @A, \Gamma \Rightarrow \Delta$, i.e., $\mathcal{M}, g \vDash @A$ and $\mathcal{M}, g \vDash B$, for all $B \in \Gamma$, but $\mathcal{M}, g \nvDash B$, for all $B \in \Delta$. From the former we get $\mathcal{M}, g \vDash A$. It follows that $\mathcal{M}, g \nvDash A, \Gamma \Rightarrow \Delta$. But this implies that $\mathcal{M}$ is not a model of $G \mid A, \Gamma \Rightarrow \Delta$.

Finally, we display the case for SPLIT@:

SPLIT@ Let $H = G \mid @\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2$ and suppose that it was derived from $G \mid @\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ by applying SPLIT@. Assume that $\mathcal{M}$ is not a model of H, i.e., for all $S \in H$ and $S \in G$, we obtain that $\mathcal{M}, g \not\models S$. In particular, we have $\mathcal{M}, g \not\models @\Gamma_1 \Rightarrow \Delta_1$ and $\mathcal{M}, g \not\models \Gamma_2 \Rightarrow \Delta_2$. That is, $\mathcal{M}, g \models @B$, for all $@B \in @\Gamma_1$, but $\mathcal{M}, g \not\models B$, for all $B \in \Delta_1$, and $\mathcal{M}, g \models B$, for all $B \in \Gamma_2$, but $\mathcal{M}, g \not\models B$, for all $B \in \Delta_2$. It follows that $\mathcal{M}, g \not\models @\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$, but this implies that $\mathcal{M}$ is not a model of $G \mid @\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

The cases for the other logical and structural rules are dealt with analogously. □

Now, we prove that **HI$^@$** is complete with respect to the axiomatization for **IPC$^@$** stated in Section 5.1.1.

Theorem 5.3.2 (Completeness)**.** If $\vdash^{\mathbf{IPC}^@} A$, then $\vdash^{\mathbf{HI}^@} A$.

Proof. We show that all axioms and rules of **IPC$^@$** are derivable in **HI$^@$**. Since axioms and the rule MP of IPC are known to be derivable, it suffices to prove that we can provide derivations of the characteristic axioms (Ax8–Ax12) and the rule RN of **IPC$^@$**.

$\vdash^{\mathbf{HI}^@} \Rightarrow @(A \to B) \to (@A \to @B)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{ID} \qquad \dfrac{}{B \Rightarrow B}\,\text{ID}}{A \to B, A \Rightarrow B}\,L\to}{@(A \to B), @A \Rightarrow B}\,L@\ (\times 2)}{@(A \to B), @A \Rightarrow @B}\,R@}{@(A \to B) \Rightarrow @A \to @B}\,R\to}{\Rightarrow @(A \to B) \to (@A \to @B)}\,R\to$$

$\vdash^{\mathbf{HI}^@} \Rightarrow @A \to A$

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{ID}}{@A \Rightarrow A}\,L@}{\Rightarrow @A \to A}\,R\to$$

$\vdash^{\mathbf{HI}^@} \Rightarrow @A \to @@A$

$$\dfrac{\dfrac{\dfrac{}{@A \Rightarrow @A}\,\text{ID}}{@A \Rightarrow @@A}\,R@}{\Rightarrow @A \to @@A}\,R\to$$

$\vdash^{\mathbf{HI}^@} \Rightarrow @A \lor (@A \to B)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$@A \Rightarrow @A$}
\RightLabel{SPLIT@}
\UnaryInfC{$@A \Rightarrow \,|\, \Rightarrow @A$}
\RightLabel{IW, R}
\UnaryInfC{$@A \Rightarrow B \,|\, \Rightarrow @A$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Rightarrow @A \rightarrow B \,|\, \Rightarrow @A$}
\LeftLabel{(x2)}
\RightLabel{$R\vee_i$}
\doubleLine
\UnaryInfC{$\Rightarrow @A \vee (@A \rightarrow B) \,|\, \Rightarrow @A \vee (@A \rightarrow B)$}
\RightLabel{EC}
\UnaryInfC{$\Rightarrow @A \vee (@A \rightarrow B)$}
\end{prooftree}
$$

$\vdash^{\mathbf{HI}^{@}} \Rightarrow @(A \vee B) \rightarrow (@A \vee @B)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$A \Rightarrow A$}
\RightLabel{IW, R}
\UnaryInfC{$A \Rightarrow A, B$}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$B \Rightarrow B$}
\RightLabel{IW, R}
\UnaryInfC{$B \Rightarrow A, B$}
\RightLabel{$L\vee$}
\BinaryInfC{$A \vee B \Rightarrow A, B$}
\RightLabel{IW, L, $L@$}
\doubleLine
\UnaryInfC{$@(A \vee B), @(A \vee B) \Rightarrow A, B$}
\RightLabel{SPLIT@}
\UnaryInfC{$@(A \vee B) \Rightarrow A \,|\, @(A \vee B) \Rightarrow B$}
\LeftLabel{(x2)}
\RightLabel{$R@$}
\doubleLine
\UnaryInfC{$@(A \vee B) \Rightarrow @A \,|\, @(A \vee B) \Rightarrow @B$}
\LeftLabel{(x2)}
\RightLabel{$R\vee_i$}
\doubleLine
\UnaryInfC{$@(A \vee B) \Rightarrow @A \vee @B \,|\, @(A \vee B) \Rightarrow @A \vee @B$}
\RightLabel{EC}
\UnaryInfC{$@(A \vee B) \Rightarrow @A \vee @B$}
\RightLabel{$R\rightarrow$}
\UnaryInfC{$\Rightarrow @(A \vee B) \rightarrow (@A \vee @B)$}
\end{prooftree}
$$

Finally, if $\vdash^{\mathbf{HI}^{@}} \Rightarrow A$, then $\vdash^{\mathbf{HI}^{@}} \Rightarrow @A$:

$$
\begin{prooftree}
\AxiomC{$\Rightarrow A$}
\RightLabel{$R@$}
\UnaryInfC{$\Rightarrow @A$}
\end{prooftree}
$$

□

5.4 Cut-elimination

In this section, we show the CUT-elimination theorem for $\mathbf{HI}^{@}$ in a systematic and uniform manner. Intuitively, the proof proceeds by shifting applications of CUT upwards in derivations according to a specific order.

Remark 7. A crucial problem for CUT-elimination proofs arises when one encounters a derivation where either the external or internal contraction rule was applied. In this case, the procedure of shifting the application of CUT over the premise where it appears, won't give as a result a CUT with a shorter derivation. In order to avoid problems with the contraction rules, CUT-elimination proofs for hypersequent calculi are usually performed either by applying a suitable procedure to trace the CUT-formula through a derivation (see, e.g., Avron [Avr87], Baaz et al. [BC02; BCF03], Ciabattoni [Cia05]) or by dealing with some *multicut* rule suitably adapted to hypersequent systems (e.g., Avron [Avr91a]). Another strategy, based on Dragalin's method, was formulated by Poggiolesi, for example, in [Pog08a], by requiring (i) invertible logical rules and (ii) admissible structural rules. Roughly, in such systems one proves the admissibility of CUT

by means of the admissibility of contraction and other structural rules. However, an adaption of Poggiolesi's strategy to **HI**$^@$ leads to serious troubles with contraction elimination.
A very general and uniform proof strategy, based on a specific notion of *substitutivity*, was presented in Metcalfe et al. [MOG08] for both single- and multiple-succedent hypersequent systems. This method was then extended by Ciabattoni et al. [CMM10] to prove cut-elimination for monoidal t-norm logic **MTL** and other fuzzy logics expanded with so-called 'truth-stresser' modalities. The proof they propose is particularly important for our purposes since it deals with rules which are *not substitutive* in the sense of [MOG08] and, as it will be discussed below, some of our rules are of this kind. Therefore, in what follows, we will present a proof based on such a methodology.[4]

Let's introduce some important concepts. We say that a *marked hypersequent* is a hypersequent with exactly one occurrence of a formula A distinguished, denoted $\Gamma, \underline{A} \Rightarrow \Delta$ or $\Gamma \Rightarrow \underline{A}, \Delta$. A *marked rule instance* is a rule with the principal formula marked (if any). Finally, we say that a hypersequent G is *appropriate for a rule* **r** if it is single-conclusion when **r** is single-conclusion. Now, suppose that G is a (possibly marked) hypersequent and H a marked hypersequent of the forms:

$$G = \Gamma_1, [A]^{\lambda_1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n \text{ and } H = H' \mid \Pi \Rightarrow \underline{A}, \Sigma$$

Let A does not occur unmarked in $\uplus_{i=1}^{n}\Gamma_i$. We define $\textsc{cut}(G, H)$ as the set containing (for $0 \leq \mu_i \leq \lambda_i$, with $i = 1, \ldots, n$):

$$H' \mid \Gamma_1, \Pi^{\mu_1}, [A]^{\lambda_1-\mu_1} \Rightarrow \Sigma^{\mu_1}, \Delta_1 \mid \cdots \mid \Gamma_n, \Pi^{\mu_n}, [A]^{\lambda_n-\mu_n} \Rightarrow \Sigma^{\mu_n}, \Delta_n$$

In a similar way, suppose that A does not occur unmarked in $\uplus_{i=1}^{n}\Delta_i$:

$$G = \Gamma_1 \Rightarrow [A]^{\lambda_1}, \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow [A]^{\lambda_n}, \Delta_n \text{ and } H = H' \mid \Pi, \underline{A} \Rightarrow \Sigma$$

So, the set $\textsc{cut}(G, H)$ is (for $0 \leq \mu_i \leq \lambda_i$, with $i = 1, \ldots, n$):

$$H' \mid \Gamma_1, \Pi^{\mu_1} \Rightarrow [A]^{\lambda_1-\mu_1}, \Sigma^{\mu_1}, \Delta_1 \mid \cdots \mid \Gamma_n, \Pi^{\mu_n} \Rightarrow [A]^{\lambda_n-\mu_n}, \Sigma^{\mu_n}, \Delta_n$$

In order to ensure that we can push applications of cut upwards in derivations, we need the following notion.

Definition 5.4.1. A rule **r** is *substitutive* if for any:

1. marked instance of $(G_1 \ldots G_n)/G$ of **r**;
2. marked hypersequent H appropriate for **r**;
3. $G' \in \textsc{cut}(G, H)$;

there exists a $G'_i \in \textsc{cut}(G_i, H)$ for $i = 1, \ldots, n$ such that $(G'_1 \ldots G'_n)/G'$ is an instance of **r**;

[4] The approach presented in [CMM10] has offered a simple solution to construct cut-free hypersequent systems for various logics, for example, in Kurokawa [Kur14], Lellmann [Lel14], Indrzejczak [Ind15], [Ind21, pp. 224–227].

As said above, substitutivity was used to provide uniform and systematic CUT-free hypsersequent-style formalizations of several logics (e.g., [MOG08; CMM10]). The idea is that by substituting some non principal formula with some multiset on the left and on the right, in both, the premises and the conclusion of a rule, gives another instance of the rule. Let **HI** be denoting the hypersequent calculus consisting only of the non-modal rules of **HI**$^{@}$:

Lemma 5.4.1 (see e.g. [MOG08])**.** The logical rules of **HI** are substitutive.

Remark 8. As noticed, e.g. in [CMM10; Kur14; Ind15], a problem with substitutivity arises in connection to some of the rules governing @. Consider, for example, the following derivations, where d' is the result of pushing CUT upwards in d:

$$d : \dfrac{\dfrac{\vdots}{A \Rightarrow @B} \qquad \dfrac{@B \Rightarrow C}{@B \Rightarrow @C}\,R@}{A \Rightarrow @C}\,\text{CUT} \quad \rightsquigarrow \quad d' : \dfrac{\dfrac{A \Rightarrow @B \qquad @B \Rightarrow C}{A \Rightarrow C}\,\text{CUT}}{?}\,?$$

However, given the restriction on the antecedent in the formulation of $R@$, there's no way to get a derivation of $@C$ on the right. In other words, this example is meant to illustrate that *arbitrary* substitutions disturb $R@$ since the rule requires that all formulas in the antecedent have @ as outermost operator.
Nonetheless, on closer inspection, the problem does not arise if one considers CUTs only on sequents in which all formulas in the antecedent are prefixed by @, i.e., with a derivation concluding with an instance of $L@$. Indeed, we might depict the solution to our previous example as follows:

$$d : \dfrac{\dfrac{\vdots}{@A \Rightarrow @B}\,L@ \qquad \dfrac{@B \Rightarrow C}{@B \Rightarrow @C}\,R@}{@A \Rightarrow @C}\,\text{CUT} \quad \rightsquigarrow \quad d' : \dfrac{\dfrac{@A \Rightarrow @B \qquad @B \Rightarrow C}{@A \Rightarrow C}\,\text{CUT}}{@A \Rightarrow @C}\,R@$$

In sum, in order to permute applications of $R@$ and other rules, it suffices to consider only sequents of the form $H \mid @\Sigma \Rightarrow A, \Pi$.

Let d be a derivation in **HI**$^{@}$. We recall that:

1. $|d|$ is the length of d, i.e., the (maximal number of applications of inference rules)+1 occurring in d;
2. $|A|$ is the *complexity* of A, i.e., the number of occurrences of its connectives;
3. $\rho(d)$ is the *cut rank* of d, i.e., (the maximal complexity of CUT-formulas in d)+1. $\rho(d) = 0$, if d is CUT-free.

Lemma 5.4.2. Let d_l and d_r be derivations in **HI**$^{@}$ such that:

1. d_l is a derivation of $G \mid \Gamma_1, [A]^{\lambda_1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n$;
2. d_r is a derivation of $H \mid \Sigma \Rightarrow A, \Pi$;
3. $\rho(d_l) \leq |A|$ and $\rho(d_r) \leq |A|$;
4. A is a complex formula occurring as principal in the conclusion of d_r.

Then, a derivation d of $G \mid H \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$, with $\rho(d) \leq |A|$, can be constructed in **HI**$^{@}$.

Proof. By induction on $|d_l|$. First of all, if d_l terminates with an axiom, then the conclusion immediately holds. For the inductive step, we distinguish several cases according to the last rule **r** applied in d_l:

1. If **r** was applied only in G, then the conclusion follows by the inductive hypothesis and an application of **r**.
2. If **r** is any of the non-modal rules of **HI**$^{@}$ not introducing A, then, applying substitutivity (Lemma 5.4.1), the claim follows by the inductive hypothesis, applications of **r** and possibly of EW.
3. Let **r** be a left non-modal rule introducing A. As an example, consider $L\wedge_1$ and let $A = B \wedge C$:

$$\dfrac{\begin{array}{c}\vdots\, d_l\\ G \mid \Gamma_1, B, [B\wedge C]^{\lambda_1 - 1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [B \wedge C]^{\lambda_n} \Rightarrow \Delta_n\end{array}}{G \mid \Gamma_1, [B\wedge C]^{\lambda_1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [B\wedge C]^{\lambda_n} \Rightarrow \Delta_n}\ L\wedge_1$$

   Furthermore, d_r ends with $R\wedge$. By applying the inductive hypothesis to the premises of d_l we get $d' : G \mid H \mid \Gamma_1, B, [\Sigma]^{\lambda_1 - 1} \Rightarrow \Delta_1, \Pi^{\lambda_1 - 1} \mid \cdots \mid \Gamma_n, [\Sigma]^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$. By applying CUT with $\Sigma \Rightarrow B, \Pi$ (one of the premises of d_r), the desired conclusion follows. Importantly, the resulting derivation has CUT rank $\leq |B \wedge C|$.
   If **r** is any other logical rule of **HI**$^{@}$ introducing A, then the proof is similar to the case just displayed.

4. Suppose that **r** is a modal rule of **HI**$^{@}$.

   (a) d_l terminates with an instance of $L@$. Let $A = @B$ be principal:

$$\dfrac{\begin{array}{c}\vdots\, d_l\\ G \mid \Gamma_1, B, [@B]^{\lambda_1 - 1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [@B]^{\lambda_n} \Rightarrow \Delta_n\end{array}}{G \mid \Gamma_1, [@B]^{\lambda_1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [@B]^{\lambda_n} \Rightarrow \Delta_n}\ L@$$

   d_r ends with $R@$ and, hence, $\Sigma = @\Sigma'$ and $\Pi = [\,]$. By applying the induction hypothesis we get a derivation d': $G \mid H \mid \Gamma_1, B, [@\Sigma']^{\lambda_1 - 1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [@\Sigma']^{\lambda_n} \Rightarrow \Delta_n,$. Then, by applying CUT with the premise of d_r (i.e., $H \mid @\Sigma' \Rightarrow B$), possibly IC, EW and EC, we obtain the desired derivation with CUT rank $\leq |@B|$.

   (b) d_l ends with an application of $L@$, with A not principal. Then d_l has the following form:

$$\dfrac{\begin{array}{c}\vdots\, d_l\\ G \mid \Gamma_1, B, [A]^{\lambda_1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n\end{array}}{G \mid \Gamma_1, @B, [A]^{\lambda_1} \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n}\ L@$$

d_r is $H \mid \Sigma \Rightarrow A, \Pi$. By applying the inductive hypothesis to d_l, we get a derivation d': $G \mid H \mid \Gamma_1, B, [\Sigma]^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, [\Sigma]^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$. Finally, an application of $L@$ gives us the desired hypersequent $G \mid H \mid \Gamma_1, @B, [\Sigma]^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, [\Sigma]^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$ with $\rho(d') \leq |A|$.

(c) Suppose that the last rule applied in d_l is $R@$. Let $\Gamma_1 = @\Gamma_1'$, $A = @B$ and $\Delta_1 = [\,]$.

$$\dfrac{\dfrac{\vdots\, d_l}{G \mid @\Gamma_1', [@B]^{\lambda_1} \Rightarrow C \mid \cdots \mid \Gamma_n, [@B]^{\lambda_n} \Rightarrow \Delta_n}}{G \mid @\Gamma_1', [@B]^{\lambda_1} \Rightarrow @C \mid \cdots \mid \Gamma_n, [@B]^{\lambda_n} \Rightarrow \Delta_n}\ R@$$

d_r also ends with $R@$, where $\Sigma = @\Sigma'$ and $\Pi = [\,]$. By applying the inductive hypothesis to the premise of d_l we get a derivation d' of the hypersequent $G \mid H \mid @\Gamma_1', [@\Sigma']^{\lambda_1} \Rightarrow C \mid \cdots \mid \Gamma_n, [@\Sigma']^{\lambda_n} \Rightarrow \Delta_n$. Then, the desired hypersequent $G \mid H \mid @\Gamma_1', [@\Sigma']^{\lambda_1} \Rightarrow @C \mid \cdots \mid \Gamma_n, [@\Sigma']^{\lambda_n} \Rightarrow \Delta_n$ follows by applying $R@$.

(d) Suppose that the last rule applied in d_l is SPLIT@ and that $A = @B$.

$$\dfrac{\dfrac{\vdots\, d_l}{G \mid @\Phi, \Gamma_1, [@B]^{\lambda_1} \Rightarrow \Delta_1, \Delta_1' \mid \cdots \mid \Gamma_n, [@B]^{\lambda_n} \Rightarrow \Delta_n}}{G \mid @\Phi, [@B]^{\lambda} \Rightarrow \Delta_1 \mid \Gamma_1, [@B]^{\lambda_1 - \lambda} \Rightarrow \Delta_1' \mid \cdots \mid \Gamma_n, [@B]^{\lambda_n} \Rightarrow \Delta_n}\ \textsc{split}@$$

d_r ends with $R@$, where $\Sigma = @\Sigma'$ and $\Pi = [\,]$. By applying the inductive hypothesis to the premise of d_l we get a derivation d' of the hypersequent $G \mid H \mid @\Phi, \Gamma_1, [@\Sigma']^{\lambda_1} \Rightarrow \Delta_1, \Delta_1' \mid \cdots \mid \Gamma_n, [@\Sigma']^{\lambda_n} \Rightarrow \Delta_n$. Then, the desired hypersequent $G \mid H \mid @\Phi, [@\Sigma]^{\lambda} \Rightarrow \Delta_1 \mid \Gamma_1, [@\Sigma]^{\lambda_1 - \lambda} \Rightarrow \Delta_1' \mid \cdots \mid \Gamma_n, [@\Sigma]^{\lambda_n} \Rightarrow \Delta_n$ follows by applying SPLIT@ and possibly EW.

(e) Suppose that the last rule applied in d_l is SPLIT@ and that A is a non-modal formula.

$$\dfrac{\dfrac{\vdots\, d_l}{G \mid @\Phi, \Gamma_1, [A]^{\lambda_1} \Rightarrow \Delta_1, \Delta_1' \mid \cdots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n}}{G \mid @\Phi \Rightarrow \Delta_1 \mid \Gamma_1, [A]^{\lambda_1} \Rightarrow \Delta_1' \mid \cdots \mid \Gamma_n, [A]^{\lambda_n} \Rightarrow \Delta_n}\ \textsc{split}@$$

d_r ends with $H \mid \Sigma \Rightarrow A, \Pi$. By applying the inductive hypothesis to the premise of d_l we get a derivation d' of the hypersequent $G \mid H \mid @\Phi, \Gamma_1, [\Sigma]^{\lambda_1} \Rightarrow \Delta_1, \Delta_1', \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, [\Sigma]^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$. Then, the desired hypersequent $G \mid H \mid @\Phi \Rightarrow \Delta_1 \mid \Gamma_1, [\Sigma]^{\lambda_1} \Rightarrow \Delta_1', \Pi^{\lambda_1} \mid \cdots \mid \Gamma_n, [\Sigma]^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$ follows by applying SPLIT@ and possibly EW.

□

Lemma 5.4.3. Let d_l and d_r be derivations in **HI$^@$** such that:

1. d_l is a derivation of $G \mid \Gamma, A \Rightarrow \Delta$;

2. d_r is a derivation of $H \mid \Sigma_1 \Rightarrow [A]^{\lambda_1}, \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow [A]^{\lambda_n}, \Pi_n$;
3. $\rho(d_l) \leq |A|$ and $\rho(d_r) \leq |A|$.

Then, a derivation d of $G \mid H \mid \Sigma_1, \Gamma^{\lambda_1} \Rightarrow \Pi_1, \Delta^{\lambda_1} \mid \cdots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi_n, \Delta^{\lambda_n}$, with $\rho(d) \leq |A|$, can be constructed in **HI@**.

Note that in the following proof, when we find a rule introducing A principally in $H \mid \Sigma_i \Rightarrow [A]^{\lambda_i}, \Pi_i$, we apply Lemma 5.4.2 to it and to d_l, and if there's no A in Γ, we obtain a derivation of $G \mid H \mid \Sigma_1, \Gamma^{\lambda_1} \Rightarrow \Pi_1, \Delta^{\lambda_1} \mid \cdots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi_n, \Delta^{\lambda_n}$, with decreased CUT rank. Otherwise, the desired hypersequent follows after some applications of structural rules.

Proof. By induction on $|d_r|$. First of all, if d_r terminates with an axiom, then the conclusion immediately holds. For the inductive step, we distinguish several cases according to the last rule **r** applied in d_r:

1. If **r** was applied only in H, then the conclusion follows by the inductive hypothesis and an application of **r**.

2. If **r** is any of the non-modal rules of **HI@** not introducing A, we apply Lemma 5.4.1 and the claim follows by the inductive hypothesis and applications of **r**.

3. Let **r** be a right non-modal rule introducing A. As an example, consider $R \rightarrow$ and let $A = B \rightarrow C$:

$$\dfrac{\begin{array}{c}\vdots\, d_r \\ H \mid \Sigma_1, B \Rightarrow C \mid \cdots \mid \Sigma_n \Rightarrow [B \rightarrow C]^{\lambda_n}, \Pi_n\end{array}}{H \mid \Sigma_1 \Rightarrow B \rightarrow C \mid \cdots \mid \Sigma_n \Rightarrow [B \rightarrow C]^{\lambda_n}, \Pi_n}\ R \rightarrow$$

   d_l ends with $L \rightarrow$. By inductive hypothesis we obtain: $G \mid H \mid \Sigma_1, \Gamma^{\lambda_1 - 1}, B \Rightarrow C \mid \cdots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi_n, \Delta^{\lambda_n}$. We apply $R \rightarrow$ in order to get the desired conclusion. Then, since $B \rightarrow C$ is principal, the conclusion follows by Lemma 5.4.2.
   If **r** is any other logical rule of **HI@** introducing A, then the proof is similar to the case just displayed.

4. Let **r** be either $L@$ or SPLIT@. As an example, take $L@$. In this case the CUT-formula A is not a principal formula:

$$\dfrac{\begin{array}{c}\vdots\, d_r \\ H \mid B, \Sigma_1 \Rightarrow [A]^{\lambda_1}, \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow [A]^{\lambda_n, \Pi_n}\end{array}}{H \mid @B, \Sigma_1 \Rightarrow [A]^{\lambda_1}, \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow [A]^{\lambda_n}, \Pi_n}\ L@$$

   d_l is a derivation of $G \mid \Gamma, A \Rightarrow \Delta$. By the inductive hypothesis we obtain: $G \mid H \mid B, \Sigma_1, \Gamma^{\lambda_1} \Rightarrow \Pi_1, \Delta^{\lambda_1} \mid \cdots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi_n, \Delta^{\lambda_n}$. An application of $L@$ gives us the requested hypersequent $G \mid H \mid @B, \Sigma_1, \Gamma^{\lambda_1} \Rightarrow \Pi_1, \Delta^{\lambda_1} \mid \cdots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi_n, \Delta^{\lambda_n}$ (with CUT rank $\leq |A|$).

5. Suppose that **r** is right modal rules of **HI$^@$**, i.e., $R@$:
   Let d_r end with $R@$. So, in this case, let $\Sigma_1 = @\Sigma_1$ and $A = @B$:

$$\dfrac{\begin{array}{c}\vdots\, d_r\\ H \mid @\Sigma_1 \Rightarrow B \mid \cdots \mid \Sigma_n \Rightarrow [@B]^{\lambda_n}, \Pi_n\end{array}}{H \mid @\Sigma_1 \Rightarrow @B \mid \cdots \mid \Sigma_n \Rightarrow [@B]^{\lambda_n}, \Pi_n}\; R@$$

   d_l is a derivation of $G \mid \Gamma, @B \Rightarrow \Delta$. By inductive hypothesis we get a derivation of: $G \mid H \mid @\Sigma_1 \Rightarrow B \mid \cdots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Delta^{\lambda_n}, \Pi_n$. Then, the desired hypersequent, $G \mid H \mid @\Sigma_1 \Rightarrow @B \mid \cdots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Delta^{\lambda_n}, \Pi_n$, follows by $R@$. Finally, apply Lemma 5.4.2.

□

Theorem 5.4.4 (CUT-elimination)**.** **HI$^@$** is a CUT-free system.

Proof. Let d be a derivation in **HI$^@$** with $\rho(d) > 0$. We perform a double induction on $\langle \rho(d), n\rho(d)\rangle$, where $n\rho(d)$ indicates the number of applications of CUT in d with CUT rank $\rho(d)$. Consider an uppermost application of CUT in d with CUT rank $\rho(d)$ and apply Lemma 5.4.3 to its premises. As a consequence, either $\rho(d)$ or $n\rho(d)$ decreases. □

As a major consequence of the CUT-elimination theorem, we obtain the following feature of **HI$^@$**:

Corollary 5.4.4.1 (Subformula property)**.** All formulas occurring in a CUT-free derivation of **HI$^@$** are subformulas of the formula to be derived.

Now, we prove two final results following from the subformula property of **HI$^@$**. Let Φ be a multiset of formulas. We denote via SUB(Φ) the multiset of subformulas of formulas in Φ.

Proposition 5.4.5 (Consistency)**.** In **HI$^@$** the empty sequent cannot be derived, i.e., $\nvdash^{\mathbf{HI}^@} \Rightarrow$.

Proof. Assume that $\vdash^{\mathbf{HI}^@} \Rightarrow$. By the subformula property, there's a derivation of $\Rightarrow$ in which only elements from SUB([ ]) occur. However, this is not possible since no rule is applicable to conclude $\Rightarrow$. Hence, $\nvdash^{\mathbf{HI}^@} \Rightarrow$. □

Proposition 5.4.6 (Decidability)**.** Given a hypersequent H, it is decidable whether $\vdash^{\mathbf{HI}^@} H$ or $\nvdash^{\mathbf{HI}^@} H$.

Proof. Let H a hypersequent. Given the subformula property of **HI$^@$**, if $\vdash^{\mathbf{HI}^@} H$, then there exists a derivation of H in **HI$^@$** (with length $\leq n$) consisting only of elements from SUB(H). Thus, performing an effective proof-search procedure is possible.

Example 5. We illustrate the method by trying to check whether $@p \to (q \vee s) \Rightarrow (@p \to q) \vee (@p \to s)$ is derivable in $\mathbf{HI}^{@}$. We generate all possible finite derivation trees with endhypersequent $@p \to (q \vee s) \Rightarrow (@p \to q) \vee (@p \to s)$, for $p, q, s \in \mathsf{At}$. We write all instance of rules that conclude it; if there is one tree all leaves of which are conclusions of ID or $L\bot$, the endsequent is derivable; if not, it is underivable.

1. Consider $L \to$ as the last rule applied and let $@p \to (q \vee s)$ be principal in the last step.

1a.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Rightarrow p}}{\Rightarrow @p}\,R@ \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,\text{ID}}{q, @p \Rightarrow q}\,\text{IW, L}}{q \Rightarrow @p \to q}\,R\to}{q \Rightarrow (@p \to q) \vee (@p \to s)}\,R\vee_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{s \Rightarrow s}\,\text{ID}}{s, @p \Rightarrow s}\,\text{IW, L}}{s \Rightarrow @p \to s}\,R\to}{s \Rightarrow (@p \to q) \vee (@p \to s)}\,R\vee_2}{q \vee s \Rightarrow (@p \to q) \vee (@p \to s)}\,L\vee}{@p \to (q \vee s) \Rightarrow (@p \to q) \vee (@p \to s)}\,L\to$$

Given the presence of an irreducible non initial sequent of the form $\Rightarrow p$, the tree doesn't lead to a terminating derivation. Permuting $L\vee$ with $R\vee_i$ is of no help to conclude the tree.

2. Consider $R\vee_i$ as the last rule applied and let $(@p \to q) \vee (@p \to s)$ be principal in the last step.

2a.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{@p \Rightarrow @p}\,\text{ID} \qquad \dfrac{\dfrac{}{q \Rightarrow q}\,\text{ID} \qquad \dfrac{\vdots}{s \Rightarrow q}}{q \vee s \Rightarrow q}\,L\vee}{@p \to (q \vee s), @p \Rightarrow q}\,L\to}{@p \to (q \vee s) \Rightarrow @p \to q}\,R\to}{@p \to (q \vee s) \Rightarrow (@p \to q) \vee (@p \to s)}\,R\vee_1$$

2b.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{@p \Rightarrow @p}\,\text{ID} \qquad \dfrac{\dfrac{\vdots}{q \Rightarrow s} \qquad \dfrac{}{s \Rightarrow s}\,\text{ID}}{q \vee s \Rightarrow s}\,L\vee}{@p \to (q \vee s), @p \Rightarrow s}\,L\to}{@p \to (q \vee s) \Rightarrow @p \to s}\,R\to}{@p \to (q \vee s) \Rightarrow (@p \to q) \vee (@p \to s)}\,R\vee_2$$

where neither of the two trees terminates (given $s \Rightarrow q$ and $q \Rightarrow s$). (Notice that applying (internal or external) contraction before $R\vee_i$ is of no help in getting a terminating tree).

2c.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Rightarrow p$}
\RightLabel{$R@$}
\UnaryInfC{$\Rightarrow @p$}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$q \Rightarrow q$}
\RightLabel{IW, L}
\UnaryInfC{$q, @p \Rightarrow q$}
\RightLabel{$R \rightarrow$}
\UnaryInfC{$q \Rightarrow @p \rightarrow q$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$s, p \Rightarrow q$}
\RightLabel{$L@$}
\UnaryInfC{$s, @p \Rightarrow q$}
\RightLabel{$R \rightarrow$}
\UnaryInfC{$s \Rightarrow @p \rightarrow q$}
\RightLabel{$L\vee$}
\BinaryInfC{$q \vee s \Rightarrow @p \rightarrow q$}
\RightLabel{$L \rightarrow$}
\BinaryInfC{$@p \rightarrow (q \vee s) \Rightarrow @p \rightarrow q$}
\RightLabel{$R\vee_1$}
\UnaryInfC{$@p \rightarrow (q \vee s) \Rightarrow (@p \rightarrow q) \vee (@p \rightarrow s)$}
\end{prooftree}
$$

2d.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Rightarrow p$}
\RightLabel{$R@$}
\UnaryInfC{$\Rightarrow @p$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$q, p \Rightarrow s$}
\RightLabel{$L@$}
\UnaryInfC{$q, @p \Rightarrow s$}
\RightLabel{$R \rightarrow$}
\UnaryInfC{$q \Rightarrow @p \rightarrow s$}
\AxiomC{}
\RightLabel{ID}
\UnaryInfC{$s \Rightarrow s$}
\RightLabel{IW, L}
\UnaryInfC{$s, @p \Rightarrow s$}
\RightLabel{$R \rightarrow$}
\UnaryInfC{$s \Rightarrow @p \rightarrow s$}
\RightLabel{$L\vee$}
\BinaryInfC{$q \vee s \Rightarrow @p \rightarrow s$}
\RightLabel{$L \rightarrow$}
\BinaryInfC{$@p \rightarrow (q \vee s) \Rightarrow @p \rightarrow s$}
\RightLabel{$R\vee_2$}
\UnaryInfC{$@p \rightarrow (q \vee s) \Rightarrow (@p \rightarrow q) \vee (@p \rightarrow s)$}
\end{prooftree}
$$

where both derivation trees do not terminate. Notice that also further permutations of rules will not change the situation.
By having considered all possible derivations of $@p \rightarrow (q \vee s) \Rightarrow (@p \rightarrow q) \vee (@p \rightarrow s)$, it follows that it is not derivable in **HI**$^{@}$.

$\square$

5.5 Conclusion and further work

In this chapter, we have provided a solution to a problem raised in [NO20] by introducing a simple, sound, complete and CUT-free hypersequent calculus for Niki and Omori's **IPC**$^{@}$.
We remark that intuitionistic logic with actuality is not the only system developed within the project of extending intuitionism from mathematical to empirical discourse and, indeed, we expect to be worth investigating also logics in the vicinity of **IPC**$^{@}$ by accommodating the framework proposed throughout this chapter. For example, if the symbol $\sim$ is used to denote so-called *empirical negation* (see, e.g., De [De13]), one could consider a hypersequent-style formulation for M. De's $\mathsf{IPC}^{\sim}$, but also for other related logics such as, for example, A. B. Gordienko's TCC_{ω} (see [NO20, Remark 5.12]).

Chapter 6

Conclusive remarks

Outlook. In this thesis, I've investigated and discussed both the logical and philosophical value of Gentzen-style proof theory, by mainly focusing the attention on generalizations of the sequent calculus and on their applications to certain non-classical logics. Such a discussion has paved the way to all results proved in part II. Each choice made in those chapters was additionally supported by our second methodological principle for which different purposes and logics motivate the choice of different sequent systems (*non-absolutistic approach*). I concluded all case studies by proposing, among other things, how the formal methodology adopted in each chapter could be successfully applied to a variety of issues connected to the central topic of the case study.

Future research. The philosophical analysis that we carried out in Chapter 1, along with the endorsement of a *non-absolutistic approach*, points us towards a broader question, i.e., is a *general* philosophy of proof systems possible?
The word "general" strictly relates to the idea for which the *practice* of proof theory gives us the primary and fundamental source for our philosophical reflections on proof systems (*practice-first view*). I strongly believe that considering the practical aspect of proof theory allows us to glimpse in a more fine-grained way the differences, as well as the relations between different proof theoretic frameworks. More precisely, the notion of practice we're referring to is not restricted only to the (fundamental) theorem-proving aspect of the proof theoretic work. Instead, the notion of practice we think might be useful to elaborate a general philosophy of proof systems is related to other aspects which underlie a proof theorist's work. The establishment of *research programmes*[1], each having its formal objectives and extra-logical motivations, precedes and motivates both, the calculus choice and the theorem-proving job. In other words, we should understand, not only *how*, but also and especially *why* we deal with certain proof theoretic structures. Therefore, a proper philosophical assessment and systematization of proof systems should include the consideration of extra-logical elements fundamental to the practical work. Such a philosophical analysis is meant to include the consideration of the *foregrounds* of different approaches

[1] The terminology is inspired by Lakatos' work in the philosophy of science, as explained, for example, in [AR09].

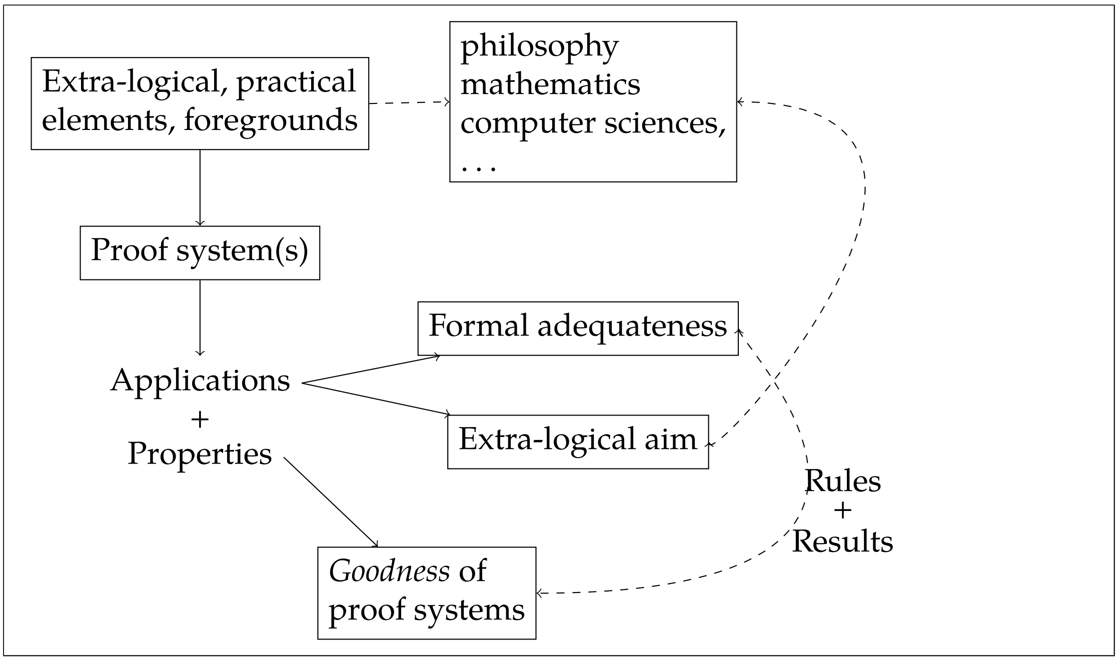

Figure 6.1: General philosophy of proof theory

to proof theory, which contain, for example, background ideas, providing motivations and justifications, as well as comparisons and critiques to alternative theories and methodologies.
In sum, the *practice-first view*, I believe, has the advantage of allowing us to grasp a more deep dimension of proof theory, where the symbolism is not only analysed and understood in terms of the properties that it satisfies, but it is seen as part of an activity, related to specific research programmes that (possibly) stand in relations with other ones. The general philosophical approach that I am trying to sketch, therefore, aims at considering proof systems not in *isolation*, but as being part of a broader context, and more or less inevitably connected to all other "players taking part in the game".

Part III

Appendices

Appendix to Chapter 3

A1 Proof of Theorem 3.3.6

Proof of Theorem 3.3.6 (cut-admissibility cont.). We finish the proof of cut-admissibility by displaying some other salient examples. We distinguish three main cases.

Case 1: If at least one of the premises of cut is an axiom, we distinguish 4 subcases:

Case 1.1: The left premise of cut is an axiom and the cut-formula is not principal. If the derivation has the following shape:

$$\dfrac{z : B, \Gamma \Rightarrow \Delta, z : B, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{z : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : B} \ (\textsc{cut})$$

It is transformed into:

$$z : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : B$$

without applications of cut.

Case 1.2: The left premise of cut is an axiom and the cut-formula is principal. The derivation:

$$\dfrac{w : A, \Gamma \Rightarrow \Delta, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{w : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ (\textsc{cut})$$

is transformed into:

$$(\text{Lemma 3.3.3}) \ \dfrac{w : A, \Gamma' \Rightarrow \Delta'}{w : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ Lw{+}Rw$$

Case 1.3: The right premise of cut is an axiom and the cut-formula is not principal. The derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad w : A, z : B, \Gamma' \Rightarrow \Delta', z : B}{z : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : B} \ (\textsc{cut})$$

It is transformed into:

$$z : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : B$$

without applications of cut.

Case 1.4: The right premise of cut is an axiom and the cut-formula is principal. The derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad w : A, \Gamma' \Rightarrow \Delta', w : A}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', w : A} \ (\textsc{cut})$$

is transformed into:

$$\text{(Lemma 3.3.3)}\ \dfrac{\Gamma \Rightarrow \Delta, w : A}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', w : A}\ \mathit{Lw{+}Rw}$$

Case 2: The cut-formula A is not principal in at least one premise. The proof proceeds by permuting the application of cut with the rule under consideration, to move the cut upwards in the transformed derivation.

Case 2.1: A is not principal in the left premise. We distinguish two subcases.
Subcase 2.1.1: Let $\Gamma = z : B \wedge_{\mathsf{d}} C, \Gamma''$:

$$\dfrac{\dfrac{zRv, z : B, v : C, \Gamma'' \Rightarrow \Delta, w : A}{z : B \wedge_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta, w : A}\ (L\wedge_{\mathsf{d}}) \qquad w : A, \Gamma' \Rightarrow \Delta'}{z : B \wedge_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ (\textsc{cut})$$

and transform it into the following one:

$$\dfrac{\dfrac{zRv, z : B, v : C, \Gamma'' \Rightarrow \Delta, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{zRv, z : B, v : C, \Gamma'' \Rightarrow \Delta}\ (\textsc{cut})}{z : B \wedge_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ (L\wedge_{\mathsf{d}})$$

where the cut-height is reduced.
Let $\Gamma = zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma''$ and consider as an example $L \rightarrow_{\mathsf{d}}$. We have the following derivation:

$$\dfrac{\dfrac{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta, w : A, v : B \qquad zRv, z : C, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta, w : A}{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta, w : A}\ (L \rightarrow_{\mathsf{d}}) \qquad w : A, \Gamma' \Rightarrow \Delta'}{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ (\textsc{cut})$$

and transform it into the following one:

$$\dfrac{\begin{matrix}\vdots\ \delta_1 \\ zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', v : B\end{matrix} \qquad \begin{matrix}\vdots\ \delta_2 \\ zRv, z : C, z : B \rightarrow_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'\end{matrix}}{zRv, z : C, z : B \rightarrow_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ (L \rightarrow_{\mathsf{d}})$$

where δ_1 is:

$$\dfrac{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta, w : A, v : B \qquad w : A, \Gamma' \Rightarrow \Delta'}{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', v : B}\ (\textsc{cut})$$

and δ_2 is:

$$\dfrac{zRv, z : C, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{zRv, z : C, z : B \rightarrow_{\mathsf{d}} C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ (\textsc{cut})$$

Subcase 2.1.2: Let $\Delta = \Delta'', z : B \wedge_{\mathsf{d}} C$:

$$\dfrac{\dfrac{zRv, \Gamma \Rightarrow \Delta'', z : B, w : A \qquad zRv, \Gamma \Rightarrow \Delta'', v : C, w : A}{\Gamma \Rightarrow \Delta'', z : B \wedge_{\mathsf{d}} C, w : A}\,(R\wedge_{\mathsf{d}}) \qquad w : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', z : B \wedge_{\mathsf{d}} C}\,(\textsc{cut})$$

it is transformed into the following application of CUT with a shorter derivation height:

$$\dfrac{\dfrac{zRv, \Gamma \Rightarrow \Delta'', z : B, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{zRv, \Gamma, \Gamma' \Rightarrow \Delta', \Delta'', z : B}\,(\textsc{cut}) \qquad \dfrac{zRv, \Gamma \Rightarrow \Delta'', v : C, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{zRv, \Gamma, \Gamma' \Rightarrow \Delta', \Delta'', v : C}\,(\textsc{cut})}{zRv, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', z : B \wedge_{\mathsf{d}} C}\,(R\wedge_{\mathsf{d}})$$

where the cut-height is reduced.
Let $\Delta = \Delta'', z : A \rightarrow_{\mathsf{d}} B$:

$$\dfrac{\dfrac{zRv, v : B, \Gamma \Rightarrow \Delta'', z : C, w : A}{\Gamma \Rightarrow \Delta'', z : B \rightarrow_{\mathsf{d}} C, w : A}\,(R \rightarrow_{\mathsf{d}}) \qquad w : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', z : B \rightarrow_{\mathsf{d}} C}\,(\textsc{cut})$$

it is transformed into the following application of CUT with a shorter derivation height:

$$\dfrac{\dfrac{zRv, v : B, \Gamma \Rightarrow \Delta'', z : C, w : A \qquad w : A, \Gamma' \Rightarrow \Delta'}{Rxbc, v : B, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', z : C}\,(\textsc{cut})}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', z : B \rightarrow_{\mathsf{d}} C}\,(R \rightarrow_{\mathsf{d}})$$

Case 2.2: A is principal in the left premise only. We distinguish two subcases.
Subcase 2.2.1: Similarly to the preceding subcase. Let $\Gamma' = z : B \wedge_{\mathsf{d}} C, \Gamma''$:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad \dfrac{zRv, w : A, z : B, v : C, \Gamma'' \Rightarrow \Delta'}{w : A, z : B \wedge_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta'}\,(L\wedge_{\mathsf{d}})}{z : B \wedge_{\mathsf{d}} C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\,(\textsc{cut})$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad zRv, w : A, z : B, v : C, \Gamma'' \Rightarrow \Delta'}{zRv, z : B, v : C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta', x^* : B}\,(\textsc{cut})}{z : B \wedge_{\mathsf{d}} C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\,(L\wedge_{\mathsf{d}})$$

with a shorter derivation height.
Let $\Gamma' = zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma''$ and consider $L \rightarrow_{\mathsf{d}}$. We have the following derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad \dfrac{w : A, zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta', v : B \qquad w : A, zRv, z : B, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta'}{w : A, zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta'}\,(L \rightarrow_{\mathsf{d}})}{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\,(\textsc{cut})$$

is reduced to the following one:

$$\dfrac{\dfrac{\vdots\,\delta_1}{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta', v : B} \quad \dfrac{\vdots\,\delta_2}{zRv, z : B \rightarrow_{\mathsf{d}} C, z : C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}}{zRv, z : A \rightarrow_{\mathsf{d}} B, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'} \; (L \rightarrow_{\mathsf{d}})$$

where δ_1 is derived by:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad w : A, zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta', v : B}{zRv, z : B \rightarrow_{\mathsf{d}} C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta', v : B} \; (\textsc{cut})$$

while δ_2 is derived by:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad w : A, zRv, z : C, z : B \rightarrow_{\mathsf{d}} C, \Gamma'' \Rightarrow \Delta'}{zRv, z : B \rightarrow_{\mathsf{d}} C, z : C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'} \; (\textsc{cut})$$

with a decreased derivation height.
Subcase 2.2.2: Let $\Delta' = \Delta'', z : B \wedge_{\mathsf{d}} C$:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad \dfrac{zRv, w : A, \Gamma' \Rightarrow \Delta'', v : B \qquad zRv, w : A, \Gamma' \Rightarrow \Delta'', z : C}{zRv, w : A, \Gamma' \Rightarrow \Delta'', z : B \wedge_{\mathsf{d}} C} \; (R\wedge_{\mathsf{d}})}{zRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', z : B \wedge_{\mathsf{d}} C} \; (\textsc{cut})$$

it is transformed into:

$$\dfrac{\dfrac{\vdots\,\delta_1}{zRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', v : B} \quad \dfrac{\vdots\,\delta_2}{zRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', z : C}}{zRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', z : B \wedge_{\mathsf{d}} C} \; (R\wedge_{\mathsf{d}})$$

where δ_1 is:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad zRv, w : A, \Gamma' \Rightarrow \Delta'', v : B}{zRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', v : B} \; (\textsc{cut})$$

and δ_2 is:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad zRv, w : A, \Gamma' \Rightarrow \Delta'', z : C}{zRv, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', z : C} \; (\textsc{cut})$$

with a shorter derivation height.
Let $\Delta' = \Delta'', z : B \rightarrow_{\mathsf{d}} C$ and the derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad \dfrac{zRv, w : A, v : B, \Gamma' \Rightarrow \Delta'', z : C}{w : A, \Gamma' \Rightarrow \Delta'', z : B \rightarrow_{\mathsf{d}} C} \; (R \rightarrow_{\mathsf{d}})}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', z : B \rightarrow_{\mathsf{d}} C} \; (\textsc{cut})$$

It is reduced to the following one:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, w : A \qquad zRv, w : A, v : B, \Gamma' \Rightarrow \Delta'', z : C}{zRv, v : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', z : C}\ (\textsc{cut})}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', z : B \rightarrow_{\mathsf{d}} C}\ (R \rightarrow_{\mathsf{d}})$$

with a shorter derivation height.
Case 3: The cases for A being $B \wedge_{\mathsf{d}} C$ or $B \rightarrow_{\mathsf{d}} C$, can be found on pp. 61 and ff.

A2 Proof of Theorem 3.4.2

Proof of Theorem 3.4.2 (Semantic Completeness cont.) In this appendix we construct a reduction tree for an arbitrary sequent $\mathcal{S}$, by applying, root-first, all rules for $\mathbf{G3D_2}$ according to a specific order. This construction is used to define a countermodel to $\mathcal{S}$ (displayed above). Importantly, recall that, to reflect the notion of validity at the actual world, we will consider derivability at 0, and not with respect to arbitrary labels.

The reduction tree is defined inductively in stages as follows: (1) $n = 0$, so $\Gamma \Rightarrow \Delta$ stands at the root of the tree. (2) If $n > 0$, we distinguish two subcases. (2.1) If every topmost sequent is an axiomatic sequent reduction the tree terminates; (2.2) If no axiomatic sequent is reached, the construction of the reduction tree does not terminate and we continue applying, root-first, all rules of $\mathbf{G3D_2}$ according to a specific order. There are $8 + j$ different stages: 8 for the rules for the propositional connectives and j for the mathematical rules. We start, for $n = 1$, with $L\neg$ and consider topmost sequents of the following form:

$$w_1 : \neg B_1, \dots, w_k : \neg B_k, \Gamma' \Rightarrow \Delta$$

where $w_1 : \neg B_1, \dots, w_k : \neg B_k$, are all formulas in Γ with $\neg$ as outermost connective. By applying, root-first, k times, $L\neg$ we obtain the following sequent:

$$\Gamma' \Rightarrow \Delta, w_1 : B_1, \dots, w_k : B_k$$

placed on top of the former.

For $n = 2$, we consider sequents of the form:

$$\Gamma \Rightarrow \Delta', w_1 : \neg B_1, \dots, w_k : \neg B_k$$

By applying, root-first, k times, $R\neg$ we obtain the following sequent:

$$w_1 : B_1, \dots, w_k : B_k, \Gamma \Rightarrow \Delta'$$

placed on top of the former.

For $n = 3$, we consider sequents of the form:

$$w_1 : B_1 \vee C_1, \dots, w_k : B_k \vee C_k, \Gamma' \Rightarrow \Delta$$

By applying, root-first, k times, $L\vee$ we obtain the following sequents:

$$w_1 : B_1, \dots, w_k : B_k, \Gamma' \Rightarrow \Delta \qquad \text{and} \qquad w_1 : C_1, \dots, w_k : C_k, \Gamma' \Rightarrow \Delta$$

For $n = 4$, we consider sequents of the form:

$$\Gamma \Rightarrow \Delta', w_1 : B_1 \vee C_1, \dots, w_k : B_k \vee C_k$$

By applying, root-first, k times, $R\vee$ we obtain the following sequents:

$$\Gamma \Rightarrow \Delta', w_1 : B_1, w_1 : C_1, \dots, w_k : B_k, w_k : C_k$$

placed on top of the former as its premises.
For $n = 5$, we consider sequents of the form:

$$w_1 : B_1 \wedge_{\mathsf{d}} C_1, \ldots, w_k : B_k \wedge_{\mathsf{d}} C_k, \Gamma' \Rightarrow \Delta$$

Let $v_1, ..., v_k$ be fresh variables, not yet used in the reduction tree. By applying, root-first, k times, $L\wedge_{\mathsf{d}}$ we obtain the following sequent:

$$w_1 R v_1, \ldots, w_k R v_k,\ w_1 : B_1,\ w_1 : B_1, v_1 : C_1, \ldots, w_k : B_k, v_k : C_k, \Gamma' \Rightarrow \Delta$$

placed on top of the former sequent.
For $n = 6$, we consider sequents of the form:

$$w_1 R v_1, \ldots, w_k R v_k,\ w_1, \Gamma \Rightarrow \Delta', w_1 : B_1 \wedge_{\mathsf{d}} C_1, \ldots, w_k : B_k \wedge_{\mathsf{d}} C_k$$

By applying, root-first, k times, $R\wedge_{\mathsf{d}}$ we obtain the following sequents:

$$w_1 R v_1, \ldots, w_k R v_k,\ w_1, \Gamma \Rightarrow \Delta', w : B_1, \ldots, w : B_k$$

and

$$w_1 R v_1, \ldots, w_k R v_k,\ w_1, \Gamma \Rightarrow \Delta', v : C_1, \ldots, v : C_k$$

placed on top of the former as its premises.
For $n = 7$, we consider topmost sequents of the following form:

$$w_1 R v_1, \ldots, w_k R v_k,\ w_1 : B_1 \rightarrow_{\mathsf{d}} C_1, \ldots, w_k : B_k \rightarrow_{\mathsf{d}} C_k,\ \Gamma' \Rightarrow \Delta$$

where labels and principal formulas are in Γ'. By applying, root-first, k times, $L \rightarrow_{\mathsf{d}}$ (with $w_1 R v_1, \ldots, w_k R v_k,\ w_1 : B_1 \rightarrow_{\mathsf{d}} C_1, \ldots, w_k : B_k, \rightarrow_{\mathsf{d}} C_k$ principal) we obtain the following sequent:

$$w_1 R v_1, \ldots, w_k R v_k,\ w_{m_1} : C_{m_1}, \ldots, w_{m_l} : C_{m_l},\ \Gamma' \Rightarrow \Delta,\ v_{j_{l+1}} : B, \ldots, v_{j_k} : B$$

where $\{m_1, \ldots, m_l\} \subseteq \{1, \ldots, k\}$ and $j_{l+1}, \ldots, j_k \in \{1, \ldots, k\} - \{m_1, \ldots, m_l\}$, and placed on top of the former as its premises.
For $n = 8$, we consider all the labelled sequents that have implications in the succedent. We consider topmost sequents of the following form:

$$\Gamma \Rightarrow \Delta', w_1 : B_1 \rightarrow_{\mathsf{d}} C_1, \ldots, w_k : B_k \rightarrow_{\mathsf{d}} C_k$$

Let $v_1, ..., v_k$ be fresh variables, not yet used in the reduction tree and apply, root-first, k times, $R \rightarrow_{\mathsf{d}}$ to obtain the following sequent:

$$w_1 R v_1, \ldots, w_k R v_k,\ v_1 : B, \ldots, v_k : B,\ \Gamma \Rightarrow \Delta',\ w_1 : C_1, \ldots, w_k : C_k$$

placed on top of the former as its premise.
Finally, we consider relational rules. If it is a rule without eigenvariable condition, we write on top of the lower sequent the result of applying the relational rule under consideration.
This construction is then used in the development of the second part of the proof displayed above (pp. 69 and ff.).

Appendix to Chapter 4

B1 Proof of Theorem 4.5.2

Proof of Theorem 4.5.2 (Syntactic Completeness). We show that (Ax1)-(Ax16) ca be derived in the calculi **G3rX**:
G3rB $\vdash \Rightarrow 0 : A \to A$

$$\text{(}a, b\text{ fresh)}\ \frac{R0ab, a : A \Rightarrow b : A}{\Rightarrow 0 : A \to A}\ R\to$$

G3rB $\vdash \Rightarrow 0 : A \wedge B \to A$ and **G3rB** $\vdash \Rightarrow 0 : A \wedge B \to B$.

$$\text{(}a, b\text{ fresh)}\ \frac{\dfrac{R0ab, a : A, a : B \Rightarrow b : A}{R0ab, a : A \wedge B \Rightarrow b : A}\ L\wedge}{\Rightarrow 0 : A \wedge B \to A}\ R\to \qquad \text{(}a, b\text{ fresh)}\ \frac{\dfrac{R0ab, a : A, a : B \Rightarrow b : B}{R0ab, a : A \wedge B \Rightarrow b : B}\ L\wedge}{\Rightarrow 0 : A \wedge B \to B}\ R\to$$

G3rB $\vdash \Rightarrow 0 : (A \to B) \wedge (A \to C) \to (A \to (B \wedge C))$. We have the following derivation:

$$\text{(}a, b\text{ fresh)}\ \frac{\dfrac{\text{(}c, d\text{ fresh)}\ \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_1 \\ Racd, \mathcal{S}, c : A, a : A \to B \Rightarrow d : B\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2 \\ Racd, \mathcal{S}, c : A, a : A \to C \Rightarrow d : C\end{matrix}}{Racd, Rbcd, R0ab, c : A, a : A \to B, a : A \to C \Rightarrow d : B \wedge C}\ R\wedge}{Rbcd, R0ab, c : A, a : A \to B, a : A \to C \Rightarrow d : B \wedge C}\ \text{R4}}{R0ab, a : A \to B, a : A \to C \Rightarrow b : A \to (B \wedge C)}\ R\to}{R0ab, a : (A \to B) \wedge (A \to C) \Rightarrow b : A \to (B \wedge C)}\ L\wedge}{\Rightarrow 0 : (A \to B) \wedge (A \to C) \to (A \to (B \wedge C))}\ R\to$$

where the conclusion of δ_1 is obtained by:

$$\frac{\dfrac{R0cc, Racd, \mathcal{S}', c : A \Rightarrow d : B, c : A}{Racd, \mathcal{S}', c : A \Rightarrow d : B, c : A}\ \text{R2} \qquad \dfrac{R0dd, Racd, \mathcal{S}', d : B, c : A \Rightarrow d : B}{Racd, \mathcal{S}', d : B, c : A \Rightarrow d : B}\ \text{R2}}{Racd, \mathcal{S}, c : A, a : A \to B \Rightarrow d : B}\ L\to$$

while the conclusion of δ_2 is derived by:

$$\frac{\dfrac{R0cc, Racd, \mathcal{S}'', c : A \Rightarrow d : C, c : A}{Racd, \mathcal{S}'', c : A \Rightarrow d : C, c : A}\ \text{R2} \qquad \dfrac{R0dd, Racd, \mathcal{S}'', d : C, c : A \Rightarrow d : C}{Racd, \mathcal{S}'', d : C, c : A \Rightarrow d : C}\ \text{R2}}{Racd, \mathcal{S}, c : A, a : A \to C \Rightarrow d : C}\ L\to$$

where $\mathcal{S} = Rbcd, R0ab$, $\mathcal{S}' = Rbcd, R0ab, a : A \to B$ and $\mathcal{S}' = Rbcd, R0ab, a : A \to C$.

G3rB $\vdash \Rightarrow 0 : A \to (A \lor B)$ and **G3rB** $\vdash \Rightarrow 0 : B \to (A \lor B)$.

$$(a, b\text{ fresh})\;\dfrac{\dfrac{R0ab, a : A \Rightarrow b : A, b : B}{R0ab, a : A \Rightarrow b : A \lor B}\,R\lor}{\Rightarrow 0 : A \to (A \lor B)}\,R\to \qquad (a, b\text{ fresh})\;\dfrac{\dfrac{R0ab, a : B \Rightarrow b : A, b : B}{R0ab, a : B \Rightarrow b : A \lor B}\,R\lor}{\Rightarrow 0 : B \to (A \lor B)}\,R\to$$

G3rB $\vdash \Rightarrow 0 : (A \to C) \land (B \to C) \to ((A \lor B) \to C)$.

The derivation is as follows:

$$(a, b\text{ fresh})\;\dfrac{(c, d\text{ fresh})\;\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_1\\ Racd, \mathcal{S}, c : A \lor B, a : B \to C \Rightarrow d : C, c : A\end{matrix} \qquad Racd, \mathcal{S}, d : C, c : A \lor B, a : B \to C \Rightarrow d : C}{Racd, Rbcd, R0ab, c : A \lor B, a : A \to C, a : B \to C \Rightarrow d : C}}{Rbcd, R0ab, c : A \lor B, a : A \to C, a : B \to C \Rightarrow d : C}\,\mathrm{R4}}{Rbcd, R0ab, c : A \lor B, a : (A \to C) \land (B \to C) \Rightarrow d : C}\,L\land}{R0ab, a : (A \to C) \land (B \to C) \Rightarrow b : (A \lor B) \to C}\,R\to}{\Rightarrow 0 : (A \to C) \land (B \to C) \to ((A \lor B) \to C)}\,R\to$$

where the conclusion of δ_1 is derived by:

$$\dfrac{\dfrac{R0cc, Racd, \mathcal{S}, c : A, a : B \to C \Rightarrow d : C, c : A}{Racd, \mathcal{S}, c : A, a : B \to C \Rightarrow d : C, c : A}\,\mathrm{R2} \qquad \begin{matrix}\vdots\,\delta_1'\\ Racd, \mathcal{S}, c : B, a : B \to C \Rightarrow d : C, c : A\end{matrix}}{Racd, \mathcal{S}, c : A \lor B, a : B \to C \Rightarrow d : C, c : A}\,L\lor$$

while δ_1' is derived by:

$$\dfrac{\dfrac{R0cc, Racd, \mathcal{S}', c : B \Rightarrow d : C, c : A, c : B}{Racd, \mathcal{S}', c : B \Rightarrow d : C, c : A, c : B}\,\mathrm{R2} \qquad \dfrac{R0dd, Racd, \mathcal{S}', d : C, c : B \Rightarrow d : C, c : A}{Racd, \mathcal{S}', d : C, c : B \Rightarrow d : C, c : A}\,\mathrm{R2}}{Racd, \mathcal{S}, c : B, a : B \to C \Rightarrow d : C, c : A}\,L\to$$

with $\mathcal{S} = Rbcd, R0ab, a : A \to C$ and $\mathcal{S}' = Rbcd, R0ab, a : A \to C, a : B \to C$.

G3rB $\vdash \Rightarrow 0 : A \land (B \lor C) \to (A \land B) \lor (A \land C)$. We obtain the following derivation:

$$(a, b\text{ fresh})\;\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_1\\ R0ab, a : A, a : B \Rightarrow b : A \land B, b : A \land C\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2\\ R0ab, a : A, a : C \Rightarrow b : A \land B, b : A \land C\end{matrix}}{R0ab, a : A, a : B \lor C \Rightarrow b : A \land B, b : A \land C}\,L\lor}{R0ab, a : A \land (B \lor C) \Rightarrow b : A \land B, b : A \land C}\,L\land}{R0ab, a : A \land (B \lor C) \Rightarrow b : (A \land B) \lor (A \land C)}\,R\lor}{\Rightarrow 0 : A \land (B \lor C) \to (A \land B) \lor (A \land C)}\,R\to$$

where the conclusion of δ_1 is derived by:

$$\dfrac{R0ab, a : A, a : B \Rightarrow b : A, b : A \land C \qquad R0ab, a : A, a : B \Rightarrow b : B, b : A \land C}{R0ab, a : A, a : B \Rightarrow b : A \land B, b : A \land C}\,R\land$$

while the conclusion of δ_2 is obtained by:

$$\dfrac{R0ab, a : A, a : C \Rightarrow b : A, b : A \wedge B \qquad R0ab, a : A, a : C \Rightarrow b : C, b : A \wedge B}{R0ab, a : A, a : C \Rightarrow b : A \wedge B, b : A \wedge C}\,R\wedge$$

G3rB $\vdash \Rightarrow 0 : {\sim}{\sim}A \to A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0a^{**}b, R0a^{**}a, R0aa^{**}, R0ab, a^{**} : A \Rightarrow b : A}{R0a^{**}a, R0aa^{**}, R0ab, a^{**} : A \Rightarrow b : A}\,\text{R3}}{R0ab, a^{**} : A \Rightarrow b : A}\,\text{R1}}{R0ab \Rightarrow b : A, a^{*} : {\sim}A}\,R{\sim}}{R0ab, a : {\sim}{\sim}A \Rightarrow b : A}\,L{\sim}}{\Rightarrow 0 : {\sim}{\sim}A \to A}\,R \to \quad (a, b \text{ fresh})$$

If **G3rB** $\vdash \Rightarrow 0 : A$, **G3rB** $\vdash \Rightarrow 0 : A \to B$, then **G3rB** $\vdash \Rightarrow 0 : B$.

$$\dfrac{\Rightarrow 0 : A \qquad \dfrac{\dfrac{\Rightarrow 0 : A \to B}{a : A \Rightarrow a : B}\,\text{(Lemma 4.5.1)}}{0 : A \Rightarrow 0 : B}\,\text{(Lemma 4.3.1) } \textsc{sub}(0/a)}{\Rightarrow 0 : B}\,\textsc{cut}$$

If **G3rB** $\vdash \Rightarrow 0 : A$, **G3rB** $\vdash \Rightarrow 0 : B$, then **G3rB** $\vdash \Rightarrow 0 : A \wedge B$.

$$\dfrac{\Rightarrow 0 : B \qquad \Rightarrow 0 : A}{\Rightarrow 0 : A \wedge B}\,R\wedge$$

If **G3rB** $\vdash \Rightarrow 0 : A \to B$, then **G3rB** $\vdash \Rightarrow 0 : (C \to A) \to (C \to B)$. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{R0bb, Rabc, a : C \to A, b : C \Rightarrow c : B, b : C}{Rabc, a : C \to A, b : C \Rightarrow c : B, b : C}\,\text{R2} \qquad \dfrac{\dfrac{\Rightarrow 0 : A \to B}{c : A \Rightarrow c : B}\,\text{(Lemma 4.5.1)}}{Rabc, c : A, a : C \to A, b : C \Rightarrow c : B}\,\text{(Lemma 4.6.1) } \textsc{lw+lw}_L}{Rabc, a : C \to A, b : C \Rightarrow c : B}\,L \to}{a : C \to A \Rightarrow a : C \to B}\,R \to \ (b, c \text{ fresh})}{\Rightarrow 0 : (C \to A) \to (C \to B)}\,\text{(Lemma 4.5.1)}$$

If **G3rB** $\vdash \Rightarrow 0 : A \to B$, then **G3rB** $\vdash \Rightarrow 0 : (B \to C) \to (A \to C)$. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 0 : A \to B}{b : A \Rightarrow b : B}\,\text{(Lemma 4.5.1)}}{Rabc, a : B \to C, b : A \Rightarrow c : C, b : B}\,\text{(Lemma 4.6.1) } \textsc{lw+rw+lw}_L \qquad \dfrac{R0cc, Rabc, c : C, a : B \to C \Rightarrow c : C}{Rabc, c : C, a : B \to C \Rightarrow c : C}\,\text{R2}}{Rabc, a : B \to C, b : A \Rightarrow c : C}\,L \to}{a : B \to C \Rightarrow a : A \to C}\,R \to \ (b, c \text{ fresh})}{\Rightarrow 0 : (B \to C) \to (A \to C)}\,\text{(Lemma 4.5.1)}$$

If **G3rB** $\vdash \Rightarrow 0 : A \to B$, then **G3rB** $\vdash \Rightarrow 0 : {\sim}B \to {\sim}A$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 0 : A \to B}{a : A \Rightarrow a : B}\ (\text{Lemma 4.5.1})}{a^* : A \Rightarrow a^* : B}\ (\text{Lemma 4.3.1})\ \textsc{sub}(a^*/a)}{a^* : A, a : {\sim}B \Rightarrow}\ L{\sim}}{a : {\sim}B \Rightarrow a : {\sim}A}\ R{\sim}}{\Rightarrow 0 : {\sim}B \to {\sim}A}\ (\text{Lemma 4.5.1})
$$

where $\mathcal{S} = R0ab, 0 : A \to B$. This completes the completeness proof for **G3rB**.
G3rDW $\vdash\Rightarrow 0 : (A \to B) \to ({\sim}B \to {\sim}A)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0d^*d^*, Rad^*c^*, \mathcal{S}, d^* : A \Rightarrow c^* : B, d^* : A}{Rad^*c^*, \mathcal{S}, d^* : A \Rightarrow c^* : B, d^* : A}\ \text{R2} \qquad \dfrac{R0c^*c^*, Rad^*c^*, \mathcal{S}, c^* : B, d^* : A \Rightarrow c^* : B}{Rad^*c^*, \mathcal{S}, c^* : B, d^* : A \Rightarrow c^* : B}\ \text{R2}}{Rad^*c^*, Rbd^*c^*, Rbcd, R0ab, d^* : A, a : A \to B \Rightarrow c^* : B}\ L\to}{Rbd^*c^*, Rbcd, R0ab, d^* : A, a : A \to B \Rightarrow c^* : B}\ \text{R4}}{Rbcd, R0ab, d^* : A, a : A \to B \Rightarrow c^* : B}\ \text{R6}}{Rbcd, R0ab, d^* : A, a : A \to B, c : {\sim}B \Rightarrow}\ L{\sim}}{Rbcd, R0ab, a : A \to B, c : {\sim}B \Rightarrow d : {\sim}A}\ R{\sim}}{R0ab, a : A \to B \Rightarrow b : {\sim}B \to {\sim}A}\ R\to\ (c, d \text{ fresh})}{\Rightarrow 0 : (A \to B) \to ({\sim}B \to {\sim}A)}\ R\to\ (a, b \text{ fresh})
$$

where $\mathcal{S} = Rbd^*c^*, Rbcd, R0ab, a : A \to B$.
G3rDJ $\vdash\Rightarrow 0 : (A \to B) \wedge (B \to C) \to (A \to C)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Racx, Raxd, \mathcal{S}', c : A \Rightarrow d : C, c : A}{Racx, Raxd, \mathcal{S}', c : A \Rightarrow d : C, c : A}\ \text{R2} \qquad \dfrac{\vdots\ \delta_1}{Racx, Raxd, \mathcal{S}, x : B, a : B \to C, \Rightarrow d : C}}{Racx, Raxd, Racd, Rbcd, R0ab, c : A, a : A \to B, a : B \to C \Rightarrow d : C}\ L\to}{Racd, Rbcd, R0ab, c : A, a : A \to B, a : B \to C \Rightarrow d : C}\ \text{R7}\ (x \text{ fresh})}{Rbcd, R0ab, c : A, a : A \to B, a : B \to C \Rightarrow d : C}\ \text{R4}}{R0ab, a : A \to B, a : B \to C \Rightarrow b : A \to C}\ R\to\ (c, d \text{ fresh})}{R0ab, a : (A \to B) \wedge (B \to C) \Rightarrow b : A \to C}\ L\wedge}{\Rightarrow 0 : (A \to B) \wedge (B \to C) \to (A \to C)}\ R\to\ (a, b \text{ fresh})
$$

and δ_1 is derived by:

$$
\dfrac{\dfrac{R0xx, Racx, Raxd, \mathcal{S}', x : B \Rightarrow d : C, x : B}{Racx, Raxd, \mathcal{S}', x : B \Rightarrow d : C, x : B}\ \text{R2} \qquad \dfrac{R0dd, Racx, Raxd, \mathcal{S}', d : C, x : B \Rightarrow d : C}{Racx, Raxd, \mathcal{S}', d : C, x : B \Rightarrow d : C}\ \text{R2}}{Racx, Raxd, \mathcal{S}, x : B, a : B \to C, \Rightarrow d : C}\ L\to
$$

where $\mathcal{S} = Racd, Rbcd, R0ab, a : A \to B$ and $\mathcal{S}' = Racd, Rbcd, R0ab, a : A \to B, a : B \to C$.
G3rTW $\vdash\Rightarrow 0 : (A \to B) \to ((B \to C) \to (A \to C))$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0ee,\mathcal{S},a:A\to B,e:A\Rightarrow e:A,f:C}{\mathcal{S},a:A\to B,e:A\Rightarrow e:A,f:C}\,\text{R2}\qquad \begin{array}{c}\vdots\,\delta_1\\ Raex,Rcxf,\mathcal{S}',x:B,c:B\to C\Rightarrow f:C\end{array}}{Raex,Rcxf,Rbex,Rdef,Rbcd,R0ab,a:A\to B,c:B\to C,e:A\Rightarrow f:C}\,L\to}{Rcxf,Rbex,Rdef,Rbcd,R0ab,a:A\to B,c:B\to C,e:A\Rightarrow f:C}\,\text{R4}}{Rdef,Rbcd,R0ab,a:A\to B,c:B\to C,e:A\Rightarrow f:C}\,\text{R8}\ (x\text{ fresh})}{Rbcd,R0ab,a:A\to B,c:B\to C\Rightarrow d:A\to C}\,R\to\ (e,f\text{ fresh})}{R0ab,a:A\to B\Rightarrow b:(B\to C)\to(A\to C)}\,R\to\ (c,d\text{ fresh})}{\Rightarrow 0:(A\to B)\to((B\to C)\to(A\to C))}\,R\to\ (a,b\text{ fresh})
$$

and δ_1 is derived by:

$$
\dfrac{\dfrac{R0xx,Raex,Rcxf,\mathcal{S}'',x:B\Rightarrow f:C,x:B}{Raex,Rcxf,\mathcal{S}'',x:B\Rightarrow f:C,x:B}\,\text{R2}\qquad \dfrac{R0ff,Raex,Rcxf,\mathcal{S}'',f:C,x:B\Rightarrow f:C}{Raex,Rcxf,\mathcal{S}'',f:C,x:B\Rightarrow f:C}\,\text{R2}}{Raex,Rcxf,\mathcal{S}',x:B,c:B\to C\Rightarrow f:C}\,L\to
$$

where $\mathcal{S} = Raex,Rcxf,Rbex,Rdef,Rbcd,R0ab,c:B\to C$,
$\mathcal{S}' = Rbex,Rdef,Rbcd,R0ab,e:A,a:A\to B$ and $\mathcal{S}'' = Rbex,Rdef,Rbcd,R0ab,e:A,a:A\to B,c:B\to C$.
G3rTW $\vdash\Rightarrow 0:(A\to B)\to((C\to A)\to(C\to B))$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0ee,Rcex,\mathcal{S},e:C,\Rightarrow f:B,e:C}{Rcex,\mathcal{S},e:C,\Rightarrow f:B,e:C}\,\text{R2}\qquad \begin{array}{c}\vdots\,\delta_1\\ Raxf,Rcex,\mathcal{S}',x:A,e:C,a:A\to B\Rightarrow f:B\end{array}}{Raxf,Rcex,Racd,Rdef,Rbcd,R0ab,e:C,a:A\to B,c:C\to A\Rightarrow f:B}\,L\to}{Racd,Rdef,Rbcd,R0ab,e:C,a:A\to B,c:C\to A\Rightarrow f:B}\,\text{R9}\ (x\text{ fresh})}{Rdef,Rbcd,R0ab,e:C,a:A\to B,c:C\to A\Rightarrow f:B}\,\text{R4}}{Rbcd,R0ab,a:A\to B,c:C\to A\Rightarrow d:C\to B}\,R\to\ (e,f\text{ fresh})}{R0ab,a:A\to B\Rightarrow b:(C\to A)\to(C\to B)}\,R\to\ (c,d\text{ fresh})}{\Rightarrow 0:(A\to B)\to((C\to A)\to(C\to B))}\,R\to\ (a,b\text{ fresh})
$$

where the conclusion of δ_1 is derived by:

$$
\dfrac{\dfrac{R0xx,Raxf,Rcex,\mathcal{S}'',x:A,e:C\Rightarrow f:B,x:A}{Raxf,Rcex,\mathcal{S}'',x:A,e:C\Rightarrow f:B,x:A}\,\text{R2}\qquad \dfrac{R0ff,Raxf,Rcex,\mathcal{S}'',f:B,x:A,e:C\Rightarrow f:B}{Raxf,Rcex,\mathcal{S}'',f:B,x:A,e:C\Rightarrow f:B}\,\text{R2}}{Raxf,Rcex,\mathcal{S}',e:C,a:A\to B\Rightarrow f:B}\,L\to
$$

with $\mathcal{S} = Raxf,Racd,Rdef,Rbcd,R0ab,a:A\to B,c:C\to A$,
$\mathcal{S}' = Racd,Rdef,Rbcd,R0ab,c:C\to A$ and $\mathcal{S}'' = Racd,Rdef,Rbcd,R0ab,c:C\to A,a:A\to B$.
G3rT $\vdash\Rightarrow 0:(A\to(A\to B))\to(A\to B)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Racx, Rxcd, \mathcal{S}, c : A \Rightarrow d : B, c : A}{Racx, Rxcd, \mathcal{S}, c : A \Rightarrow d : B, c : A}\,\text{R2} \qquad \begin{array}{c}\vdots\,\delta_1\\ Racx, Rxcd, \mathcal{S}, x : A \to B, c : A \Rightarrow d : B\end{array}}{Racx, Rxcd, Racd, Rbcd, R0ab, a : A \to (A \to B), c : A \Rightarrow d : B}\,L\to}{Racd, Rbcd, R0ab, a : A \to (A \to B), c : A \Rightarrow d : B}\,\text{R10}\ (x\text{ fresh})}{Rbcd, R0ab, a : A \to (A \to B), c : A \Rightarrow d : B}\,\text{R4}}{R0ab, a : A \to (A \to B) \Rightarrow b : A \to B}\,R\to\ (c, d\text{ fresh})}{\Rightarrow 0 : (A \to (A \to B)) \to (A \to B)}\,R\to\ (a, b\text{ fresh})
$$

and δ_1 is derived by:

$$
\dfrac{\dfrac{R0cc, Racx, Rxcd, \mathcal{S}', c : A \Rightarrow d : B, c : A}{Racx, Rxcd, \mathcal{S}', c : A \Rightarrow d : B, c : A}\,\text{R2} \qquad \dfrac{R0dd, Racx, Rxcd, \mathcal{S}', d : B, c : A \Rightarrow d : B}{Racx, Rxcd, \mathcal{S}', d : B, c : A \Rightarrow d : B}\,\text{R2}}{Racx, Rxcd, \mathcal{S}, x : A \to B, c : A \Rightarrow d : B}\,L\to
$$

where $\mathcal{S} = Racd, Rbcd, R0ab, a : A \to (A \to B)$ and $\mathcal{S}' = Racd, Rbcd, R0ab, a : A \to (A \to B), x : A \to B$.

G3rT $\vdash \Rightarrow 0 : (A \wedge (A \to B)) \to B$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0aa, Raaa, \mathcal{S}, a : A \Rightarrow b : B, a : A}{Raaa, \mathcal{S}, a : A \Rightarrow b : B, a : A}\,\text{R2} \qquad Raaa, R0ab, a : B, a : A, a : A \to B \Rightarrow b : B}{Raaa, R0ab, a : A, a : A \to B \Rightarrow b : B}\,L\to}{R0ab, a : A, a : A \to B \Rightarrow b : B}\,\text{R11}}{R0ab, a : A \wedge (A \to B) \Rightarrow b : B}\,L\wedge}{\Rightarrow 0 : (A \wedge (A \to B)) \to B}\,R\to\ (a, b\text{ fresh})
$$

where $\mathcal{S} = R0ab, a : A \to B$.

G3rT $\vdash \Rightarrow 0 : (A \to {\sim}A) \to {\sim}A$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0b^*a^*, R0ab, \mathcal{S}, b^* : A, \Rightarrow a^* : A}{R0ab, \mathcal{S}, b^* : A, \Rightarrow a^* : A}\,\text{R5}}{R0ab, \mathcal{S} \Rightarrow a^* : A, b : {\sim}A}\,R{\sim} \qquad \dfrac{\dfrac{\dfrac{R0b^*a^*, R0ab, \mathcal{S}, b^* : A \Rightarrow a^* : A}{R0ab, \mathcal{S}, b^* : A \Rightarrow a^* : A}\,\text{R5}}{R0ab, \mathcal{S} \Rightarrow b : {\sim}A, a^* : A}\,R{\sim}}{R0ab, \mathcal{S}, a : {\sim}A \Rightarrow b : {\sim}A}\,L{\sim}}{Raa^*a, R0ab, a : A \to {\sim}A \Rightarrow b : {\sim}A}\,L\to}{R0ab, a : A \to {\sim}A \Rightarrow b : {\sim}A}\,\text{R12}}{\Rightarrow 0 : (A \to {\sim}A) \to {\sim}A}\,R\to\ (a, b\text{ fresh})
$$

where $\mathcal{S} = Raa^*a, a : A \to {\sim}A$.

G3rT $\vdash \Rightarrow 0 : A \vee {\sim}A$.

$$
\dfrac{\dfrac{\dfrac{R00^*0, 0^* : A \Rightarrow 0 : A}{0^* : A \Rightarrow 0 : A}\,\text{R15}}{\Rightarrow 0 : A, 0 : {\sim}A}\,R{\sim}}{\Rightarrow 0 : A \vee {\sim}A}\,R\vee
$$

G3rRW $\vdash \Rightarrow 0 : (A \to (B \to C) \to (B \to (A \to C))$.

$$
\begin{prooftree}
\AxiomC{$R0ee, Raey, Rycf, \mathcal{S}, e : A, c : B \Rightarrow f : C, e : A$}
\RightLabel{R2}
\UnaryInfC{$Raey, Rycf, \mathcal{S}, e : A, c : B \Rightarrow f : C, e : A$}
\AxiomC{$\vdots\ \delta_1$}
\noLine
\UnaryInfC{$Raey, Rycf, \mathcal{S}, y : B \to C, e : A, c : B \Rightarrow f : C$}
\RightLabel{$L \to$}
\BinaryInfC{$Raey, Rycf, Racd, Rdef, Rbcd, R0ab, e : A, c : B, a : A \to (B \to C) \Rightarrow f : C$}
\LeftLabel{(y fresh)}
\RightLabel{R14}
\UnaryInfC{$Racd, Rdef, Rbcd, R0ab, e : A, c : B, a : A \to (B \to C) \Rightarrow f : C$}
\RightLabel{R4}
\UnaryInfC{$Rdef, Rbcd, R0ab, e : A, c : B, a : A \to (B \to C) \Rightarrow f : C$}
\LeftLabel{(e, f fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$Rbcd, R0ab, c : B, a : A \to (B \to C) \Rightarrow d : A \to C$}
\LeftLabel{(c, d fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$R0ab, a : A \to (B \to C) \Rightarrow b : B \to (A \to C)$}
\LeftLabel{(a, b fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$\Rightarrow 0 : (A \to (B \to C)) \to (B \to (A \to C))$}
\end{prooftree}
$$

and δ_1 is derived by:

$$
\begin{prooftree}
\AxiomC{$R0cc, Raey, Rycf, \mathcal{S}', e : A, c : B \Rightarrow f : C, c : B$}
\RightLabel{R2}
\UnaryInfC{$Raey, Rycf, \mathcal{S}', e : A, c : B \Rightarrow f : C, c : B$}
\AxiomC{$R0ff, Raey, Rycf, \mathcal{S}', f : C, e : A, c : B \Rightarrow f : C$}
\RightLabel{R2}
\UnaryInfC{$Raey, Rycf, \mathcal{S}', f : C, e : A, c : B \Rightarrow f : C$}
\RightLabel{$L \to$}
\BinaryInfC{$Raey, Rycf, \mathcal{S}, y : B \to C, e : A, c : B \Rightarrow f : C$}
\end{prooftree}
$$

where $\mathcal{S} = Racd, Rdef, Rbcd, R0ab, a : A \to (B \to C)$ and $\mathcal{S}' = Racd, Rdef, Rbcd, R0ab, a : A \to (B \to C), y : B \to C$.

G3rRW $\vdash \Rightarrow 0 : A \to ((A \to B) \to B)$.

$$
\begin{prooftree}
\AxiomC{$R0aa, Rcad, \mathcal{S}, a : A \Rightarrow d : B, a : A$}
\RightLabel{R2}
\UnaryInfC{$Rcad, \mathcal{S}, a : A \Rightarrow d : B, a : A$}
\AxiomC{$R0dd, Rcad, \mathcal{S}, d : B, a : A \Rightarrow d : B$}
\RightLabel{R2}
\UnaryInfC{$Rcad, \mathcal{S}, d : B, a : A \Rightarrow d : B$}
\RightLabel{$L \to$}
\BinaryInfC{$Rcad, Racd, Rbcd, R0ab, a : A, c : A \to B \Rightarrow d : B$}
\RightLabel{R13}
\UnaryInfC{$Racd, Rbcd, R0ab, a : A, c : A \to B \Rightarrow d : B$}
\RightLabel{R4}
\UnaryInfC{$Rbcd, R0ab, a : A, c : A \to B \Rightarrow d : B$}
\LeftLabel{(c, d fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$R0ab, a : A \Rightarrow b : (A \to B) \to B$}
\LeftLabel{(a, b fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$\Rightarrow 0 : A \to ((A \to B) \to B)$}
\end{prooftree}
$$

where $\mathcal{S} = Racd, Rbcd, R0ab, c : A \to B$.

G3rR $\vdash \Rightarrow 0 : ((A \to A) \to B) \to B$.

$$
\begin{prooftree}
\AxiomC{$R0cd, Ra0a, R0ab, c : A \Rightarrow b : A, d : A$}
\LeftLabel{(c, d fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$Ra0a, R0ab \Rightarrow b : B, 0 : A \to A$}
\AxiomC{$Ra0a, R0ab, a : B \Rightarrow b : B$}
\RightLabel{$L \to$}
\BinaryInfC{$Ra0a, R0ab, a : (A \to A) \to B \Rightarrow b : B$}
\RightLabel{R16}
\UnaryInfC{$R0ab, a : (A \to A) \to B \Rightarrow b : B$}
\LeftLabel{(a, b fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$\Rightarrow 0 : ((A \to A) \to B) \to B$}
\end{prooftree}
$$

G3rRM $\vdash \Rightarrow 0 : A \to (A \to A)$.

$$
\begin{prooftree}
\AxiomC{$R0ad, Racd, \mathcal{S}, a : A, c : A \Rightarrow d : A$}
\AxiomC{$R0cd, Racd, \mathcal{S}, a : A, c : A \Rightarrow d : A$}
\RightLabel{R17}
\BinaryInfC{$Racd, Rbcd, R0ab, a : A, c : A \Rightarrow d : A$}
\RightLabel{R4}
\UnaryInfC{$Rbcd, R0ab, a : A, c : A \Rightarrow d : A$}
\LeftLabel{(c, d fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$R0ab, a : A \Rightarrow b : A \to A$}
\LeftLabel{(a, b fresh)}
\RightLabel{$R \to$}
\UnaryInfC{$\Rightarrow 0 : A \to (A \to A)$}
\end{prooftree}
$$

where $\mathcal{S} = Rbcd, R0ab$. $\square$

B2 Proof of Theorem 4.5.3

Proof of Theorem 4.5.3 (Semantic Completeness cont.) In this appendix we construct a reduction tree for an arbitrary sequent $\mathcal{S}$, by applying, root-first, all rules for **G3rX** according to a specific order. This construction is used to define a countermodel to $\mathcal{S}$ (displayed above). Importantly, recall that, to reflect the notion of validity at the actual world, we will consider derivability at 0.
The reduction tree is defined inductively in stages as follows: (1) If $n = 0$, then $\Gamma \Rightarrow \Delta$ stands at the root of the tree. (2) If $n > 0$, we distinguish two subcases. (2.1) If every topmost sequent is an axiomatic sequent reduction the tree terminates; (2.2) If no axiomatic sequent is reached, the construction of the reduction tree does not terminate and we continue applying, root-first, all rules of **G3rX** according to a specific order. There are $8 + j$ different stages: 8 for the rules for the propositional connectives and j for the mathematical rules. We start, for $n = 1$, with $L{\sim}$ and consider topmost sequents of the following form:

$$0 : {\sim}B_1, \dots, 0 : {\sim}B_k, \Gamma' \Rightarrow \Delta$$

where $0 : {\sim}B_1, \dots, 0 : {\sim}B_k$, are all formulas in Γ with $\sim$ as outermost connective. By applying, root-first, k times, $L{\sim}$ we obtain the following sequent:

$$\Gamma' \Rightarrow \Delta, 0^* : B_1, \dots, 0^* : B_k$$

placed on top of the former.
For $n = 2$, we consider sequents of the form:

$$\Gamma \Rightarrow \Delta', 0 : {\sim}B_1, \dots, 0 : {\sim}B_k$$

By applying, root-first, k times, $R{\sim}$ we obtain the following sequent:

$$0^* : B_1, \dots, 0 :^*: B_k, \Gamma \Rightarrow \Delta'$$

placed on top of the former.
For $n = 3$, we consider sequents of the form:

$$0 : B_1 \wedge C_1, \dots, 0 : B_k \wedge C_k, \Gamma' \Rightarrow \Delta$$

By applying, root-first, k times, $L\wedge$ we obtain the following sequent:

$$0 : B_1, 0 : C_1, \dots, 0 : B_k, 0 : C_k, \Gamma' \Rightarrow \Delta$$

The case for $n = 6$, with $R\vee$ is symmetric.
For $n = 4$, we consider sequents of the form:

$$\Gamma \Rightarrow \Delta', 0 : B_1 \wedge C_1, \dots, 0 : B_k \wedge C_k$$

By applying, root-first, k times, $R\wedge$ we obtain the following sequents:

$$\Gamma \Rightarrow \Delta', 0 : B_1, \dots, 0 : B_k \qquad \text{and} \qquad \Gamma \Rightarrow \Delta', 0 : C_1, \dots, 0 : C_k$$

placed on top of the former as its premises. The case for $n = 5$, with $L\vee$ is symmetric.
For $n = 7$, we consider topmost sequents of the following form:

$$R0a_1b_1, \ldots, R0a_kb_k,\ 0 : B_1 \rightarrow C_1, \ldots, 0 : B_k, \rightarrow C_k,\ \Gamma' \Rightarrow \Delta$$

where labels and principal formulas are in Γ'. By applying, root-first, k times, $L \rightarrow$ (with $R0a_1b_1, \ldots, R0a_kb_k, 0 : B_1 \rightarrow C_1, \ldots, 0 : B_k, \rightarrow C_k$ principal) we obtain the following sequent:

$$R0a_1b_1, \ldots, R0a_kb_k,\ b_{m_1} : C_{m_1}, \ldots, b_{m_l} : C_{m_l},\ \Gamma' \Rightarrow \Delta,\ a_{j_{l+1}} : B, \ldots, a_{j_k} : B$$

where $\{m_1, \ldots, m_l\} \subseteq \{1, \ldots, k\}$ and $j_{l+1}, \ldots, j_k \in \{1, \ldots, k\} - \{m_1, \ldots, m_l\}$, and placed on top of the former as its premises.
For $n = 8$, we consider all the labelled sequents that have implications in the succedent. We consider topmost sequents of the following form:

$$\Gamma \Rightarrow \Delta', 0 : B_1 \rightarrow C_1, \ldots, 0 : B_k, \rightarrow C_k$$

Let $a_1, ..., a_k$ and $b_1, \ldots, b_k$ be fresh variables, not yet used in the reduction tree and apply, root-first, k times, $R \rightarrow$ to obtain the following sequent:

$$R0a_1b_1, \ldots, R0a_kb_k,\ a_1 : B, \ldots, a_k : B,\ \Gamma \Rightarrow \Delta',\ b_1 : C_1, \ldots, b_k : C_k$$

placed on top of the former as its premise.
Finally, we consider relational rules. If it is a rule without eigenvariable condition, we write on top of the lower sequent the result of applying the relational rule under consideration. For relational rules with eigenvariable condition, the proof proceeds analogously to the proof at stage $n = 8$. As an example, consider R7 and a topmost sequent of the following form:

$$Ra_1b_1c_1, \ldots, Ra_kb_kc_k, \Gamma' \Rightarrow \Delta$$

Let $x_1, \ldots, x_k$ be variables not yet used in the reduction tree. By applying k times, root-first, R7, we obtain the following sequent, placed on top of the former:

$$Ra_1b_1x_1, Ra_1x_1c_1, Ra_1b_1c_1, \ldots, Ra_kb_kx_k, Ra_kx_kc_k, Ra_kb_kc_k, \Gamma' \Rightarrow \Delta$$

This construction is then used in the development of the second part of the proof displayed in Section 4.5 (pp. 87 and ff.).

B3 Proof of Theorem 4.6.4

Proof of Theorem 4.6.4 (CUT-admissibility cont.). We finish the proof of CUT-admissibility by displaying some other salient examples.We distinguish three main cases.
Case 1: If at least one of the premises of CUT is an axiom, we distinguish 4 subcases:

Case 1.1: The left premise of CUT is an axiom and the CUT-formula is not principal. If the derivation has the following shape:

$$\dfrac{R0bc, b : B, \Gamma \Rightarrow \Delta, c : B, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B}\ \textsc{cut}$$

It is transformed into:

$$R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B$$

without applications of CUT.
Case 1.2: The left premise of CUT is an axiom and the CUT-formula is principal. The derivation:

$$\dfrac{R0ba, b : A, \Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{R0ba, b : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

is transformed into:

$$\text{(Lemma 4.6.1)}\ \dfrac{\text{(Lemma 4.3.1)}\ \dfrac{a : A, \Gamma' \Rightarrow \Delta'}{b : A, \Gamma' \Rightarrow \Delta'}\ \textsc{sub}(b/a)}{R0ba, b : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{lw+rw+lw}_L$$

Case 1.3: The right premise of CUT is an axiom and the CUT-formula is not principal. The derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, R0bc, b : B, \Gamma' \Rightarrow \Delta', c : B}{R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B}\ \textsc{cut}$$

It is transformed into:

$$R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B$$

without applications of CUT.
Case 1.4: The right premise of CUT is an axiom and the CUT-formula is principal. The derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad R0ab, a : A, \Gamma' \Rightarrow \Delta', b : A}{R0ab, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b : A}\ \textsc{cut}$$

is transformed into:

$$\text{(Lemma 4.6.1)}\ \dfrac{\text{(Lemma 4.3.1)}\ \dfrac{\Gamma \Rightarrow \Delta, a : A}{\Gamma \Rightarrow \Delta, b : A}\ \textsc{sub}(b/a)}{R0ab, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b : A}\ \textsc{lw+rw+lw}_L$$

Case 2: The CUT-formula A is not principal in at least one premise. The proof proceeds by permuting the application of CUT with the rule under consideration, to move the CUT upwards in the transformed derivation.

Case 2.1: A is not principal in the left premise. We distinguish two subcases.
Subcase 2.1.1: Let $\Gamma = x : {\sim}B, \Gamma''$:

$$\dfrac{\dfrac{\Gamma'' \Rightarrow \Delta, a : A, x^* : B}{x : {\sim}B, \Gamma'' \Rightarrow \Delta, a : A}\,L{\sim} \qquad a : A, \Gamma' \Rightarrow \Delta'}{x : {\sim}B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,\textsc{cut}$$

and transform it into the following one:

$$\dfrac{\dfrac{\Gamma'' \Rightarrow \Delta, a : A, x^* : B \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma'', \Gamma' \Rightarrow \Delta, \Delta', x^* : B}\,\textsc{cut}}{x : {\sim}B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,L{\sim}$$

where the CUT-height is reduced.
Let $\Gamma = Rxbc, x : B \rightarrow C, \Gamma''$ and consider as an example $L \rightarrow$. We have the following derivation:

$$\dfrac{\dfrac{Rxbc, x : B \rightarrow C, \Gamma'' \Rightarrow \Delta, a : A, b : B \qquad Rxbc, c : C, x : B \rightarrow C, \Gamma'' \Rightarrow \Delta, a : A}{Rxbc, x : B \rightarrow C, \Gamma'' \Rightarrow \Delta, a : A}\,L\rightarrow \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, x : B \rightarrow C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,\textsc{cut}$$

and transform it into the following one:

$$\dfrac{\dfrac{Rxbc, x : B \rightarrow C, \Gamma'' \Rightarrow \Delta, a : A, b : B \quad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, x : B \rightarrow C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', b : B}\,\textsc{cut} \qquad \dfrac{Rxbc, c : C, x : B \rightarrow C, \Gamma'' \Rightarrow \Delta, a : A \quad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, c : C, x : B \rightarrow C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,\textsc{cut}}{Rxbc, x : B \rightarrow C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\,L\rightarrow$$

with two CUTS of lower height.
Subcase 2.1.2: Let $\Delta = \Delta'', x : {\sim}B$:

$$\dfrac{\dfrac{x^* : B, \Gamma \Rightarrow \Delta'', a : A}{\Gamma \Rightarrow \Delta'', x : {\sim}B, a : A}\,R{\sim} \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : {\sim}B}\,\textsc{cut}$$

it is transformed into the following application of CUT with a shorter derivation height:

$$\dfrac{\dfrac{x^* : B, \Gamma \Rightarrow \Delta'', a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{x^* : B, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta'}\,\textsc{cut}}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : {\sim}B}\,R{\sim}$$

Let $\Delta = \Delta'', x : A \rightarrow B$:

$$\dfrac{(b, c \text{ fresh})\ \dfrac{Rxbc, b : B, \Gamma \Rightarrow \Delta'', c : C, a : A}{\Gamma \Rightarrow \Delta'', x : B \rightarrow C, a : A}\,R\rightarrow \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : B \rightarrow C}\,\textsc{cut}$$

it is transformed into the following application of CUT with a shorter derivation height:

$$\dfrac{\dfrac{Rxbc, b : B, \Gamma \Rightarrow \Delta'', c : C, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, b : B, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', c : C}\ \textsc{cut}}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : B \to C}\ R\to \ (b, c \text{ fresh})$$

As an example for the relational rules, we deal with R7 (with variable condition). Let $\Gamma = Rabc, \Gamma''$:

$$\dfrac{\dfrac{Rabx, Raxc, Rabc, \Gamma'' \Rightarrow \Delta, a : A}{Rabc, \Gamma'' \Rightarrow \Delta, a : A}\ \text{R7} \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rabc, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

(x is a fresh variable) It is transformed in the following one:

$$\dfrac{\dfrac{Rabx, Raxc, Rabc, \Gamma'' \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rabx, Raxc, Rabc, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}}{Rabc, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \text{R7}$$

The other cases for relational rules are dealt with analogously.
Case 2.2: A is principal in the left premise only. We distinguish two subcases.
Subcase 2.2.1: Similarly to the preceding subcase. Let $\Gamma' = x : {\sim}B, \Gamma''$:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad \dfrac{a : A, \Gamma'' \Rightarrow \Delta', x^* : B}{a : A, x : {\sim}B, \Gamma'' \Rightarrow \Delta'}\ L{\sim}}{x : {\sim}B, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma'' \Rightarrow \Delta', x^* : B}{\Gamma, \Gamma'' \Rightarrow \Delta, \Delta', x^* : B}\ \textsc{cut}}{x : {\sim}B, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ L{\sim}$$

with a shorter derivation height.
Let $\Gamma' = Rxbc, x : B \to C, \Gamma''$ and consider $L \to$. We have the following derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad \dfrac{a : A, Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta', b : B \qquad a : A, Rxbc, c : C, x : B \to C, \Gamma'' \Rightarrow \Delta'}{a : A, Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta'}\ L\to}{Rxbc, x : B \to C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

is reduced to the following one:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta', b : B}{Rxbc, x : B \to C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta', b : B}\ \textsc{cut} \qquad \dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, Rxbc, c : C, x : B \to C, \Gamma'' \Rightarrow \Delta'}{Rxbc, c : C, x : B \to C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ \textsc{cut}}{Rxbc, x : A \to B, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ L\to$$

with two cuts of lower height.
Subcase 2.2.2: Let $\Delta' = \Delta'', x : {\sim}B$:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad \dfrac{x^* : B, a : A, \Gamma' \Rightarrow \Delta''}{a : A, \Gamma' \Rightarrow \Delta'', x : {\sim}B}\ R{\sim}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : {\sim}B}\ \textsc{cut}$$

it is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad x^* : B, a : A, \Gamma' \Rightarrow \Delta''}{x^* : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta''}\ \text{CUT}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : {\sim}B}\ R{\sim}$$

with a shorter derivation height.
Let $\Delta' = \Delta'', a : B \rightarrow C$ and the derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad (b, c \text{ fresh})\ \dfrac{Rxbc, a : A, b : B, \Gamma' \Rightarrow \Delta'', c : C}{a : A, \Gamma' \Rightarrow \Delta'', x : B \rightarrow C}\ R\rightarrow}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : B \rightarrow C}\ \text{CUT}$$

It is reduced to the following one:

$$(b, c \text{ fresh})\ \dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad Rxbc, a : A, b : B, \Gamma' \Rightarrow \Delta'', c : C}{Rxbc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', c : C}\ \text{CUT}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : B \rightarrow C}\ R\rightarrow$$

with a shorter derivation height.
Case 3: The procedure for A being ${\sim}B$ or $B \rightarrow C$, can be found on pp. 91 and ff.

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Curriculum Vitae
Fabio De Martin Polo

1. Generalities & Contacts

Name:	Fabio
Surname:	De Martin Polo
Date of birth:	19/12/1995
Place of birth:	Brunico, Italy
Nationality:	Italian

2. Current Position

10/2019 - 09/2023	**PhD in Philosophy** Ruhr University Bochum, Department of Philosophy I Thesis: *Topics in the Proof Theory of Non-classical Logics. Philosophy and Applications*. Defence: July 17th, 2023. Committee: Heinrich Wansing (supervisor), Hitoshi Omori (co-supervisor), Francesca Poggiolesi, Christian Straßer.

3. Education

07/2019	**M.A. in Philosophical Sciences** University Ca' Foscari Venice (Italy) Thesis: *Numbers, Objects and Abstraction. Notes on a philosophical interpretation of mathematics.* Supervisors: Dr. Enrico Jabara, Prof. Dr. Paolo Pagani. Grade: 110/110 CUM LAUDE.
SoSe 2018	**Visiting Semester** Munich Center for Mathematical Philosophy, Ludwig-Maximilian-University Munich (Germany) Areas of study: Modal Logic, Incompleteness Theorems, Proof Theory, Axiomatic Metaphysics.
07/2017	**B.A. in Philosophy** University Ca' Foscari Venice (Italy) Thesis: *From Logic to Philosophy. Kurt Gödel's Ontological Argument.* Supervisor: Prof. Dr. Luigi Perissinotto. Grade: 110/110 CUM LAUDE.
07/2014	**High School Degree** High School 'Dante Alighieri', Bressanone (Italy) Areas of study: Foreign languages, Humanities, Latin, Sciences.

4. Publications

Journal articles

2023	**De Martin Polo, F.**: "Modular Labelled Calculi for Relevant Logics". *The Australasian Journal of Logic*. Vol. 20, No. 1, pp. 47–87.

Book Chapers/Conference Proceedings

Forthcoming	**De Martin Polo, F.**: "Discussive Logic. A Short History of the First Paraconsistent Logic ". In: Ingolf Max and Jens Lemanski (eds.), *Historia Logicae*, vol. 1, College Publications, London. (Accepted: November 2021)
2021	**De Martin Polo, F.**: "A CUT-free Hypersequent Calculus for Intuitionistic Modal Logic **IS5**". In: Butler, A. (ed.), *Proceedings of the 18th International Workshop of Logic and Engineering of Natural Language Semantics 18 (LENLS18), JSAI-isAI2021* pp. 217-230.

PhD Thesis

Forthcoming	**De Martin Polo, F.**: *Topics in the Proof Theory of Non-classical Logics. Philosophy and Applications*, Ruhr University Bochum.

Under review

De Martin Polo, F.: "A note on CUT-elimination for intuitionistic logic with 'actuality'", under review for *The Logic Journal of the IGPL. Special issue on Non-classical Modal and Predicate Logics*. (Submitted: October 2022)

De Martin Polo, F.: "Fusion, Fission and Ackermann's Truth Constant in Relevant Logics. A proof theoretic investigation", In *New Directions in Relevant Logics*, Springer. (Submitted: March 2023)

De Martin Polo, F.: "Beyond Semantic Pollution: Towards a Practiced-Based Philosophical Analysis of Labelled calculi", *Erkenntnis*. (Submitted: June 2023)

5. Presentations

01/2023	"Semantic Proof Systems: Why and How to Stop Worrying". Workshop on *Perspectives on Logic and Philosophy*, Ruhr University Bochum (Germany).
11/2022	"On the Proof Theory for Relevant Logics". *Workshop on New Directions in Relevant Logic*, online event organised by Igor Sedlár, Shawn Standefer and Andrew Tedder.
11/2022	"Reduced Routley-Meyer Models and Labelled Sequent Calculi for Relevant Logics". *Autumn Workshop on Proof Theory and its Applications 2022*, Utrecht University (Netherlands).
09/2022	"Notes on the Proof Theory for Relevant Logics. From Routley-Meyer Semantics to Labelled Sequent Systems". *Logica 2022*, The Premonstratensian Monastery of Teplá (Czech Republic).
09/2022	"Remarks on Jaśkowski's Discussive Logic. Origins, Development and a Little Proof Theory". *The 6th World Congress of Paraconsistency – The 2nd Stanisław Jaśkowski Memorial Symposium*, University of Toruń (Poland).
09/2022	"CUT-elimination for Intuitionistic Logic with Actuality Resolved via Hypersequents". *PhDs in Logic XIII*, University of Turin (Italy).
06/2022	"Proof Analysis in Jaśkowski's Discussive Logic. CUT-admissibility and Embedding theorems". *5th SILFS Postgraduate Conference on Logic and Philosophy of Science*, University Bicocca Milan (Italy).
11/2021	"A CUT-free Hypersequent Calculus for Intuitionistic Modal Logic **IS5**". *Logic and Engineering of Natural Language Semantics 18 (LENLS18), Workshop of the JSAI International Symposia on AI 2021*, Keio University, Yokohama, Kanagawa (Japan).
10/2018	"The Art of Paradox". *Seminar*, Italian Institute of Sciences, Literature and Arts, Venice (Italy).

6. Teaching experiences

WiSe 2022	*Philosophy of alternative mathematics*, Reading seminar, Ruhr University Bochum (principal instructor, with Franci Mangraviti).
SoSe 2021	*History of connexive logic*, Reading seminar, Ruhr University Bochum (some lessons).

Schools:

11/2022	Autumn School on 'Proof Theory and its Applications', Utrecht University (participant).
10/2020	Hilbert-Bernays Summer School on 'Logic and Computation', Georg August Universität Göttingen, Connexive Logic Class (tutor).

7. Service to the profession

Refereeing (Journals) :	*Logic Journal of the IGPL, Bulletin of the Section of Logic, Studia Logica*.
Refereeing (Conferences):	*AiML2022*.

Organised Events: Co-organizer, Workshop on *Perspectives on Logic and Philosophy*, Ruhr University Bochum (Germany), January 9–11, 2023 (Link).

Organizing committee member, *Trends in Logic XXI*, Ruhr University Bochum (Germany), December 6–8, 2021 (Link).

Organizing committee member, *NCMPL 2021* (Non-Classical Modal and Predicate Logics), Ruhr University Bochum (Germany), November 23–26, 2021 (Link).