A Note on Graded Modal Logic

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Abstract

We introduce a notion of bisimulation for graded modal logic. Using these bisimulations the model theory of graded modal logic can be developed in a uniform manner. We illustrate this by establishing the finite model property, and proving invariance and definability results.

1 Introduction

The language of graded modal logic (GML) has modal operators $\Diamond^{\geqslant n}$ (for $n \in \mathbb{N}$) that can count the number of successors of a given state: a state w in a model (W, R, V) satisfies $\Diamond^{\geqslant n}\varphi$ iff there exist at least n R-related states that satisfy φ. Originally introduced in the early 1970s [9, 10], the language has enjoyed an increased interest during the past few years, especially because of its considerable expressive power. Formal logical and algebraic results on axiomatizability, decidability, and expressive completeness over bounded trees have been reported in a number of papers [2, 3, 5, 7, 8, 12, 20], and the language has shown up in various guises in knowledge representation, generalized quantifier theory, algebraic logic, and fuzzy reasoning [6, 13, 14, 17, 18].

This note is concerned with graded modal logic as a description language for reasoning about models. It is part of a larger enterprise to study the model theory — and in particular, the expressive power — of restricted description languages such as modal and temporal languages, terminological logics and feature logics (cf. [1, 15, 16, 19]). Bisimulations have proved to be a very powerful tool in this area, but so far a version of bisimulation that is appropriate for graded modal logic has not been proposed.[1] As a consequence, the model theory of graded modal logic is not as well developed as the model theory of, say, standard modal or temporal logic. In this note we propose a notion of bisimulation, called g-bisimulation that 'fits' GML exactly in the sense that a first-order formula is invariant under g-bisimulations iff it is equivalent to a graded modal formula (cf. Theorem 4.3 below).

The remainder of this note is organized as follows. The next section introduces the main notions needed. In Section 3 g-bisimulations are defined. In Section 4 we first give a quick and intuitive proof for the finite model property of GML using g-bisimulations, and then prove the above invariance theorem, as well as two results on definability. Section 5 contains some concluding comments.

2 Basic definitions

Graded modal formulas are built up using propositional variables $p, q, \ldots$, the constants $\top$ and $\bot$, boolean connectives $\neg$, $\wedge$, and the unary modal operators $\Diamond^{\geqslant n}$, for $n \geqslant 1$. We use $\mathcal{L}_{\mathsf{GML}}$ to denote this language.

[1] Typist's comment: In fact, such a notion has been proposed by van der Hoek in [12], but apparently has been overlooked by the author. However, the cited paper did not establish the invariance and definability results presented here, and the definition itself was slightly different. in fact, simpler. The question remains whether the results of the present note can be obtained with the definition of graded bisimulations proposed by van der Hoek.

A *model* is a triple $M = (W, R, V)$, where W is a non-empty set of states, R is a binary relation on W, and V is a valuation, that is: a function assigning a subset of W to every proposition letter. The *satisfaction relation* is defined in the familiar way for the atomic and boolean cases, while for the modal operators we put

$$M, w \models \Diamond^{\geqslant n}\varphi \iff \exists^{\neq} v_1 \dots v_n \bigwedge_{1\leqslant i\leqslant n}(Rwv_i \wedge M, v_i \models \varphi),$$

where $\exists^{\neq} v_1 \dots v_n \, \Phi \equiv \exists v_1 \dots v_n \, (\bigwedge_{1\leqslant i<j\leqslant n}(v_i \neq v_j) \wedge \Phi)$. If X is a set of states, we write $X \models \varphi$ to denote that $v \models \varphi$ for all $v \in X$.

The *graded modal type* of a state is the set of all graded modal formulas it satisfies: $\mathsf{tp}(w) = \{\varphi \in \mathcal{L}_{\mathsf{GML}} \mid w \models \varphi\}$; if necessary we record the model M in which w lives as a subscript: $\mathsf{tp}_M(w)$. Two states w, v are *graded modally equivalent* if $\mathsf{tp}(w) = \mathsf{tp}(v)$ (notation: $w \equiv_{\mathsf{g}} v$).

Let $\mathcal{L}_1$ be the first-order language with unary predicate symbols corresponding to the proposition letters in $\mathcal{L}_{\mathsf{GML}}$, and with one binary relation symbol R. Models can be viewed as $\mathcal{L}_1$-structures in the usual first-order sense. The *standard translation* takes graded modal formulas φ into equivalent formulas $\mathrm{ST}_x(\varphi)$ in $\mathcal{L}_1$. It maps proposition letters p to unary predicate symbols Px, commutes with the booleans, and the modal cases are given by

$$\mathrm{ST}_x(\Diamond^{\geqslant n}\varphi) = \exists^{\neq} y_1 \dots y_n \bigwedge_{1\leqslant i\leqslant n}(Rxy_i \wedge \mathrm{ST}_{y_i}(\varphi)).$$

We sometimes write $\varphi^*(x)$ as a shortcut for $\mathrm{ST}_x(\varphi)$. For any model M and state w in M, we have $M, w \models \varphi$ iff $M \models \varphi^*[w]$, where the latter denotes first-order satisfaction of $\varphi^*(x)$ under the assignment of $x := w$.

3 G-bisimulations

In this section we introduce the main notion of this note: g-bisimulations. In [19] bisimulations are advocated as the central tool in the model theory of modal logic; see [15, 16] for case studies implementing this strategy for Since, Until logic, and for negation-free modal logics. In Section 4 below we will use g-bisimulations to establish the finite model property, and to prove invariance and definability results for graded modal logic, thus showing that g-bisimulations can play a similar central role in the model theory of graded modal logic.

By way of introduction we first consider bisimulations.

Definition 3.1. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models. A *bisimulation* between M and M' is a relation $Z \subseteq W \times W'$ satisfying the following requirements:

1. Z is non-empty;
2. if xZx', then $x \models p$ iff $x' \models p$, for all proposition letters p;
3. if xZx' and xRy, then there exists $y' \in W'$ with yZy' and $x'R'y'$;
4. if xZx' and $x'R'y'$, then there exists $y \in W$ with yZy' and xRy.

We write $Z\colon (M, w) \leftrightarrows (M', w')$ to denote that Z is a bisimulation between M and M' with wZw'.

Ordinary modal formulas are preserved under bisimulations: if $(M, w) \leftrightarrows (M', w')$, then for any modal formula φ, we have $M, w \models \varphi$ iff $M', w' \models \varphi$. On the contrary, graded modal formulas are not preserved under bisimulations. To see this, consider the following two models M_1 and M_2, where $M_1 = (\{0, 1, 2\}, \{(0, 1), (0, 2)\}, V_1)$, $M_2 = (\{3, 4\}, \{(3, 4)\}, V_2)$, and V_1, V_2 verify all proposition letters true in all states. The relation $Z = \{(0, 3), (1, 4), (2, 4)\}$ is a bisimulation between M_1 and M_2. But $0 \not\equiv_{\mathsf{g}} 3$, as $0 \models \Diamond^{\geqslant 2}\top$ and $3 \not\models \Diamond^{\geqslant 2}\top$.

To define a truth-preserving notion of bisimulation for graded modal logic, we need the following definitions. If X is a set, we write $|X|$ to denote its cardinality, and 2^X_n to denote the collection of all subsets of X of cardinality $n \geqslant 1$. Also, we write $xR_{\bullet}Y$ to denote that xRy for all $y \in Y$. In this notation we have: $M, w \models \Diamond^{\geqslant n}\varphi$ iff there is $Y \in 2^W_n$ with $wR_{\bullet}Y$ and $Y \models \varphi$.

Definition 3.2. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models. A g-*bisimulation* between M and M' is a tuple $Z = (Z_1, Z_2, \dots)$ of relations $Z_n \subseteq 2^W_n \times 2^{W'}_n$ satisfying the following requirements:

1. Z_1 is non-empty;
2. if $\{x\}\, Z_1\, \{x'\}$, then $x \models p$ iff $x' \models p$, for all proposition letters p;
3. if $\{x\}\, Z_1\, \{x'\}$ and $xR_\bullet Y$, where $Y \in 2^W_n$, then there exists $Y' \in 2^{W'}_n$ with $Y\, Z_n\, Y'$ and $x'R'_\bullet Y'$;
4. if $\{x\}\, Z_1\, \{x'\}$ and $x'R'_\bullet Y'$, where $Y' \in 2^{W'}_n$, then there exists $Y \in 2^W_n$ with $Y\, Z_n\, Y'$ and $xR_\bullet Y$;
5. if $X\, Z_n\, X'$, then
   (a) for every $x \in X$ there exists $x' \in X'$ with $\{x\}\, Z_1\, \{x'\}$, and
   (b) for every $x' \in X'$ there exists $x \in X$ with $\{x\}\, Z_1\, \{x'\}$.

We write $M, w \leftrightarroweq_{\mathsf{g}} M', w'$ if there is a g-bisimulation Z between M and M' with $\{w\}\, Z_1\, \{w'\}$.

To grasp the intuition behind Def. 3.2, reconsider the definition of a (normal) bisimulation. There, bisimilar states satisfy the same (ordinary) modal formulas because they satisfy the same proposition letters (Def. 3.1, item 2), and because the relevant relational patterns present in one model are mirrored in the other (Def. 3.1, items 3 and 4). To guarantee that g-bisimilar states satisfy the same graded modal formulas, one requires, firstly, that they satisfy the same proposition letters (Def. 3.2, item 2). Next, to preserve formulas of the form $\Diamond^{\geqslant n}\varphi$, sets of successors of size n present in one model should be mirrored in the other (Def. 3.2, items 3 and 4). If two such sets 'mirror' each other, and all the states in the one set agree on a formula, then all the states in the other should do so as well (Def. 3.2, item 5(a,b)). Formally, this is expressed in the following proposition.

Proposition 3.3. *If* $M, w \leftrightarroweq_{\mathsf{g}} M', w'$*, then* $w \equiv_{\mathsf{g}} w'$*.*

Proof. Let Z be a g-bisimulation between (M, w) and (M', w'). The proof is by induction on formulas. The atomic and boolean cases are trivial. For the modal case, assume that $w \models \Diamond^{\geqslant n}\varphi$. Then there exists $Y \in 2^W_n$ with $wR_\bullet Y$ and $Y \models \varphi$. By item 3 of Def. 3.2, there exists $Y' \in 2^{W'}_n$ with $Y\, Z_n\, Y'$ and $w'R'_\bullet Y'$. We are done once we have shown that $Y' \models \varphi$, for then $w' \models \Diamond^{\geqslant n}\varphi$. To this end, pick any $y' \in Y'$. By item 5(b) of Def. 3.2, there exists $y \in Y$ with $\{y\}\, Z_1\, \{y'\}$. As $Y \models \varphi$, we get $y \models \varphi$ and, by induction hypothesis, this implies $y' \models \varphi$. □

As a corollary, the models M_1 and M_2 considered above are not g-bisimilar.

By restricting the definition of g-bisimulation to just a finite tuple $(Z_1, \dots, Z_k)$ we arrive at the notion of g_k-bisimulation; we write $M, w \leftrightarroweq_{\mathsf{g}_k} M', w'$ if there is a g_k-bisimulation between w and w'. This notion of bisimulation is appropriate for the fragment $\mathcal{L}_{\mathsf{GML}}$ in which all modal operators $\Diamond^{\geqslant n}$ have subscripts $n \leqslant k$. In particular, for $k = 1$ we get a notion that is equivalent to the standard notion of bisimulation defined in Def. 3.1.

Let us introduce another restriction, which does not limit the length of the tuple $(Z_1, \dots)$, but rather the number of times the clauses in Def. 3.2 can be applied starting from a given pair of points.

Definition 3.4. Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models, and let $m \geqslant 0$ be a natural number. A g-*bisimulation up to* m between M and M' is a sequence of tuples $Z^i = (Z^i_1, Z^i_2, \dots)$, $0 \leqslant i \leqslant m$, of relations $Z^i_n \subseteq 2^W_n \times 2^{W'}_n$ satisfying the following requirements:

1. Z^0_1 is non-empty, and $Z^m_n \subseteq \ldots \subseteq Z^0_n$ for each $n \geqslant 1$;
2. if $\{x\}\, Z^0_1\, \{x'\}$, then $x \models p$ iff $x' \models p$, for all proposition letters p;
3. if $\{x\}\, Z^{i+1}_1\, \{x'\}$ and $xR_\bullet Y$, where $i < m$ and $Y \in 2^W_n$, then $\exists Y' \in 2^{W'}_n$ with $Y\, Z^i_n\, Y'$ and $x'R'_\bullet Y'$;
4. similar to item 3;
5. like item 5 in Def. 3.2, but with Z^i_n and Z^i_1 instead of Z_n and Z_1, for all $0 \leqslant i \leqslant m$.

We write $M, w \leftrightarroweq^m_{\mathsf{g}} M', w'$ if there is a g-bisimulation up to m between M and M', say $Z^0, \dots, Z^m$, with $\{w\}\, Z^0_1\, \{w'\}$. The notion of a g_k-*bisimulation up to* m is defined similarly. The notation $\leftrightarroweq^m_{\mathsf{g}_k}$ has the obvious meaning.

Let $M = (W, R, V)$ be a model, and assume $w \in W$. For each $i \in \mathbb{N}$ we define the *i-hull* $H_i(w)$ around w in M as follows. The 0-hull $H_0(w)$ is simply $\{w\}$; the $(i+1)$-hull is the set $H_{i+1}(w) = R(H_i(w)) = \{y \in W \mid \exists x \in H_i(w)\colon xRy\}$.

We write M_w to denote the submodel of M that is generated by w. That is, M_w is the submodel of M whose domain is $\bigcup_{i\geqslant 0} H_i(w)$. Clearly, for any model M and state w in M, $(M, w) \leftrightarroweq_{\mathsf{g}} (M_w, w)$.

If M is generated by w, we define the *restriction of M to depth m*, notation: $M{\restriction}m$, to be the submodel of M whose domain is the set $\bigcup_{0\leqslant i\leqslant m} H_i(w)$.

Proposition 3.5. *Let M be a model generated by a world w. Then $(M, w) \leftrightarroweq^m_{\mathsf{g}} (M{\restriction}m, w)$.*

The *degree* of a graded modal formula is the largest number of nested modal operators occurring in it. The *index* of a formula is the highest n such that the operator $\Diamond^{\geqslant n}$ occurs in the formula. Let $w \equiv^m_{\mathsf{g}} w'$ (resp. $w \equiv^m_{\mathsf{g}_k} w'$) denote that w and w' verify the same graded modal formulas of degree at most m (and index at most k).

Proposition 3.6. *Let M, M' be two models, and let $w \in W$, $w' \in W'$, $m \geqslant 0$.*
If $(M, w) \leftrightarroweq^m_{\mathsf{g}} (M', w')$, then $w \equiv^m_{\mathsf{g}} w'$.
If $(M, w) \leftrightarroweq^m_{\mathsf{g}_k} (M', w')$, then $w \equiv^m_{\mathsf{g}_k} w'$?

In Theorem 4.5 we will establish, in particular, the converse of the latter implication.

4 Results

In this section we first give a new and intuitive proof of the finite model property for graded modal logic using g-bisimulations. We then use g-bisimulations to prove the main results of this note: invariance and definability.

4.1 Finite model property

The finite model property for graded modal logic was first established in [11]; see also [3, 12]. The proof presented below is attractive because it clearly brings out the two obvious reasons why $\mathcal{L}_{\mathsf{GML}}$ has the finite model property: to determine the truth or falsehood of a graded modal formula, only R-paths $wR \dots Rv$ of finite length are needed, and every state on such a path only needs finitely many successors.

Let's get to work. Fix a satisfiable formula φ with degree m and index k. Let M and w be such that $M, w \models \varphi$. We will construct a finite submodel of M that is still a model for φ. First, we may assume that $M = M_w$. Consider $M{\restriction}m$; it only has finite R-paths, and $(M{\restriction}m, w), w \models \varphi$. Note that $M{\restriction}m$ need not be finite, as it may be infinitely branching.

Consider the sublanguage $\mathcal{L}_{\mathsf{GML}}(\varphi)$ in which all formulas are built up using only proposition letters that occur in φ. It is easily verified that there are only finitely many non-equivalent formulas in $\mathcal{L}_{\mathsf{GML}}(\varphi)$ with degree at most m and index at most k.

Our final model $(M{\restriction}m)^{\leqslant k}$ is defined as follows. Its domain is the union of certain subsets $H'_0, \dots, H'_m$ of the domain of $M{\restriction}m$, where $H'_i \subseteq H_i(w)$. Here $H'_0 = H_0(w) = \{w\}$, and to define H'_i (for $1 \leqslant i \leqslant m$) do the following:

set $H'_i = \varnothing$
  for all $x \in H'_{i-1}$
    for each of the finitely many non-equivalent $\mathcal{L}_{\mathsf{GML}}(\varphi)$-formulas ψ
    of degree at most $(m-i)$ and index at most k
      select as many as possible (but at most k) R-successors y of x with $y \models \psi$
      add these states to H'_i
end.

The accessibility relation and valuation of the model $(M{\restriction}m)^{\leqslant k}$ are simply the restrictions to its the domain. Clearly, $(M{\restriction}m)^{\leqslant k}$ is finite, and $(M{\restriction}m)^{\leqslant k}, w \leftrightarroweq^{m}_{\mathsf{g}_k} M{\restriction}m, w$. Putting things together, we arrive at the following result:

Theorem 4.1. $\mathcal{L}_{\mathsf{GML}}$ *has the finite model property.*

4.2 Invariance

We need the following notion. A model M is *ω-saturated* (in the sense of first-order logic) if whenever Δ is a set of $\mathcal{L}'_1$-formulas, where $\mathcal{L}'_1$ extends $\mathcal{L}'$ by the addition of finitely many new individual constants, and Δ is finitely satisfiable in an $\mathcal{L}'_1$-expansion of M, then Δ is satisfiable in this expansion.

Lemma 4.2. *Let M and M' be two ω-saturated models, and let $w \in W$, $w' \in W'$.*
Then $w \equiv_{\mathsf{g}} w'$ iff $w \leftrightarroweq_{\mathsf{g}} w'$.

Proof. The '$\Leftarrow$' implication is Proposition 3.3. For '$\Rightarrow$' implication, assume that $w \equiv_{\mathsf{g}} w'$, and define a series of relations $Z = (Z_1, \ldots)$ between the finite subsets of W and W' by putting (for $n \geqslant 1$):

$$\begin{aligned} X\, Z_n\, X' \quad \text{iff} \quad & |X| = |X'| = n \quad \text{and} \\ & \forall x \in X\ \exists x' \in X'\colon\ x \equiv_{\mathsf{g}} x' \quad \text{and} \\ & \forall x' \in X'\ \exists x \in X\colon\ x \equiv_{\mathsf{g}} x'. \end{aligned}$$

Let us check that Z is a g-bisimulation between w and w'. First, as $w \equiv_{\mathsf{g}} w'$, we have $\{w\}\, Z_1 \{w'\}$, so Z_1 is non-empty. Condition 2 from Def. 3.2 is trivially fulfilled.

As to condition 3, assume $\{v\}\, Z_1\, \{v'\}$ and $vR_{\bullet}Y$, where $Y \subseteq W$ and $|Y| = n$. We need to find a finite set $Y' \subseteq W'$ with $v'R'_{\bullet}Y'$ and $Y Z_n Y'$. Consider the graded modal types of the states in Y; clearly, some of them may coincide, so let

$$\{T_1, \ldots, T_s\} = \{\, \mathsf{tp}(y) \mid y \in Y \,\}, \qquad \text{where } s \leqslant n.$$

Next, we need to record, for each type T_i, how many states in Y have this type:

$$n_i = |\{y \in Y \mid \mathsf{tp}(y) = T_i\}|, \qquad \text{for each } 1 \leqslant i \leqslant s.$$

Then all $n_i > 0$ and $n_1 + \ldots + n_s = n$. Now consider the following collection Δ of first-order formulas with free variables x and y^i_k, for $1 \leqslant i \leqslant s$, $1 \leqslant \ell \leqslant n_i$:

$$\bigcup_{1 \leqslant i \leqslant s} \Big(\{y^i_k \neq y^i_\ell \mid 1 \leqslant k < \ell \leqslant n_i\} \cup \{R(x, y^i_\ell) \mid 1 \leqslant \ell \leqslant n_i\} \cup \{\varphi^*(y^i_\ell) \mid \varphi \in T_i,\ 1 \leqslant \ell \leqslant n_i\} \Big).$$

We want to show that the set of formulas Δ is satisfied at $x := v'$ in M'. If we succeed in doing so, then, for each type T_i we have found n_i successors of v' satisfying T_i. Putting these successors together gives us a set Y' of size n. Indeed, there will be n_i successors of type T_i due to the conjuncts $y^i_k \neq y^i_\ell$, and successors of different types will be distinct, as types are maximal. Moreover, it is obvious that for each state $y \in Y$, there will be a state $y' \in Y'$ with $\{y\}\, Z_1\, \{y'\}$, and conversely. Thus $Y\, Z_n\, Y'$, and we have established condition 3.

Let us see why Δ is satisfiable at $x := v'$. Since M' is ω-saturated, it suffices to show that Δ is *finitely* satisfiable at v'. Assume for the sake of contradiction that this is not the case. Then there exist finite sets $\Phi_i \subseteq T_i$, $1 \leqslant i \leqslant s$, (or even formulas, since T_i are closed under finite conjunctions) such that

$$M' \models \neg \bigwedge_{1 \leqslant i \leqslant s} \exists^{\neq} y^i_1 \ldots y^i_{n_i} \bigwedge_{1 \leqslant \ell \leqslant n_i} (R(x, y^i_\ell) \wedge \Phi^*_i(y^i_\ell))[v']$$

This is equivalent to

$$M', v' \models \neg(\Diamond^{\geqslant n_1}\Phi_1 \wedge \ldots \wedge \Diamond^{\geqslant n_s}\Phi_s).$$

But as $M, v \models \Diamond^{\geqslant n_1}\Phi_1 \wedge \ldots \wedge \Diamond^{\geqslant n_s}\Phi_s$, this contradicts to $\{v\}\, Z_1\{v'\}$, since the latter implies $v \equiv_{\mathsf{g}} v'$. Hence, Δ is (finitely) satisfiable at v', as required.

Finally, condition 4 is proved analogously to condition 3, and condition 5 is immediate from the construction of Z. □

An $\mathcal{L}_1$-formula $\alpha(x)$ is *invariant under* g*-bisimulations* if for all models M and M', all states w in M and w' in M', whenever $(M, w) \leftrightarrows_{\mathsf{g}} (M', w')$, we have $M \models \alpha[w]$ iff $M' \models \alpha[w']$.

Below, if $\Delta(x) \cup \{\alpha(x)\}$ is a set of first-order formulas with free variable x, then we write $\Delta \models \alpha$ if, for all models M and states w in M, $M \models \Delta[w]$ implies $M \models \alpha[w]$.

Theorem 4.3 (Invariance)**.** *Assume that $\mathcal{L}_1$ is countable. An $\mathcal{L}_1$-formula $\alpha(x)$ is (equivalent to the translation of) a graded modal formula iff it is invariant under* g*-bisimulations.*

Proof. The '⇒' implication is simply Proposition 3.3. For the other direction, assume that $\alpha(x)$ is invariant under g-bisimulations. Consider the set of graded modal consequences of α:

$$\mathsf{GML}(\alpha) \; = \; \{\, \varphi^*(x) \;\mid\; \alpha(x) \models \varphi^*(x) \,\}.$$

In order to prove that $\alpha(x)$ is equivalent to the translation of some graded modal formula, it suffices to show that $\mathsf{GML}(\alpha) \models \alpha$. Indeed, if $\mathsf{GML}(\alpha) \models \alpha$, then, by compactness, $\Delta(x) \models \alpha(x)$ for some finite subset $\Delta \subseteq \mathsf{GML}(\alpha)$. Hence $\models \bigwedge \Delta(x) \to \alpha(x)$. Trivially, $\models \alpha(x) \to \bigwedge \Delta(x)$, so $\models \alpha(x) \leftrightarrow \bigwedge \Delta(x)$ and we are done, as $\Delta(x)$ is a finite set of translations of graded modal formulas.

It remains to show that $\mathsf{GML}(\alpha) \models \alpha$. Take any M and w and assume that $M \models \mathsf{GML}(\alpha)[w]$. We need to show that $M \models \alpha[w]$. Consider the following set of $\mathcal{L}_1$-formulas:

$$\Gamma(x) \; = \; \{\, \varphi^*(x) \mid M, w \models \varphi \,\}.$$

Obviously, $M \models \Gamma[w]$.

Claim 1. The set of formulas $\Gamma(x) \cup \{\alpha(x)\}$ is satisfiable.

► Assume not, then by compactness, for some finite subset $\Gamma' \subseteq \Gamma$, we have $\models \alpha \to \neg \bigwedge \Gamma'$. Hence $\neg \bigwedge \Gamma' \in \mathsf{GML}(\alpha)$. This implies $M \models \neg\Gamma'[w]$, in contradiction with $M \models \Gamma[w]$. ◄

So, there is a model N and its state v such that $N \models \alpha[v]$ and $N \models \Gamma[v]$. The latter is equivalent to $N, v \models \mathsf{tp}(w)$ and hence to $w \equiv_{\mathsf{g}} v$. Now, to conclude the proof, we want to 'lift' α from (N, v) to (M, w). To do so, take ω-saturated elementary extensions (N^+, v) and (M^+, w) of (N, v) and (M, w), respectively (cf. [4, Theorem 6.1]). Then $\mathsf{tp}_{M^+}(w) = \mathsf{tp}_{N^+}(v)$, and so by Lemma 4.2 we get that $(M^+, w) \leftrightarrows_{\mathsf{g}} (N^+, v)$.

Therefore, from $N \models \alpha[v]$ it follows that $N^+ \models \alpha[v]$ by elementary extension, then $M^+ \models \alpha[w]$ by invariance of $\alpha(x)$ under g-bisimulations, and finally $M \models \alpha[w]$, as required. □

4.3 Definability

To simplify the presentation, we will work with *pointed models*; these are structures of the form (M, w), where w is a state in M, called the *distinguished point* of (M, w). We will assume that g-bisimulations between two pointed models link the singletons containing their distinguished points.

Let $\mathbb{K}$ be a class of pointed models. Then $\mathbb{K}$ is *definable* by a set of graded modal formulas if there exists a set of formulas Δ such that $\mathbb{K} = \{(M, w) \mid (M, w) \models \Delta\}$; $\mathbb{K}$ is *definable by a single formula* if it is definable by means of a singleton set; $\overline{\mathbb{K}}$ denotes the class of pointed models outside $\mathbb{K}$.

$\mathbb{K}$ is *closed under ultraproducts* (*ultrapowers*) if every ultraproduct (ultrapower) of models in $\mathbb{K}$ is itself in $\mathbb{K}$; $\mathbb{K}$ is *closed under* g*-bisimulations* if every model g-bisimilar to a model in $\mathbb{K}$ is in $\mathbb{K}$.

Below, $\mathrm{Th}(M, w) = \{\varphi \in \mathcal{L}_{\mathsf{GML}} \mid (M, w) \models \varphi\}$ and $\mathrm{Th}(\mathbb{K}) = \{\varphi \in \mathcal{L}_{\mathsf{GML}} \mid \mathbb{K} \models \varphi\}$.

Theorem 4.4 (Definability 1)**.** *Assume that the graded modal language $\mathcal{L}_{\mathsf{GML}}$ is countable, and let $\mathbb{K}$ be a class of pointed models. Then*

1. *$\mathbb{K}$ is definable by a set of graded modal formulas iff $\mathbb{K}$ is closed under g-bisimulations and ultraproducts, while $\overline{\mathbb{K}}$ is closed under ultrapowers.*

2. *$\mathbb{K}$ is definable by a single graded modal formula iff $\mathbb{K}$ is closed under g-bisimulations and ultraproducts, while $\overline{\mathbb{K}}$ is closed under ultraproducts.*

Proof. 1. The *only if* direction is easy. For the converse, we can 'bisimulate' familiar arguments from first-order model theory. Assume $\mathbb{K}$ is closed under ultraproducts and g-bisimulations, while $\overline{\mathbb{K}}$ is closed under ultrapowers. We will show that $\Delta := \mathrm{Th}(\mathbb{K})$ defines $\mathbb{K}$. First, $\mathbb{K} \models \Delta$.

Second, assume that $(M, w) \models \Delta$; we need to show $(M, w) \in \mathbb{K}$. Let I be the set of all finite subsets of $\mathrm{Th}(M, w)$. Each $i \in I$ has a model $(N_i, v_i) \models i$ in $\mathbb{K}$. By standard model-theoretic arguments there exists an ultraproduct $(N, v) = \prod_U (N_i, v_i)$ such that $\mathrm{Th}(N, v) = \mathrm{Th}(M, w)$. As $\mathbb{K}$ is closed under ultraproducts, $(N, v) \in \mathbb{K}$.

Now, let U' be a countably incomplete[2] ultrafilter, and consider the ultrapowers

$$(N^*, v^*) := \textstyle\prod_{U'} (N, v) \quad \text{and} \quad (M^*, w^*) := \textstyle\prod_{U'} (M, w).$$

Both models are ω-saturated (cf. [4, Theorem 6.1]), and moreover $w^* \equiv_{\mathsf{g}} v^*$. Hence, by Lemma 4.2, $(N^*, v^*) \leftrightarroweq_{\mathsf{g}} (M^*, w^*)$. So, $(N^*, v^*) \in \mathbb{K}$ by closure under ultraproducts, and $(M^*, w^*) \in \mathbb{K}$, by closure under g-bisimulations. Finally, $(M, w) \in \mathbb{K}$, since $\overline{\mathbb{K}}$ is closed under ultrapowers.

2. Again, the *only if* direction is easy. Assume $\mathbb{K}, \overline{\mathbb{K}}$ satisfy the stated conditions. Then both are closed under ultrapowers, hence, by item 1, there are sets of graded modal formulas Δ, Δ' defining $\mathbb{K}$ and $\overline{\mathbb{K}}$, respectively. Obviously, $\Delta \cup \Delta' \models \bot$, so by compactness for some $\varphi_1, \ldots, \varphi_n \in \Delta$ and $\varphi'_1, \ldots, \varphi'_m \in \Delta'$, we have $\bigwedge_i \varphi_i \models \bigvee_j \neg \varphi'_j$. Then $\mathbb{K}$ is defined by $\bigwedge_i \varphi_i$. □

[2] An ultrafilter is *countably incomplete* if it is *not* closed under countable intersections (of course, it is still closed under finite intersections).

To conclude this, section we present an alternative and more manageable characterization of the properties definable in graded modal logic.

Theorem 4.5 (Definability 2). *Assume that the graded modal language $\mathcal{L}_{\mathsf{GML}}$ contains only finitely many proposition letters, and let $\mathbb{K}$ be a class of pointed models. Then $\mathbb{K}$ is definable by a single graded modal formula iff, for some $k, m \in \mathbb{N}$, $\mathbb{K}$ is closed under $\leftrightarrows^m_{\mathsf{g}_k}$ (g_k-bisimulations up to m).*

Proof. Clearly, if $\mathbb{K}$ is definable by a single formula of index k and degree m, then it is closed under $\leftrightarrows^m_{\mathsf{g}_k}$. To prove the converse, assume that k, m are such that $\mathbb{K}$ is closed under $\leftrightarrows^m_{\mathsf{g}_k}$. Let $(M, w) \in \mathbb{K}$, and define $\varphi^{k,m}_{M,w}$ to be the conjunction of all formulas in $\mathrm{Th}(M, w)$ of degree at most m and index at most k — there are only finitely many non-equivalent such formulas, as we are working in a finite-variable language. Hence we may assume $\varphi^{k,m}_{M,w}$ to be a (finitary) formula in $\mathcal{L}_{\mathsf{GML}}$.

Using the finite character of the language again, we find that there are only finitely many non-equivalent formulas of the form $\varphi^{k,m}_{M,w}$ for all $(M, w) \in \mathbb{K}$. Let $\Phi^{k,m}$ be their disjunction. Then $\Phi^{k,m}$ defines $\mathbb{K}$. For, assume that $(M, w) \models \Phi^{k,m}$; we need to show that $(M, w) \in \mathbb{K}$. From $(M, w) \models \Phi^{k,m}$ it follows that there is a pointed model $(M', w') \in \mathbb{K}$ that agrees with (M, w) on all graded modal formulas of degree at most m and index at most k, i.e., $w \equiv^m_{\mathsf{g}_k} w'$. The latter fact implies that $(M, w) \leftrightarrows^m_{\mathsf{g}_k} (M', w')$. To see this, define tuples of relations $Z^i = (Z^i_1, \ldots, Z^i_k)$, $0 \leqslant i \leqslant m$, by

- $\{x\}\, Z^i_1\, \{x'\}$, for $0 \leqslant i \leqslant m$, iff $x \equiv^i_{\mathsf{g}_k}$; and
- $X\, Z^i_n\, X'$, for $2 \leqslant n \leqslant k$ and $0 \leqslant j < m$, iff $|X| = |X'| = n$ and $\forall x \in X\ \exists x' \in X'$: $\{x\}\, Z^i_1\, \{x'\}$ and $\forall x' \in X'\ \exists x \in X$: $\{x\}\, Z^i_1\, \{x'\}$.

This defines a g_k-bisimulation up to m between (M, w) and (M', w'). As $(M', w') \in \mathbb{K}$ and $\mathbb{K}$ is closed under $\leftrightarrows^m_{\mathsf{g}_k}$, this implies that $(M, w) \in \mathbb{K}$, and we are done. □

5 Conclusion

In this note g-bisimulations were introduced as a tool for exploring the model theory of graded modal logic. Their usefulness was demonstrated by their use in obtaining both known results (the finite model property) and new ones (invariance and definability).

Now that a working notion of bisimulation is available for graded modal logic, it may be used to obtain further results on the model (and frame) theory of graded modal logic. Obvious questions to be answered next include the following: Can g-bisimulations be used to prove a Goldblatt-Thomason style result about the classes of frames definable in $\mathcal{L}_{\mathsf{GML}}$? What is the appropriate kind of Ehrenfeucht-Fraïssé style games needed to prove analogs of the results in this note for the class of finite models? Fragments of $\mathcal{L}_{\mathsf{GML}}$ have been used in terminological reasoning [6]; can these fragments be characterized by adapting the notion of g-bisimulation?

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