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A SYSTEM OF DYNAMIC MODAL LOGIC

ABSTRACT. In many logics dealing with information one needs to make statements not only about cognitive states, but also about transitions between them. In this paper we analyze a dynamic modal logic that has been designed with this purpose in mind. On top of an abstract information ordering on states it has instructions to move forward or backward along this ordering, to states where a certain assertion holds or fails, while it also allows combinations of such instructions by means of operations from relation algebra. In addition, the logic has devices for expressing whether in a given state a certain instruction can be carried out, and whether that state can be arrived at by carrying out a certain instruction.

This paper deals mainly with technical aspects of our dynamic modal logic. It gives an exact description of the expressive power of this language; it also contains results on decidability for the language with 'arbitrary' structures and for the special case with a restricted class of admissible structures. In addition, a complete axiomatization is given. The paper concludes with a remark about the modal algebras appropriate for our dynamic modal logic, and some questions for further work.

The paper only contains some sketchy examples showing how the logic can be used to capture situations of dynamic interest, far more detailed applications are given in a companion to this paper (De Rijke [33]).

1. Introduction

Over the past decade logicians have paid more and more attention to dynamic aspects of reasoning. Motivated by examples taken from such diverse disciplines as natural language semantics, linguistic analysis of discourse, the philosophy of science, artificial intelligence and program semantics, a multitude of logical systems have been proposed, each of them equipped with the predicate "dynamic". At present it is not clear at all what it is that makes a logical system a *dynamic* system. One of the very few general perspectives on dynamic matters is due to Van Benthem [6]. This paper studies a dynamic modal language $(\mathcal{DML})$ designed within this perspective by Van Benthem [4, 6]. Before introducing the formal aspects of $\mathcal{DML}$, let me sketch the main ideas underlying it.

Nowadays many logical systems focus on the structure and processing of information. Often these calculi do not aim at dealing with what is true at information *states*, but rather with *transitions* between such states. Cognitive notions, however, have a dual character. Actual inference, for instance, is a mixture of more dynamic short-term effects and long-term

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static ones. Thus, in a logical analysis of dynamic matters it is desirable to have two levels of propositions co-existing. In addition, the two levels may mutually influence each other; the effects of transitions are often couched in static terms, and the processing of pieces of static information may give rise to instructions as to getting from one cognitive state to another. The general format for $\mathcal{DML}$, then, is one of two levels, of *states* and of *transitions*, plus systematic *interactions* between them.

Given this choice of basic ingredients we are faced with a number of questions, including:

1. what are states and transitions?
2. what are the appropriate connectives?
3. which relations model the interaction between states and transitions?
4. do we evaluate formulas only at states, or also at transitions?

In $\mathcal{DML}$ we opt for the following We abstract from any particular choice of states, and take them to be primitive objects without further structure. Although recent years have witnessed the emergence of calculi in which transitions are primitive objects too (in [5, 37] they are called *arrows*), here our transitions will simply be ordered pairs of states. In our choice of connectives we will be rather conservative: we use propositions with the usual Boolean operations to talk about states, and we use the usual relation algebraic operations (including converse) to combine procedures that denote sets of transitions. Among our procedures there will be a relation $\sqsubseteq$ denoting an abstract notion of *information growth* or *change*, which we will assume to be a preorder; given this choice one can sometimes think of the elements of our structures as the cognitive states an agent passes through searching for knowledge.

As to the interaction between states and transitions, states are linked to transitions via *modes*, and transitions are linked to states via *projections*, as in Figure 1 below. The choice of projections and modes will, of course, depend on the particular application one has in mind; here, we chooce a very basic set. The projections we consider return, given a procedure as input, its *domain*, *range* and *fix points*. Given our interests in dynamic matters here, they form a natural choice, expressing, for instance, whether or not in a given state a certain change is at all possible. The modes we consider take a formula ϕ as input, and return the procedure consisting of all moves along the information ordering to states where ϕ holds, or all moves *backwards* along the ordering to states where ϕ fails; in addition there is the simple "test-for-ϕ" relation.

The issue whether we evaluate formulas at states, transitions, or both, is a subtle one. Our language has syntactic items referring to relations, but the notions of validity and consequence are couched solely in terms of

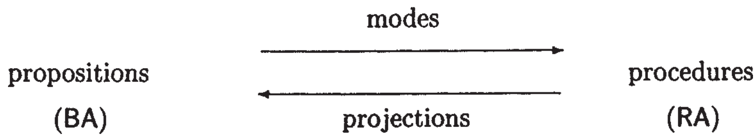

Figure 1. Propositions and procedures.

formulas denoting sets of states; thus $\mathcal{DML}$ cannot express the *identity* of two relations directly – only the effects of making transitions can be measured. That is: $\mathcal{DML}$-formulas can only be evaluated at states, not at pairs. De Rijke [34] deals with a truly two-sorted language in which states and transitions have, so to say, equal rights.

I believe that $\mathcal{DML}$ is not just another device for reasoning about dynamics and change, but, rather, that it provides a more general framework in which other proposals can be described and compared. A number of such descriptions and comparisons have been given by Van Benthem [4, 6] and De Rijke [33]; §3 below contains a brief survey. Jaspars [21], and Jaspars and Krahmer [22] use $\mathcal{DML}$ as a medium for *comparing* different systems of dynamic semantics, and making them compatible.

The main purpose of this paper is to study the language $\mathcal{DML}$ in precise and formal detail. After some initial definitions in §2, §3 contains examples of the uses of the $\mathcal{DML}$; these include Theory Change, Update Semantics, and Dynamic Inference. In §4 the expressive power of the language is studied; a precise syntactic description is given of the first-order counterpart of $\mathcal{DML}$, as well as a characterization in terms of bisimulations. In §5 (un-)decidability results for satisfiability in $\mathcal{DML}$ are given. §6 provides a complete axiomatization of validity in the language of $\mathcal{DML}$. Some quick remarks about the kind of modal algebras appropriate for the language studied here are made in §7. Finally, §8 contains concluding remarks.

2. Some definitions

Let Φ be a set of proposition letters. We define the dynamic modal language $\mathcal{DML}(\Phi)$, or just $\mathcal{DML}$ for short. Its formulas and procedures (typically denoted by ϕ and α, respectively) are built up from proposition letters $(p \in \Phi)$ according to the following rules

$$\begin{aligned} \phi &::= p \mid \bot \mid \top \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \mathsf{do}(\alpha) \mid \mathsf{ra}(\alpha) \mid \mathsf{fix}(\alpha), \\ \alpha &::= \mathsf{exp}(\phi) \mid \mathsf{con}(\phi) \mid \alpha_1 \cap \alpha_2 \mid \alpha_1 \,;\, \alpha_2 \mid -\alpha \mid \alpha\breve{} \mid \phi?. \end{aligned}$$

$\mathcal{DML}$-formulas are assumed to live in a set $Form(\Phi)$, the procedures in a set $Proc(\Phi)$, and the elements of $Form(\Phi) \cup Proc(\Phi)$ are referred to as *$\mathcal{DML}$-expressions*.

At several occasions we will refer to a version of $\mathcal{DML}$ with multiple base relations $\sqsubseteq_i$ taken from a set Ω, with corresponding modes exp_i; we use $\mathcal{DML}(\Phi, \Omega)$ to refer to this language.

The intended interpretation of the above connectives and operators is the following. A formula $\mathsf{do}(\alpha)$ ($\mathsf{ra}(\alpha)$) is true at a state x iff x is in the domain (range) of α, and $\mathsf{fix}(\alpha)$ is true at x if x is a fixed point of α. The interpretation of $\mathsf{exp}(\phi)$ (read: expand with ϕ) is the set of all moves along the information ordering $\sqsubseteq$ leading to a state where ϕ holds; the interpretation of $\mathsf{con}(\phi)$ (read: contract with ϕ) consists of all moves *backwards* along the ordering to states where ϕ *fails*. As usual, ϕ? is the "test-for-ϕ" relation, while the intended interpretation of the operators left unexplained should be clear.[1]

The *models* for $\mathcal{DML}$ are structures of the form $\mathfrak{M} = (W, \sqsubseteq, [\![\cdot]\!], V)$, where $\sqsubseteq \subseteq W^2$ is transitive and reflexive relation (the *information ordering*) $[\![\cdot]\!]$: $Proc(\Phi) \to 2^{W\times W}$, and V: $\Phi \to 2^W$ is a *valuation* assigning subsets of W to proposition letters.[2] The interpretation of the projections is the following:

$$\begin{array}{lll} \mathfrak{M}, x \models \mathsf{do}(\alpha) & \text{iff} & \exists y((x,y) \in [\![\alpha]\!]), \\ \mathfrak{M}, x \models \mathsf{ra}(\alpha) & \text{iff} & \exists y((y,x) \in [\![\alpha]\!]), \\ \mathfrak{M}, x \models \mathsf{fix}(\alpha) & \text{iff} & (x,x) \in [\![\alpha]\!]. \end{array}$$

A model $\mathfrak{M}$ is *standard* if it interprets the relational part of the language as follows:

$$\begin{array}{rcl} [\![\mathsf{exp}(\phi)]\!] & = & \lambda xy.\,(x \sqsubseteq y \wedge \mathfrak{M}, y \models \phi), \\ [\![\mathsf{con}(\phi)]\!] & = & \lambda xy.\,(x \sqsupseteq y \wedge \mathfrak{M}, y \not\models \phi), \\ [\![\alpha \cap \beta]\!] & = & [\![\alpha]\!] \cap [\![\beta]\!], \\ [\![\alpha\,;\beta]\!] & = & [\![\alpha]\!]\,;[\![\beta]\!], \\ [\![-\alpha]\!] & = & -[\![\alpha]\!], \\ [\![\alpha\breve{}]\!] & = & \{(x,y) : (y,x) \in [\![\alpha]\!]\}, \\ [\![\phi?]\!] & = & \{(x,x) : \mathfrak{M}, x \models \phi\}. \end{array}$$

As usual, we say that a formula ϕ is a *consequence* of a set of formulas Δ if for every (standard) model $\mathfrak{M}$ and every x in $\mathfrak{M}, \mathfrak{M}, x \models \psi$, for all $\psi \in \Delta$, implies $\mathfrak{M}, x \models \phi$.

Observe that ra and fix are definable using the other operators; the contraction mode $\mathsf{con}(\cdot)$ is equivalent to $\mathsf{exp}(\neg\phi)\breve{}$. Whenever this is convenient we will assume that exp only has $\top$ as its argument; this is justified by the equivalence $[\![\mathsf{exp}(\phi)]\!] = [\![\mathsf{exp}(\top)\,;\phi?]\!]$.

The original definition of $\mathcal{DML}$ as given in Van Benthem [4] included the *minimal* projections μ-$\mathsf{exp}(\cdot)$ and μ-$\mathsf{con}(\cdot)$ whose definitions read

$$[\![\mu\text{-}\mathsf{exp}(\phi)]\!] \;=\; \lambda xy.\,(x \sqsubseteq y \wedge \mathfrak{M}, y \models \phi \wedge \neg\exists x(x \sqsubseteq z \sqsubset y \wedge \mathfrak{M}, z \models \phi)), \text{ and}$$

$$[\![\mu\text{-}\mathsf{con}(\phi)]\!] \;=\; \lambda xy.\,(x \sqsupseteq y \wedge \mathfrak{M}, y \not\models \phi \wedge \neg\exists x(x \sqsupseteq z \sqsupset y \wedge \mathfrak{M}, z \not\models \phi)).$$

They have been left out because they are definable:

$$(x, y) \;\in\; [\![\mu\text{-}\mathsf{exp}(\phi)]\!] \text{ iff}$$
$$(x, y) \;\in\; [\![\mathsf{exp}(\phi) \cap -(\mathsf{exp}(\phi)\,; (\mathsf{exp}(\top) \cap -\top?))]\!],$$

and similarly for μ-$\mathsf{con}(\phi)$.

There are obvious connections between $\mathcal{DML}$ and *Propositional Dynamic Logic* ($\mathcal{PDL}$, [20]). The "old diamonds" $\langle\alpha\rangle$ from $\mathcal{PDL}$ can be simulated in $\mathcal{DML}$ by putting $\langle\alpha\rangle\phi := \mathsf{do}(\alpha\,;\phi?)$. And conversely, the expansion and contraction operators are definable in a particular mutation of $\mathcal{PDL}$ where taking converses of program relations is allowed and a name for the information ordering is available: $[\![\mathsf{exp}(\phi)]\!] = [\![\sqsubseteq\,;\phi?]\!]$. The domain operator $\mathsf{do}(\alpha)$ can be simulated in standard $\mathcal{PDL}$ by $\langle\alpha\rangle\top$. A difference between the two is that (standard) $\mathcal{PDL}$ only has the regular program operations $\cup$, ; and the Kleene star *, while $\mathcal{DML}$ has the full relational repertoire $\cup, -, \breve{}$ and ;, but not *. Another difference is not a technical difference, but one in emphasis: whereas in $\mathcal{PDL}$ the Boolean part of the language clearly is the primary component of the language and the main concern lies with the *effects* of programs, in $\mathcal{DML}$ one focuses on the *interaction* between the static and dynamic component.

A related formalism whose relational apparatus is more alike that of $\mathcal{DML}$ is the *Boolean Modal Logic* ($\mathcal{BML}$) studied by Gargov and Passy [16]. This system has atomic relations $\rho_1, \rho_2, \ldots$, a constant for the universal relation ∇, and relation-forming operators $\cap, \cup$ and $-$. Relations are referred to within $\mathcal{BML}$ by means of the $\mathcal{PDL}$-like diamonds $\langle\alpha\rangle$. Since the language of $\mathcal{BML}$ does *not* allow either ; or $\breve{}$ as operators on relations, it is a strict subset of $\mathcal{DML}(\Phi, \Omega)$.

3. Some examples

It is high time for an example or two. Here's a simple-minded one. Suppose you're sitting in a room, waiting for the start of a talk by a famous logician who is known for his lively presentation, and who has done a lot of work on non-monotonic logic. So, after some time the lights are dimmed and logician comes in (l). You can see that he's carrying a birds cage with a bird in it, although you can not see what kind of bird it is. Having read the relevant literature you conclude that the bird must be a penguin (p) called Tweety (t). However, the first thing the speaker says, while holding up the cage and pointing at the bird in it, is: "This bird is not called Tweety". In that case, you think, it's probably not a penguin either. The speaker continues: "I want to do a little experiment with you. I want you to think of a name for this bird; any name will do, as long as it's not Tweety". Being a cooperative member of the audience you think of a name other than Tweety, say Bob $(b)\ldots$ Some of the changes brought about in your initial informational state during this story may schematically be represented as

$$\mathsf{exp}(l)\,;(\mathsf{exp}(p)\cap\mathsf{exp}(t))\,;\mathsf{con}(t)\,;\mathsf{con}(p)\,;(\mathsf{exp}(b)\cap-\mathsf{exp}(t)),$$

where ; is the usual relational composition as defined in §2.

Theory Change

One of the original motivations for the invention of $\mathcal{DML}$ was to obtain a formalism for reasoning about the cognitive moves an agent makes while searching for new knowledge or information; possible moves one should be able to formulate included acquiring new information, and giving up information, as was illustrated in the above example. It later turned out that along similar lines $\mathcal{DML}$ can be used to model postulates for Theory Change. I will briefly sketch this.

Consider a set of beliefs or a knowledge set T. As our perception of the world as described by T changes, the knowledge set may have to be modified. In the literature on theory change a number of such modifications have been identified [1, 24], including expansions, contractions and revisions. If we acquire information that does not contradict T, we can simply *expand* our knowledge set with this piece of information. When a sentence ϕ previously believed becomes questionable and has to be abandoned, we *contract* our knowledge with ϕ. Somewhat intermediate between expansion and contraction is the operation of *revision*: the operation of resolving the conflict that arises when the newly acquired information contradicts our old beliefs. The revision of T by a sentence ϕ is often thought of as consisting of first making changes to T, so as to then be able to expand with ϕ. According to general wisdom on theory change, *as little as possible* of the old theory is to be given up in order to accommodate for newly acquired information.

Gärdenfors and others have proposed an influential set of rationality postulates that the revision operation must satisfy. By defining revision and expansion operators inside $\mathcal{DML}$ all of the postulates (except one) can be modeled inside $\mathcal{DML}$. We briefly sketch how this may be done. First, one represents theories T as nodes in a model, and statements of the form "$\phi \in T$" as modal formulas $[\sqsubseteq]\phi$ (i.e. $\neg\mathsf{do}(\mathsf{exp}(\top)\,;\neg\phi?)$). Then, following the above maxim to change as little as possible of the old theory, one defines an expansion operator $[+\phi]\psi$ ("ψ belongs to every theory that results from expanding with ϕ") as

$$[+\phi]\psi := \neg\mathsf{do}(\mu\text{-}\mathsf{exp}([\sqsubseteq]\phi)\,;\neg[\sqsubseteq]\psi?).$$

So, $[+\phi]\psi$ is true at a node x if in every minimal $\sqsubseteq$-successor y of x where $[\sqsubseteq]\phi$ holds (i.e. where ϕ has been added to the theory), the formula $[\sqsubseteq]\psi$ is true (i.e. ψ is in the theory). Next, one defines a revision operator $[*\phi]\psi$ ('ψ belongs to every theory resulting from revising by ϕ') by first minimally removing possible conflicts with ϕ, than minimally adding ϕ, and subsequently testing whether ψ belongs to the result:

$$[*\phi]\psi := \neg\mathsf{do}((\mu\text{-}\mathsf{con}([\sqsubseteq]\neg\phi)\,;\mu\text{-}\mathsf{exp}([\sqsubseteq]\phi))\,;\neg[\sqsubseteq]\psi?).$$

Given this modeling the Gärdenfors postulates can be translated into $\mathcal{DML}$. As an example we consider the 3rd postulate, also known as the inclusion postulate: "the result of revising T by ϕ is included in the expansion of T with ϕ", or $T * \phi \subseteq T + \phi$. Its translation reads: $[*\phi]\psi \rightarrow [+\phi]\psi$. It is easily verified that this translation is valid on all $\mathcal{DML}$-models. In fact, nearly all of Gärdenfors [15]'s postulates for revision and contraction come out true in this modeling. The only one that fails is the 8th postulate, also known as "conjunctive vacuity" Fuhrmann [12]; its failure is caused by the information ordering $\sqsubseteq$ in $\mathcal{DML}$-models not being a function. De Rijke [33] provides further details.

Update Semantics

Further formalisms to which $\mathcal{DML}$ has been linked include conditional logic and other systems that somehow involve a notion of change. But, whereas the applications to Theory Change and conditionals do not require the states in $\mathcal{DML}$-models to have any particular structure, others do.

For example, one version of Frank Veltman's Update Semantics [36] may be seen as a formalism for reasoning about models of the modal system **S5** (where each **S5**-model represents a possible information state of a single agent) and certain transitions between such models. By imposing the structure of **S5**-models on the individual states in a $\mathcal{DML}$-model, the latter becomes a class of **S5**-models in which the $\mathcal{DML}$-apparatus can be used to reason about *global* transitions between **S5**-models, while the language of **S5** can be used to reason about the *local* structure of the **S5**-models. The global transitions can then be interpreted as various kinds of updates; [33] shows how Veltman's `might`-operator and sequential conjunction can be accounted for in this way. Furthermore, notions of consequence considered by Veltman for Update Semantics can be modeled using the $\mathcal{DML}$-apparatus.

Dynamic Connectives; Dynamic Inference

Many of the dynamic operators that have been proposed in the literature can be defined in $\mathcal{DML}$. The underlying reason for this is that most dynamic proposals have some kind of two-dimensional structures in common as their underlying models, and that the operators considered are usually only concerned with certain pre- and postconditions of transitions in such structures – $\mathcal{DML}$ is strong enough to reason about the pre- and postconditions of all transitions defined by the standard operations on binary relations, and many more besides. For instance, the *residuals* of Vaughan Pratt's action logic [29] can be defined in $\mathcal{DML}$:

$$\begin{aligned} \alpha \Rightarrow \beta &= \{(x,y) : \forall z((z,x) \in [\![\alpha]\!] \to (z,y) \in [\![\beta]\!])\} \\ &= -(\alpha\breve{}\,; -\beta), \\ \alpha \Leftarrow \beta &= \{(x,y) : \forall z((y,z) \in [\![\alpha]\!] \to (x,z) \in [\![\beta]\!])\} \\ &= -(-\beta\,; \alpha\breve{}). \end{aligned}$$

Let δ denote the diagonal relation; i.e., $\delta := \top?$. As pointed out by Van Benthem [6] the *test negation* proposed by Groenendijk and Stokhof [17] becomes

$$\sim \alpha = \{(x,x) \colon \neg\exists y((x,y) \in [\![\alpha]\!])\} = \delta \cap -(\alpha\,; \top?).$$

A logical system is sometimes dubbed dynamic because it has dynamic connectives as in the above examples, and sometimes because it has a dynamic notion of inference. Quite often the latter can also be simulated in $\mathcal{DML}$. Here are some examples taken from Van Benthem [6]. The standard notion of inference $\models_1$ ("every state that models all of the premises, should also model the conclusion") may represented as

$$\phi_1, \ldots, \phi_n \models_1 \psi \text{ iff } \mathsf{fix}(\phi_1?) \wedge \cdots \wedge \mathsf{fix}(\phi_n?) \to \mathsf{fix}(\psi?).$$

A more dynamic notion $\models_2$ taken from Groenendijk and Stokhof [17], which may be paraphrased as "process all premises consecutively, then you should be able to move to a state where the conclusion holds", has the following transcription in $\mathcal{DML}$:

$$\phi_1, \ldots, \phi_n \models_2 \psi \text{ iff } \mathsf{ra}\big(\mathsf{exp}(\phi_1) ; \cdots ; \mathsf{exp}(\phi_n)\big) \rightarrow \mathsf{do}(\mathsf{exp}(\psi)).$$

A third notion of inference, $\models_3$, found for example in Van Eijck and De Vries [10] which reads "whenever it is possible to consecutively expand with all premises, then it should be possible to expand with the conclusion", can be given the following representation:

$$\phi_1, \ldots, \phi_n \models_3 \psi \text{ iff } \mathsf{do}\big(\mathsf{exp}(\phi_1) ; \cdots ; \mathsf{exp}(\phi_n)\big) \rightarrow \mathsf{do}(\mathsf{exp}(\psi)).$$

4. The connection with classical logic

When interpreted on models ordinary modal formulas are equivalent to a special kind of first order formulas. To be precise, these first order translations form a restricted 2-variable fragment of the full first order language, one that can easily be described syntactically, and for which a semantic characterization can be given in terms of so-called p-relations or bisimulations (cf. Van Benthem [3, 35] for details). Likewise, the first order transcriptions of modal formalisms used to reason about relation algebras live in a 3-variable fragment of the full first order language; they too can be given precise syntactic and semantic descriptions (cf. De Rijke [35]).

Of course, the above two are special cases of a much more general phenomenon, namely the relation between patterns or important features of structures and bisimulations that precisely preserve these patterns on the one hand, and (extended) modal formulas whose validity is invariant under such bisimulations on the other hand (again, cf. [35]). In the present case of the $\mathcal{DML}$-language it is also possible to give a precise syntactic description of its first order transcriptions (this will be done in §4.1), and the notion of bisimulation can be adapted to obtain a semantic characterization of these first order transcriptions (in §4.2).

4.1. *Translation into First Order Logic*

The usual translation $(\cdot)^*$ taking modal formulas to first order ones (over a vocabulary $\{R,\ P_1,\ P_2, \ldots\}$) can be extended to the full $\mathcal{DML}$-language without too much trouble (cf. [3] for the standard modal case). However, whereas standard modal formulas translate into formulas having one free variable in a two-variable fragment, expressions in the $\mathcal{DML}$-language translate into formulas of a three-variable fragment that may contain up to two free variables.

My approach will be a bit more general than the one suggested by the truth definition given in §2; instead of $\sqsubseteq$ I will use an abstract binary relation symbol R to translate the modal operators and the 'dynamic' constructs.

DEFINITION 4.1. Let τ be the (first order) vocabulary $\{R,\ P_1,\ P_2,\ldots\}$, with R a binary relation symbol, and the P_i's unary relation symbols. Let $L(\tau)$ be the set of all first order formulas over τ (with identity). Define a translation $(\cdot)^*$ taking $\mathcal{DML}$-formulas to formulas in $L(\tau)$ as Table 1, where $[y/x]\alpha$ denotes the result of substituting y for all free occurrences of x in α.

PROPOSITION 4.2. *Let θ be an expression in $\mathcal{DML}(\Phi)$. Then, for any $\mathfrak{A}$, and for any $x, y \in A$, we have*

1. $\mathfrak{A}, x \models \theta$ *iff* $\mathfrak{A} \models \theta^*[x]$, *if* $\theta \in Form(\Phi)$, *and*
2. $(x, y) \in [\![\theta]\!]_{\mathfrak{A}}$ *iff* $\mathfrak{A} \models \theta^*[x, y]$, *in case* $\theta \in Proc(\Phi)$.

The $(\cdot)^*$-translations of $\mathcal{DML}$-formulas can be described exactly using the following definition.

DEFINITION 4.3. Fix individual variables $x_1,\ x_2,\ x_3$ as before, and let $\tau = \{R,\ P_1,\ P_2,\ \ldots\}$ be as before. Let $x,\ y$ range over $\{x_1,\ x_2\}$, with the understanding that $x \not\equiv y$. The set of first order formulas (with identity) $L_3^{1,2}(\tau)$ is the smallest set X such that

1. $x_1 = x_1,\ P_i x_1 \in X$;
2. if $\phi(x_1), \psi(x_1) \in X$, then so are their conjunction, disjunction, and negations;
3. $Rx_1x_2, (x_1 = x_2) \in X$;
4. if $\phi(x, y), \psi(x, y) \in X$, then so are their conjunction, disjunction, and negations;
5. if $\phi(x, y) \in X$, then so is $\phi(y, x)$;
6. if $\phi(x, y) \in X$, then so is $\exists x_2 \phi(x, y)$;
7. if $\phi(x, y), \psi(x, y) \in X$, then so is $\exists x_3(\phi(x, x_3) \wedge \psi(x_3, y))$;
8. if $\phi(x, y), \psi(x_1) \in X$, then so is $\phi(x, y) \wedge \psi(x_2)$.

PROPOSITION 4.4. *Every expression in the $\mathcal{DML}$-language translates into a formula in $L_3^{1,2}(\tau)$ via $(\cdot)^*$. And conversely, for every $\phi \in L_3^{1,2}(\tau)$ there is an expression $\theta \in \mathcal{DML}(\Phi)$ such that $\models \theta^* \leftrightarrow \phi$.*

Proof. One may use an inductive argument to see that every expression in $\mathcal{DML}(\Phi)$ translates into a formula in $L_3^{1,2}(\tau)$ via the mapping $(\cdot)^*$. For

TABLE 1.
The standard translation.

$(\top)^*$	$=$	$(x = x)$	$(p)^*$	$=$	$P(x)$
$(\neg\phi)^*$	$=$	$\neg\phi^*$	$(\phi \wedge \psi)^*$	$=$	$\phi^* \wedge \psi^*$
$(\mathsf{do}(\alpha))^*$	$=$	$\exists y(\alpha^*)$	$(\mathsf{ra}(\alpha))^*$	$=$	$\exists y[y/x, x/y](\alpha^*)$
		$(\mathsf{fix}(\alpha))^* = [x/y](\alpha^*)$			
$(\alpha \cap \beta)^*$	$=$	$\alpha^* \wedge \beta^*$	$(-\alpha)^*$	$=$	$\neg(\alpha^*)$
$(\alpha\check{})^*$	$=$	$[y/x, x/y]\alpha^*$	$(\alpha\, ;\beta)^*$	$=$	$\exists z([z/y]\alpha^* \wedge [z/x]\beta^*)$
$(\mathsf{exp}(\phi))^*$	$=$	$(xRy) \wedge [y/x]\phi^*$	$(\phi?)^*$	$=$	$(x = y) \wedge \phi^*$
		$(\mathsf{con}(\phi))^* = (yRx) \wedge \neg[y/x]\phi^*$			

the converse, define a mapping $(\cdot)^\dagger\colon\ L_3^{1,2}(\tau) \to \mathcal{DML}(\Phi)$ as follows:

$$
\begin{aligned}
(x_1 = x_1)^\dagger &= \top \\
(Px_1)^\dagger &= p \\
(\phi(x_1) \wedge \psi(x_1))^\dagger &= \phi(x_1)^\dagger \wedge \psi(x_1)^\dagger \\
(\phi(x_1) \vee \psi(x_1))^\dagger &= \phi(x_1)^\dagger \vee \psi(x_1)^\dagger \\
(\neg\phi(x_1))^\dagger &= \neg\phi(x_1)^\dagger \\
(Rx_1x_2)^\dagger &= \mathsf{exp}(\top) \\
(Rx_2x_1)^\dagger &= \mathsf{con}(\bot) \\
(x_1 = x_2)^\dagger &= \top? \\
(\phi(x,y) \wedge \psi(x,y))^\dagger &= \phi(x,y)^\dagger \cap \psi(x,y)^\dagger \\
(\neg\phi(x,y))^\dagger &= -\phi(x,y)^\dagger \\
(\phi(x,y) \vee \psi(x,y))^\dagger &= \phi(x,y)^\dagger \cup \psi(x,y)^\dagger \\
(\exists x_2\phi(x_1,x_2))^\dagger &= \mathsf{do}(\phi(x_1,x_2)^\dagger) \\
(\exists x_2\phi(x_2,x_1))^\dagger &= \mathsf{ra}(\phi(x_1,x_2)^\dagger) \\
\exists x_3(\phi(x,x_3) \wedge \psi(x_3,y))^\dagger &= \phi(x,y)^\dagger\, ; \psi(x,y)^\dagger \\
(\phi(x,y) \wedge \psi(x_2))^\dagger &= \phi(x,y)^\dagger\, ; (\top? \cap (\psi(x_1)^\dagger?)).
\end{aligned}
$$

Then,
for all $\phi \in L_3^{1,2}(\tau)$, and all $\mathfrak{M}, \vec{x}$, we have $\mathfrak{M}, \vec{x} \models \phi$ iff $\mathfrak{M}, \vec{x} \models \phi^\dagger$. □

Let X be a set of (first order) formulas, and let K be a class of models. Then the $\mathcal{DML}$-language is called *expressively complete with respect to X over* K if for all $\chi \in X$ there is a $\mathcal{DML}$-expression ϕ such that $\mathsf{K} \models \phi^* \leftrightarrow \chi$. If K is the class of all models I will suppress ‘over K’.

The *two-variable fragment* $L_2(\tau)$ is the set of all first order formulas over τ using only two variables.

COROLLARY 4.5. *The $\mathcal{DML}$-language is expressively complete with respect to the two-variable fragment $L_2(\tau)$ of first order logic over $\{R,\ P_1,\ldots\}$ with identity.*

Proof. This is immediate from 4.4: since the two-variable fragment $L_2(\tau)$ over $\{R,\ P_1,\ldots\}$ (with identity) is contained in $L_3^{1,2}(\tau)$, it follows that $\mathcal{DML}(\Phi)$ is expressively complete for that fragment. □

An alternative proof for Corollary 4.5 can be given, by defining an explicit algorithm for transforming $L_2(\tau)$ in $\mathcal{DML}$-expressions. Since this would take up too much space here without yielding additional insights, I omit the details.

What about expressive completeness of the $\mathcal{DML}$-language with respect to the full first order language? It may amuse the reader to check that the temporal operator $UNTIL$, whose truth definition is

$$\begin{aligned}&\mathfrak{M}, x \models UNTIL(p,q) \text{ iff}\\ &\exists y(xRy \wedge Py \wedge \neg\exists z(xRzRy \wedge z \neq y \wedge \neg Qz)),\end{aligned}$$

can be defined by

$$\mathsf{do}(\mathsf{exp}(p) \cap -[\mathsf{exp}(\neg q)\,;(R \cap -\delta)]),$$

where δ is the diagonal relation, defined by $\delta = \top?$. Of course, the definition of $SINCE$, the backward-looking version of $UNTIL$, is similar. Hence, by Kamp's theorem (cf. Kamp [23]), the $\mathcal{DML}$-language is expressively complete with respect to the full first order language over continuous linear orders.

An obvious question here is whether the Stavi connectives $SINCE'$ and $UNTIL'$ are definable in the $\mathcal{DML}$-language, and, thus, by a result of Jonathan Stavi, whether the $\mathcal{DML}$-language is expressively complete with respect to the language of first order logic over *all* linear orders (cf. Gabbay [14]). Here, $UNTIL'(p,q)$ is defined by

(1) $\exists y(xRy \wedge \forall z(xRzRy \to Qz))\wedge$

$\forall y(xRy \wedge \forall z(xRzRy \to Qz) \to (Qy \wedge \exists x(yRx \wedge$

(2) $\quad\forall z(yRzRx \to Qz)))) \wedge$

$\exists y(xRy \wedge \neg Qy \wedge Py \wedge \forall z(xRzRy \wedge \exists y(xRyRz \wedge$

(3) $\quad\neg Qy) \to Pz))$.

Of course $SINCE'(p, q)$ is the "backward-looking" version of $UNTIL'(p, q)$. In the $\mathcal{DML}$-language the operator $UNTIL'(p, q)$ can be defined as follows:

(4) $\mathsf{do}\big(R \cap -[\mathsf{exp}(\neg q)\,;R]\big) \wedge$

(5) $\neg\mathsf{do}\big(R \cap -[\mathsf{exp}(\neg q)\,;R] \cap$
$\quad - \big[\mathsf{exp}(q) \cap \mathsf{do}\big(R \cap -[\mathsf{exp}(\neg q)\,;R]\big)?\big]\big) \wedge$

(6) $\mathsf{do}\big(\mathsf{exp}(\neg q \wedge p) \cap -\big[\big(\mathsf{exp}(\neg p) \cap (\mathsf{exp}(\neg q)\,;R)\big)\,;R\big]\big).$

I leave it as an exercise to check that (1), (2) and (3) are defined by the $\mathcal{DML}$-formulas (4), (5) and (6), respectively.

4.2. *Bisimulations*

I will now characterize $L_3^{1;2}(\tau)$, and hence, by 4.4, the $\mathcal{DML}$-language, semantically. The key notion here will be an appropriate kind of bisimulations, generalizing the so-called *p-relations* of [3, Theorem 3.9] and [32, Theorem 4.7].

A 2-*partial isomorphism* f from $\mathfrak{M}$ to $\mathfrak{N}$ is simply an isomorphism $f\colon \mathfrak{M}_0 \cong \mathfrak{N}_0$, where $\mathfrak{M}_0$, $\mathfrak{N}_0$ are substructures of $\mathfrak{M}$ and $\mathfrak{N}$, respectively, whose domains have cardinality at most 2. A set I of 2-partial isomorphisms from $\mathfrak{M}$ into $\mathfrak{N}$ has the *back and forth property* if

> for every $f \in I$ with $|f| \leq 1$, and every $x \in \mathfrak{M}$ (or $y \in \mathfrak{N}$) there is a $g \in I$ with $f \subseteq g$ and $x \in$ domain(g) (or $y \in$ range(g)).

I write $I\colon \mathfrak{M} \cong_2 \mathfrak{N}$ if I is a non-empty set of 2-partial isomorphisms and I has the back and forth property.

By 4.5 the full 2-variable fragment of $L(\tau)$ is contained in the $\mathcal{DML}$-language. Hence any relation between models that is to preserve truth of $\mathcal{DML}$-formulas should 'act' like a (partial) isomorphism on sequences of length at most 2. Indeed, modulo one additional requirement the latter completely characterizes the $\mathcal{DML}$-language (cf. 4.11).

DEFINITION 4.6. A *bisimulation* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is a relation $\mathcal{B} \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2)$ such that

1. $\mathcal{B} \neq \emptyset$,
2. $\vec{x}\mathcal{B}\vec{y}$ implies lh$(\vec{x})$ = lh$(\vec{y})$, where lh$(\vec{x})$ is the length of $\vec{x}$,
3. if $x_1x_2\mathcal{B}y_1y_2$ then $x_1\mathcal{B}y_1$ and $x_2\mathcal{B}y_2$,
4. if $x_1x_2\mathcal{B}y_1y_2$ and $x_3 \in \mathfrak{M}_1$ then there is a $y_3 \in \mathfrak{M}_2$ such that $x_1x_3\mathcal{B}y_1y_3$ and $x_3x_2\mathcal{B}y_3y_2$, and similarly in the opposite direction,
5. for $I = \{\emptyset\} \cup \mathcal{B}$ we have $I\colon \mathfrak{M}_1 \cong_2 \mathfrak{M}_2$.

EXAMPLE 4.7. The conditions in Definition 4.6 are rather strong, as is witnessed, for instance, by the fact that two finite linear models are isomorphic iff they are bisimilar, in the sense of 4.6. The truth of this claim may be seen as follows: any two finite linear models that have the same first order theory are isomorphic, and on linear models the two notions of first order equivalence and of being equivalent for all $\mathcal{DML}$-formulas coincide, by our remarks in §4.1; furthermore, by Proposition 4.10 two models that are bisimilar verify the same $\mathcal{DML}$-formulas.

However, on the class of all finite models bisimilarity and isomorphism do not coincide. Here are two models establishing this claim:

$$\mathfrak{M}: \boxed{\bullet_1 \ \bullet_2 \ \bullet_3 \ \bullet_4} \qquad \mathfrak{M}': \boxed{\bullet_{1'} \ \bullet_{2'} \ \bullet_{3'}},$$

where for *all* points (in $\mathfrak{M}$ and $\mathfrak{M}'$) have the same valuation. Define $\mathcal{B} \subseteq (W \times W') \cup (W^2 \times W'^2)$ by putting

$$\begin{aligned}\mathcal{B} \;=\; &\{(i,i')\colon\ 1 \le i,i' \le 3\}\\ &\cup \{(ij,i'j')\colon\ 1 \le i,j,i',j' \le 3,\ \text{and}\\ &\quad ((i=j \text{ and } i'=j') \text{ or } (i \ne j \text{ and } i' \ne j'))\}\\ &\cup \{(i4,j'k'),(4i,j'k')\colon\ 1 \le i \le 3, 1 \le j' \ne k' \le 3\}\\ &\cup \{(44,i'i')\colon\ 1 \le i' \le 3\}.\end{aligned}$$

The reader may verify that this is indeed a bisimilation between $\mathfrak{M}$ and $\mathfrak{M}'$; hence, bisimulations and isomorphisms do not coincide on all finite models.

EXAMPLE 4.8. Given two finite models $\mathfrak{M}_1,\ \mathfrak{M}_2$ with $\vec{x} \in \mathfrak{M}_1,\ \vec{y} \in \mathfrak{M}_2$ such that for all $\phi \in L(\tau)$, $\mathfrak{M}_1,\ \vec{x} \models \phi$ iff $\mathfrak{M}_2,\ \vec{y} \models \phi$, one may define a ‘canonical’ bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ that connects $\vec{x}$ and $\vec{y}$, by putting

$$\vec{u}\mathcal{B}\vec{v} \text{ iff for all } \phi \in \mathcal{DML}(\Phi),\ \mathfrak{M}_1, \vec{u} \models \phi \text{ iff } \mathfrak{M}_2, \vec{v} \models \phi.$$

(That this does indeed define a bisimulation is essentially because $\mathfrak{M}_1$ and $\mathfrak{M}_2$, being finite, are saturated, cf. the proof of 4.11.) It follows that two finite $\mathcal{DML}$-models are bisimilar iff they satisfy the same formulas.

PROPOSITION 4.9. *Let $\mathcal{B}$ be a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$. Then*

1. domain$(\mathcal{B}) = \mathfrak{M}$, range$(\mathcal{B}) = \mathfrak{N}$
2. *if $x \in \mathfrak{M},\ y \in \mathfrak{N},\ x\mathcal{B}y$ and $x' = \mathfrak{M}$, then there is a $y' \in \mathfrak{N}$ with $xx'\mathcal{B}yy'$, and similarly in the opposite direction.*

An $L(\tau)$-formula $\phi(\vec{x})$ is *invariant for bisimulations* if for all models $\mathfrak{M}_1$, $\mathfrak{M}_2$ and all bisimulations $\mathcal{B}$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and all $\vec{x} \in W_1$, $\vec{y} \in W_2$ such that $\vec{x}\mathcal{B}\vec{y}$, we have $\mathfrak{M}_1, \vec{x} \models \phi$ iff $\mathfrak{M}_2, \vec{y} \models \phi$.

PROPOSITION 4.10. *$L_3^{1,2}(\tau)$-formulas are invariant for bisimulations.*

Proof. By induction on $\mathcal{DML}$-expressions plus an application of 4.4. Here are some cases in the inductive proof. Let $\mathcal{B}$ be a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$.
$\mathsf{do}(\alpha)$. Suppose $x\mathcal{B}y$. Then

$$\begin{aligned} \mathfrak{M}, x \models \mathsf{do}(\alpha) &\implies \text{for some } x'((x,x') \in [\![\alpha]\!]_{\mathfrak{M}}), \\ &\implies \exists y'(xx'\mathcal{B}yy' \wedge (y,y') \in [\![\alpha]\!]_{\mathfrak{N}}), \\ &\qquad \text{by 4.9(2) and IH,} \\ &\implies \mathfrak{N}, y \models \mathsf{do}(\alpha). \end{aligned}$$

$\mathsf{exp}(\phi)$. Suppose $xx'\mathcal{B}yy'$. Then

$$\begin{aligned} (x,x') \in [\![\mathsf{exp}(\phi)]\!]_{\mathfrak{M}} &\implies (x,x') \in R_{\mathfrak{M}} \text{ and } \mathfrak{M}, x' \models \phi, \\ &\implies (y,y') \in R_{\mathfrak{N}} \text{ and } \mathfrak{N}, y' \models \phi \\ &\qquad \text{by 4.6(5) and IH,} \\ &\implies (y,y') \in [\![\mathsf{exp}(\phi)]\!]_{\mathfrak{N}}. \end{aligned}$$

$\alpha\,;\beta$. Suppose $xx'\mathcal{B}yy'$. Then

$$\begin{aligned} (x,x') \in [\![\alpha\,;\beta]\!]_{\mathfrak{M}} &\implies \text{for some } x''((x,x'') \in [\![\alpha]\!]_{\mathfrak{M}} \text{ and} \\ &\qquad (x'',x') \in [\![\beta]\!]_{\mathfrak{M}}), \\ &\implies \exists y''(xx''\mathcal{B}yy'' \wedge x''x'\mathcal{B}y''y' \wedge \\ &\qquad (y,y'') \in [\![\alpha]\!]_{\mathfrak{N}} \wedge (y'',y') \in [\![\beta]\!]_{\mathfrak{N}}), \\ &\qquad \text{by 4.6(4) and IH,} \\ &\implies (y,y') \in [\![\alpha\,;\beta]\!]_{\mathfrak{N}}. \end{aligned}$$

□

It's the converse of 4.10 that is more interesting:

THEOREM 4.11. *A first order formula $\phi(\vec{x})$ in $L(\tau)$ is equivalent to an $L_3^{1,2}(\tau)$-formula only if it is invariant for bisimulations.*

Proof. The proof is an extension of [32, Theorem 4.7]; see also [35]. Define

$$E(\phi) = \{\psi \in L_3^{1,2}(\tau)\colon\ \phi \models \psi \text{ and } FV(\psi) \subseteq FV(\phi)\}.$$

We show that $E(\phi) \models \phi$. Then, by compactness the result follows.

So assume $\mathfrak{M}, \vec{w} \models E(\phi)$. Introduce new constants $\underline{\vec{w}}$ to stand for the objects $\vec{w}$. Set $L^*(\tau) = L(\tau) \cup \{\underline{\vec{w}}\}$, and expand $\mathfrak{M}$ to an $L^*(\tau)$-model in the obvious way. For $\{\psi\} \cup T \subseteq L(\tau), \psi^*$ and T^* have the obvious meaning.

Define $T = \{\psi \in L_3^{1,2}(\tau)\colon\ \mathfrak{M}, \vec{w} \models \psi, FV(\psi) \subseteq FV(\phi)\}$. By compactness there is an $L^*(\tau)$-model $\mathfrak{N}^*$ such that $\mathfrak{N}^* \models T^* \cup \{\phi^*\}$. By standard model theory there are ω-saturated extensions $\mathfrak{M}_1^* = (W_1, \ldots, \vec{w}_1) \succ \mathfrak{M}^*$ and $\mathfrak{N}_1^* = (W_2, \ldots, \vec{w}_2) \succ \mathfrak{N}^*$ such that $\vec{w}_1$ and $\vec{w}_2$ both realize T, and $\mathfrak{N}_1^* \models \phi^*$.

Define a relation $\mathcal{B} \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2)$ between (the $L(\tau)$-reducts of) $\mathfrak{M}_1^*$ and $\mathfrak{N}_1^*$, by putting

$$
\begin{array}{rll}
x_1\mathcal{B}y_1 & \text{iff} & \text{for all } \psi(x) \in L_3^{1,2}(\tau), \text{ and } \mathfrak{M}_1, x_1 \models \psi \\
 & \text{iff} & \mathfrak{N}_1, y_1 \models \psi, \text{ and} \\
x_1x_2\mathcal{B}y_1y_2 & \text{iff} & \text{for all } \psi(x,y) \in L_3^{1,2}(\tau),\ \mathfrak{M}_1, x_1, x_2 \models \psi \\
 & \text{iff} & \mathfrak{N}_1, y_1, y_2 \models \psi.
\end{array}
$$

I claim that $\mathcal{B}$ is in fact a bisimulation between $\mathfrak{M}_1$ and $\mathfrak{N}_1$. To see this, let us check that the conditions of 4.6 hold. Firstly, we have $\mathcal{B} \neq \emptyset$ because $\vec{w}_1\mathcal{B}\vec{w}_2$ holds. For, suppose that $\psi(\vec{x}) \in L_3^{1,2}(\tau)$; then $\psi \in T$, hence $\mathfrak{M}, \vec{w}_2 \models \psi$; and similarly in the opposite direction.

Conditions 2 and 3 are trivial, and to see that 4 is fulfilled, assume that $x_1x_2\mathcal{B}y_1y_2$ and $x_3 \in \mathfrak{M}_1$. What I need to show is: $\exists y_3(x_1x_3\mathcal{B}y_1y_3 \wedge x_3x_2\mathcal{B}y_3y_2)$. To this end set

$$
\begin{aligned}
\Psi(x,y) &= \{\psi(x,y) \in L_3^{1,2}(\tau)\colon\ \mathfrak{M}_1^*, x_1, x_3 \models \psi\}, \\
\Xi(x,y) &= \{\psi(x,y) \in L_3^{1,2}(\tau)\colon\ \mathfrak{M}_1^*, x_3, x_2 \models \psi\}.
\end{aligned}
$$

Then $\Psi(\underline{y}_1, y) \cup \Xi(y, \underline{y}_2)$ is finitely satisfiable in $(\mathfrak{N}_1^*, \underline{y}_1, \underline{y}_2)$. Hence, since $\mathfrak{N}_1^*$ is ω-saturated, it is satisfiable in $(\mathfrak{N}_1^*, \underline{y}_1, \underline{y}_2)$. But this means that for some $y_3 \in W_2,\ x_1x_3\mathcal{B}y_1y_3$ and $x_3x_2\mathcal{B}y_3y_2$, as required. The other half of Condition 4 may be established in a similar way.

Next, we have to check that for $I = \{\emptyset\} \cup \{(\vec{x}, \vec{y}) : \vec{x}\mathcal{B}\vec{y}\}$ we have $I\colon \mathfrak{M}_1 \cong_2 \mathfrak{N}_1$. Now obviously, since each of $(\neg)P_ix, (\neg)x = y, (\neg)Rxy$ and $(\neg)Ryx$ is in $L_3^{1,2}(\tau)$, any $f \in I$ must be a 2-partial isomorphism. So all we have left to do, is show that I has the back and forth property. But this may done along the lines of the proof that Condition 4 is satisfied.

To conclude, $\mathcal{B}$ is a bisimulation between $\mathfrak{M}_1$ and $\mathfrak{N}_1$. So, by invariance for bisimulations $\mathfrak{N}_1^* \models \phi^*$ implies $\mathfrak{M}_1^* \models \phi$. Since $\mathfrak{M}_1^* \succ \mathfrak{M}^*$ it follows that $\mathfrak{M}^* \models \phi^*$, and so $\mathfrak{M}, \vec{w} \models \phi$. □

Using 4.11 some results about definability of classes of $\mathcal{DML}$-models can easily be derived. For an elegant formulation of these results it is convenient to consider so-called pointed models as our fundamental structures (as in Kripke's original publications) . Here, a *pointed* model is a structure of the form $(W, \sqsubseteq, [\![\cdot]\!], V, w)$, where $(W, \sqsubseteq, [\![\cdot]\!], V)$ is an ordinary $\mathcal{DML}$-model, and $w \in W$.

COROLLARY 4.12. *Let* M *be a class of pointed models. Then* M *is definable by means of a* $\mathcal{DML}$*-formula iff it is closed under bisimulations and ultraproducts, while its complement is closed under ultraproducts.*

Proof. Similar to [32, Theorem 4.8]; see also [35, §6]. □

5. Decidability

In the preceding sections we have seen several examples showing that the $\mathcal{DML}$-language is an expressive one. Of course, this power does not come without a price: we will show that satisfiability in the $\mathcal{DML}$-language is not decidable. After that we show that decidability may be restored either by restricting the language, or by restricting the class of structures used to interpret the $\mathcal{DML}$-language.

5.1. *The Full Language Interpreted on Pre-Orders*

As a language, $\mathcal{DML}$ is somewhere in between the language of **S4**$_t$, the temporal analogue of the modal logic of pre-orders **S4**, and full relational algebra. It is well-known that the latter is undecidable. Since in the intermediate case of $\mathcal{DML}$ we only have the operations of relation algebra on top of a single relation, it may be hoped that we are closer to **S4**$_t$ than to relational algebra, and hence that $\mathcal{DML}$ is decidable.

But here is already an important difference between the two: **S4**$_t$ enjoys the finite model property, while $\mathcal{DML}$ does not. To see this, recall that δ is the diagonal relation defined by $\top$?, and define an operator E by putting $E\phi := \mathsf{do}((\delta \cup -\delta)\,;\phi?)$; so E behaves like an existential quantifier; the operator A is defined as the dual of E: $A\phi := \neg E \neg \phi$.

- $R := \mathsf{exp}(\top)$,
- $\infty := \neg E\mathsf{do}((R \cap -\delta) \cap R\breve{})$.

Then, since ∞ forces the absence of loops, the formula $A\mathsf{do}(R \cap -\delta) \wedge \infty$ is satisfiable only on infinite $\mathcal{DML}$-models. And in fact we have the following result:

THEOREM 5.1. *Satisfiability in* $\mathcal{DML}$ *is* $\prod_1^0$*-hard.*

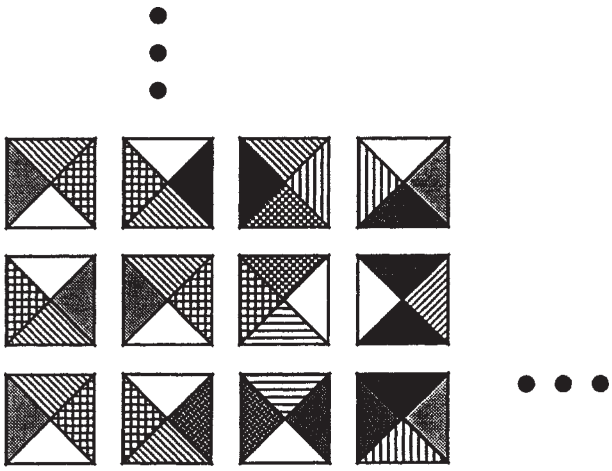

Figure 2. The unbounded tiling problem.

Proof. This is a reduction of a known $\prod_1^0$-complete problem, a so-called *unbounded tiling problem* (UTP), to satisfiability in $\mathcal{DML}$. The version of the UTP that I will use here is given by the following data. Given a set of tiles $T = \{d_0, \ldots, d_m\}$, each having 4 sides whose colors are in $C = \{c_0, \ldots, c_k\}$, is there a tiling of $\mathbb{N} \times \mathbb{N}$? The rules of the tiling game are

1. every point in the grid is associated with a single tile,
2. adjacent edges have the same color.

Now, the version of the UTP presented here is known to be $\prod_1^0$-complete (cf. Harel [19]). So to prove the theorem it suffices to define, for a given set of tiles T, a formula ϕ_T in the $\mathcal{DML}$-language such that

1. its models look like grids,
2. it says that every point is covered by a tile from T,
3. and that colors match right and above neighbors,

and show that ϕ_T is satisfiable iff T can tile $\mathbb{N} \times \mathbb{N}$. Let's get to work now. To make a grid, define

- LEAVE$(\phi) := (\phi? \,;R)$,
- ONE $:= (R \cap -\delta) \cap -[(R \cap -\delta)\,; (R \cap -\delta)]$;
  then, for all $\mathfrak{M}$, and for all $x, y \in \mathfrak{M}$,
  $(x, y) \in [\![\text{ONE}]\!]_{\mathfrak{M}}$ iff $xRy \wedge x \neq y \wedge \neg\exists z(xRz \wedge x \neq z \wedge zRy \wedge z \neq y)$,

- $\text{UP} := [\text{ONE} \cap \text{LEAVE}(p \wedge q) \cap \mathsf{exp}(p \wedge \neg q)]$
  $\cup [\text{ONE} \cap \text{LEAVE}(p \wedge \neg q) \cap \mathsf{exp}(p \wedge q)]$
  $\cup [\text{ONE} \cap \text{LEAVE}(\neg p \wedge q) \cap \mathsf{exp}(\neg p \wedge \neg q)]$
  $\cup [\text{ONE} \cap \text{LEAVE}(\neg p \wedge \neg q) \cap \mathsf{exp}(\neg p \wedge q)]$,

- $\text{RIGHT} := [\text{ONE} \cap \text{LEAVE}(p \wedge q) \cap \mathsf{exp}(\neg p \wedge q)]$
  $\cup [\text{ONE} \cap \text{LEAVE}(\neg p \wedge q) \cap \mathsf{exp}(p \wedge q)]$
  $\cup [\text{ONE} \cap \text{LEAVE}(p \wedge \neg q) \cap \mathsf{exp}(\neg p \wedge \neg q)]$
  $\cup [\text{ONE} \cap \text{LEAVE}(\neg p \wedge \neg q) \cap \mathsf{exp}(p \wedge \neg q)]$,

- LEFT is defined similarly,
- $\text{EQUAL}(\alpha, \beta) := \neg E\mathsf{do}(\alpha \cap -\beta) \wedge \neg E\mathsf{do}(\beta \cap -\alpha)$, where E is the existential modality defined above:
  $E\phi := \mathsf{do}((\delta \cup -\delta)\,;\phi?)$
- $\text{CR} := \text{EQUAL}\big((\text{UP}\,;\text{RIGHT}), (\text{RIGHT}\,;\text{UP})\big) \wedge$
  $\text{EQUAL}\big((\text{UP}\,;\text{LEFT}), (\text{LEFT}\,;\text{UP})\big)$.

Here, finally, is the formula that will force our models to contain a copy of $\mathbb{N} \times \mathbb{N}$:

- $\text{GRID} := (p \wedge q) \wedge A\mathsf{do}(\text{UP}) \wedge A\mathsf{do}(\text{RIGHT}) \wedge \text{CR} \wedge \infty$,

where A is the dual of the modality E.

Next we have to define formulas that force 2 and 3. Let $T = \{d_0, \ldots, d_m\}$ and $C = \{c_0, \ldots, c_k\}$ be given. For each color $c_i \in C$ introduce four proposition letters, suggestively denote by $up = c_i,\ right = c_i,\ down = c_i, left = c_i$. Identifying each tile $d \in T$ with its four sides I assume that each tile d is represented as

$$(up = c_{i_1} \wedge right = c_{i_2} \wedge down = c_{i_3} \wedge left = c_{i_4}) \\ \wedge\Big(\bigwedge_{c \in T\setminus\{c_{i_1}\}} \neg up = c \wedge \cdots \wedge \bigwedge_{c \in T\setminus\{c_{i_4}\}} \neg left = c\Big).$$

Then, put

- $\text{COVER} := A \bigvee_{d \in T} d$,

and

- $\text{MATCH} := A\Big(\bigwedge_{c \in C}(up = c \to [\text{UP}]\,down = c)$
  $\wedge \bigwedge_{c \in C}(right = c \to [\text{RIGHT}]\,left = c)\Big)$.

Put $\phi_T :=$ GRID$\wedge$COVER$\wedge$MATCH. Then ϕ_T is satisfiable in a $\mathcal{DML}$-model iff T can tile $\mathbb{N} \times \mathbb{N}$. The *if*-direction is trivial, since if a tiling exists ϕ_T is satisfiable in $\mathbb{N} \times \mathbb{N}$, simply by verifying $(p \wedge q)$ in $(0, 0)$, switching the truth values of p and q while going right and up through the grid, respectively, while the tiling will tell you how to satisfy COVER and MATCH. Conversely, the domain of any $\mathcal{DML}$- model in which ϕ_T is satisfied in some point x, must contain a copy of $\mathbb{N} \times \mathbb{N}$ with x as its origin; as COVER and MATCH are satisfied in x there must be a tiling of this copy of $\mathbb{N} \times \mathbb{N}$. □

COROLLARY 5.2. *Satisfiability in $\mathcal{DML}$ is $\prod_1^0$-hard.*

One may get a reduction of the UTP to $\mathcal{DML}$-satisfiability with somewhat less than what I have used in the proof of 5.1. For instance, it is not necessary to actually have a real grid inside models satisfying the 'reduction formula' ϕ_T; instead it suffices to have structures satisfying a Church-Rosser like property like $\forall xyz(Rxy \wedge Rxz \rightarrow \exists u(Ryu \wedge Rzu))$.

5.2. *Fragments and Special Frame Classes*

In the literature on arrow logics and other modal logics related to relation algebra one quite find various strategies for overcoming undecidability results (cf. for example [2]). Examples include relativization, restricted the language, and restricting the class of models considered. Let us briefly consider what the latter two strategies might give us; in the setting of dynamic logics such as the dynamic modal logic of this paper, the first strategy is pursued in [27].

What are natural and reasonably large fragments of the $\mathcal{DML}$-language that are decidable? To answer this, let's step back a second and see what made the proof of 5.1 work. Essentially, we were able to build a grid there, thanks to the availability of $;, \cap$ and $-$. Thus, when looking for reasonably large *decidable* fragments of the $\mathcal{DML}$-language, giving up some of these three might get us results. Indeed, giving up ; (and $\breve{}$, by the way) restricts $\mathcal{DML}$ to the Boolean modal logic of Gargov and Passy as mentioned in §2, and this is a decidable system (cf. [16]). Alternatively, giving up – again yields decidability by Danecki [9]. Of course, in these fragments some of the more complex operators like E, A, and μ-$\mathsf{exp}(\cdot)$, μ-$\mathsf{con}(\cdot)$ will no longer be definable, thus it remains to be seen whether adding any of these to the above fragments preserves decidability.

Another approach towards obtaining decidability is not to restrict the language, but to restrict the structures used to interpret the language. As an example I will consider the class of all trees. Just to be precise, by a *tree* is a meant a structure $(W, \sqsubseteq)$ with $\{\sqsubseteq\} \subseteq W^2$ a transitive, asymmetric relation such that for each $x \in W$ the set of $\sqsubseteq$-predecessors of x is linearly ordered by $\sqsubseteq$.

Let $\mathrm{Th}_{\mathcal{DML}}(\mathsf{TREES})$ denote the set of $\mathcal{DML}$-formulas valid on the class TREES of all trees. In §6.3 I will axiomatize $\mathrm{Th}_{\mathcal{DML}}(\mathsf{TREES})$, but for the moment all we need to know about it, is that it lacks the finite model property. (To see this simply consider the formula $A\mathsf{do}(R) \wedge \infty$ from §5.1.) Thus, to establish decidability of this theory some other tools will have to be employed. Of course, one obvious candidate is Rabin's Theorem [31]; to apply this result the semantics of $\mathrm{Th}_{\mathcal{DML}}(\mathsf{TREES})$ has to be embedded in $S\omega S$, the monadic second order theory of infinitely many successor functions. Here, I will take an easier way out by appealing to a result by Gurevich and Shelah [18]. Let L_{GS} be the language of monadic second order with additional unary predicates, i.e. it has individual variables and unary predicate variables (ranging over branches) as well as a binary relation symbol $<$ and unary predicate constants $P_0, P_1, \ldots$. And let $\mathrm{Th}_{GS}(\mathsf{TREES})$ be the set of L_{GS} formulas valid on all trees. Then obviously, the question whether a given $\mathcal{DML}$ formulas ϕ is valid on all trees, boils down to the question whether its standard translation ϕ^* is a theorem of $\mathrm{Th}_{GS}(\mathsf{TREES})$. But by [18] the latter question is decidable.

THEOREM 5.3. *Given a $\mathcal{DML}$ formula ϕ, the question*

$$\text{“Is } \mathrm{Th}_{\mathcal{DML}}(\mathsf{TREES}) \cup \{\phi\} \text{ satisfiable?”}$$

is decidable.

Several variations on the above, variations, moreover, that will still yield decidable theories, are quite natural and worth considering. They include, for example, the set of $\mathcal{DML}$ formulas valid all trees of finite depth, or the $\mathcal{DML}$ formulas valid on all well-founded trees.

6. COMPLETENESS

To provide a complete axiomatization of validity in $\mathcal{DML}$ we will use a special modal operator D, called the *difference operator*, that may be defined by $D\phi := \mathsf{do}(-\delta\,;\phi?)$ in $\mathcal{DML}$; that is, $D\phi$ is true at a state if ϕ is true at a *different* state. We will first sketch a construction for a completeness proof in a language containing D only; full details of this construction are presented in Venema [38], generalizing constructions found in Gabbay and Hodkinson [13]. Then, in §6.2, we present a complete calculus **DML** for our dynamic modal language. Finally, in §6.3, **DML** is proved complete using the completeness construction for the difference operator.

6.1. *How to Use the D Operator*

Let me first present the logic **DL** governing the D operator:

DEFINITION 6.1. Let $\overline{D}$ abbreviate $\neg D \neg$. Besides the classical tautologies **DL** has the following axioms

$$\begin{array}{ll} (\mathrm{D1}) & \overline{D}(p \to q) \to (\overline{D}p \to \overline{D}q), \\ (\mathrm{D2}) & p \to \overline{D}Dp, \\ (\mathrm{D3}) & DDp \to p \vee Dp. \end{array}$$

Its rules of inference are Modus Ponens, Universal Generalization for $\overline{D}$, Substitution and a special Irreflexivity Rule for D:

$$\begin{array}{ll} (\mathrm{MP}) & \phi \to \psi, \phi/\psi, \\ (\mathrm{UG}_{\neq}) & \phi/\overline{D}\phi, \\ (\mathrm{SUB}) & \phi/\sigma\phi, \text{ for any substitution instance } \sigma\phi \text{ of } \phi, \\ (\mathrm{IR}_D) & p \wedge \neg Dp \to \phi/\phi, \text{ provided } p \text{ does not occur in } \phi. \end{array}$$

Let $\mathcal{O} = \{D\} \cup \{\Diamond_1, \Diamond_2, \ldots, \Diamond_1\breve{}, \Diamond_2\breve{}, \ldots\}$ be a collection of unary modal operators. I will write $\Box_i$ for the dual $\neg\Diamond_i\neg$ of $\Diamond_i$, and I suppose that for every $\Diamond \in \mathcal{O}$ we have its converse $\Diamond\breve{}$ available in $\mathcal{O}$ (the converse of D is D itself). For the time being I assume the language does not contain any operators like do, ra, or fix.

Let Λ be a logic which contains the axioms of **DL** plus $\Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$, $\phi \to \Box_i\Diamond_i\breve{}\phi$, $\phi \to \Box_i\breve{}\Diamond_i\phi$, and $\Diamond_i\phi \to \phi \vee D\phi$, for every $\Diamond_i \in \mathcal{O}$, and which has MP, $\mathrm{UG}_{\Box_i}$, IR_D, and SUB as its rules of inference.

DEFINITION 6.2. Let Λ be a logic as specified above. A theory Δ is Λ-*consistent* if $\Delta \nvdash_\Lambda \bot$.

Let Φ be a collection of proposition letters. A theory Δ is called a Φ-*theory* if all proposition letters occurring in formulas in Δ are elements of Φ. Δ is called a *complete* Φ-theory if $\phi \in \Delta$ or $\neg\phi \in \Delta$, for all formulas built up using proposition letters in Φ.

Δ is a *distinguishing* Φ-theory if (i) for some proposition letter p, $p \wedge \neg Dp \in \Delta$, and (ii) whenever $\Diamond_1(\phi_1 \wedge \Diamond_2(\phi_2 \wedge \cdots \wedge \Diamond_m\phi_m)\cdots) \in \Delta$, then for some proposition letter p,

$$\Diamond_1(\phi_1 \wedge \Diamond_2(\phi_2 \wedge \cdots \wedge \Diamond_m(\phi_m \wedge p \wedge \neg Dp))\cdots) \in \Delta.$$

LEMMA 6.3. *Let Σ be a consistent theory in Λ, and let p be a proposition letter not occurring in any formula in Σ. Suppose $\phi_1 \wedge \Diamond_1(\phi_2 \wedge \Diamond_2(\cdots \wedge \Diamond_{m-1}\phi_m)\cdots) \in \Sigma$. Then the union of Σ and $\{\phi_1 \wedge \Diamond_1(\phi_2 \wedge \Diamond_2(\cdots \wedge \Diamond_{m-1}(\phi_m \wedge p \wedge \neg Dp))\cdots)\}$ is consistent.*

Proof. Cf. [13, Corollary 2.2.3]. □

LEMMA 6.4 (Extension lemma). *Let Σ be a Λ-consistent Φ-theory. Let $\Phi' \supseteq \Phi$ be an extension of Φ by a countably infinite set of proposition letters. Then there is a complete, Λ-consistent, distinguishing Φ'-theory Δ containing Σ.*

Proof. This is similar to the proof of [13, Theorem 2.3.1] or [38, Lemma 4.6]. Nevertheless, it is short enough to be included here.

Define $\Delta = \bigcup_{n<\omega} \Delta_n$, where each Δ_n is a consistent Φ'-theory, satisfying $|\Delta_n \setminus \Delta_1| < \omega$, for all n. To define these $\Delta_n s$, let $p \in \Phi' \setminus \Phi$. Then, by 6.3, $\Sigma \cup \{p \wedge \neg Dp\}$ is Λ-consistent. Set $\Delta_1 = \Sigma \cup \{p \wedge \neg Dp\}$.

Let $\phi_2,\ \phi_3, \ldots$ be an enumeration of all Φ'-formulas, and suppose that Δ_n has been defined and has the properties cited. Define $\Delta_{n+1} = \Delta_n \cup E_n$, where

$(*)$ $E_n = \{\neg\phi_n\}$, if $\Delta_n \cup \{\phi_n\}$ is not Λ-consistent,

$(**)$ $E_n = \{\phi_n\}$, if $\Delta_n \cup \{\phi_n\}$ is Λ-consistent, and ϕ is not of the form $\Diamond_1(\psi_1 \wedge \Diamond(\psi_2 \wedge \cdots \wedge \Diamond_m\psi_m)\cdots)$,

$(***)$ if ϕ_n does have this form, then since $|\Delta_n \setminus \Delta_1| < \omega$, there are proposition letters $p_1, \ldots, p_m \in \Phi' \setminus \Phi$ that do not occur in Δ_n. Set
$E_n = \{\phi_n, \Diamond_1(\psi_1 \wedge p_1 \wedge \neg Dp_1 \wedge \Diamond_2(\cdots \wedge \Diamond_m(\psi_m \wedge p_m \wedge \neg Dp_m))\cdots)\}$.

It is obvious that Δ_{n+1} is Λ-consistent if it has been defined according to $(*)$ or $(**)$. But, by repeated applications of 6.3 it is also consistent when defined according to $(***)$. I leave it to the reader to check that Δ is complete, Λ-consistent and distinguishing. □

DEFINITION 6.5. Let Φ be a countably infinite set of proposition letters. Let W_c be the set of all complete, Λ-consistent, distinguishing Φ-theories. On W_c we define relations $R_{c\Diamond}$, for $\Diamond \in \mathcal{O}$, by putting $\Delta_1 R_{c\Diamond} \Delta_2$ iff for all Φ-formulas ϕ, if $\Box\phi \in \Delta_1$, then $\phi \in \Delta_2$ (or equivalently: if $\phi \in \Delta_2$ then $\Diamond\phi \in \Delta_1$, or equivalently: if $\breve{\Box}\phi \in \Delta_2$ then $\phi \in \Delta_1$).

We use R_{cD} to denote the relation defined using the D operator.

LEMMA 6.6 (Successor lemma). *Let $\Delta_1 \in W_c$. Assume $\Diamond_i\phi \in \Delta_1$. Then there is a $\Delta_2 \in W_c$ with $\Delta_1 R_{c\Diamond_i} \Delta_2$ and $\phi \in \Delta_2$.*

Proof. Cf. [13, Proposition 2.3.2] or [38, Lemma 4.7]. □

We now turn to defining a model in which the interpretation of the D operator is real inequality.

Define $\Delta \sim_D \Delta_2$ if $\Delta_1 = \Delta_2$ or $\Delta_1 R_{cD} \Delta_2$. By [32, Theorem 3.2] or [38, Lemma 4.9] $\sim_D$ is an equivalence relation. A subset of W_c is called *connected* if it is a $\sim_D$-equivalence class. By an easy argument one can show that R_{cD} is *real* inequality when restricted to a connected subset of W_c (cf. [32, Theorem 3.2] or [38, Lemma 4.11]). Also, since Λ contains the axioms $\Diamond_i \phi \rightarrow \phi \vee D\phi$, any connected subset of W_c must be closed under $R_{c\Diamond_i}$.

DEFINITION 6.7. A d-*canonical frame* for Λ is a tuple $\mathfrak{F}_d = (W_d, \{R_{d\Diamond}\colon \Diamond \in \mathcal{O}\})$, where W_d is a connected subset of W_c, and $R_{d\Diamond} = R_{c\Diamond} \upharpoonright (W_c \times W_c)$.

A d-*canonical model* for Λ is a tuple $\mathfrak{M}_d = (\mathfrak{F}_d, V_d)$, where $\mathfrak{F}_d$ is d-canonical frame, and V_d is given by $\Delta \in V_d(p)$ iff $p \in \Delta$.

LEMMA 6.8 (Truth Lemma). *For all formulas ϕ in the language containing the modal operators in $\mathcal{O}$, and all $\Delta \in \mathfrak{M}_d$, we have $\mathfrak{M}_d, \Delta \models \phi$ iff $\phi \in \Delta$.*

Proof. We argue by induction on ϕ, and only treat the case $\phi \equiv \Diamond\psi$. If $\mathfrak{M}_d, \Delta_1 \models \Diamond\psi$, then there is a $\Delta_2 \in W_d$ with $\Delta_1 R_{d\Diamond} \Delta_2$ and $\mathfrak{M}_d, \Delta_2 \models \psi$. By the induction hypothesis $\psi \in \Delta_2$. Since $\Delta_1 R_{d\Diamond} \Delta_2$ this implies $\Diamond\psi \in \Delta_1$.

Conversely, by the Successor Lemma and the remarks preceding Definition 6.7 $\Diamond\psi \in \Delta_1$ implies that for some $\Delta_2 \in W_d$, $\Delta_1 R_{d\Diamond} \Delta_2$ and $\psi \in \Delta_2$. By the induction hypothesis this gives $\mathfrak{M}_d,\ \Delta \models \psi$, and hence $\mathfrak{M}_d,\ \Delta_1 \models \Diamond\psi$. □

6.2. *DML*

The next step is to define a logic in $\mathcal{DML}$ and see how we can adapt the techniques of the previous subsection to it. In order to apply the completeness construction involving the irreflexivity rule (IR_D), we need to have an Extension Lemma and a Successor Lemma. And to obtain those, it is essential that for every modality we have a converse modality available. We will use the modal operators D and $\mathsf{do}(\alpha\,;\cdot ?)$, for α a procedure in $\mathcal{DML}$, as input for the construction. The converse of the latter is $\mathsf{do}(\alpha\breve{}\,;\cdot ?)$; D is its own converse.

We will also add *inclusion axioms* stating that any binary relation is included in ∇, the universal relation. As pointed out above, the inclusion axioms will allow us to generate along the relation R_{cD} in the provisional canonical model to arrive at the final canonical model without destroying the Truth Lemma. We add inclusion axioms for each of our modal operators $\mathsf{do}(\alpha\,;\cdot ?)$.

With these modifications we can define a canonical model for a dynamic modal logic **DML**, and establish an Extension and Successor Lemma, and, finally, a Completeness Theorem for this logic.

DEFINITION 6.9. Let $[\alpha]\phi?$ abbreviate the formula $\neg\mathsf{do}(\alpha\,;\neg\phi?)$; $E\phi$ is short for $\phi \vee D\phi\,;\mathsf{con}(\phi)$ abbreviate $(\mathsf{exp}(\neg\phi))\breve{}$, $\mathsf{exp}(\phi)$ is short for $(\mathsf{exp}(\top)\,;\phi?)$; and δ is short for $\top?$.

Besides enough classical tautologies, and the axioms of **DL** (taken as axioms over $\mathcal{DML}(\Phi)$) the system **DML** has the following axioms:

Definitions

(DML0) $\mathsf{do}(\alpha) \leftrightarrow \mathsf{do}(\alpha\,;\delta)$,

(DML1) $\mathsf{ra}(\alpha) \leftrightarrow \mathsf{do}(\breve{\alpha})$,

(DML2) $\mathsf{fix}(\alpha) \leftrightarrow \mathsf{do}(\alpha \cap \delta)$.

Basic axioms

(DML3) $[\alpha](p \to q) \to ([\alpha]p \to [\alpha]q)$,

(DML4) $\mathsf{do}(\alpha\,;p?) \to p \vee Dp$.

Relation connectives

(DML5) $\mathsf{do}(\alpha \cap \beta\,;p?) \to \mathsf{do}(\alpha\,;p?) \wedge \mathsf{do}(\beta\,;p?)$,

(DML6) $E(p \wedge \neg Dp) \to \big(\mathsf{do}(\alpha\,;p?) \wedge \mathsf{do}(\beta\,;p?)$
$\to \mathsf{do}(\alpha \cap \beta\,;p?)\big)$,

(DML7) $\mathsf{do}((\alpha\,;\beta)\,;p?) \leftrightarrow \mathsf{do}(\alpha\,;\mathsf{do}(\beta\,;p?)?)$,

(DML8) $E(p \wedge \neg Dp) \to (\mathsf{do}(\alpha\,;p?) \leftrightarrow \neg\mathsf{do}(-\alpha\,;p?))$,

(DML9) $p \to [\alpha]\mathsf{do}(\breve{\alpha};p?)$,

(DML10) $p \to [\breve{\alpha}]\mathsf{do}(\alpha\,;p?)$.

Test

(DML11) $\mathsf{do}(p?\,;q?) \leftrightarrow (p \wedge q)$.

Structure

(DML12) $\mathsf{do}\big(\mathsf{exp}(\top)\,;\big(\mathsf{do}(\mathsf{exp}(\top)\,;p?)\big)?\big) \to \mathsf{do}(\mathsf{exp}(\top)\,;p?)$,

(DML13) $p \to \mathsf{do}(\mathsf{exp}(\top)\,;\top?)$.

Besides those of **DL**, the rules of inference of **DML** are:

(UG_α) $\phi/[\alpha]\phi$, for $\alpha \in Proc(\Phi)$.

Observe that for every relational connective in our language **DML** has one or two axioms describing its behaviour *implicitly*, that is, in the context of a formula of the form $\mathsf{do}(\cdot\,;\cdot?)$. As $\mathcal{DML}$-formulas are evaluated at points, not at pairs or transitions, it is impossible to state explicitly how the relational connectives should behave.

6.3. *Completeness of DML*

Now that we have defined our logic we can prove a completeness result; as announced before we will use the construction from §6.1. As input for the construction we will use the modal operators D and $\mathsf{do}(\alpha\,;\cdot?)$, for α a procedure in $\mathcal{DML}$ in which all occurrences of exp are of the form $\mathsf{exp}(\top)$ only.

THEOREM 6.10. *The system* **DML** *is sound and complete with respect to its standard models.*

Proof. Proving soundness is left to the reader. To prove completeness, define $\mathcal{O}$ to be $\{D, \mathsf{do}(\alpha\,;\cdot?) : \alpha \in Proc(\Phi')\}$, where it is assumed that all occurrences of exp are of the form $\mathsf{exp}(\top)$ only.

We build up the proof in a number of steps.

Canonical Relations. The canonical relation R_{cD} is defined as in 6.5. For the modal operators $\mathsf{do}(\alpha\,;\cdot?)$ the canonical relation $R_{c\alpha}$ is defined by putting $R_{c\alpha}\Delta_1\Delta_2$ if for all ϕ, $\phi \in \Delta_2$ implies $\mathsf{do}(\alpha\,;\phi?) \in \Delta_1$.

Successor Lemma. Let Δ_1 be a maximal consistent distinguishing theory. If Δ_2 contains a formula of the form $D\phi$ or $\mathsf{do}(\alpha\,;\phi?)$, then the required R_{c_D}-successor or $R_{c\alpha}$-successor exists: if $D\phi \in \Delta_1$, then there is a maximal consistent distinguishing Δ_2 with $\phi \in \Delta_2$ and $R_{cD}\Delta_1\Delta_2$, and if $\mathsf{do}(\alpha\,;\phi?) \in \Delta_1$, then there is a maximal consistent distinguishing Δ_2 with $\phi \in \Delta_2$ and $R_{c\alpha}\Delta_1\Delta_2$.

Provisional Canonical Model. A provisional canonical model $\mathfrak{M}_c$ is defined by putting $\mathfrak{M}_c = (W_c, R_{cD}, R_{c\mathsf{exp}}, [\![\cdot]\!]_c, V_c)$, where W_c is the set of all maximal **DML**-consistent distinguishing theories, R_{cD} is as defined before; $R_{c\mathsf{exp}} = R_{c\alpha}$, for $\alpha = \mathsf{exp}(\top)$, is the informational ordering; $[\![\alpha]\!]_c = R_{c\alpha}$, and $V_c(p) = \{\Delta : p \in \Delta\}$.

Observe that the provisional canonical model may still be a *non-standard* model for **DML**: R_{cD} need not connect every two different point in W_c, even though it is symmetric and irreflexive.

Final Canonical Model. To obtain a final canonical model which is based on a standard frame for **DML** we generate along the relation R_{cD}. More precisely, take any Δ_1 in W_c, and consider all Δ_2 with $R_{cD}\Delta_1\Delta_2$; let W^f be the resulting subset of W_c. For all procedures α, let R^f_α be the restriction of $R_{c\alpha}$ to W^f.

A *final canonical model* for **DML** is a tuple $\mathfrak{M}^f = (W^f, R_D^f, R_{\mathsf{exp}}^f, [\![\cdot]\!]^f, V^f)$, with W^f, R_D^f as above, $R_{\mathsf{exp}}^f = R_\alpha^f$ for $\alpha = \mathsf{exp}(\top)$, $[\![\alpha]\!]^f = R_\alpha^f$, and $V^f(p) = V_c(p) \cap W^f$.

Structure Lemma. Any final canonical model $\mathfrak{M}^f$ for **DML** is a standard model for **DML**.

Proof. To show that $\mathfrak{M}^f$ is standard, we have to show that R_D^f is real inequality, that the relational connectives behave properly, and that R_{exp}^f is transitive and reflexive. A useful feature of the canonical model that is worth recalling before we start off, is that by construction for any Δ in $\mathfrak{M}^f$ there is a proposition letter p_Δ such that $p \wedge \neg Dp \in \Delta$.

First of all, R_D^f is real inequality in $\mathfrak{M}^f$. As to the relational connectives, consider $\cap$. By (DML5)

$$[\![\alpha \cap \beta]\!]^f = R_{\alpha\cap\beta}^f \subseteq R_\alpha^f \cap R_\beta^f = [\![\alpha^f]\!] \cap [\![\beta]\!]^f.$$

For the converse inclusion, assume that $(\Delta, \Sigma) \in [\![\alpha]\!]^f \cap [\![\beta]\!]^f$. Let p be a unique proposition letter in Σ. Then $E(p \wedge \neg Dp)$, $\mathsf{do}(\alpha\,;p?)$, $\mathsf{do}(\beta\,;p?) \in \Delta$. Hence, by axiom (DML6), $\mathsf{do}(\alpha \cap \beta\,;p?) \in \Delta$. But this is possible only if $(\Delta, \Sigma) \in [\![\alpha \cap \beta]\!]^f$, as required.

By using axiom (DML7) it is easily verified that $[\![\alpha\,;\beta]\!]^f = [\![\alpha]\!]^f\,;[\![\beta]\!]^f$.

To see that $[\![-\alpha]\!]^f = -[\![\alpha]\!]^f$, argue as follows. Assume $(\Delta, \Sigma) \in [\![-\alpha]\!]^f$. Let p be a unique proposition letter with $p \in \Sigma$. Then $E(p \wedge \neg Dp)$, $\mathsf{do}(-\alpha\,;p?) \in \Delta$. Therefore, $\neg\mathsf{do}(\alpha\,;p?) \in \Delta$, by axiom (DML8). It follows that $(\Delta, \Sigma) \notin [\![\alpha]\!]^f$. For the converse inclusion, assume $(\Delta, \Sigma) \notin [\![\alpha]\!]^f$. Choose a unique proposition letter p in Σ. Then, by the Successor Lemma, $E(p \wedge \neg Dp)$, $\neg\mathsf{do}(\alpha\,;p?) \in \Delta$, and, by axiom (DML8), $\mathsf{do}(-\alpha\,;p?) \in \Delta$. But this is possible only if $(\Delta, \Sigma) \in [\![-\alpha]\!]^f$.

To prove that the converse operation $\breve{}$ is standard, use axioms (DML9) and (DML10). For the test operation ? one uses (DML11).

Finally, to see that R_{exp}^f has the right structural properties, viz. that it is transitive and reflexive, use axioms (DML12) and (DML13). As we have assumed that the only argument of exp is $\top$, we don't have to establish any further properties of R_{exp}^f. □

Truth Lemma. Let $\mathfrak{M}^f$ be a final canonical model. For all $\Delta \in W^f$ and all $\mathcal{DML}^+$-formulas ϕ, we have $\mathfrak{M}, \Delta \models \phi$ iff $\phi \in \Delta$.

Proof. The proof is by induction on ϕ; the only interesting cases are $D\phi$ and $\mathsf{do}(\alpha)$, for α any procedure. We only treat the case $\mathsf{do}(\alpha)$.

If $\mathsf{do}(\alpha) \in \Delta_1$ then, by (DML0), $\mathsf{do}(\alpha\,;\top?) \in \Delta_1$. By the Successor Lemma there exists Δ_2 with $(\Delta_1, \Delta_2) \in [\![\alpha]\!]$. Hence, $\Delta_1 \models \mathsf{do}(\alpha)$.

Conversely, if $\Delta_1 \models \mathsf{do}(\alpha)$, choose Δ_1 such that $(\Delta_1, \Delta_2) \in [\![\alpha]\!]$. As $\top \in \Delta_2$ it follows that $\mathsf{do}(\alpha\,;\top?) \in \Delta_1$. So, by another application of (DML0), $\mathsf{do}(\alpha) \in \Delta_1$. □

Of course, from the Truth Lemma one can derive the completeness of **DML** using a standard argument. □ Two remarks are in order. First, I want to stress that nothing in the proof of Theorem 6.10 depends in an essential way on the relation underlying relation exp being a pre-order. Also, the proof and result easily generalize to a logic **DML**$(\Phi\,;\Omega)$ in an extension $\mathcal{DML}(\Phi\,;\Omega)$ of $\mathcal{DML}$, where one has propositions Φ as before, and multiple base relations $\sqsubseteq_i$ $(i \in \Omega)$, none of which needs to be a pre-order.

Second, building on the proof of Theorem 6.10 one look at special model classes. As an example, the class of trees is axiomatized by adding the following axioms to **DML**. Let $\langle\sqsubset\rangle\phi$ abbreviate $\mathsf{do}((\mathsf{exp}(\top)\cap -\delta)\,;\phi?)$, and let $[\sqsubset]$ be the dual of $\langle\sqsubset\rangle$. (And similarly for $\langle\sqsupset\rangle$ and $[\sqsupset]$.)

(T1) $\quad p \wedge \neg Dp \to [\sqsubset][\sqsupset]\neg p$,

(T2) $\quad \langle\sqsupseteq\rangle(p \wedge \neg Dp) \to [\sqsupset](\langle\sqsubset\rangle p \vee p \vee \langle\sqsubset\rangle p)$.

Axiom T1 will make sure that in the canonical model the relation $\sqsubset$ is asymmetric, while axiom T2 will guarantee that sets of predecessors in the canonical model are linearly ordered by $\sqsubset$.

7. WHICH ALGEBRAS?

In this section I will define modal algebras appropriate for the dynamic modal language $\mathcal{DML}$. I will need one or two preliminary definitions. First, a *Boolean module* is a structure $\mathfrak{M} = (\mathfrak{B}, \mathfrak{R}, \Diamond)$, where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation algebra and $\Diamond$ is a mapping $\mathfrak{R} \times \mathfrak{B} \to \mathfrak{B}$ such that

M1 $\quad \Diamond(r, a + b) = \Diamond(r, a) + \Diamond(r, b)$,

M2 $\quad \Diamond(r + s, a) = \Diamond(r, a) + \Diamond(s, a)$,

M3 $\quad \Diamond(r, \Diamond(s, a)) = \Diamond((r\,;s), a)$,

M4 $\quad \Diamond(\delta, a) = a$,

M5 $\quad \Diamond(0, a) = 0$,

M6 $\quad \Diamond(r\breve{}, \Diamond(r, a)') \leq a'$.

Just as Boolean algebras formalize reasoning about sets, and relation algebras formalize reasoning about relations, Boolean modules formalize reasoning about sets interacting with relations through $\Diamond$. In the full

Boolean module $\mathfrak{M}(U) = (\mathfrak{B}(U), \mathfrak{R}(U), \Diamond)$ over a set $U \neq \emptyset$ the operation $\Diamond$ is defined by

$$\Diamond(R, A) = \langle R \rangle A = \{x : \exists y((x, y) \in R \wedge y \in A)\}.$$

(See Brink [7] for a formal definition of Boolean modules and some examples.)

Now, Boolean modules are almost, but not quite, the modal algebras appropriate for $\mathcal{DML}$. To obtain a perfect match, what we need in addition to the *set* forming operation or *projection* $\Diamond$, is an operation that forms new *relations*, i.e., a *mode*. This brings us to the notion of a *Peirce algebra*, which is a two-sorted algebra $\mathfrak{P} = (\mathfrak{B}, \mathfrak{R}, \Diamond, {}^c)$ with $(\mathfrak{B}, \mathfrak{R}, \Diamond)$ a Boolean module, and $(\cdot)^c : \mathfrak{B} \to \mathfrak{R}$ a mapping, called (*left*) *cylindrification*, such that for every $a \in \mathfrak{B},\ r \in \mathfrak{R}$ we have

- $\Diamond(a^c, 1) = a$, and
- $\Diamond(r, 1)^c = r\,;1$.

In the full Peirce algebra $\mathfrak{P}(U)$ over a set $U \neq \emptyset,\ (\cdot)^c$ is defined as $A^c = \{(x, y) : x \in A\}$. The algebraic apparatus of Peirce algebras has been used as an inference mechanism in terminological representation (cf. Brink, Britz and Schmidt [8]).

The precise connection between the $\mathcal{DML}$ and Peirce algebras is:

the modal algebras for the dynamic modal language $\mathcal{DML}(\Phi)$ are the Peirce algebras generated by a single relation R and the "propositions" Φ.

To see this, it suffices to show that $\Diamond$ and $(\cdot)^c$ are definable in $\mathcal{DML}$, and that $\mathsf{do},\ \mathsf{ra},\ \mathsf{fix}$ and $\mathsf{exp},\ \mathsf{con}$, ? are definable in full Peirce algebras generated by R and Φ:

$$\begin{aligned} \Diamond(\alpha, \phi) &= \{x\colon\ \exists y((x, y) \in [\![\alpha]\!]_{\mathfrak{M}} \wedge \mathfrak{M}, y \models \phi)\} \\ &= \mathsf{do}(\alpha\,; \phi?), \\ \phi^c &= \{(x, y)\colon\ \mathfrak{M}, x \models \phi\} = \phi?\,; (\delta \cup -\delta), \end{aligned}$$

and

$$\begin{aligned} \mathsf{do}(\alpha) &= \Diamond(\alpha, 1), & \mathsf{exp}(\phi) &= R \cap \phi^{c\smile}, \\ \mathsf{ra}(\alpha) &= \Diamond(\alpha^{\smile}, 1), & \mathsf{con}(\phi) &= R^{\smile} \cap (\neg\phi)^{c\smile}, \\ \mathsf{fix}(\alpha) &= \Diamond((\alpha \cap \delta), 1), & \phi? &= \delta \cap \phi^c. \end{aligned}$$

Given Theorem 6.10, the connection between $\mathcal{DML}$ and Peirce algebras established here may be interpreted as saying that (the obvious algebraic counterpart of) **DML** completely axiomatizes the set identities (i.e., identities $s = t$ between terms denoting sets) valid in all representable Peirce algebras over a single relation R, and propositions Φ.

Of course, as $\mathcal{DML}$ is closely related to propositional dynamic logic $\mathcal{PDL}$ (cf. §2), modal algebras for $\mathcal{DML}$ are closely related to the *dynamic algebras* $\mathfrak{D} = (\mathfrak{B}, \mathfrak{K}, \Diamond)$ of Kozen [26] and Pratt [30]. These too are structures that serve to interpret a two-sorted language: propositions are represented in a Boolean algebra $\mathfrak{B}$ as in our case, but relations (or programs) are represented in a *Kleene algebra* $\mathfrak{K} = (K, \cup, ;, 0, ^*)$, where * is the Kleene star. However Kleene algebras need not be Boolean ones, and in most definitions they don't include a converse operation $\breve{}$. Like Boolean modules dynamic algebras have a projection $\Diamond : \mathfrak{K} \times \mathfrak{B} \to \mathfrak{B}$; but in most definitions they are not equipped with any modes.

8. WHAT'S NEXT?

In this paper we analyzed a dynamic modal language $\mathcal{DML}$ whose distinctive aspect is its attention for the interplay between static objects and dynamic transitions. The dynamic language turned out to be a powerful one, and to have a number of applications in other areas of logic. The expressive power of $\mathcal{DML}$ was exemplified by the fact that it coincides with a large fragment of first-order logic, that its satisfiability problem is undecidable, and that we needed a difference operator D and an irreflexivity rule to match its expressive power. Nevertheless, the language of **DML** could still be analyzed with general tools and techniques from modal logic.

Several natural extensions of the language studied here present themselves. Given the close connections between $\mathcal{DML}$ and $\mathcal{PDL}$, it may seem natural to add the Kleene star * that is present in $\mathcal{PDL}$ to $\mathcal{DML}$. But $\mathcal{DML}$ with Kleene star has a Σ^1_1-complete satisfiability problem; this may be proved by using a *recurrent tiling problem* (RTP): given a finite set of tiles T, and a tile $d_1 \in T$, can T tile $\mathbb{N} \times \mathbb{N}$ such that d_1 occurs infinitely often on the first row? The RTP is a Σ^1_1-complete problem [19]. To obtain a reduction of the RTP to satisfiability in $\mathcal{DML}$ plus Kleene star, we define a formula ϕ_{RT} as the conjunction of the formula ϕ_T used in the proof of Theorem 5.1 and a formula REC to be defined shortly. We use a new propositional symbol row_0 which can only be true at nodes on the bottom row of a grid; we will ensure that there exists an infinite number of worlds where row_0 holds and the tile d_1 is placed. Now, define REC to be the conjunction of row_0, $A[\text{UP}]\neg row_0$ and

$$[\text{RIGHT}^*](row_0 \to \mathsf{do}(\text{RIGHT}^* ; (row_0 \wedge d_1)?)).$$

As in the proof of Theorem 5.1 it may be shown that T recurrently tiles $\mathbb{N} \times \mathbb{N}$ iff ϕ_{RT} is satisfiable. This proves a Σ^1_1 lower bound. A Σ^1_1 upper bound is found by observing that a formula in $\mathcal{DML}$ plus Kleene star is satisfiable iff it is satisfiable on a countable model.

As pointed out in Section 7, **DML** axiomatizes the set identities valid in all representable Peirce algebras. A natural next question is: what about the two-sorted modal language that deals with set identities as well as relation identities? De Rijke [34] provides an answer: among other things that paper uses a completeness construction similar to the one used in Section 6 of the present paper to come up with a complete axiomatization for this two-sorted modal language.

As pointed out in Sections 1 and 3, for some applications it may be necessary to be more precise about the structure of the states in $\mathcal{DML}$-models, rather than treating them as some kind of 'black boxes'. Using a result by Finger and Gabbay [11], if we have a complete 'local' logic governing what happens inside these boxes, this local system can be amalgamated with the $\mathcal{DML}$ as a global system on top of it – while preserving such properties as completeness. Likewise, it may be useful to add (more) structure to the transitions or changes as well; one can think of formalisms involving intricate plans or processes here as an area where this could be of use.

To conclude, here are some technical questions.

1. In Section 5 we mentioned a number of systems closely related to $\mathcal{DML}$ with a decidable satisfiable problem. An obvious question is to locate the boundary (in terms of fragments of $\mathcal{DML}$) where the satisfiability problem becomes undecidable more precisely, and to identify as large as possible a decidable fragment of $\mathcal{DML}$. In particular, what if we start with as few relational connectives as possible but with the minimal expansion and contraction operators μ-exp and μ-con – will we still have decidability?
2. From De Rijke [32] it is known that adding the irreflexivity rule (IR_D) to the basic modal logic of the D-operator does not add any new consequences. What about **DML**? With **DML** it is not clear whether the $(\mathrm{IR})_D$-rule adds anything to **DML** in terms of new consequences.
3. In Section 5 we found several fragments of $\mathcal{DML}$ with a decidable satisfiability problem. What is their complexity?

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NOTES

[1] This terminology $\mathsf{exp}(\cdot)$ and $\mathsf{con}(\cdot)$ derives from one of the uses of $\mathcal{DML}$, viz. as a setting in which the basic operations studied in Theory Change, *expansions* and *contractions*, are modeled. See §3 for some details.

[2] A quick remark about the properties of $\sqsubseteq$. It seems a reasonable minimal requirement to let this abstract relation of information growth or change be a pre-order. Pre-orders have a long tradition as information structures, viz. their use as models for intuitionistic logic. Of the technical results presented below none hinges on $\sqsubseteq$ being a pre-order.

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