Against the Tyranny of 'Pure States' in Quantum Theory

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Abstract

In this paper we provide arguments against the dominant role played by the notion of *pure sate* within the orthodox account of quantum theory. Firstly, we will argue that the origin of this notion is intrinsically related to the widespread empirical-positivist understanding of physics according to which 'theories describe actual observations of subjects (or agents)'. Secondly, we will show how within the notion of pure state there is a scrambling of two mutually incompatible definitions. On the one hand, a contextual definition which attempts to provide an intuitive physical grasp in terms of the *certain prediction* of a measurement outcome; and on the other hand, a non-contextual purely abstract mathematical definition which has no clear physical content. We will then turn our attention to the way in which pure states and mixtures have been considered by two categorical approaches to QM, namely, the topos approach originally presented by Chris Isham and Jeremy Butterfield [27, 28, 29] and the more recent logos categorical approach presented by the authors of this article [11, 12]. While the first approach presents serious difficulties in order to produce an orthodox understanding of pure states and mixtures, the latter presents a new scheme in which the distinction between pure states and mixtures becomes completely irrelevant.

Keywords: *pure state, mixture, quantum mechanics, graphs.*

1 Pure States in Quantum Mechanics

The notion of *pure state* plays an essential role within the many debates that take place today within the orthodox literature discussing about Quantum Mechanics (QM). Its role established since the axiomatic formulation of the theory has become increasingly dominant establishing a primacy over the so called *mixed states.* As explained by David Mermin [33, p. 758]: "[P]eople distinguish between pure and mixed states. It is often said that a system is in a pure state if *we* have maximum *knowledge* of the system, while it is in a mixed state if *our knowledge* of the system is incomplete." The explicit reference to "our knowledge" is strictly related to the following operational definition of pure states: 'If a quantum system is prepared in such way that one can devise a maximal test yielding with certainty (probability = 1) a particular outcome, then it is said that the quantum system is in a *pure state.*' In turn, the notion of *maximal test* allows to interpret a quantum observable as being an *actual property* —i.e., a property that will yield the answer *yes* when being measured [36]. It is

1

then stated that the pure state of a quantum system is described by a unit vector in a Hilbert space which in Dirac's notation is written as $|\psi\rangle$.

Depending on the basis, a pure state in $\mathcal{H}$ is also represented by a superposition of states:

$$|\psi\rangle = \sum a_i |\varphi\rangle_i$$

As a consequence, depending on the choice of the basis, a pure state will yield uncertain results. Asher Peres explains this important point in the following manner:

> "According to quantum theory, we have a choice between different, mutually incompatible tests. For example, we may orient the Stern-Gerlach magnet in any direction we please. Why then is such a Stern-Gerlach test called complete? The reason can be stated as the following postulate:
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> **A. Statistical determinism.** *If a quantum system is prepared in such a way that it certainly yields a predictable outcome in a specified maximal test, the various outcomes of* **any** *other test have definite* **probabilities.** *In particular, these probabilities do not depend on the details of the procedure used for preparing the quantum system, so that it yields a specific outcome in the given maximal test. A system prepared in such a way is said to be in a* **pure** *state.*" [34, p. 66]

The notion of pure state can be also extended to density operators. Let $\mathcal{H}$ be a Hilbert space. A density operator ρ (i.e. a positive trace class operator with trace 1) is called a *state*. Being positive (and self-adjoint), the eigenvalues of ρ are non-negative and real and it is possible to diagonalize it. If the rank of ρ is equal to 1, this diagonal matrix is given by $(1, 0, \ldots, 0)$ and ρ is equal to $vv^\dagger$ for some normalized vector $v \in \mathcal{H}$. In this case, ρ is called a *pure state*. If the rank of ρ is grater than 1 (or equivalently if $\mathrm{Tr}(\rho^2) < 1$), the state is called a *mixed state*; or in short, *mixture*. For example, the vector $\alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, gives the following density matrix:

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix}$$

Notice that, if ρ is a pure state (i.e., $\mathrm{Tr}(\rho^2) = 1$), there always exists a basis in which the matrix can be diagonalized as:

$$\rho_{pure} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Contrary to the case of pure states, when considering mixed states, all observables are *uncertain*; they all possess a probability which pertains to the open interval $(0, 1)$. These properties are referred to in the literature as *indeterminate* or *potential* properties (see e.g., [36]). Indeterminate properties might, or might not become actualized in a future instant of time; they are *uncertain* properties which cannot be considered as elements of physical reality (in the EPR sense [24]). As an example of a mixed state (i.e., $\mathrm{Tr}(\rho^2) < 1$) we can consider the following diagonal matrix,

$$\rho_{mixed} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Here, the observables related to the diagonal elements have probability $\frac{1}{2}$. Such states are referred to in te literature as providing *minimal knowledge*. So while pure states guarantee the existence of an observable which, if measured, will be obtained with *certainty* (probability equal to 1), mixed states do not. This means there will exist no single context for which a mixed state will predict with certainty a *yes-no* answer for a specific measurement.

The just mentioned distinction between *pure states* and *mixtures* was introduced in order to support a twofold foundation. On the one hand, an empirical-positivist understanding of physics as

an algorithmic mathematical device capable of predicting observations; and on the other, an atomist metaphysical understanding of physical reality in terms of systems constituted by definite valued properties. These two interconnected presuppositions have severe inconsistencies when related to the orthodox mathematical formalism of QM. Not only the notion of pure state is ill defined, but the reference to 'mixtures' —as contraposed to 'pure states'— is extremely problematic for it erases the fundamental distinction between *quantum mixture* and *classical mixture* —a known distinction in the specialized literature which Bernard d'Espagnat termed *proper* and *improper* (see [13, chap. 6]). However, we will show that even the distinction between pure state and improper mixed state becomes irrelevant when seriously considering the Born rule as providing the invariant condition which allows us to describe an objective (non-classical) state of affairs.

Historically, the definition of *pure state* was not introduced by mere chance. This notion is a fundamental cornerstone of a specific viewpoint regarding the understanding of physical theories in general, and of QM in particular. So before considering the intrinsic technical difficulties of the definition of pure state, in order to understand in depth the already rotten roots of "purity" we require some historical context to which we will now turn our attention.

2   The 20th Century Positivist Re-Foundation of Physics

In the 20th Century, physics as a discipline was subject of a deep re-foundation, mainly due to the coming into power of positivism and its empiricist anti-metaphysical agenda. Physics had been, since the ancient Greeks, always understood as a discipline which attempted to describe or express *physis* —a kernel Greek concept later on translated as 'reality' [4]. The attempt of physicists was to capture aspects of reality through theories; i.e. the aim was to theoretically *represent* physical reality. Of course, the nature and meaning of this *representation* was not unproblematic. In the 17th Century, Immanuel Kant, a physicist himself, inaugurated a critical account of *representation* through which the naive idea of 'unveiling' *reality as it is* was severely questioned. As part of the revolt against those who naively believed in the possibility of discovering *the thing in itself*, Machian positivism deconstructed the very foundation of classical mechanics. At the end of the 19th Century, Ernst Mach argued a critical analysis of classical mechanics as related to atomist metaphysics and absolute space and time. His investigations led him to the conclusion that science is nothing but the systematic and synoptical recording of data of experience. In his *Analysis of Sensations*, Mach concluded that primary sensations constitute the ultimate building blocks of science, inferring at the same time that scientific concepts are only admissible if they can be defined in terms of sensations. From this empiricist standpoint he argued strongly against the existence of *atoms*. Metaphysical speculation —understood now as a discourse attempting to go beyond the observed phenomena— should be erased from scientific inquiry and research. The crisis produced by Mach turned physics away from (classical) metaphysical presuppositions and closer back to "common sense" human experience. The result was the coming into being of a completely new idea of physical understanding. Theories would not be regarded anymore as describing or expressing —in some way— reality. This would be too metaphysical, too pretentious. Instead, theories had to be understood in a seemingly more modest manner; i.e., as a simple 'economy of (human) experience'.

The critical Machian attack against classical Newtonian metaphysics, played also an essential role in the development of both Relativity and the theory of quanta. But even though QM was developed taking into account 19th Century positivists anti-metaphysical ideas,[1] it remained anyhow strongly linked to atomism. This scrambling produced very soon a paradoxical entanglement between two mutually incompatible positions, namely, atomist substantivalism that maintained —in metaphysical

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[1] Heisenberg's matrix mechanics is an excellent example.

3

terms— the existence of unobservable atoms, and Machian empirical-positivism which grounding itself in observed phenomena affirmed the need to eradicate all *a priori* metaphysical notions from physics —including of course that of 'atom'. However, regardless of the obvious inconsistencies, very soon, the critical analysis of the Newtonian metaphysical picture was forgotten and atomist metaphysics became regarded —even by positivists— as part of our "common sense" understanding of the world.

Joining forces with positivism, after the IIWW, instrumentalism helped to distance and replace the original foundation of physics —grounded on the old metaphysical notion of *physis*— by a more human foundation relative to actual observations. The shout: "shut up and calculate!" meant a lot more: "shut up, calculate, and stop talking about metaphysics and a reality we cannot observe!" Today, the entanglement between, on the one hand, an empiricist (anti-metaphysical) instrumentalist account of physics as a discipline making exclusive reference to "common sense" observations, and on the other, a deeply rooted classical language making reference to (unobservable) microscopic particles has crated what might be called a curious "sophistic substantialism" (see for a detailed discussion [9]). This strange paradoxical conjunction finds its Archimedean point in the notion of *actuality* which plays a double role within the debates about the philosophical foundation of the theory of quanta. Indeed, actuality has two different —not necessarily compatible— meanings and uses which have been confused and scrambled within the orthodox literature. Firstly, there is an empiricist understanding of actuality as the *hic et nunc* experience of an individual agent. According to Bas van Fraassen, one the most prominent contemporary empiricists:

> "the only believe involved in accepting a scientific theory is belief that it is empirically adequate: all that is *both* actual *and* observable finds a place in some model of the theory. So far as empirical adequacy is concerned, the theory would be just as good if there existed nothing at all that was either unobservable or not actual. Acceptance of the theory does not commit us to belief in the reality of either sort of thing." [38, p. 197]

This first meaning of actuality can be resumed in the following manner:

Definition 2.1 (Empiricist Actuality) *Actuality as making reference to hic et nunc observations of subjects (or agents).*

Secondly, actuality is also —implicitly— understood in metaphysical terms as characterizing a *mode of existence* independent of observations. In the XVII Century, within the Newtonian mechanical description of the world, any indetermination —related in the Aristotelian scheme to the potential realm of being— was erased from the physical representation of reality. In fact, within classical mechanics, every physical system could be described exclusively by means of its actual properties. As remarked by Dennis Dieks:

> "In classical physics the most fundamental description of a physical system (a point in phase space) reflects only the actual, and nothing that is merely possible. It is true that sometimes states involving probabilities occur in classical physics: think of the probability distributions in statistical mechanics. But the occurrence of possibilities in such cases merely reflects our ignorance about what is actual. The statistical states do not correspond to features of the actual system (unlike the case of the quantum mechanical superpositions), but quantify our lack of knowledge of those actual features." [16, p. 124]

This second understanding of actuality which can be defined without any reference whatsoever to observability is of course purely formal and metaphysical. As discussed in [11], an *Actual State of Affairs* (ASA) can be defined as a closed system considered in terms of a set of actual (definite valued) properties which can be thought as a map from the set of properties to the $\{0, 1\}$. Specifically, an

ASA is a function $\Psi : \mathcal{G} \to \{0, 1\}$ from the set of properties to $\{0, 1\}$ satisfying certain compatibility conditions. We say that the property $P \in \mathcal{G}$ is *true* if $\Psi(P) = 1$ and $P \in \mathcal{G}$ is *false* if $\Psi(P) = 0$. The evolution of an ASA is formalized by the fact that the morphism f satisfies $\Phi f = \Psi$. Diagrammatically,

$$\mathcal{G}_{t_1} \xrightarrow{f} \mathcal{G}_{t_2}$$
$$\Psi \searrow \qquad \swarrow \Phi$$
$$\{0, 1\}$$

Then, given that $\Phi(f(P)) = \Psi(P)$, the truth of $P \in \mathcal{G}_{t_1}$ is equivalent to the truth of $f(P) \in \mathcal{G}_{t_2}$. This formalization comprises the idea that the properties of a system remain existent through the evolution of the system. The model allows then to claim that the truth or falsity of a property is independent of particular observations. Or in other words, binary-valuations are a formal way to capture the classical actualist (metaphysical) representation of physics according to which the properties of objects preexist to their measurement.

Definition 2.2 (Metaphysical Actuality) *Actuality as making reference to a mode of existence defined in terms of definite binary valuedness of properties which evolve completely independently of subjects and their measurements.*

Maybe the best exposure of this scrambling present within QM is the definition of *element of physical reality* presented in the famous 1935 paper by Einstein, Podolsky and Rosen [24].

Element of Physical Reality: *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity.*

As remarked by Diederik Aerts and Massimiliano Sassoli de Bianchi [2, p. 20]: "An element of reality is a state of prediction: a property of an entity that we know is actual, in the sense that, should we decide to observe it (i.e., to test its actuality), the outcome of the observation would be certainly successful." It is in this way that the relation between observation and reality is pasted together.

The notion of *pure state* in QM has an analogous role to the one played by actuality within the present widespread empiricist understanding and analysis of physical theories. Just like the notion of actuality has a double reference, the notion of pure state scrambles contextual measurements with a non-contextual mathematical definition. As we shall see in the next section, the tension threatening inconsistency within the definition itself of 'pure state' is found not only at the philosophical level of analysis, it is also —maybe more importantly— already present within its formal definition itself.

3   The (Non-)Contextual Definition(s) of 'Pure State'

An exposure of the tension found within the just mentioned incompatible reference to the notion of 'actuality' is also present in QM within the definition(s) of *pure state*. As we mentioned above, there is an operational definition of pure state grounded on a specific context of inquiry, but there is also a non-contextual definition of pure state provided in purely abstract mathematical terms. Let us analyze these two different definitions in some detail.

The operational contextual definition of pure state rests —as already discussed— on a specific type of measurement called *maximal test*. Such a test is maximal in the case we obtain with certainty (probability = 1) the observable in question: if we measure the state $|\psi\rangle$ (in its correspondent basis) we are *certain* that we will obtain the observable related to this state. This definition rests on the explicit reference to the particular basis (or context) in which the vector in Dirac's notation can

be written as a single term, namely, as $|\psi\rangle$. We say the definition is 'contextual' because it makes explicit reference to only one context between the many possible ones. Exactly the same actualist intuition appears in the case of density operators where the state ρ is a pure state, if there exists a basis in which the matrix can be diagonalized as:

$$\rho_{pure} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

This contextual definition has the purpose to secure the existence of an observable which will be *certain*, and consequently *actual*, if measured.

Definition 3.1 (Operational Contextual Definition of Pure State) *Given a quantum system in the state $|\psi\rangle$, there exists an experimental situation (or context) in which the test of it will yield with certainty (probability = 1) its related outcome.*

Instead, mathematical physicists tend to use a seemingly different non-contextual definition of pure state. This definition has no physical counterpart and makes reference to a purely abstract mathematical feature of vectors, namely, that when considered in terms of density operators their norm is 1, that is, ρ is a pure state if $\mathrm{Tr}(\rho^2) = 1$, or equivalently[2] when $\rho = \rho^2$.

Definition 3.2 (Mathematical Non-Contextual Definition of Pure State) *A vector in Hilbert space Ψ.[3] Or in terms of density operators, an operator ρ which is a projector, i.e. such that the $Tr(\rho^2) = 1$ or $\rho = \rho^2$.*

In this case the notion of pure state is obviously non-contextual and makes reference simultaneously to the abstract vector Ψ, to the state $|\psi\rangle$, but also to the state $\sum a_i|\varphi\rangle_i$ and to any other rotation of the mentioned "pure state" (see for discussion [5]).

To sum up, the notion of pure state has a double definition, on the one hand, in terms of a contextual measurement, and on the other, in terms of a purely non-contextual abstract feature of vectors. We find here a shift with no relation of continuity between the contextual "common sense" operational account and its non-contextual abstract mathematical counterpart. While the operational definition makes reference to a specific physical situation, the mathematical one goes clearly beyond this constraint and considers pure states as invariants (basis independent). Threatening inconsistency, pure states are used and defined in a seemingly paradoxical manner, both contextual and non-contextual.

4 Mathematical Invariance and Physical Objectivity

The relation between a mathematical formalism, a conceptual scheme and experience is not a "self evident" *given* within theories. Indeed, this interrelation is one of the most complex and subtle aspects present within the development of physical theories. Albert Einstein addressed this relation explicitly when developing his special theory of relativity, arguing that every physical concept must be able to provide an explicit operational connection to both physical reality and experience; something which —he also stressed— pure mathematics lacks completely.

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[2]A density matrix can be diagonalized, thus giving a set of eigenvalues $0 \leq \lambda_1 \leq \dots < \lambda_n \leq 1$ with $\sum_i \lambda_i = 1$. If $\mathrm{Tr}(\rho^2) = 1$, then $\lambda_1 = \dots = \lambda_{n-1} = 0$ and $\lambda_n = 1$. Hence, $\mathrm{rk}(\rho) = 1$ and then $\rho = |v\rangle\langle v|$ and $\rho = \rho^2$. Conversely, if $\rho = \rho^2$ it has eigenvalues 0 or 1, but from $\sum_i \lambda_i = 1$ it follows $\lambda_1 = \dots = \lambda_{n-1} = 0$ and $\lambda_n = 1$. Hence, $\mathrm{Tr}(\rho^2) = 1$.

[3]Like in [12] we distinguish here between the purely abstract vector Ψ and its specific representation in a basis $|\psi\rangle$.

"We cannot ask whether it is true that only one straight line goes through two points. We can only say that Euclidean geometry deals with things called 'straight lines,' to each of which is ascribed the property of being uniquely determined by two points situated on it. The concept 'true' does not tally with the assertions of pure geometry, because by the word 'true' we are eventually in the habit of designating always the correspondence with a 'real' object; geometry, however, is not concerned with the relation of the ideas involved in it to objects of experience, but only with the logical connection of these ideas among themselves." [23, p. 2]

Einstein remarked the fundamental distinction between purely abstract mathematical notions and physical concepts. When discussing the concept of *simultaneity* he explained the following:

"The concept does not exist for the physicist until he has the possibility of discovering whether or not it is fulfilled in an actual case. We thus require a definition of simultaneity such that this definition supplies us with the method by means of which, in the present case, he can decide by experiment whether or not both the lightning strokes occurred simultaneously. As long as this requirement is not satisfied, I allow myself to be deceived as a physicist (and of course the same applies if I am not a physicist), when I imagine that I am able to attach a meaning to the statement of simultaneity. (I would ask the reader not to proceed farther until he is fully convinced on this point.)" [23, p. 26]

Thus, there must always exist within a physical theory something like an "operational methodology" which allows to connect its physical concepts and mathematical formalism with physical experience.[4]

As remarked by Max Born [3]: "the idea of invariant is the clue to a rational concept of reality, not only in physics but in every aspect of the world." In physics, it is the invariance present in the mathematical formalism of the theory which allows us to determine what is to be considered *the same* irrespectively of the perspective from which it is being represented. Invariants capture the objective non-contextual content of a theory. Consequently, only invariant notions can be considered independently and beyond a particular experimental situation (or context). In physics, invariants are quantities having the same value for any reference frame. The transformations that allow us to consider the physical magnitudes from different frames of reference have the property of forming a *group*. In the case of classical mechanics we have the Galilei transformations which keep space and time apart, while in relativity theory we have the Lorentz transformations which introduce an intimate connection between space and time coordinates. Of course, restricting ourselves to physical magnitudes that remain always the same, independently of the reference frame, does not provide a dynamical picture of the world, instead such description only provides a static table of data. Obviously such description is completely uninteresting for physics, which always attempts to describe, not only how the world *is* but —far more importantly— how the world *changes*. Thus, that which matters the most for physical description is the *invariant variations* of physical magnitudes, that is, the dynamical magnitudes which vary but can be considered still the same (e.g., position, velocity, momentum, energy, etc.). *The difference within the identity.* More specifically, in physics it is not only important to consider magnitudes that vary with respect to a definite reference frame (S), but also the consistent translation that allows us to consider that same variation with respect to a different frame of reference (S'). This relation (of the values between S and S') is also provided via the transformation laws. Such transformations include not only the dynamics of the observables but also the dynamics between the different observers (see also [10]).

Even though the values of physical magnitudes might also vary from one reference frame to the other —due to the dynamics between reference frames—, in both classical physics and relativity

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[4]In [7] we provided the following related definition of **Meaningful Operational Statement:** *Every operational statement within a theory capable of predicting the outcomes of possible measurements must be considered as meaningful with respect to the representation of physical reality provided by that theory.*

theory there is a *consistent translation* between the values of magnitudes of different frames secured by the transformation laws. The position of a rabbit running through the fields and observed by a distant passenger of a high speed train can be translated to the position of that same rabbit taken from the perspective of another passenger waiting on the platform of the station. The fact that the values of observables (position, momentum, etc.) can be consistently translated from one reference frame to the other allows us to assume that such physical observables also bear an objective real existence completely independent of the specific choice of the reference frame pertaining to each observer. It is this consistency within translation which allows the physicist to claim that: the rabbit has a set of dynamical properties (position, a momentum, etc.) independently of his observers in the train and on the platform. The observables of the physical system are independent of the observers; i.e., they are non-contextual. We can thus claim that such properties are *dynamical variations* that pertain to the physical system itself.[5] The same reasoning can be applied to coordinate transformations in the phase space Γ. If we consider a set of observables in a coordinate system, S, and perform a transformation of coordinates (e.g., a rotation) to a new system, S', then the values of the observables will be also consistently translated from the system S to the system S'. Such consistency, which is again secured by the transformation, is the *objectivity condition* which allows us to consider the observables as preexistent to the choice of the coordinate system (i.e., the mathematical representation from which we choose to describe our system). It is in this way, that mathematical invariance allows us to detach the empirical subject —the particular observer— from the objective representation of physical reality. Now, returning to our previous analysis, a *pure state* is strictly related to a single basis. Consequently, its operational definition only makes sense for an observer traveling in the train, but not for one standing in the platform!

The problem we are discussing here is intrinsically related to the orthodox manner in which the formalism of the theory has been constrained to binary values. As discussed in [10], a Corollary of the Kochen-Specker theorem [32] is that in quantum theory there is no invariance of observables when considering the *binary valuations* of properties pertaining to different contexts; i.e., there does not exist a *Global Binary Valuation*. The orthodox conclusion is that the restrictions imposed by the formalism must be regarded as contextual; and this means that "the properties of a system are different whether you look at them or not." QM does not describe an objective (preexistent) state of affairs, measurements do not discover or unveil reality. Instead of staying close to the mathematical formalism, orthodoxy has preferred to retain the restrictions imposed by the atomist metaphysical picture of binary properties constituting systems. And it is the introduction of these (metaphysical) constraints within the mathematical formalism of the theory which leads to these very weird conclusions —at least for a realist. However, as we have argued in detail in [6, 11], if we give up the *binary* restriction applied to the representation of the mathematical formalism and advance towards an *intensive* definition of physical quantities, it is then possible to restore a *Global (Intensive) Valuation* for all projection operators without inconsistencies. The key which opens this possibility is the invariant character of the Born rule itself. If we take seriously this rule, which can be derived from the mathematical formalism itself, we must conclude that the elements of physical reality described by QM are not of a binary nature; instead, they must be regarded in intensive terms. Thus, if we are willing to pay the price of giving up the metaphysical picture of binary properties, objectivity can be easily restored and the subjects (or agents) performing experiments can be once again regarded as completely detached from the theoretical representation of physical

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[5]In more general terms, as discussed in [11], it is exactly this formal aspect which allows us to talk in terms of an *Actual State of Affairs (ASA)* that evolves in time; i.e., a dynamical description in terms of the variation of (objective) definite valued observables (or 'dynamical properties') independent of the (subjective choice of the) perspective (or reference frame) from which they are being observed. Even in relativity theory, due to the Lorentz transformations, one can still consider 'events' as the building blocks of physical reality.

reality provided by QM.

5 Pure States and Mixtures from a Categorical Viewpoint

Going back to QM, we can now begin to understand the serious difficulties present within the definition(s) of pure state. It is only the contextual definition of *pure state* which provides the possibility to find out in a concrete case if the concept is true or false —or in other words, it is only one basis between the infinitely many existent basis which contains a physical operational content. However, the mathematical non-contextual definition of *pure state*, which is not equivalent, lacks completely such operational reference. This is clearly problematic, for there is no obvious link between the contextual and the non-contextual definitions of *pure state*. Things become even more complicated when we shift our attention to *mixed states* in which case there is no physical counterpart related to QM —beyond the mere reference to measurement outcomes.[6] Given a physical situation, it seems difficult to even tell the difference between a classical (proper) mixture and a quantum (improper) mixture without invoking —once again— the reference to a pure state (see [13, Chap. 6]).

In this section we would like to turn our attention to two categorical approaches which have addressed in radically different ways the meaning of pure and mixed states. On the one hand, the topos approach, originally proposed by Chirs Isham, Jeremy Butterfield and Andreas Döring [20, 22, 21, 27, 28, 29, 30, 31], and on the other, the more recent logos approach presented by the authors of this paper [11, 12]. But before entering this discussion, let us provide some basic mathematical notions.

First of all, a *category* consists of a collection of *objects* (often denoted as X, Y, A, B), a collection of *morphisms* (or *arrows*, denoted f, g, p, q) and four operations,

- To each arrow f, there exists an object $dom(f)$, called its *domain*.

- To each arrow f, there exists an object $codom(f)$, called its *codomain*.

- To each object X, there exists an arrow 1_X, called the *identity map* of X.

- To each pair of arrows f, g such that $dom(g) = codom(g)$ there exists a *composition map*, fg such that $dom(fg) = dom(g)$ and $codom(fg) = codom(f)$.

An arrow f is often denoted as $f : X \to Y$ to empathizes the fact that $dom(f) = X$ and $codom(f) = Y$. We say that an arrow $f : X \to Y$ is invertible if there exists an arrow $g : Y \to X$ such that $fg = 1_Y$ and $gf = 1_X$. The collection of arrows between X and Y is denoted hom(X, Y).

Example 5.1 *The first example of a category is the category of sets $\mathcal{S}ets$. Another example is the category of graphs. The category of graphs, denoted $\mathcal{G}ph$, extends naturally the category of sets. A (simple) graph is a set with a reflexive and symmetric relation. More formally, $\mathcal{G}$ is a graph if*

- *Reflexivity: $P \sim P$ for all $P \in \mathcal{G}$.*

- *Symmetry: if $P \sim Q$, then $Q \sim P$ for all $P, Q \in \mathcal{G}$.*

Elements of the graph are called nodes *and an* edge *between two nodes is present if these two nodes are related. Arrows between graphs send nodes to nodes and edges to edges.*

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[6]While classical mixtures make reference to the ignorance of an underlying preexistent actual state of affairs, quantum mixtures —due to the non-existence of a joint probability distribution [37]— are simply incompatible with such a (classical) ignorance interpretation; and even worse —just like quantum superpositions [7]—, quantum mixtures lack a reference beyond measurement outcomes and mathematical structures.

A remarkable fact is that the collection of categories has itself a structure of a category. The arrows are called *functors*. A functor $F : \mathcal{C} \to \mathcal{D}$ assigns objects to objects, arrows to arrows and is compatible with the four operations (domain, codomain, identity and composition).

Let us present three standard constructions in category theory, the *comma category*, the *graph of a functor* and the *category over an object*. The second construction is a particular case of the first and the third of the second. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be two functors with the same codomain. The *comma category* $F|G$ is a category whose objects are arrows in $\mathcal{C}$ of the form

$$f : F(A) \to G(B),$$

where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. An arrow between f and g is a commutative square. The *graph of a functor* $F : \mathcal{A} \to \mathcal{C}$ is defined as the comma category $F|1$, where $1 = 1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor. In the special case where the functor F is equal to $\hom(-, C)$ for some object $C \in \mathcal{C}$, the graph of this functor is called the category over C and is denoted $\mathcal{C}|_C$. This is our main structure. Objects in $\mathcal{C}|_C$ are given by arrows to C, $p : X \to C$, $q : Y \to C$, etc. Arrows $f : p \to q$ are commutative triangles,

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
& {\scriptstyle p}\searrow \quad \swarrow {\scriptstyle q} & \\
& C &
\end{array}
$$

Example 5.2 *Let $\mathcal{S}ets|_{\mathbf{2}}$ be the category of sets over $\mathbf{2}$, where $\mathbf{2} = \{0, 1\}$ and $\mathcal{S}ets$ is the category of sets. Objects in $\mathcal{S}ets|_{\mathbf{2}}$ are functions from a set to $\{0, 1\}$ and morphisms are commuting triangles,*

$$
\begin{array}{ccc}
\mathcal{G}_1 & \xrightarrow{\ f\ } & \mathcal{G}_2 \\
& {\scriptstyle \Psi}\searrow \quad \swarrow {\scriptstyle \Phi} & \\
& \{0, 1\} &
\end{array}
$$

In the previous triangle, Ψ and Φ are objects of $\mathcal{S}ets|_{\mathbf{2}}$ and f is a function satisfying $\Phi f = \Psi$.

As we mention before, this category is relevant in classical logic. We can assign a true/false value to every element of $\mathcal{G}_1$ and/or $\mathcal{G}_2$ in a consistent manner. We say that $P \in \mathcal{G}_1$ (assume P is a proposition in the space $\mathcal{G}_1$) is true if $\Psi(P) = 1$, else we say that P is false. Even more so, assume that we have a map $f : \mathcal{G}_1 \to \mathcal{G}_2$ such that $\Phi f = \Psi$, then the truth or falsity of P is unchanged via f, that is $f(P)$ is true if and only if P is true,

$$f(P) \text{ is true} \iff 1 = \Phi(f(P)) = \Psi(P) \iff P \text{ is true}$$

In the logos approach we generalize the category $\mathcal{S}ets|_{\mathbf{2}}$ extending $\mathcal{S}ets$ to $\mathcal{G}ph$ and the set $\mathbf{2}$ to the interval $[0, 1]$.

Now that we have some basic facts and constructions from category theory, let us review the topos approach and how it can handle mixed states. For an introduction to the topos approach, see [11] or [25]. The topos approach makes use of the definition of *context category* and of *spectral presheaf*. Let $\mathcal{H}$ be a Hilbert space and consider $\mathcal{V}(\mathcal{H})$ the set of commutative subalgebras $V \subseteq B(\mathcal{H})$ of bounded operators (with identity). Using the natural order in $\mathcal{V}(\mathcal{H})$, we can consider it as a category. We call $\mathcal{V}(\mathcal{H})$ the *context category*. To each $V \in \mathcal{V}(\mathcal{H})$ ($V \subseteq B(\mathcal{H})$ is a commutative subalgebra with identity), we assign its Gelfand spectrum (a compact topological space). This assignment, denoted $\underline{\Sigma}$, is called the *spectral presheaf*. The topos approach is particulary interested in some particular subobjects of $\underline{\Sigma}$. A subobject $\underline{S}$ is called *clopen* if $\underline{S}(V)$ is a clopen[7] subset of $\underline{\Sigma}(V)$ for all $V \in \mathcal{V}(\mathcal{H})$. We

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[7]A clopen subset in a topological space is a set both open and closed.

denote the set of clopen-subobjects as $Sub_{cl}(\underline{\Sigma})$. The authors construct a map $\delta : \mathcal{P}(\mathcal{H}) \to Sub_{cl}(\underline{\Sigma})$ called *daseinisation of projection operators* which sends each projector P_i to a clopen subobject $\delta(P)$, where $\mathcal{P}(\mathcal{H})$ denotes the set of projectors in $\mathcal{H}$. The basic idea behind this construction is to recover classical physics. For each $V \in \mathcal{V}(\mathcal{H})$, the space $\underline{\Sigma}(V)$ has to be interpreted as a state space and for each projector P, the subset $\delta(P)(V)$ has to be interpreted as a proposition in the state space $\underline{\Sigma}(V)$. In summary, according to the slogan *quantum physics is equivalent to classical physics in the appropriate topos*, [22], the topos approach defines for each $V \in \mathcal{V}(\mathcal{H})$, a state space and a Boolean logic (all subject to compatibilities conditions).

In [19, 22] Isham and Doëring make an attempt to incorporate to the topos approach the notions of probability and density matrices. In order to do so, they extend their previous constructions [22, p. 6]. As they [22, p. 3] argue: "Probabilities are thereby built into the mathematical structures in an intrinsic manner. They are tied up with the internal logic of the topos and do not show up as external entities to be introduced when speaking about experiments" The authors define the presheaf $[0,1]^{\geq}$ given by $[0,1]$-valued, nowhere-increasing functions on $\mathcal{V}(\mathcal{H})$. In other words, if $p \in [0,1]^{\geq}$, then $p(V) \in [0,1]$ for all $V \in \mathcal{V}(\mathcal{H})$ and if $V' \subseteq V$ we have $p(V') \geq p(V)$. Now, given a density matrix ρ, the authors constructed a map $\mu^{\rho} : Sub_{cl}(\underline{\Sigma}) \to [0,1]^{\geq}$ such that, when restricted to the image of δ and taking minimum over $\mathcal{V}(\mathcal{H})$, they recover Born's rule,

$$\min_{V \in \mathcal{V}(\mathcal{H})} \mu^{\rho}(\delta(P))(V) := \mathrm{Tr}(\rho P)$$

The general definition of μ^{ρ} to the whole set $Sub_{cl}(\underline{\Sigma})$ is rather technical and non-trivial. In fact, several alternative constructions are needed in order to prove the previous formula. The basic idea behind their construction is that a density matrix ρ defines a probability measure on each state space $\underline{\Sigma}(V)$.

At this point it becomes important to make some remarks about the topos program. Even though the authors of the present paper believe the topos approach is a very interesting and original proposal, it has several mathematical and philosophical drawbacks. From a mathematical point of view, it is evident from the new mathematical constructions, that it becomes necessary to adapt all the previous formalism in order to be able to incorporate mixed states. The result of this process is the creation of a very complex and elaborated theory which is difficult to follow even for an expert in the field. Furthermore, it is not even clear if the results in the previous formulation are still valid. From a philosophical point of view there seems to exist a tension between, on the one hand, a supposedly realist approach to physics which attempts to talk about systems and well defined properties,[8] and on the other, a neo-Bohrian scheme[9] which makes explicit use of several anti-realist ideas such as the idea that 'truth' can be 'partial' or understood in 'degrees', that 'reality is contextual' and must be understood in purely inter-subjective terms. As Bernard D'Espagnat has made clear, "Bohr was not a realist".[10]

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[8] As remarked by Döring and Isham in [18]: "When dealing with a closed system, what is needed is a realist interpretation of the theory, not one that is instrumentalist. The exact meaning of 'realist' is infinitely debatable, but, when used by physicists, it typically means the following: (1) *The idea of "a property of the system" (i.e. "the value of a physical quantity") is meaningful, and representable in the theory.* (2) *Propositions about the system are handled using Boolean logic. This requirement is compelling in so far as we humans think in a Boolean way.* (3) *There is a space of "microstates" such that specifying a microstate leads to unequivocal truth values for all propositions about the system. The existence of such a state space is a natural way of ensuring that the first two requirements are satisfied.* The standard interpretation of classical physics satisfies these requirements, and provides the paradigmatic example of a realist philosophy in science. On the other hand, the existence of such an interpretation in quantum theory is foiled by the famous Kochen-Specker theorem."

[9] The topos approach has been developed explicitly as a neo-Bohrian attempt to understand QM. A reference to the Danish physicist which has become completely explicit not only in the works of Chris Heunen, Klaas Landsman and Bas Spitters, but also in the works of Vasilios Karakostas and Elias Zafiris.

[10] In [14, p. 98] D'Espagnat quoted Bohr arguing that: "The description of atomic phenomena has [...] a perfectly

In the logos approach, taking as a standpoint the orthodox mathematical formalism of QM, we attempt to restore an objective account in which subjects are —as Einstein wanted— completely detached from the theoretically represented state of affairs. We do so by willingly paying the price of abandoning the classical metaphysical account of reality in terms of 'systems' and 'properties' —which the topos approach wants to retain— and introducing a new non-classical representation in which the physical notions of *power* and *potentia* play an essential role. Taking as a standpoint the invariance of the Born rule, our main interest becomes the category $\mathcal{G}ph|_{[0,1]}$ of graphs over the interval $[0,1]$. Let us begin by reviewing some properties of the category of graphs.

First, we give an example of a graph coming from the quantum formalism,

Example 5.3 *Let $\mathcal{H}$ be Hilbert space and let Ψ be a vector, $\|\Psi\| = 1$. Take $\mathcal{G}$ as the set of observables with the commutation relation given by QM, the* quantum commutation relation*. This relation is reflexive, symmetric but **not** transitive, hence $\mathcal{G}$ is a non-complete[11] graph.*

Definition 5.4 *Let $\mathcal{G}$ be a graph. A* context *is a complete subgraph (or aggregate) inside $\mathcal{G}$. A* maximal context *is a context not contained properly in another context. If we do not indicate the opposite, when we refer to contexts we will be implying maximal contexts.*

For example, let P_1, P_2 be two nodes of a graph $\mathcal{G}$. Then, $\{P_1, P_2\}$ is a context if P_1 is related to P_2, $P_1 \sim P_2$. Saying differently, if there exists an edge between P_1 and P_2. In general, a collection of nodes $\{P_i\}_{i \in I} \subseteq \mathcal{G}$ determine a *context* if $P_i \sim P_j$ for all $i, j \in I$. Equivalently, if the subgraph with nodes $\{P_i\}_{i \in I}$ is complete.

Theorem 5.5 *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{G}$ be the graph of immanent powers with the commutation relation given by QM. It then follows that:*

1. *The graph $\mathcal{G}$ contains all the contexts (or quantum situations).*

2. *Each context is capable of generating the whole graph $\mathcal{G}$.*

Proof: See [12].   □

In the logos approach we work with the category $\mathcal{G}ph|_{[0,1]}$. An object in $\mathcal{G}ph|_{[0,1]}$ consists of a map $\Psi : \mathcal{G} \to [0,1]$, where $\mathcal{G}$ is a graph. Intuitively, Ψ assigns a *potentia* to each node of the graph $\mathcal{G}$. Specifically, to each node $P \in \mathcal{G}$, we assign a number $\Psi(P)$, but this time, $\Psi(P)$ is a number between 0 and 1. Then, in order to provide a map to the graph of immanent powers, we use the Born rule. We remark that in the logos scheme the Born rule is not an axiom added to the theory which would require an independent derivation —as argued by Deutsch, Wallace and Dieks [15, 39]— but a consequence of the orthodox mathematical formalism itself. Gleason's theorem [26] is just an answer to the mathematical problem of defining all measures on the closed subspaces of a Hilbert space. Gleason's theorem derives the Born as the natural measure for QM, and at the same time precludes the possibility of two valued measures (see [37] for a detailed analysis.). Thus, to each *power $P \in \mathcal{G}$*, we assign through the Born rule the number $p = \Psi(P)$, where p is a number between 0 and 1 called *potentia*. As discussed in detail in [11], we call this map $\Psi : \mathcal{G} \to [0,1]$ a *Potential State of Affairs* (PSA for short). Summarizing, we have the following:

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objective character, in the sense that no explicit reference is made to any individual observer and that therefore... no ambiguity is involved in the communication of observation." He then explains this quotation in the following manner: "That Bohr identified objectivity with intersubjectivity is a fact that the quotation above makes crystal clear. In view of this, one cannot fail to be surprised by the large number of his commentators, including competent ones, who merely half-agree on this, and only with ambiguous words. It seem they could not resign themselves to the ominous fact that Bohr was not a realist."

[11]A graph is complete if there is an edge between two arbitrary nodes.

Definition 5.6 *Let $\mathcal{H}$ be a Hilbert space and let ρ be a density matrix. Take $\mathcal{G}$ as the graph of immanent powers with the quantum commutation relation. To each immanent power $P \in \mathcal{G}$ apply the Born rule to get the number $\Psi(P) \in [0,1]$, which is called the potentia (or intensity) of the power P. Then, $\Psi : \mathcal{G} \to [0,1]$ defines an object in $\mathcal{G}ph|_{[0,1]}$. We call this map a* Potential State of Affairs *(or a* PSA *for short).*

Intuitively, we can picture a PSA as a table,

$$\Psi : \mathcal{G}(\mathcal{H}) \to [0,1], \quad \Psi : \begin{cases} P_1 & \to & p_1 \\ P_2 & \to & p_2 \\ P_3 & \to & p_3 \\ & \vdots & \end{cases}$$

Thus, an abstract vector in Hilbert space (or a density matrix) provides a table of intensive powers describing an objective PSA. The Born rule —contrary to the orthodox interpretation and the topos interpretation [22]— acquires an objective reference, namely, the intensive measure (i.e., the potentia) of each power. In this case, the epistemic account of such intensive quantification must be produced through a statistical analysis. Obviously, a single measurement is not enough in order to find out the specific value of the potentia of a power; we require many measurements of the same power in order to learn about its potentia.

In this way, the logos approach departs from the well known positivist idea according to which physical theories predict measurements that can be restricted to *yes-no* questions. As Peres explains [34, p. 202]: *"There are 'elementary tests' (yes-no experiments) labelled A, B, C, ... Their outcomes are labelled a, b, c, ... = 1 (yes) or 0 (no).* In quantum theory, these elementary tests are represented by projection operators." It is this idea —considered by many as "not controversial"— which implies the (metaphysical) imposition of a binary reference to physical existence. After this definition, there is an implicit —very controversial— shift from the empirical finding of actual measurements (observables) to the metaphysical reference of projection operators now understood as preexistent properties (See section 2 and also [8]). As Peres continues to explain:

> "The simplest observables are those for which all the coefficients a_r are either 0 or 1. These observables correspond to tests which ask yes-no questions (yes = 1, no = 0). They are called *projection operators*, or simply projectors , for the following reason: For any normalized vector v, one can define a matrix $P_v = vv^\dagger$, with the properties $P_v^2 = P_v$ and $P_v u = vv^\dagger u = v\langle v, u \rangle$ (3.52) The last expression is a vector parallel to v, for any u, unless $\langle v, u \rangle = 0$. In geometric terms, $P_v u$ is the projection of u along the direction of v." [34, p. 66]

It is by imposing this binary restriction to the values of projection operators that, as explicitly shown by the Kochen-Specker theorem [32], one reaches a contradiction. Like many others today, Peres [34, p. 14] concludes that therefore: "Quantum physics [...] is incompatible with the proposition that measurements discover some unknown but preexisting reality." This conclusion goes back to Bohr's analysis of QM and his insistence that the most important (epistemological) lesson to be learnt from QM is that, we subjects, are not only spectators but also actors in the great drama of (quantum) existence. However, as we have demonstrated in [11] through the explicit development of an intensive non-contextuality theorem, this is simply not true. When considering the Born rule as computing *intensive values*, objectivity can be restored and QM becomes compatible with the proposition that (statistical) measurements discover an unknown but preexistent (potential) reality. Of course, the price we have willingly paid is to give up the classical (metaphysical) representation of reality in terms of actual 'systems' and 'properties'.

To sum up, some important remarks go in order:

I. Our approach makes explicit the existence of two distinct levels of mathematical representation regarding vectors. On the one hand, we have the PSA, i.e., as an abstract vector in Hilbert space Ψ; and on the other hand, we have the particular basis-representation of the PSA in a specific context; i.e., the vector written in a basis $|\psi\rangle$ which we call a quantum situation. While the first level is obviously non-contextual, the second level is explicitly contextual (see for a detailed analysis [12]). In the logos approach we have not only different names for these different concepts but also a notation which makes explicit this fundamental distinction right from the start.

II. The interpretation of the Born rule in intensive terms allows to bypass the need of a *binary valuation*. But more importantly, it provides an invariant quantification of projection operators and a *global intensive valuation* which escapes the constraints of the Kochen-Specker theorem restoring an objective reference to the formalism [8, 11].

III. The logos approach embraces the shift from a *binary* understanding of *certainty* to an *intensive* one. The number that we find by applying the Born rule is not a measure of 'lack of knowledge' of an inaccurate representation of an *Actual State of Affairs*; it is on the very contrary an objective account of the potentia of the powers constituting an objective *Potential State of Affairs*. As a direct consequence, the distinction between *pure state* and *mixed state* becomes completely irrelevant. Pure states are states which, when considered from within a specific basis, bear an intensity equal to 1.

An important feature of our logos approach is that it allows us to distinguish between the notions of PSA, superpositions and intensive powers. Indeed, given a PSA, Ψ, defined by a unit vector, v, and given a basis (or context), $\mathcal{C} = \{|w_1\rangle, \ldots, |w_k\rangle\}$, we can write v as a *Quantum Situation*:

$$QS_{\Psi,\mathcal{C}} := \sum_{i=1}^{k} c_i |w_i\rangle.$$

In fact, we can assign to Ψ a multiplicity of different superpositions (or quantum situations):

$$QS_{\Psi,\mathcal{C}_1}, QS_{\Psi,\mathcal{C}_2}, \ldots, QS_{\Psi,\mathcal{C}_n}$$

one for each context $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. Even more, as remarked in Theorem 5.2, each superposition can generate ($\rightsquigarrow$) not only the other superpositions (by simply making a change of basis) but also the whole PSA,

$$QS_{\Psi,\mathcal{C}_1} \rightsquigarrow QS_{\Psi,\mathcal{C}_2} \rightsquigarrow \ldots \rightsquigarrow QS_{\Psi,\mathcal{C}_n} \rightsquigarrow \Psi.$$

It is also true that there is a class of equivalence between the different representations which allow us to write the following:

$$QS_{\Psi,\mathcal{C}_1} = QS_{\Psi,\mathcal{C}_2} = \ldots = QS_{\Psi,\mathcal{C}_n} = \Psi.$$

However, this equivalence relation does not mean that these different quantum situations are making reference to a physical system constituted by properties (for a more detailed discussion see [12]).

A useful visual representation is provided in the logos approach through the use of graphs with partially filled nodes representing each *power* and its respective *potentia*. Graphs allow us to picture simultaneously the whole PSA, Ψ, the different context dependent quantum situations, $QS_{\Psi,\mathcal{C}_i}$, as well as each different power with its respective potentia.
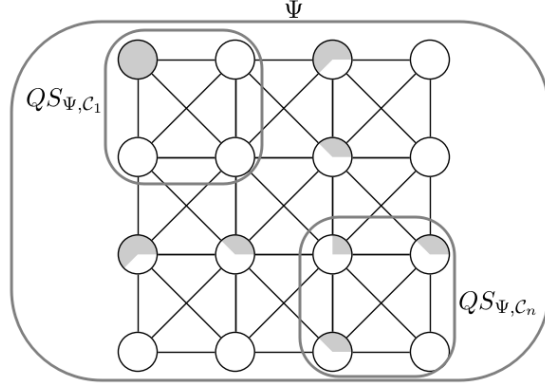
Fig. 1: A PSA Ψ with two different quantum situations (or contexts) pointed.

In figure 1 we can clearly see that even though there is a sense in which $QS_{\Psi,\mathcal{C}_1} = QS_{\Psi,\mathcal{C}_n}$, there is also an obvious sense in which $QS_{\Psi,\mathcal{C}_1} \neq QS_{\Psi,\mathcal{C}_n}$. An obvious difference between them is that the quantum situations $QS_{\Psi,\mathcal{C}_1}$ and $QS_{\Psi,\mathcal{C}_n}$ are not making reference to the same section of the graph Ψ. Furthermore, different quantum situations are related to different *meaningful operational statements* (see footnote 4) providing a specific information of the state of affairs.

The following theorem guarantees that the PSA representation is equivalent to the density matrix representation:

Theorem 5.7 *The knowledge of a particular PSA, Ψ, is equivalent to the knowledge of the density matrix ρ_Ψ. In particular, if Ψ is defined by a normalized vector v_Ψ, $\|v_\Psi\| = 1$, then we can recover the vector from Ψ.*

Proof: See [12].   □

Through the use of graphs the logos representation is capable to account for both non-contextual and contextual levels simultaneously. While the whole graph provides an account of the non-contextual PSA, Ψ, the specific context makes reference to the particular experimental (quantum) situation, $QS_{\Psi,\mathcal{C}}$, in which the nodes (powers) and their intensive values (potentia) computed through the Born rule can be exposed through a statistical analysis. This is the most important point for the ongoing discussion. The logos approach allows to clearly distinguish between the non-contextual and the contextual parts of the mathematical formalism.

Now that we have the mathematical definition of a PSA, let us go back to the analysis of *pure states*. For simplicity, let us work in $\mathbb{C}^2$. The following analysis can be carried out without difficulties to any dimension. As we defined mathematically (Definition 3.2), a pure state is a unit vector $v \in \mathbb{C}^2$ or in terms of density matrices, it is a 2×2 hermitian matrix ρ of the form $|v\rangle\langle v|$,

$$\rho = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}, \quad v = (a,b), \quad |a|^2 + |b|^2 = 1.$$

Notice that

$$\rho^2 = |v\rangle\langle v|v\rangle\langle v| = |v\rangle\langle v| = \rho$$

and

$$\mathrm{Tr}(\rho^2) = |a|^4 + |a|^2|b|^2 + |a|^2|b|^2 + |b|^4 = |a|^2(|a|^2 + |b|^2) + (|a|^2 + |b|^2)|b|^2 = 1.$$

Let us translate this representation to our formalism. First, the graph of immanent powers $\mathcal{G}$ in $\mathbb{C}^2$ can be pictured as follows,
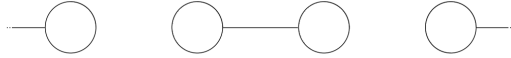
Fig. 2: Graph of immanent powers in dimension two.

The previous graph continues to the left and right indefinitely.

Let us choose the basis $v_1 = (1,0)$ and $v_2 = (0,1)$. This is represented as choosing a maximal context, that is, a *complete set of commuting observables* $\mathcal{C} = \{|v_1\rangle\langle v_1|, |v_2\rangle\langle v_2|\}$,

Fig. 3: Graph with the context $\mathcal{C}$ marked.

Now, we define the PSA $\Psi : \mathcal{G} \to [0,1]$ by using the Born rule. In this example,

$$\mathrm{Tr}(\rho \cdot |v_1\rangle\langle v_1|) = |a|^2, \quad \mathrm{Tr}(\rho \cdot |v_2\rangle\langle v_2|) = |b|^2.$$

We picture the restriction of Ψ to the context $\mathcal{C}$ as

Fig. 4: The context $\mathcal{C}$ with the assigned potentia to each power.

But of course, we can choose another basis. In fact, we can choose the orthonormal basis given by $v = (a,b)$ and $w = (-b,a)$. Over this context, the representation of Ψ is rather easy,
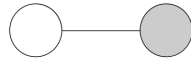

Fig. 5: Another context showing a pure state.

As we mentioned above, it is only this particular basis which contains a clear physical operational counterpart relating 'the state' to the '*certain prediction* of a measurement outcome'. Indeed, since —following the empiricist-positivist agenda— it is only the actual *and* observable which can find a place in some model of the theory, certain knowledge becomes restricted to actual observable values. From the logos approach none of these states is problematic. All states provide objective intensive knowledge of the state of affairs described by QM [6].

Through the use of graphs we can now visualize very easily the fundamental equivocity present within the different —both contextual and non-contextual— definition(s) of *pure state* (section 3). As we discussed above, while the mathematical definition makes reference to an abstract context-independent vector (i.e. an invariant), the operational counterpart is clearly context-dependent and restricts itself to a particular basis (i.e., the basis in which there exists one power with potentia equal to 1). The following graph (figure 6) shows the simultaneous reference of the notion of *pure state*, first, to an abstract vector in Hilbert space, second, to a vector represented in a specific basis, and third, to a single *eigenvector* whose *eigenvalue* is 1. Clearly, each of these mathematical elements possesses not only a distinct mathematical definition, they also codify a completely different set of meaningful operational statements containing the physical information.

In the logos approach, through the use of graphs we understand visually the confusion present in the orthodox literature according to which a *pure state* makes reference, at the same time, firstly, to the whole PSA; secondly, to the single filled node; and thirdly, also to any maximal context containing this node. The scrambling of these three distinct levels of mathematical representation is clearly problematic since we have explicitly shown in the logos approach that there is obviously a difference between considering the whole graph (i.e., a PSA, Ψ), , a particular section of the graph (i.e., a quantum situation, $QS_{\Psi,\mathcal{C}}$), and a particular node of the graph (i.e. an intensive power, P_i).

16

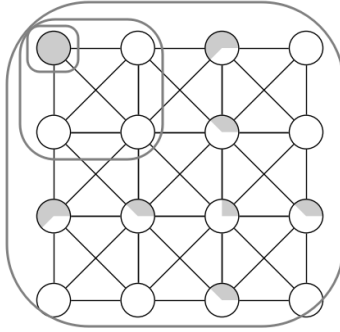Fig. 6: The orthodox referential equivocity
present within the definition of pure state(s).

To summarize, while in the topos approach it is necessary to redefine the formalism to incorporate mixed states, in the logos approach, any PSA is treated in a natural and simple way without having to abandon the orthodox formalism or adding anything "by hand". While the topos approach makes reference to classical mixtures interpreting them in terms of ignorance, the logos approach makes reference to quantum mixtures right from the start giving them an objective probabilistic interpretation in terms of intensive values and dissolving in this way the orthodox distinction between pure and mixed states. While the topos approach remains tied to a contextual and intersubjective reference to a (binary) actual state of affairs, the logos approach restores the possibility to think of QM as a theory capable of producing a non-contextual, objective representation of a (potential) state of affairs. Finally, while the topos is extremely complicated, both formally and conceptually, the logos approach presents a simple introduction to QM through the theory of graphs as well as an intuitive conceptual understanding of what is really going on through the notion of *intensive power*.

6   The Democracy of States in the Logos Approach to QM

In a truly Spinozian spirit, we might say that, in the logos approach —contrary to orthodoxy— there are no states which can be considered as more important or fundamental than others; all states in QM are *as* important. Our representation in terms of graphs makes explicit the democratic nature of a potential state of affairs in which different quantum states co-exist. An intensity of a node (a power) equal to 0.5 and an intensity equal to 1, both provide the same complete accurate type of certain knowledge. The so called *actual* properties become just a particular case of *potential* or *indefinite* properties, just like probability equal to 1 is a particular value of probability, not essentially different from probability equal to 0.5 or 0.77. Actual properties are just a particular case of potential properties, those with potentia =1. Consequently, also from a purely mathematical perspective, a (pure) state $\rho = \rho^2$ and a (impure) state $\rho \neq \rho^2$, are regarded as equivalent. Both states provide particular graphs with different tables of powers and potentia, the fact that in the first case there exists a power which has a potentia = 1 is completely irrelevant both from a physical and mathematical perspective.

Carlo Rovelli has recently argued in [35] that Schödinger introduced "the notion of 'wave function' ψ, soon to be evolved into the notion of 'quantum state' ψ, endowing it with heavy ontological weight. This conceptual step was wrong, and dramatically misleading. We are still paying the price for the confusion it has generated." Indeed, as we have discussed above, the deep confusion and misunderstanding comes, partly, from the equivocity introduced by the orthodox notation which is unable to account for the different levels of mathematical representation present within the formalism of the theory. But this equivocity has been created by the inadequate idea according to which 'QM obviously talks about systems'. The problem is not that ψ is understood in ontological terms, the

17

problem is that its understanding has been dogmatically restricted to the classical atomist representation in terms of space-time systems. In the logos approach we have provided not only a notation which makes explicit the distinction between the different mathematical levels of representation, we have also provided a conceptual framework in which the mathematical formlaism finds a natural connection to operationally well defined physical concepts. While the non-contextual aspect of abstract vectors is described in terms of a PSA, the contextual nature of quantum superpositions is clearly stressed through the reference to the notion of 'quantum situation'. In this respect, an important aspect of our logos approach is that all these new (non-classical) notions possess a physical operational counterpart. And just like Einstein required, these newly introduced physical concepts contain the operational conditions allowing to discover whether or not they are fulfilled in an actual case.

Conclusion

In this paper we have discussed the untenability of the notion of *pure state* in the orthodox formalism of QM. Through the aid of graphs we have shown the equivocity present within the different definitions confused and scrambled in the present literature. We have also shown that through the application of an intensive analysis it is possible to restore an objective theoretical representation of QM. In this new scheme it becomes explicit why the distinction between *pure state* and *mixed state* is completely irrelevant both from a mathematical and a physical perspective.

Acknowledgements

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