# Mathematical Justification without Proof

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**ASBTRACT**
According to a widely held view in the philosophy of mathematics, direct inferential justification for mathematical propositions (that are not axioms) requires proof. I challenge this view while accepting that mathematical justification requires arguments that are put forward as proofs. I argue that certain fallacious putative proofs considered by the relevant subjects to be correct can confer mathematical justification. But mathematical justification doesn't come for cheap: not just any argument will do. I suggest that to successfully transmit justification an argument must satisfy specific standards, some of which are social. I contrast my view with Huemer's inferential conservatism, which makes mathematical justification too easy to get. Although in this article I focus on mathematical inferential beliefs, the view on offer generalizes to other inferential beliefs.

**KEYWORDS**
Mathematical justification, inferential justification, fallibilism, proof, social epistemology.

## 1. Introduction

According to a widely held view, inferential justification requires at least two things, namely, (i) that the premises support the conclusion and (ii) that the premises are justified – this is because justification should be truth-conducive. For example, Chris Tucker explains:

> One requirement that seems necessary to secure the truth connection is that *E therefore P* justifies its conclusion only if E evidentially supports P. Another is that *E*

> *therefore P* justifies its conclusion only if one's belief in E is justified. These two requirements are commonly imposed because they seem needed to guarantee that belief in the conclusion is likely true in the relevant sense. (2012, 328)

In the case of deductive inferences, requirement (i) implies that the premises entail the conclusion and therefore that if the premises are true, then the conclusion is also true. In this article, considering examples arising from the practice of mathematics, I argue that this requirement is too strong. I suggest that inferential justification in mathematics does not require a relation of entailment between premises and conclusion. More generally, I want to deny that a genuine logical relation is necessary for inferential justification. This does not mean, however, that everything goes – I will indicate that other constraints should replace requirement (i).

Let *mathematical justification* be a particular brand of inferential justification in which the premises are mathematical, and the inference is supposed to be a valid deductive inference. It is what is generally required in mathematical research to justify beliefs that are not about axioms, but about propositions that necessitate proofs. To be sure, other forms of justification in mathematics exist. Axioms can be justified in a variety of ways, appealing to mathematical intuition or extrinsic reasons, for example. Moreover, heuristic arguments can play a justificatory role in mathematics. For instance, a subject can be justified in believing that a conjecture is true, but if her belief is based on a non-deductive argument, the conjecture will not be promoted to the status of theorem until mathematical justification is provided.

Contrary to the received view in the philosophy of mathematics, I contend that mathematical justification does not require proof.  More specifically, I suggest that putative proofs that contain imprecisions, gaps, and even damning mistakes can confer justification. This implies that a subject can hold a mathematically justified belief that is false – therefore, in my view, mathematical justification is not factive.  This is not only because, as it has been emphasized in the literature, we could go wrong with respect to the premises of our proofs (i.e., the choice of axioms); it also follows from the fact some fallacious arguments can confer justification.

My proposal has some common tenets with Michael Huemer's (2016) inferential conservatism (that is, phenomenal conservatism applied to inferential beliefs), according to which, under certain conditions, intellectual appearances provide prima facie justification. However, pace Huemer, I suggest that inferential justification is not an entirely internal affair. Contrary to unalloyed versions of inferential conservatism, in my view, not just any argument will be poised to confer inferential justification – as it will become clear later, I do integrate an externalist constraint in my account of mathematical justification.

I show that in order to confer justification, a putative proof might be flawed but must satisfy specific constraints.  These could take different forms.  One option I explore is to appeal to social considerations.  I elaborate on previous work (De Toffoli 2021) to sketch a view in which mathematical justification is cashed out in terms of arguments that are in line with the inferential practices of mathematical communities in good epistemic standing.

Although I focus on the case of mathematical inferences, my results generalize to other inferential beliefs.  My proposal can thus be seen as a first step towards a new account of inferential justification in general, one in which no genuine logical relation is required, but social constraints are at play.

The plan of the paper is as follows. In Section 2, I present some cases in which it is plausible to think that fallacious arguments that seem genuine proofs to the relevant practitioners provide justification. In Section 3, I consider the objection to the effect that such arguments might confer other positive epistemic properties, but not epistemic justification. I offer four replies to this objection. In Section 4, I show that a form of phenomenal conservatism for inferential beliefs is at first plausible but turns out to be inadequate because it makes mathematical justification too cheap. In Section 5, I outline a new account of mathematical justification, one that combines internalist and externalist components. In Section 6, I consider the role of defeaters in such an account. Finally, in Section 7, I sum up the discussion and indicate how my results can be generalized to inferential beliefs beyond mathematics.

## 2. Mathematical Justification: Motivations for a New Conception

According to a widely accepted position, proofs are the only road to genuine knowledge of new and substantial mathematical facts that are not axioms. Still, there are other ways to attain knowledge in mathematics. We can know theorems by testimony, and we can come to know axioms or, arguably, certain mathematical facts by non-deductive means.[1] However, *mathematical knowledge,* in the limited sense of *knowledge by proof* is what matters most to

[1] See (Paseau 2015).

mathematicians. In this article, I focus on the type of inferential justification that generally gives rise to such knowledge, *mathematical justification*.

Mathematical justification is a type of inferential justification in which the premises are mathematical, and the inference is supposed to be a proof, that is, a valid deductive argument from some accepted premises.[2] Moreover, the inference is *direct* – that is, the subject's belief that P is based on a mathematical argument and not on other reasons, such as the proposition that she proved P. Cases as the following are therefore excluded. S justifiably believes herself to have proved *E implies P* and consequently acquires the justified belief that *E implies P*. Moreover, she has independent justification for E. She can then perform the simple deductive inference: E implies P, E, therefore P. This is a case in which S has inferential (deductive) justification for P that is not direct because it is not based on a mathematical argument from E, and thus it is not mathematical in my sense.

Mathematical justification is arguably the most important type of justification at play in professional mathematical contexts and is therefore central to an epistemology of mathematics that is sensitive to actual mathematical practice. Focusing exclusively on it rather than on justification for mathematical beliefs in general, allows me to sidestep two issues that would act as distractors. First, I sidestep the issue of whether we can be justified in believing mathematical propositions by virtue of arguments that are not put forward as proofs, that is, of whether various types of non-deductive arguments can provide justification. Even if this turns

[2] For the purpose of this article, I will work with the assumption that proof is a success-term. This assumption is common (but not entirely uncontroversial) among philosophers of mathematics.

out to be the case, mathematical justification in my sense remains a central epistemological notion.

Second, I do not discuss the issue concerning how axioms and logical systems are justified. To be sure, these are thorny topics in the philosophy of mathematics and the philosophy of logic, but they need not concern us here. As a matter of fact, in the context of this article I am interested in investigating the epistemic norms at play in mathematical practice, and mathematicians themselves rarely address foundational issues.

Focusing on mathematical justification alone allows me to direct my attention exclusively to the proper basis of justified beliefs in mathematical practice. According to the received view, such a proper basis is given by mathematical proofs. Proofs yield indefeasible mathematical justification that always constitutes knowledge, or so the story goes. We can label this view, which emerged from the work of Frege and was later endorsed by the logical empiricists, the *logical conception*. While I focus on mathematics, this view is supposed to include other domains as well.[3] According to it, the analysis of knowledge boils down to

> identifying which logical properties of and/or logical relations among propositions suffice for justification (or for whatever other epistemic property is taken to turn mere true belief into knowledge). (Kitcher 1992, 56)

[3] For a contemporary endorsement of a version of the logical conception for the a priori domain, see (Smithies 2015).

For the logical conception, epistemic justification is to be understood in an entirely apsychological way. Philip Kitcher (1992, 57) explains:

> following both Frege and the Wittgenstein of the *Tractatus*, they pursue epistemological questions in an apsychologistic way – logic, not psychology, is the proper idiom for epistemological discussion[.]

According to the logical conception, justification is then considered to be objective and to perfectly track evidential relations that hold between propositions. When applied to mathematics, justification so understood is given by proofs, which are conceived as deductions from axioms. Proofs are thus often considered in this context to be equivalent to formal proofs.

One worry with the logical conception is that it prevents us from appreciating epistemic features that emerge when mathematics is conceived as a *human* activity. For instance, it struggles to make sense of cases—less rare than one would like to think—in which a well-functioning subject gets things wrong or gets things right for defective reasons, as in the following example:

> **KEMPE.** The 4-color conjecture[4] was already a focus of mathematicians' attention when Alfred B. Kempe published an argument for it in 1879. This was a careful and convincing

[4] The 4-color conjecture roughly states that four colors suffice to add color to a planar map (such as a map of the world) in such a way that no two neighboring regions have the same color. For the details of the conjecture and of Kempe's attempt to prove it, see (Sipka 2002).

argument that divided the problem into several subcases. But in 1890, Percy Heawood found a counterexample to one of the subcases. It took roughly a century to find a genuine proof – and it still used Kempe's ideas.

When Kempe published his argument, it was reasonable for him to believe (and take himself to know) the 4-color conjecture on the basis of it. After all, not only did he check his own argument, but other members of the mathematical community, starting with his reviewers, checked it as well, with none dissenting until eleven years later.[5]

We can contrast Kempe with two imagined mathematicians with respect to epistemic standing: 1) Mr. Absent, a sloppy, absent-minded mathematician who believes in the truth of the 4-color conjecture by virtue of a perfunctory argument marred by glaring mistakes, and 2) Ms. Acid, a brilliant, meticulous mathematician, who, unknowingly drugged by a jealous rival, ends up believing the conjecture on the basis of the same perfunctory argument – which, in her compromised state, she mistook for a proof. Although none of the three mathematicians know the 4-color conjecture on the basis of their putative proofs, a plausible desideratum for an epistemological account that does justice to mathematical practice is to deliver different verdicts in each of these three cases. For the logical conception, however, in terms of justification, they stand (or, better, they fall) together. This is because none of them has a proof for the conjecture, and therefore none of them ends up being justified.

[5] This shows that Kempe had both first-order and higher-order reasons. In the context of this article, I mainly focus on first-order reasons, but do not exclude that these two types of reasons might be related to each other.

In Section 5, I sketch an alternative conception in which mathematical justification requires arguments appropriately related to proofs but does not require (genuine) proofs. By accepting that we can be justified by virtue of a fallacious argument, we can open up the conceptual space – inhabited by Kempe, but not by Mr. Absent or Ms. Acid – outside mathematical knowledge but within mathematical justification. Besides, there is a further tool that can serve to differentiate these different failures. While both Mr. Absent and Ms. Acid hold unjustified beliefs, their epistemic standing is different in so far as Mr. Absent is blameworthy, while Ms. Acid is excused due to mitigating circumstances. Separating these different types of epistemic failures is key to making sense of mathematical practice from an epistemological perspective.

## 3. One Objection and Four Replies

While knowledge implies truth, justification does not – at least according to a widely held view among epistemologists. Even in mathematics, the case of KEMPE suggests that it is plausible to think that a subject can be mathematically justified in believing a false proposition or in believing a true proposition by virtue of a fallacious argument. This type of fallibilism about mathematical justification – one that does not concern axioms[6] – goes against the logical conception.

[6] One could also extend similar considerations to the justification of axioms. This is, however, beyond the scope of this article.

Here is a position a proponent of the logical conception could take to resist this suggestion. She could insist that in cases like Kempe's, it was plausible to believe that he was justified before the error was found. However, he was not. Similarly, it was plausible to think that he had knowledge, but he did not. Indeed, the proponents of the logical conception must say that Kempe was not justified since they cash out the notion of mathematical justification in terms of proofs, and Kempe did not have a proof.[7] One way to distinguish Kempe's belief from other unjustified beliefs would then be to refer to other epistemic notions, such as the notion of excusability.[8] This is a significant challenge. Let me offer four replies.

First, one might think that Kempe's belief, even if it is based on an incorrect argument, deserves a positive epistemic evaluation that goes beyond being merely excused. After all, Kempe not only acted in an acceptable way; he acted in an exemplary way. His argument was meticulous, which is why it took so long to catch the error in it – it is therefore plausible that he met the relevant normative standard. He was careful in articulating his argument, and it is reasonable to think that he had first-order evidence of its correctness. Moreover, he also had higher-order evidence given by the favorable verdict of other members of the mathematical community.

Second, using only one epistemic notion – excusability – rather than two – justification *and* excusability – does not allow us to differentiate the relative epistemic standing of each of our three subjects: Kempe, Ms. Acid, and Mr. Abstract. In fact, according to the proponent of

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[7] This is also the position that any infallibilist about mathematical justification must take.

[8] This move is what Jessica Brown (2018, 70) calls the "excuse maneuver," which she criticizes as well— in her discussion, she focuses on the knowledge norm of justification, which obviously leads to infallibilism about justification.

the logical conception, none of the three subjects would be justified, and the notion of excuse could be merely used to single out Kempe.[9]

Third, when we consider the history of mathematics, it is reasonable to attribute justification to subjects who did not have proofs proper. I want to vindicate the intuition that mathematicians in possession of arguments in keeping with the epistemic standards of their well-functioning mathematical communities were generally justified[10] – even if their beliefs turned out to be based on arguments that are not genuine proofs in our modern sense of the term (which appeals to the notion of formal proof). It is well-known, for example, that Euclidean geometry was not logically impeccable since some inferences are not supported by the axioms stated in advance but rely on a form of visual imagination or diagrams' inspection. Toward the end of the 19th century, Pasch and Hilbert proposed ways to fill the logical gaps present in Euclid. Just to give an idea of the situation, consider that Pasch introduced an axiom, now known as *Pasch's axiom*, that basically says that if a line enters a triangle from a side, then it has to exit it from a different side. Because of the appeal to forms of reasoning (e.g., involving diagrams or visual imagination) that went beyond what the postulates and common notions granted, Euclidean putative proofs are not proofs in the modern sense of the term. Undoubtedly, the reason why Euclidean arguments failed to amount to proofs is different than the reason why Kempe's argument failed to amount to a proof. In the former, the problem is

[9] The proponent of the logical conception could appeal to yet another notion to distinguish the epistemic status of Ms. Acid and Mr. Absent: the one of *exemption*. According to Littlejohn, "Exemptions (e.g., insanity, infancy, or incapacitation) work by showing that the agent lacked the rational capacities needed to be answerable for their responses" (2022, 2690). Accordingly, all three characters would remain unjustified, Kempe would be excused, and Ms. Acid would be exempt.

[10] In Section 5, I will say more about how to characterize well-functioning mathematical communities.

with the underlying system, while in the latter, there is a significant mistake in the argument – still, in both cases, those arguments fall short of what the proponents of the logical conception call *proof*.

The problem with Euclidean geometry generalizes to all mathematics before the turn to rigor that culminated at the end of the 19th century. To deny that arguments that do not satisfy the modern standards of rigor can confer justification introduces the threat of a form of skepticism, namely, the conclusion that nobody was mathematically justified before the 19th century. This might be an attractive position for proponents of the logical conception. Here is Russell, for instance: "there probably did not exist, in the eighteenth century, any single logically correct piece of mathematical reasoning" (1903, §434). Still, it presents serious drawbacks for someone who wants to articulate a notion of mathematical justification sympathetic to the history and practice of mathematics.

Fourth and final, tying mathematical justification to an idealized notion of proofs (basically the one of formal proof), as the proponents of the logical conception might want to do, leads to meta-inductive skeptical worries that threaten our contemporary and future mathematical beliefs as well.[11] Given the history of refinements in the notion of proof, it is reasonable to think that further refinements will be implemented.[12] Acknowledging this fact should make the proponent of the logical conception doubt that we currently have mathematical justification for our mathematical beliefs. This should *not* lead us to believe that

[11] Thanks to Hilary Kornblith for this point.

[12] For instance, in the not so far future, machine checkable formal proofs, obtained with tools like interactive theorem provers might be necessary for mathematical justification. For a survey on the use of such tools in contemporary mathematics, see (Avigad 2024).

such mathematical beliefs are not true. After all, most of the refinements to the notion of proof in the past had little, if any, effect on our views about the truths at issue. For example, the redefinition of geometrical proof did not change our view of the truths of Euclidean geometry. It is true that, thanks to the definition of formal proof, we have a clear account of what idealized mathematical arguments are. However, the refinements to our notion of proof would not concern such idealizations but those arguments that mathematicians actually use to justify their results in practice. If these refinements show that we didn't have genuine proofs, and genuine proofs are required for justification, then the meta-inductive argument does yield broad skepticism about both mathematical knowledge and mathematical justification.

I concede that these objections will not persuade the convinced proponent of the logical conception or the convinced infallibilist. However, they provide reasons for the rest of us to consider an epistemological framework supporting fallibilism about mathematical justification. This is what the next sections set out to do.

## 4. Inferential Appearances

*Phenomenal conservatism* applied to inferential beliefs is a position that allows us to make sense of cases of defeasible inferential justification, even in the case of mathematics. Roughly, it is the view that appearances confer prima facie justification. The primary target of this view is perceptual beliefs. However, one of its leading proponents, Michael Huemer (2013, 339), explicitly includes inferential beliefs:

My account of inferential justification makes possible cases in which a subject is inferentially justified in believing P on the basis of E, even though the inference from E to P is fallacious. This can happen because a fallacious inference may seem cogent to a subject, while the subject lacks grounds for doubting its cogency.

Huemer's position is relevant here because it is motivated precisely by appealing to a case in which it is plausible to think that a fallacious putative proof confers justification. Here it is.

> **UNFORTUNATE MATHEMATICIAN.** S is a skilled mathematician who has just completed an apparent proof that P, starting from the single premise E. Though the reasoning was not especially difficult, nor was the conclusion especially surprising, S checked every step very carefully, and S has no reason to suspect that anything went wrong. But suppose (as is compatible with all the foregoing) that E does not actually support P; there is a subtle error that renders the proof neither valid nor cogent. (2016, 147)

For our concerns, the case is analogous to KEMPE.[13] Huemer explains:

[13]The main differences are that in the case of the UNFORTUNATE MATHEMATICIAN the argument is explicitly simple, has just one single premise, and the premise does not in fact support the conclusion.

> It is an advantage of PC [Phenomenal Conservatism] that it enables us to explain this otherwise odd result. The mathematician is justified in believing P because his apparent proof makes it seem that P must be correct, given his premises. (*ibid.*)

Phenomenal conservatism thus provides us with an alternative to the logical conception, one in which mathematical justification does not require proof. It is here crucial to note that the Unfortunate Mathematician is not just inferentially justified, but he is *mathematically* justified. This is because his belief is based on a putative proof.

In former work, Huemer (2002) interpreted the case differently. Although he wanted to accommodate the intuition that the Unfortunate Mathematician is justified, he did not allow for a fallacious argument to generate inferential justification. Here is his previous take on the case:

> our description of the case makes it clear that S would be justified in believing that he has just proven P. The proposition that S has just proven P, although false, does confirm that P. So S can […] be justified in believing P, on the basis of the proposition that he has just proven P. (336)

According to this description, S would be inferentially justified since he would infer P from the proposition that he has just proven P. However, S would not be *mathematically* justified – since his belief is not based on the putative proof, and thus the inference is not direct. This description is misleading, however. Plausibly, the Unfortunate Mathematician's belief that P that we are evaluating is not based on the proposition that he has just proven P, but rather on the putative proof he produced (or its premise). That is why, although it gives the right verdict, this description of the case will not do.

So, in keeping with most recent Huemer, I submit that genuine inferential justification (in this case, mathematical justification) can arise from flawed arguments. In cases like this, it seems clear that, from his perspective, S should believe the proposition that he takes himself to have proven. However,

> his evidence for P does not logically support P.[14] So it is possible to be justified in believing P on the basis of evidence that does not objectively support it. (Huemer 2013, 11)

So far, I agree. However, on further inspection, it becomes apparent that his position is too generous, making inferential justification too easy to acquire. As a matter of fact, Huemer does not impose any condition on the quality of the argument. But it is unreasonable to think that *all* fallacious arguments can confer inferential justification, even if this is only *prima facie* justification (I will discuss defeaters in Section 6). Further, it is not plausible that all fallacious

[14] In the case of Kempe, the premises did logically support the conclusion since the 4-color conjecture was eventually proven and it is now commonly referred to as the 4-color theorem. However, his argument did not establish it – that is what matters.

arguments *that seem correct to a subject* can confer prima facie inferential justification – recall Mr. Absent or Ms. Acid. There must be other constraints in place to make this view plausible.[15]

Huemer's case of the Unfortunate Mathematician is compelling because, although we seem to evaluate it from an entirely internalist perspective, *an externalist constraint is already in place* – that is, a constraint whose satisfaction is not always first-person accessible. Here is why. Although with "careful consideration," Huemer might simply mean "seems careful to the subject doing the considering," the subject he focuses on is, like Kempe but unlike Ms. Acid and Mr. Absent, a well-functioning, cognitively diligent subject who produces an argument that is quite good albeit it contains a subtle mistake. This is why it is reasonable to interpret "careful consideration" as a stronger constraint, one that includes an externalist component that is not necessarily accessible to the subject. That is, a subject like Ms. Acid could believe she is careful, while she is not. Huemer discusses the very same example in three different articles (2002, 2013, 2016), and in all instances, he emphasizes that the mathematician's behavior is "careful" and that the error is a "subtle" one – the mathematician is *unfortunate* precisely because his error was not at all obvious and could have been missed by other mathematicians as well – but this is not something that can always be evaluated from the first-person perspective.

However, for Huemer, prima facie inferential justification does not require any such externalist constraint. But then he should have more vividly described the consequences of his account by considering someone like Mr. Absent or Ms. Acid rather than someone like Kempe. It is here that our accounts part ways. It is my contention that, at least for the case of

[15] See (Chudnoff 2024) for a more general critique of Huemer's account of inferential justification where it is suggested that the way Huemer conceives of inferential appearances makes it hard to understand how they can play a justificatory role at all.

mathematical justification (which is *the* motivating case for Huemer's account of inferential justification), an externalist constraint must be met for an argument to confer justification.

## 5. Mathematical Justification and Simil-Proofs

There are various types of constraints that one could articulate to distinguish between putative proofs that are poised to confer justification and those that are not. My suggestion, outlined below, is to draw on social considerations and impose a reliability constraint at the level of the epistemic community in which the subject operates.[16] Alternatives (which I cannot discuss in this context) would be to recur to an individual reliability constraint or to devise an objective way of measuring how close a fallacious argument is to a valid one.[17]

The basic motivation for appealing to social consideration is the thought that in order to confer justification, a putative proof cannot be *obviously* mistaken but needs to be the sort of argument that, if understood, would be accepted as correct by reasonable subjects with the appropriate training. This would explain why Kempe's belief is mathematically justified while Mr. Absent's and Ms. Acid's beliefs are not. The key observation is that Kempe's flawed argument did justify his belief in the 4-color conjecture not only because it looked good to him but because it satisfied the standards of acceptability for rigorous proofs of the mathematical

[16] This is in line with recent proposals that have emphasized the importance of social considerations even to individual justification. See, for example, (Goldberg 2018).
[17] This would be hard, however. In fact, there are many ways in which an argument can be flawed. Kempe was *on the right track* because an improved version of his argument was later used to prove the 4-color theorem. However, there are other cases in which is it plausible to think that fallacious arguments that are not in the right track in that way confer justification as well – one such case is perhaps the one of Vladimir Voevodsky's fallacious results (2014). Thanks to Jessica Wilson and Christopher Peacocke for suggestions along these directions. I leave it to further work to investigate this line of thought.

practice to which it was addressed, which was a well-functioning mathematical practice. On the other hand, the argument Mr. Absent and Ms. Acid came up with was convincing to them (they had, after all, what Huemer calls an inferential appearance), but due to the presence of glaring mistakes (that is, *glaring* for other practitioners), they were not the sort of things other mathematicians would accept as correct.

In short, my proposal is that although mathematical justification does not require *ideally* good arguments (i.e., genuine proofs), it requires *intersubjectively* good arguments, that is, arguments that average members of the relevant community (working without impairments and with enough time) would not find wanting. In order to spell out what "intersubjectively good arguments" are, in previous work, I introduced the notion of *simil-proofs*. Here is a slightly modified characterization, in which simil-proofs are indexed to mathematical communities:[18]

> (SIMIL-PROOF FOR M) An argument A is a *simil-proof* for mathematical community M at time t if it satisfies the standards of acceptability for proofs at play in community M at time t.

This is an in-principle condition – a rough practical way to determine what the standards of acceptability governing a specific mathematical community are, is checking what is actually accepted by the community as a proof, for instance, by consulting relevant scientific journals.

[18] This new definition is in line with the original one but extends it to putative proofs in general, not only to arguments that have been accepted as proofs by actual subjects.

However, the standards of acceptability are normative and not descriptive so they do not necessarily determine what is *actually* accepted by a community at a given time. If, for instance, all members of a community happen to be drugged at time *t*, then they might accept arguments that do not satisfy their community's standards of acceptability for proof or fail to accept arguments that do satisfy them.

A few clarifications are in order. First, simil-proofs are indexed to specific mathematical communities. However, for the sake of ease of expression, I will omit to mention the community when this creates no ambiguities. Second, communities can be characterized at different levels of generality – but this is not something that has to be fixed in this context. For instance, the community of contemporary algebraists includes the one formed by contemporary algebraists working on noncommutative algebra. Third, simil-proof is a time-sensitive notion. Consider Kempe's case. He initially had a simil-proof. However, after Heawood found a mistake, Kempe's argument was no longer a simil-proof. Fourth, having a simil-proof is an externalist criterion, that is, it is not always possible for a subject to know whether she has a simil-proof. A distracted mathematician or a woman under the influence (like Ms. Acid) could wrongly believe that her argument is a simil-proof for her mathematical practice.

With the notion of simil-proof at hand, we are ready to turn to mathematical justification: the suggestion is that simil-proofs, even flawed ones, can confer justification. In order to keep the connection between justification and truth, however, we have to impose a condition on the mathematical community at issue. The mathematical community must be *well-functioning*; that is, the standard of acceptability for proofs governing the community must be such that most arguments that satisfy them will have a true conclusion and will not contain damning mistakes (i.e., mistakes, like Kempe's, that invalidate the proof).[19] These are mathematical communities in *good epistemic standing*.

To be sure, it is difficult to determine which mathematical communities are well-functioning, and borderline cases are to be expected. Moreover, mathematical communities are not static and defined once and for all but evolve in time, and their epistemic standing at a given time might be hard to evaluate. For instance, the Italian School of Algebraic Geometry that straddled the 19th and 20th centuries began as a very successful, well-functioning mathematical community but later degenerated, as witnessed by the fact that many incomplete or plainly wrong arguments were accepted and published (De Toffoli and Fontanari 2023).

What is important, however, is that there are some clear cases. For contemporary mathematics, the situation is fairly straightforward. Most mathematical communities, such as the ones investigating combinatorics or general topology, are ongoing and recognized communities and have an established track record that can certify their reliability. The situation might not be so clear in cases of mathematicians investigating new, fast-developing areas. Consider symplectic geometry. Although research in this area is growing, its methods have been criticized, and it is not clear whether it is a reliable practice or not (Hartnett 2017).[20]

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[19] A clarification is in order. Putative proofs can contain all sort of mistakes and logical gaps. Local errors are common and do not pose a particular threat because they can be easily fixed. Damning mistakes, on the other hand, are more problematic because they show that the putative proof does not in fact support its conclusion. Kempe's argument contained a damning mistake. However, Euclid's proof did not; still, they did contain certain logical gaps.

[20] Another mathematical practice whose legitimacy is up to discussion is the one in which Japanese mathematician Shinichi Mochizuki is involved. Mochizuki claimed to have proven an important number theoretic result, the abc Conjecture, but other mathematicians disagree. See (Klarreich 2018).

With respect to the history of mathematics, mathematical communities practicing Euclidean geometry were certainly well-functioning, although, as I mentioned above, Euclidean geometry is not logically flawless. Although its methodology was later criticized, it was a well-functioning community, as witnessed by the fact that its results have been incorporated into analytic geometry.

Here, then, is my proposal. Let S be a mathematician participating in a well-functioning mathematical community M:

> PRIMA FACIE MATHEMATICAL JUSTIFICATION: S's belief that P based on a putative proof is prima facie *mathematically justified* if and only if the putative proof is a simil-proof for M.

To be sure, this is a condition of prima facie, defeasible justification. I will discuss a no-defeaters condition in the next section.

One could object that this definition is unsatisfactory since it merely applies to subjects participating in a legitimate mathematical community. However, from the outset, I decided to focus on mathematical justification, which is the type of justification that is generally required in professional contexts. Therefore, the main target of analysis is constituted by mathematicians operating within well-functioning mathematical communities. It is because of this reason that it is not overly restrictive to focus on such subjects.[21]

[21] It may be possible to extend this definition to subjects who are isolated or operate within defective communities by linking them to the closest (according to some evaluations) well-functioning mathematical communities.

I proposed that for a subject operating within a well-functioning mathematical community, mathematical justification requires a simil-proof.  This constraint allows us to make sense of why Kempe is justified, but Mr. Absent and Ms. Acid are not.  This is also in line with the experience of many mathematicians.

A last remark is in order.  In the account on offer, not only it is possible for a subject to be mathematically justified in the absence of a proof, it is also possible for a subject to have a proof and fail to be mathematically justified.  For instance, this can happen if the subject is operating within a well-functioning mathematical community and has a proof but not a simil-proof because she fails to articulate it in a way other people can understand and check (given adequate time and lack of impairments) – that is, in a way that complies to the standard of acceptability of the community.  According to mathematician Alf van der Poorten, this is the correct verdict.  Mathematicians, he argues, have the duty of making their arguments understandable to their peers – this is because only with the collective effort of checking each other's arguments can we overcome our individual shortcomings.  In his words,

> [A] recognized mathematician, had best have clear arguments written in the language of the majority—the language expected by other mathematicians—if her surprising

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Similarly, in the case of a genius mathematician ahead of her time, one could think of the experts of her community as ideal agents with the relevant background (which is not possessed by actual mathematicians). However, for the purposes of this paper, I prefer restricting my analysis to the relatively simple case of subjects belonging to well-functioning mathematical communities.

arguments are to get a proper hearing. That's not unfair; it's our playing the odds. (van der Poorten 1996)[22]

In this context, I find it plausible that the norms governing mathematical justification are aligned with the norms of assertion in a given community. This is roughly because I am attracted to the idea that, in order to be justified to believe a certain mathematical result, a subject participating in a specific mathematical community must be able to articulate shareable reasons in favor of it.[23] Obviously, this would require many more reasons in support, which is something I leave for further work.

However, we should be careful not to overgeneralize van der Poorten's idea. For instance, if a more-than-average expert comes up with an argument that an average expert would not be able to grasp at a time, this does not mean *ipso facto* that her argument is not a simil-proof. As a matter of fact, the argument might well satisfy the standards of acceptability governing the relevant community. If the argument is good, plausibly, with enough time and without impairments, an average expert would come to understand it and accept it.

Let me now turn to the issue of defeaters.

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[22] Here is some context. Van der Poorten discusses the case of Kurt Heegner. Roberts explains: "A private scholar in Germany, Heegner published a paper in 1952 claiming to provide a solution to the class number problem in number theory. His proof was regarded as fatally flawed. "The arguments were sufficiently obscurely written," Alf van der Poorten remarked, "to leave considerable doubt about their completeness, even in essence." As it turned out, Heegner was correct. His real contribution, van der Poorten adds, "is by now well recognized." A poignant question remains. "[W]as it a disgraceful scandal that his contribution was not recognized in his lifetime?" Van der Poorten thinks not" (Roberts 2019).

[23] In (De Toffoli 2021) I hinted at the fact that is it only through *shareable* arguments that we can mitigate our individual fallibility.

## 6. The No-Defeater Condition

Prima facie mathematical justification for P can be defeated in cases in which the subject has evidence that P is false (rebutting defeater) or evidence that her justification for P is defective (undermining defeater), such as evidence that her belief-forming processes are unreliable. We have:

> MATHEMATICAL JUSTIFICATION: S's belief that P based on a putative proof is *mathematically justified* if and only if (i) the putative proof is a simil-proof for M, and (ii) S does not possess any relevant defeaters.

Let's now go back to Huemer's proposal. As we saw, he imposes no requirement on the quality of a putative proof that is poised to confer justification. However, he can appeal to a no-defeater condition to give the right verdict to Mr. Absent. In Huemer's view, Mr. Absent has an inferential appearance, and thus he is prima facie justified. However, he also has evidence to the effect that his belief-forming process is unreliable, and so his justification is defeated. I want to suggest that this move is problematic. First, a similar solution is not available in the case of Ms. Acid. For a reasonable description of the case, one in which Ms. Acid is not in the position to find out that she has been drugged, she has no defeaters, and so her belief would turn out to be justified. Second, Huemer's account of inferential justification remains entirely insensitive to the quality of the argument. In my view, this is a problem. The way I see it, the primary issue with Mr. Absent is not that he has a defeater (although I agree that he does), but that, not having a simil-proof, he is not even prima facie justified. Therefore, the presence of defeaters is not crucial to explain why Mr. Absent is not justified.

Still, defeaters are needed to understand related cases such as the one of Huemer's *Irresponsible Mathematician*.

> **IRRESPONSIBLE MATHEMATICIAN**. Suppose, this time, that S has just constructed a valid argument for P from the sole premise, E, and that S is adequately justified in accepting E. Suppose, however, that he has not checked this proof over. Furthermore—just to make the intuitions about justification quite clear—let's suppose that S knows he has frequently made mistakes in calculation in the past, and that the present proof consists in 20 pages of complex calculations. Is S, now, justified in accepting P? (Huemer 2002, 336)

In this case, S articulates what turns out to be a proof, but it is a matter of luck because he does not check it appropriately, and he is prone to making mistakes. We can imagine the Irresponsible Mathematician to be a lucky version of Mr. Absent; his belief-forming process is utterly unreliable, but he gets lucky and ends up producing a proof.

According to Huemer, the Irresponsible Mathematician is in the very same situation as Mr. Absent. They both have an inferential seeming. Consequently, they both have prima facie mathematical justification. But in both cases, they end up not being justified because of the presence of a defeater. It is my contention that although the final verdict is correct in the two cases, it is a desirable feature of any account of inferential justification to differentiate them. In

Mr. Absent's case, the argument is obviously flawed (it is neither a proof nor a simil-proof), while in the Irresponsible Mathematician's case, the argument is good (it is both a proof and a simil-proof). This difference should matter. In my view, it is reflected in the fact that only the Irresponsible Mathematician is prima facie justified.

The situation is different when we tweak Ms. Acid's case. One consequence of my proposal is that a lucky version of Ms. Acid (one in which, notwithstanding her altered state, she manages to come up with a simil-proof) ends up being mathematically justified. Like Mr. Absent's lucky counterpart, she is prima facie justified. Unlike him, however, she has no relevant defeaters, at least for some descriptions of the case. Although this might appear at first counterintuitive, I take it to be a desirable outcome. This is because, in the account on offer, what matters for mathematical justification is not the cognitive process with which a subject arrives at a belief but rather whether the belief is based on an intersubjectively good argument, that is, an argument in keeping with the inferential practices of a mathematical community in good epistemic standing: a simil-proof for such community.[24] This means that a reliability constraint is imposed at the level of the mathematical practice rather than at the level of the subject's belief-forming processes.

That said, if one still wanted to exclude such cases and deny that a subject can hold a mathematically justified belief she arrived at in an unusual or unreliable way, one could impose a third condition on mathematical justification – or else substitute my social constraint given by

[24] A way to differentiate this case from standard cases where a subject comes up with a simil-proof is to say that Lucky Ms. Acid has propositional justification for the conclusion of her simil-proof but, unlike what happens in the standard case, fails to form a belief that is appropriately based on it and thus ends up not being doxastically justified.

simil-proof with a different constraint altogether. One such alternative condition could be developed imposing a reliability constraint at the level of the subject. For instance, fallibilist versions of knowledge-first virtue epistemologies do just that. For example, Christoph Kelp (2017) has argued that in order for a belief to be justified, it must be *competent*. In turn, a competent belief is roughly understood as a belief produced by an exercise of an ability to know – that is, by a belief-forming mechanism that reliably produces knowledge. Certainly, Ms. Acid's lucky counterpart is in no position to form competent beliefs.

## 7. Conclusion

Appealing to the notion of simil-proof, we can block the collapse of mathematical justification into mathematical knowledge. This move opens enough epistemological space for mathematicians whose arguments comply with the inferential practices of their well-functioning mathematical communities but fail to constitute (genuine) proofs. This is crucial for an analysis of mathematical justification capable of tracking mathematical practice.

In the account I delineated, mathematical justification presents a social dimension since it requires a simil-proof, and whether an argument is a simil-proof for a mathematical community M depends on the standards of acceptability for proofs governing M.

The proposed take on mathematical justification can be extended to other types of inferences and therefore to a more general account of inferential justification. The moral of the above considerations is that a genuine logical relation is not a necessary requirement for inferential justification. However, simply eliminating such condition (as Huemer does) leads to counterintuitive results and makes inferential justification too easy to get. I suggested adding a social condition. The idea is to consider subjects engaged within an epistemic community in good epistemic standing (mathematical or otherwise) and to link inferential justification to arguments that satisfy the standards of acceptability within such a practice.

With respect to deductive arguments, the case of mathematics is particularly interesting because deductions in many other domains tend to be relatively simple, and therefore, fallacious legitimate deductions are extremely rare. We can also consider non-deductive arguments. Still, the case of deductions is privileged because genuine deductions are truth-preserving. Thus, the difference between ideally good arguments and merely intersubjectively good arguments is starker in the case of deductive arguments compared to non-deductive arguments.

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## References

Avigad, Jeremy. 2024. "Mathematics and the formal turn." *BULLETIN (New Series) OF THE AMERICAN MATHEMATICAL SOCIETY* 61 (2): 225-240.

Brown, Jessica. 2018. *Fallibilism: Evidence and Knowledge*. Oxford University Press.

Chudnoff, Elijah. 2024. "Inferential Seemings." In *Oxford Studies in Philosophy of Mind (Vol. 4)*, edited by Uriah Kriegel. Oxford University Press.

De Toffoli, Silvia. 2021. "Groundwork for a fallibilist account of mathematics." *The Philosophical Quarterly* 71 (4): 823-844.

De Toffoli, Silvia, and Claudio Fontanari. 2023. "Recalcitrant Disagreement in Mathematics: an Endless and Depressing Controversy in the History of Italian Algebraic Geometry." *Global Philosophy* 33, no. 38: 1-29.

Goldberg, Sanford. 2018. *To the Best of Our Knowledge: Social Expectations and Epistemic Normativity*. Oxford University Press.

Hartnett, Kevin. 2017. "A Fight to Fix Geometry's Foundations." *Quanta Magazine*.

Huemer, Michael. 2002. "Fumerton's Principle of Inferential Justification." *Journal of Philosophical Research* 27: 329-340.

---. 2013. "Phenomenal Conservatism Über Alles." In *Seemings and Justification*, edited by Chris Tucker, 328-350. Oxford University Press.

---. 2016. "Intellectual Appearances." In *Intellectual Assurance: Essays on Traditional Epistemic Internalism*, edited by Brett Coppenger and Michael Bergmann, 144-160. Oxford University Press.

Kelp, Christoph. 2017. "Knowledge First Virtue Epistemology." In *Knowledge First: Approaches in Epistemology and Mind*, edited by Adam Carter, Emma Gordon and Benjamin Jarvis. Oxford University Press.

Kitcher, Philip. 1992. "The Naturalists Return." *Philosophical Review* 101 (1): 53-114.

Klarreich, Erica. 2018. "Titans of Mathematics Clash Over Epic Proof of ABC Conjecture." *Quanta Magazine*.

Littlejohn, Clayton. 2022. "A justification for excuses: Brown's discussion of the knowledge view of justification and the excuse manoeuvre." *Philosophical Studies* 179 (8): 2683-2696.

Paseau, Alexander. 2015. "Knowledge of Mathematics without Proof." *The British Journal for the Philosophy of Science* 66 (4): 775-799.

Roberts, David Michael. 2019. "A Crisis of Identification." *Inference* 4 (3).

Russell, Bertrand. 1903. *The Principles of Mathematics*. Cambridge University Press.

Sipka, Timothy. 2002. "Alfred Bray Kempe's "proof" of the Four Color Theorem." *Math Horizons* 10 (2): 21-23, 26.

Smithies, Declan. 2015. "Ideal rationality and logical omniscience." *Synthese* 192 (9): 2769-2793.

Tucker, Christopher. 2012. "Movin' on up: higher-level requirements and inferential justification." *Philosophical Studies* 157 (3): 323-340.

van der Poorten, Alf. 1996. *Notes on Fermat's Last Theorem*. Vol. 181.*Canadian Mathematical Society Series of Monographs and Advanced Texts*. New York: John Wiley & Sons.

Voevodsky, Vladimir. 2014. "The Origins and Motivations of Univalent Foundations." *The Institute Letter (The Institute for Advanced Studies)*.