The Epistemic Roles of Diagrams

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Diagrams are common tools of communication and reasoning. Although they are used in many different contexts, perhaps the most interesting and philosophically puzzling uses of diagrams are in science and mathematics. Focusing on how diagrams are used in these fields, many questions arise: Can diagrams function as representational models in science? Can diagrams be used to prove mathematical propositions? Can there be formal diagrammatic systems in logic? Why are diagrams at times such effective cognitive tools?

Properly epistemological questions on diagrams have mostly to do with their role in justification and knowledge. One vexed issue is whether mathematical diagrams can be part of proofs or are better seen as devices used for discovery exclusively—i.e., whether they can play a genuine justificatory role or not. The same issue has not attracted similar attention in science since it is generally assumed that scientific diagrams play pragmatic rather than justificatory roles (Bueno 2006). For this reason, in this entry, I will focus on the epistemology of mathematical diagrams.

While it is well-known that mathematicians use (and have been using since antiquity) a wide range of visual representations, what epistemic role these visual representations play remains controversial. Until recently, mathematical diagrams were thought to play an important heuristic role but not to be the kind of thing that could contribute to the justification of mathematical propositions. But skepticism towards the use of diagrams to justify mathematical claims is waning, and recent works highlight the possibility of using diagrams to prove in mathematics.

1. The Background: How Diagrams Became Irrelevant

But what are mathematical diagrams? The most popular (and most discussed) mathematical diagrams are the ones that feature in ancient Greek geometry, like the ones that accompany the Propositions of Euclid's *Elements* (c.a. 300 BCE). Let us consider a simple example.

The first Proposition of Euclid's *Elements* tells us how to construct an equilateral triangle on a given segment AB. To do so, we construct a diagram as follows (see Figure 1): we first draw a circle with center A and radius AB, then we draw another circle with radius AB, this time centered in B. The circles cut one another at C, which we join to A and B. It is then easy to check that all segments AB, AC, and BC are equal to each other and, therefore, that the triangle ABC is equilateral.

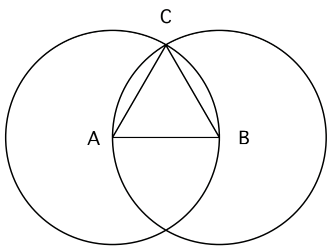

Figure 1. A Euclidean diagram

In order for Euclid's first proof to go through, we need to use the fact that the two circles intersect. How do we know that? Well, we can look at the diagram!

Diagrams, written or imagined, seem to be necessary for some of Euclid's proofs. This came to be seen as a problem towards the end of the nineteenth century. After having been considered to be the gold standard of rigorous mathematics for about two millennia, Euclid's geometry was considered to be defective. Pasch (1926, 43) expresses this critical attitude:

> For the appeal to a figure is, in general, not at all necessary. It does facilitate essentially the grasp of the relations stated in the theorem and the constructions applied in the proof. Moreover, it is a fruitful tool to discover such relationships and constructions. However, if one is not afraid of the sacrifice of time and effort involved, then one can omit the figure in the proof of any theorem; indeed, the theorem is only truly demonstrated if the proof is completely independent of the figure.

And if Euclid's arguments do indeed appeal to figures, then they are not genuine proofs, at least according to Pasch. This is why he came up with an alternative system for Euclidean geometry – in particular, he introduced an axiom that takes his name. *Pasch's Axiom* basically says that if a line enters a triangle from a side without going through a vertex, then it must exit the triangle from another side. It does seem obvious, but it is also obvious that the two circles in Figure 1 cut each other! The problem is that mathematics is not about feelings of obviousness but rather about logical relations between propositions.

Pasch was not alone in his attack on the use of diagrams in proofs. As it is clear from his *Geometry and the Imagination* (Hilbert and Cohn-Vossen 1932), Hilbert was well aware of the importance of diagrams in mathematics, but, like Pasch, he did not think they could play a genuine justificatory role.

The ban of diagrams and intuition from proofs is tied to a complex phenomenon that is usually referred to as the *rigorization of mathematics*. It started at the end of the nineteenth century and led to the modern articulation of analysis. Additionally, the discovery of non-Euclidean geometries further challenged the reliability of geometric intuition, which was associated with the use of diagrams. Diagrams started to be regarded as, at best, helpful heuristic devices and, at worse, pernicious representations.

It is therefore not surprising that in the late nineteenth century and early twentieth century, philosophers of mathematics did not pay attention to diagrams at all. The main topic on the agenda was to investigate the foundations of mathematics – that is, to find a way to justify mathematical theories.

Moreover, for most of the twentieth century, other epistemological questions that occupied philosophers of mathematics were framed at a high level of generality and mainly had to do with *Benacerraf's problem*, that is, the problem of access to abstract objects. If mathematical objects are abstract and thus are causally inert, how can we know anything about them? This problem affects mathematics at all levels, and actual mathematical thinking (mediated or not by diagrams) did not seem to matter at all to solve it.

2. The Philosophy of Mathematical Practice: Diagrams Are Back in the Game

However, towards the end of the twentieth century, a new approach to the study of mathematics gained ground, namely, the *Philosophy of Mathematical Practice* – for a representative sample of contributions belonging to this approach, see (Mancosu 2008). This is a turn to the inner working of mathematicians, which puts into focus questions that have to do with individual justification and knowledge. There are at least four consequences of taking the practice seriously that are relevant to the study of diagrams.

The first has to do with appreciating that the epistemology of mathematics should not be reduced to the study of proofs. Philosophers of mathematical practice, following the steps traced by Lakatos in *Proofs and Refutations* (1976), started to take into consideration not only how mathematical propositions are established but also how they are discovered. This gave rise to a series of studies highlighting the role of diagrams in mathematical discovery and understanding (Giaquinto 2007; Carter 2019; Giardino 2018).

The second consequence is to endorse the non-skeptical position according to which mathematicians before the rigorization of the end of the nineteenth century possessed mathematical knowledge. Understanding the history of mathematics requires coming to terms with the role played by diagrams in various practices—Euclidean geometry is a prime example.

The third is to realize that mathematical practice is heterogeneous and different areas of mathematics should be investigated. For instance, by looking at representations used in category theory or homological algebra, it is clear that diagrams are not only geometric but can also be algebraic. Commutative diagrams used in these areas are akin to algebraic notations extending in two dimensions and therefore do not pose the problems typical of geometric diagrams.

The fourth consequence is to analyze the impact of new technology on mathematics. Towards the end of the twentieth century, diagrams regained importance in mainstream mathematics. Mancosu (2005) talks of a 'renaissance' of visualization. This is partially due to the novel technological possibilities that allow mathematicians to create digital models of complicated geometric structures.

3. Euclidean Diagrams, Knot Diagrams, and Commutative Diagrams

The literature on diagrams has grown considerably in the last decade. It now includes a vast amount of detailed case studies aimed at capturing fine features of the use of diagrams in various mathematical contexts – the most systematic study on the topic being (Giaquinto 2007). To survey some of the main results that have been produced, I will briefly mention three case studies: Euclidean diagrams, knot diagrams, and commutative diagrams.

Manders (2008) explained the success of Euclidean geometry by assigning an inferential role to diagrams. He articulated how Greek mathematics consists of an interaction between diagrams and text. His crucial insight was that for diagrams to be used rigorously, only those diagrammatic features that are invariant under slight perturbations could carry relevant information. When a diagram is reproduced, slight variations are inevitably introduced and should be disregarded. Manders distinguished between *exact* and *co-exact* diagrammatic features. Only the latter can be read off a diagram. Co-exact features roughly track topological relations. A prototypical co-exact feature is the existence of an intersection point as required by Proposition (I,1): if we change the exact metric properties of the two circles, the intersection point will change position, but its existence would not be threatened. Other co-exact features are part/whole relationships between regions (such as enclosures), line segments, and angles. Exact features are precise metric properties such as lengths of segments, their straightness, and the equality of angles or segments. Manders's analysis is also at the base of a formal diagrammatic system for Euclidean geometry (Mumma 2012). Other important accounts of diagrams in the context of Euclidean geometry that emphasize the idea that diagrams play a crucial role in proofs are (Netz 1998) and (Macbeth 2010).

Independently of its historical accuracy, Manders's account is important because it gives specific conditions that must be satisfied for diagrams to be used rigorously. Building on the basic idea behind the distinction between exact and co-exact properties, Larvor (2019) and De Toffoli (2022) have proposed new general criteria for diagrams to be used in proofs.

But diagrams are not only a trace of the past; they are also used in contemporary mathematics. For example, knot diagrams (see Figure 2) are used in contemporary mathematics – often within proofs as well. A mathematical knot can be characterized as a closed simple curve in space. A knot diagram is a regular projection of a knot in the plane.

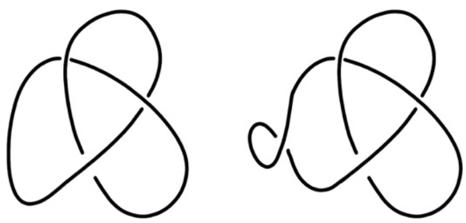

Figure 2. Two equivalent knot diagrams

According to Brown (2008), knot diagrams are particularly interesting because they support calculations. This shows that diagrams should be seen as special kinds of mathematical notation.

De Toffoli and Giardino (2014) point out the fact that knot diagrams are particularly effective because they trigger *enhanced manipulative imagination*. This is a kind of imagination that is enhanced with mathematical training and aids practitioners to mentally imagine manipulating knots in space – for instance, it is through this imagination that we can recognize that the two diagrams in Figure 2 represent the same knot. By supporting specific types of reasoning that have a precise mathematical interpretation, knot diagrams are thus seen to play a justificatory role.

Although most prototypical diagrams are geometric (like Euclidean diagrams) or topological (like knot diagrams), diagrams can also be algebraic. Commutative diagrams were introduced in the second half of the twentieth century and are akin to algebraic notations in two dimensions. For this reason, they do not threaten the reliability of the proofs in which they figure. For example, in Figure 3 is a simple diagram composed of three nodes and three arrows – the fact that the diagram is commutative means that the composition of arrows *f* and *g* is equal to arrow *h*. Commutative diagrams clearly show that diagrams do not have to be associated with visual intuition. Nevertheless, their two-dimensional layout makes them particularly effective mathematical notations (Corfield 2003).

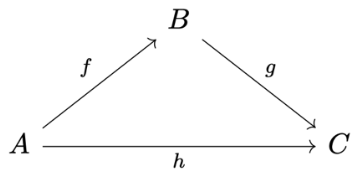

Figure 3. A commutative diagram

The case of commutative diagrams makes it clear that diagrams can enter the inferential structure of proofs. However, it remains to establish whether diagrams can be *essential* to the proofs in which they figure. Analyzing two diagrammatic proofs, involving topological and algebraic diagrams, De Toffoli (2023) argues that they can. This is not to say that diagram-free proofs of the same result could not be found. Rather, this implies that there are plausible criteria of identity for proofs such that any such diagram-free proof would not be a different presentation of the same proof but a different proof altogether.

4. Conclusion

We saw that although diagrams were used in ancient Greek geometry, from the rigorization of mathematics until recently, they were banned from the context of proofs and neglected by philosophers. The renewed interest in mathematical practice is what contributed to bringing diagrams to the center of philosophical discussions about mathematics.

From an epistemological perspective, it is significant that many types of diagrams, such as diagrams used in topology and algebra, not only play a role in discovery, but a justificatory role as well. These diagrams can even be essential to the proofs in which they figure, depending on our criteria for individuating proofs.

The question of whether mathematical diagrams can play a justificatory role has significant ramifications. For one, it is related to the nature of mathematical rigor and to how deductive proofs should be characterized. Diagrams have been used as counterexamples to the *Standard View* according to which mathematical rigor should be cashed out in terms of formalizability in an appropriate formal system (Tanswell 2015; Rav 2007). Some scholars, however, emphasize the compatibility between the Standard View and the use of diagrams in proofs (De Toffoli 2021).

A further ramification has to do with the debate on the a priori. If it is indeed possible to acquire mathematical justification (and knowledge) by visually inspecting a diagram, does this show that mathematical knowledge is not a priori after all? And if it does not, does this force us to endorse a particular position on the a priori debate? According to Giaquinto (2015), the use of diagrams shows that many mathematical propositions can be justified a posteriori. But crucially, this holds also when we use non-diagrammatic mathematical notations. It is therefore not clear whether diagrams themselves threaten the a priori nature of mathematics or the use of notations in general.

To conclude, mathematical diagrams pose significant questions about mathematical justification and knowledge and their analysis forces us to think about general issues such as how mathematical understanding is promoted, the nature of rigorous mathematical proofs, and the a priori status of mathematics.

See also: A PRIORI KNOWLEDGE; GEOMETRY; MATHEMATICAL KNOWLEDGE.

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