Maxwell gravitation

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Abstract

This paper gives an explicit presentation of Newtonian gravitation on the backdrop of Maxwell spacetime, giving a sense in which acceleration is relative in gravitational theory. However, caution is needed: assessing whether this is a robust or interesting sense of the relativity of acceleration depends upon subtle questions over how to identify the spacetime structure of a theory.

The following two observations are well-known to philosophers of physics:

1. Newtonian gravitation admits, in addition to the well-known velocity-boost and potential-shift symmetries, a "gravitational gauge symmetry" in which the gravitational field is altered.

2. Newtonian gravitation may be presented in a "geometrised" form known as Newton-Cartan theory,[1] in which the dynamically allowed trajectories are the geodesics of a non-flat connection.

Moreover, it is widely held that these two observations are intimately related. However, aspects of this relationship remain somewhat obscure. In particular, there is

[1]Due originally to [Trautman, 1965].

widespread disagreement over the sense in which the symmetry of observation 1 motivates the move from a non-geometrised formulation to the geometrised formulation of observation 2; and over the extent to which such motivation ought to be regarded as analogous to the use of the velocity-boost symmetry to motivate the move from Newtonian to Galilean spacetime, or to the use of the potential-shift symmetry to motivate the move from a formulation in terms of gravitational potentials to a formulation in terms of gravitational fields.

In this paper, I seek to clarify this relationship. First, I consider the symmetry from point 1 above, in the context of Newtonian gravitation set on Galilean spacetime. I then briefly review the geometrised formulation of the theory, and discuss some puzzling aspects concerning the relativity of acceleration. This motivates an exploration of Maxwell spacetime, and—the main contribution of this paper—the presentation of a Newtonian theory of gravitation set on Maxwell spacetime. I then show that there is a natural sense in which this theory may be regarded as equivalent to Newton-Cartan theory. I conclude with some discussion of how this illuminates the conceptual issues we began with, and how it relates to the broader literature.

I will assume familiarity with the differential-geometric architecture standardly used to present classical gravitational theories.[2] All the theories we will consider postulate at least as much structure as that of Leibnizian spacetime, which comprises data (M, t_a, h^{ab}): here, M is a differential manifold which is diffeomorphic to $\mathbb{R}^4$; t_a is a smooth, curl-free 1-form, of signature $(+,0,0,0)$; and h^{ab} is a smooth, symmetric rank-$(0,2)$ tensor, of signature $(0,+,+,+)$. t_a and h^{ab} are orthogonal, i.e., they satisfy

$$t_a h^{ab} = 0 \tag{1}$$

Given our topological assumptions, t_a induces a foliation of M into three-dimensional

[2]See [Friedman, 1983], [Earman, 1989], and—especially—[Malament, 2012].

hypersurfaces; we require that each such hypersurface is diffeomorphic to $\mathbb{R}^3$. h^{ab} induces a three-dimensional metric on each hypersurface. We require that each hypersurface is complete relative to this induced metric, and that the induced metric is flat.[3] We will use L to denote a Leibnizian spacetime. If $L = \langle M, t_a, h^{ab} \rangle$ is a Leibnizian spacetime, then a connection ∇ on M is said to be *compatible* with L just in case it satisfies

$$\nabla_a t_b = 0 \tag{2a}$$

$$\nabla_a h^{bc} = 0 \tag{2b}$$

We will only consider compatible connections in this paper.

A *Galilean spacetime* is a Leibnizian spacetime equipped with a flat compatible connection. The first theory we will consider is that of Newtonian gravitation on Galilean spacetime—for short, "Galilean gravitation". Each model of such a theory comprises the following data:

- A Galilean spacetime $\langle L, \nabla \rangle$
- A spacelike vector field G^a
- A scalar field ρ
- A unit timelike vector field ξ^a

satisfying the following equations:

$$\nabla_a G^a = -4\pi\rho \tag{3a}$$

$$\nabla^{[c} G^{a]} = 0 \tag{3b}$$

$$\xi^n \nabla_n \xi^a = G^a \tag{3c}$$

[3] For more detail on the above, see [Malament, 2012, §4.1].

The scalar field ρ represents the mass density, the vector field G^a represents the gravitational field, and the vector field ξ^a represents the tangents to the trajectories of material bodies. Given a model of Galilean gravitation, we will refer to the integral curves of ξ^a as the *dynamical trajectories*. The gravitational field is related to the mass density by the source equation for this theory, equation (3a), whilst the dynamical trajectories must obey the equation of motion for this theory, equation (3c). I have chosen to work with a gravitational field, rather than the gravitational potential. This is simply in order to remove the gauge symmetries of the potential, so that we can focus on those symmetries that alter the field itself. Equation (3b), the condition that the gravitational field is *twist-free*, ensures that this decision is harmless: given our assumptions about the topology of L, it holds of G^a if and only if there is a scalar field φ such that $G^a = \nabla^a \varphi$.[4] It will be helpful to have a term for a structure $\langle L, \nabla, G^a, \rho, \xi^a \rangle$ which does not necessarily satisfy equations (3).[5] We will refer to such a structure as a model-candidate for Galilean gravitation.[6]

This theory is a toy theory: the mass density ρ is just represented as a phenomenological background, in the sense that there is nothing constraining the motion of the matter whose density ρ allegedly represents—in particular, nothing requiring that it flow along dynamical trajectories. The most straightforward way to correct this is to

[4] See [Malament, 2012, Proposition 4.1.6]. Note that this is analogous to the role played the equation $\nabla \times \mathbf{E} = 0$ in electrostatics.

[5] That is, what in e.g. [Belot, 2007] is referred to as a "kinematical possibility".

[6] The metaphysically inlined may think of model-candidates as representing worlds which are metaphysically possible according to Galilean gravitation (they contain the right ontological ingredients), and of models as representing worlds which are physically possible according to Galilean gravitation (they contain the right ontological ingredients, arranged in the right way).

supplement the equations (3) with some form of continuity equation,[7] such as

$$\xi^a \nabla_a \rho + \rho(\nabla_a \xi^a) = 0 \tag{4}$$

As we will see, the analysis of this paper applies equally well to the toy theory (3), and to the theory obtained by supplementing (3) with (4).

Our concern in this paper is with a certain transformation one can make of the models of this theory (whether or not (4) is included)—specifically, one obtained by altering the connection and gravitational field as follows:

$$\nabla \mapsto (\nabla, \eta^a t_b t_c) \tag{5a}$$

$$G^a \mapsto G^a + \eta^a \tag{5b}$$

where η^a is any spacelike vector field such that $\nabla^a \eta^b = 0$. The notation $(\nabla, \eta^a t_b t_c)$ follows Malament: [Malament, 2012, Proposition 1.7.3] shows that given any connection ∇ on a manifold M, any other connection ∇' may be expressed in the form (∇, C^a_{bc}) (for some symmetric tensor field C^a_{bc}), meaning that for any tensor field $T^{a_1 \dots a_r}_{b_1 \dots b_s}$ on M:

$$\begin{aligned} \nabla'_c T^{a_1 \dots a_r}_{b_1 \dots b_s} = & \nabla_c T^{a_1 \dots a_r}_{b_1 \dots b_s} \\ & - C^{a_1}_{cn} T^{n a_2 \dots a_r}_{b_1 \dots b_s} - \dots - C^{a_r}_{cn} T^{a_1 \dots a_{r-1} n}_{b_1 \dots b_s} \\ & + C^n_{cb_1} T^{a_1 \dots a_r}_{n b_2 \dots b_s} + \dots + C^n_{cb_s} T^{a_1 \dots a_r}_{b_1 \dots b_{s-1} n} \end{aligned} \tag{6}$$

It is straightforward to show that the transformation (5) is a symmetry of Galilean gravitation, in the following sense: if $\nabla' = (\nabla, \eta^a t_b t_c)$ and $G'^a = G^a + \eta^a$ are substituted into the equations (3), we get the same equations out again; the same goes for

[7] Alternatively, one could represent the matter by a mass-momentum field T^{ab}, and derive equation (4) by requiring that $T^{ab} t_a t_b > 0$ whenever $T^{ab} \neq 0$, and that $\nabla_a T^{ab} = 0$ (see [Malament, 2012, pp. 265–266]). I have not done that here, because I want to keep the equation of motion (3c) and the continuity equation (4) logically independent from one another: employing a mass-momentum density, however, leads to the derivation of *both* the continuity equation and the equation of motion.

the continuity equation (4). Consequently, any model-candidate $\langle L, \nabla, G^a, \rho, \xi^a \rangle$ is a model of Galilean gravitation if and only if $\langle L, \nabla', G'^a, \rho, \xi^a \rangle$ is also a model of Galilean gravitation.

Now, if we read the theory literally, then these two models would appear to represent distinct possibilities (since the two models are not isomorphic to one another). That is, if all the mathematical structures present in the models are taken to represent physical structure, then the two models disagree over what the world is like: they disagree over the magnitude of the gravitational field, for instance, and over the acceleration of matter. Yet this is a problematic judgment, since it seems that two such possibilities would be epistemically indistinguishable from one another: all seemingly observationally accessible quantities, such as relative distances, are the same in the two models. Such epistemic underdetermination gives us some reason to think that we should seek another theory which, read literally, does not give rise to such a problem (whilst still capturing the "good" content of Galilean gravitation, i.e., the content that is invariant under (5)).[8]

The standard view is that such a theory is provided by *Newton-Cartan gravitation*. A *Newton-Cartan spacetime* consists of a Leibnizian spacetime, together with a (compatible) connection $\widetilde{\nabla}$ obeying the *homogeneous Trautman conditions*:

$$\widetilde{R}^{ab}{}_{cd} = 0 \tag{7a}$$

$$\widetilde{R}^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{c}{}_{d}{}^{a}{}_{b} \tag{7b}$$

[8]The above kind of argument is an instance of a more general one: the claim that that the differences between symmetry-related models of a theory are (in some sense) not differences that should be taken seriously, and which should motivate us either to interpret the theory in such a way that it is not committed to that structure, or to replace the theory by a more parsimonious one (for discussion, see [Møller-Nielsen, 2016]). However, it is controversial both how exactly the notion of "symmetry" should be defined, and how (or whether) this general interpretational maxim should apply (see [Saunders, 2003], [Brading and Castellani, 2003], [Baker, 2010], [Dasgupta, 2014], [Dewar, 2015], [Caulton, 2015], and references therein). Since the general debate is tangential to our purposes, I pass over it here.

Note that all flat connections obey the conditions (7); as such, Galilean spacetime is a Newton-Cartan spacetime. A model of Newton-Cartan gravitation then comprises

- A Newton-Cartan spacetime $\langle L, \widetilde{\nabla} \rangle$
- A scalar field $\rho : L \to \mathbb{R}$
- A unit timelike vector field ξ^a

satisfying the following equations:

$$\widetilde{R}_{bd} = 4\pi\rho t_b t_d \tag{8a}$$

$$\xi^n \widetilde{\nabla}_n \xi^a = 0 \tag{8b}$$

Again, this theory may be supplemented by the continuity equation (4), if a more realist dynamics for the matter is desired. The relationship between Galilean gravitation and Newton-Cartan gravitation is captured in what are known as the *geometrisation* and *recovery* theorems.[9] The former states that from any model of Galilean gravitation, one can obtain a unique model of Newton-Cartan gravitation: namely, that given by taking $\widetilde{\nabla} = (\nabla, G^a t_b t_c)$. Note that two models of Galilean gravitation which are related by the transformation (5) will generate the same model of Newton-Cartan gravitation. The latter asserts that given a model of Newton-Cartan gravitation, there is a model of Galilean gravitation related to it by $\widetilde{\nabla} = (\nabla, G^a t_b t_c)$; several models, in fact, related by (5). It is in this sense that Newton-Cartan gravitation captures the invariant content of Galilean gravitation: there is a systematic one-to-one correspondence between models of Newton-Cartan gravitation and equivalence classes of (5)-related models of Galilean gravitation.

At the same time, however, there is something potentially puzzling about this case. As mentioned above, the acceleration of the matter represented by ξ^a is not invariant

[9] See [Malament, 2012, Propositions 4.2.1, 4.2.5], [Trautman, 1965].

under the transformations (5). If models related by such a transformation correspond to the same physical situation, then the natural reading would seem to be that accelerations are not a real, or objective, or absolute feature of the world (according to Newtonian gravitational theory). This notion is supported by reflection on the transition from setting Newtonian gravitation on Newtonian spacetime (wherein there is a standard of absolute rest) to setting it on Galilean spacetime. Here, we observe that applying a "boost" transformation is a symmetry of the dynamics. In Newtonian spacetime, trajectories have (absolute) velocities, relative to absolute space; but those velocities are not invariant under boosts. This is generally taken to licence the claim that such velocities are not real, or objective, or absolute features of the world (according to the best interpretation of the theory). This claim is supported by the fact that we can set the theory instead on Galilean spacetime, in which there is not the structure required to impute absolute velocities to trajectories. So if this transition involves the repudiation of absolute velocities (since they are not invariant under boosts), analogous reasoning would suggest that the move from Galilean gravitation to Newton-Cartan gravitation should involve the repudation of absolute accelerations (since they are not invariant under (5)).

However, the orthodox view is that this is decisively *not* the case. The reason for this is straightforward: any model of Newton-Cartan gravitation *does* have enough structure to make pronouncements on the accelerations of trajectories, since it contains a privileged connection $\tilde{\nabla}$. As such, in transitioning from Galilean to Newton-Cartan gravitation,

> We eliminate the notions of absolute acceleration and rotation relative to ∇, but we replace them with new notions of absolute acceleration and rotation relative to $\tilde{\nabla}$. Hence, the move from [Galilean gravitation] to [Newton-Cartan gravitation] does not involve a relativization of acceleration parallel

to the relativization of velocity [...][10]

Why is this? The essence of the disanalogy is as follows. In performing a boost transformation, we only transform the spatiotemporal structure (or, equivalently, only transform all the non-spatiotemporal structure). Hence, the structure common to any two boost-related models of gravitation on Newtonian spacetime is just the dynamical structure, together with whatever aspects of Newtonian spacetime structure are invariant under boosts. Those aspects are exactly the structure of Galilean spacetime.

By contrast, the transformation (5) transforms *both* a piece of spatiotemporal structure (the connection) *and* a piece of "material" or "dynamical" structure (the gravitational field). Thus, some piece of structure can be common to any pair of (5)-related models of Galilean gravitation, and yet go beyond the structure of Galilean spacetime which is invariant under (5). And indeed, this is exactly what happens. To make this clearer, let us look in more detail at what aspects of Galilean spacetime are invariant under (5).

Given a Galilean spacetime $\langle L, \nabla \rangle$, the structure which is invariant under a transformation of the form (5a) goes by the moniker of *Maxwell spacetime*.[11] Intuitively, the idea is that a Maxwell spacetime contains a "standard of rotation", but no "standard of acceleration". More precisely,[12] we say that a pair of connections ∇ and ∇' compatible with a given Leibnizian spacetime are *rotationally equivalent* if, for any unit timelike field θ^a on L, $\nabla^{[a}\theta^{b]} = 0$ iff $\nabla'^{[a}\theta^{b]} = 0$. Then, a *Maxwell spacetime* comprises

- A Leibnizian spacetime L
- A *standard of rotation* $[\nabla]$: an equivalence class of rotationally equivalent flat affine connections (compatible with L)

[10][Friedman, 1983, p. 122]; I have modified Friedman's notation to fit with that used in this paper.
[11][Earman, 1989, chap. 2]
[12]This definition follows [Weatherall, 2015b].

The following proposition demonstrates the invariance of Maxwell spacetime under (5a):

Proposition 1. Let $\langle L, [\nabla]\rangle$ be a Maxwell spacetime, and consider any $\nabla \in [\nabla]$. For any other connection ∇', $\nabla' \in [\nabla]$ (i.e. ∇' is flat and rotationally equivalent to ∇) iff $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^a \eta^b = 0$.

Proof. The "only if" direction follows immediately from the proof of Proposition 3 in [Weatherall, 2015b]. □

So given a pair of models of Galilean gravitation related by (5), the structure shared by their Galilean spacetimes $\langle L, \nabla\rangle$ and $\langle L, \nabla'\rangle$ is that of their common Maxwell spacetime $\langle L, [\nabla]\rangle$; yet, the process of fusing the Galilean connection and gravitational field into a Newton-Cartan connection yields the same results in both models, as the geometrisation theorem demonstrates. So in this sense, the Newton-Cartan connection is invariant under (5), even though it is not part of the structure of Galilean spacetime invariant under (5).

Recently, however, [Saunders, 2013] has queried whether we really should regard Newton-Cartan theory as the spacetime theory that properly encodes the lessons of the symmetry canvassed above: he argues that we can "interpret [Newton's] laws [...] directly as concerning the relative motions of particle pairs",[13] and hence, as describing a theory set on Maxwell spacetime rather than Galilean spacetime.[14] Saunders' analysis concerns the point-particle formulation of Newtonian gravitation, but he continues:

> There remain important questions, above all, moving over to a manifold formulation: What is the relation between a theory of gravity (and other

[13][Saunders, 2013, p. 41]

[14]Strictly, against the backdrop of a spacetime structure equivalent to it, which Saunders refers to as "Newton-Huygens spacetime".

forces) formulated in Maxwell space-time and one based on Newton-Cartan space-time?[15]

Obviously, assessing that relationship requires us to first present such a theory set on Maxwell spacetime in the required fashion. I now do so, although the theory to be presented is *only* a theory of gravity, rather than of other forces. Extending it to include non-gravitational interactions would be valuable, but is not something I undertake here.

Without further ado, then, a model of *Maxwell gravitation* comprises

- A Maxwell spacetime $\langle L, [\nabla] \rangle$
- A scalar field ρ
- A unit timelike vector field ξ^a

satisfying the following equations:

$$\nabla_a(\xi^n \nabla_n \xi^a) = -4\pi\rho \tag{9a}$$

$$\nabla^c(\xi^n \nabla_n \xi^a) - \nabla^a(\xi^n \nabla_n \xi^c) = 0 \tag{9b}$$

where ∇ is an arbitrary element of $[\nabla]$. This is only well-specified if the choice of ∇ is indeed arbitrary; the following proposition shows that it is. It also shows that the theory obtained by including the continuity equation (4) is similarly well-defined.

Proposition 2. Let $\langle L, [\nabla], \rho, \xi^a \rangle$ be a model-candidate for Maxwell gravitation, and consider any $\nabla, \nabla' \in W$. Then equations (9) and (4) hold with respect to ∇ iff they hold with respect to ∇'.

[15] [Saunders, 2013, p. 46]

Proof. By Proposition 1, $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^a \eta^b = 0$. It follows that

$$\xi^n \nabla'_n \xi^a = \xi^n \nabla_n \xi^a - \eta^a \tag{10}$$

So first, by equations (10) and (6),

$$\begin{aligned} \nabla'_a(\xi^n \nabla'_n \xi^a) &= \nabla'_a(\xi^n \nabla_n \xi^a - \eta^a) \\ &= \nabla_a(\xi^n \nabla_n \xi^a - \eta^a) - \eta^a t_a t_r(\xi^n \nabla_n \xi^r - \eta^r) \\ &= \nabla_a(\xi^n \nabla_n \xi^a) - \nabla_a \eta^a \end{aligned}$$

Since $\nabla^a \eta^b = 0$, $\nabla_a \eta^b = t_a \theta^n \nabla_n \eta^b$, where θ^n is any future-directed unit timelike field; it follows that $\nabla_a \eta^a = 0$.[16] So (9a) holds with respect to ∇ iff it holds with respect to ∇'. Second,

$$\begin{aligned} \nabla'^c(\xi^n \nabla'_n \xi^a) &= \nabla'^c(\xi^n \nabla_n \xi^a - \eta^a) \\ &= \nabla^c(\xi^n \nabla_n \xi^a - \eta^a) - h^{dc} \eta^a t_d t_e(\xi^n \nabla_n \xi^e - \eta^e) \\ &= \nabla^c(\xi^n \nabla_n \xi^a) \end{aligned}$$

And so equation (9b) also holds with respect to ∇ iff it holds with respect to ∇'.

Finally, consider equation (4). Since

$$\begin{aligned} \nabla'_a \xi^a &= \nabla_a \xi^a - \eta^a t_a t_r \xi^r \\ &= \nabla_a \xi^a \end{aligned}$$

and $\nabla'_a \rho = \nabla_a \rho$, it is immediate that (4) holds with respect to ∇ iff it holds with respect to ∇'. □

[16] This observation is adapted from [Malament, 2012, p. 277].

We now consider the relationship between Maxwell gravitation and Newton-Cartan gravitation. First, we say that a connection is *compatible* with a given Maxwell spacetime if it is compatible with the Leibnizian substructure of the Maxwell spacetime, and rotationally equivalent to the members of $[\nabla]$. We now prove an intermediate proposition, giving the relationship between different Newton-Cartan connections compatible with a given standard of rotation.

Proposition 3. Let $\langle L, [\nabla]\rangle$ be a Maxwell spacetime, and let $\widetilde{\nabla}$ be any Newton-Cartan connection compatible with $[\nabla]$. Then for any other connection $\widetilde{\nabla}'$, $\widetilde{\nabla}'$ is a Newton-Cartan connection compatible with $[\nabla]$ if and only if $\widetilde{\nabla}' = (\widetilde{\nabla}, \eta^a t_b t_c)$, for some spacelike field η^a such that $\widetilde{\nabla}^{[a}\eta^{b]} = 0$.

Proof. First, suppose that $\widetilde{\nabla}' = (\widetilde{\nabla}, \eta^a t_b t_c)$ for such a field η^a. Then for any timelike θ^a,

$$\begin{aligned}
\widetilde{\nabla}'^{[a}\theta^{b]} &= h^{n[a}\widetilde{\nabla}'_n\theta^{b]} \\
&= h^{n[a}\widetilde{\nabla}_n\theta^{b]} - h^{n[a}\eta^{b]}\theta^m t_m t_n \\
&= \widetilde{\nabla}^{[a}\theta^{b]}
\end{aligned}$$

So clearly, $\widetilde{\nabla}'^{[a}\theta^{b]} = 0$ iff $\widetilde{\nabla}^{[a}\theta^{b]} = 0$, i.e., $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are rotationally equivalent. It remains to show that $\widetilde{\nabla}'$ satisfies the homogeneous Trautman conditions (7). Applying the standard condition relating two Riemann tensors,[17] we obtain

$$\widetilde{R}'^a{}_{bcd} = \widetilde{R}^a{}_{bcd} + 2t_b t_{[d}\widetilde{\nabla}_{c]}\eta^a \tag{11}$$

It is then a straightforward computation to show that

$$\widetilde{R}'^{ab}{}_{cd} = \widetilde{R}^{ab}{}_{cd} \tag{12}$$

[17] [Malament, 2012, Equation 1.8.2]

So clearly, $\widetilde{R}'^{ab}{}_{cd} = 0$ iff $\widetilde{R}^{ab}{}_{cd} = 0$.

Next, suppose that $\widetilde{R}^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{c}{}_{d}{}^{a}{}_{b}$. Again, a straightforward computation (together with the twist-freedom of η^a) yields

$$\widetilde{R}'^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}'^{c}{}_{d}{}^{a}{}_{b} \tag{13}$$

where the third equality uses our supposition, and the twist-freedom of η^a. Showing that if $\widetilde{R}'^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}'^{c}{}_{d}{}^{a}{}_{b}$ then $\widetilde{R}^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{c}{}_{d}{}^{a}{}_{b}$ proceeds similarly.

The converse half of the proof is adapted from [Weatherall, 2015b]. Suppose that $\widetilde{\nabla}'$ is a Newton-Cartan connection compatible with $[\nabla]$. Since $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are both compatible with L, there is some antisymmetric tensor field κ_{ab} such that $\widetilde{\nabla}' = (\widetilde{\nabla}, 2h^{an}t_{(b}\kappa_{c)n})$.[18] Now let θ^a be some unit timelike field such that $\widetilde{\nabla}^{[a}\theta^{b]} = 0$ (some such field is guaranteed to exist, since $\widetilde{\nabla}$ obeys the homogeneous Trautman conditions).[19] Using the fact that $\widetilde{\nabla}'^{[a}\theta^{b]} = 0$, we can show that $\widetilde{\nabla}' = (\widetilde{\nabla}, \eta^a t_b t_c)$ for some spacelike field η^a (see [Weatherall, 2015b, p. 91] for details of the computation).

It remains to show that η^a is twist-free. By using equation (11), we obtain

$$\widetilde{R}'^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{a}{}_{b}{}^{c}{}_{d} + 2t_b t_d \widetilde{\nabla}^c \eta^a \tag{14}$$

So by exchange of indices, and applying the second homogeneous Trautman condition,

$$t_b t_d \widetilde{\nabla}^c \eta^a = t_b t_d \widetilde{\nabla}^a \eta^c \tag{15}$$

Since $t_a \neq 0$, $\widetilde{\nabla}^{[c}\eta^{a]} = 0$. □

We can now show that there is an intimate relationship between Maxwell gravitation and Newton-Cartan gravitation: more specifically, that each model of Newton-

[18][Malament, 2012, Proposition 4.1.3]
[19]See [Malament, 2012, Proposition 4.3.7].

Cartan gravitation is naturally associated with a unique model of Maxwell gravitation, and vice versa. This provides a sense in which the two theories might be regarded as equivalent, since the mutual pair of associations might be regarded as showing how the two theories are intertranslatable with one another.[20]

Proposition 4. Let $\langle L, \tilde{\nabla}, \rho, \xi^a\rangle$ be a model of Newton-Cartan gravitation. Then there is a unique standard of rotation $[\nabla]$ such that $\tilde{\nabla}$ is compatible with $[\nabla]$; and $\langle L, [\nabla], \rho, \xi^a\rangle$ is a model of Maxwell gravitation.

Proof. First, define $[\nabla]$ as consisting of all and only those connections which are flat, and which are rotationally equivalent to $\tilde{\nabla}$. By the Trautman recovery theorem, there is at least one such connection, so $[\nabla]$ is nonempty. Hence, it is indeed a standard of rotation with which $\tilde{\nabla}$ is compatible—and it is manifestly unique in this regard.

It remains to show that $\langle L, [\nabla], \rho, \xi^a\rangle$ is a model of Maxwell gravitation. Let ∇ be an arbitrary element of $[\nabla]$. ∇ is a Newton-Cartan connection,[21] and is evidently compatible with $[\nabla]$; so by Proposition 3, $\tilde{\nabla} = (\nabla, \eta^a t_b t_c)$ where $\tilde{\nabla}^{[a}\eta^{b]} = 0$. Since $\tilde{\nabla}$ and ∇ are rotationally equivalent, we also have that $\nabla^{[a}\eta^{b]} = 0$. By equations (8b) and (10),

$$\eta^a = \xi^n \nabla_n \xi^a \tag{16}$$

So, first,

$$\begin{aligned}\nabla^c(\xi^n\nabla_n\xi^a) - \nabla^a(\xi^n\nabla_n\xi^c) &= \nabla^{[c}\eta^{a]} \\ &= 0\end{aligned}$$

[20]cf. [Glymour, 1970], [Glymour, 1977], [Barrett and Halvorson, MS].

[21]As remarked earlier, any flat connection trivially satisfies the homogeneous Trautman conditions.

So equation (9b) is satisfied. Second, using equations (8a) and (11),

$$
\begin{aligned}
4\pi\rho t_b t_d &= \widetilde{R}_{bd} \\
&= 2t_b t_{[a}\nabla_{d]}\eta^a \\
&= -t_b t_d \nabla_a(\xi^n \nabla_n \xi^a)
\end{aligned}
$$

Since $t_a \neq 0$, it follows that equation (9a) is satisfied. □

Proposition 5. Let $\langle L, [\nabla], \rho, \xi^a\rangle$ be a model of Maxwell gravitation. Then there is a unique Newton-Cartan connection $\widetilde{\nabla}$ compatible with $[\nabla]$ such that $\langle L, \widetilde{\nabla}, \rho, \xi^a\rangle$ is a model of Newton-Cartan gravitation.

Proof. First, we show existence. Let ∇ be an arbitrary element of $[\nabla]$, and define

$$\widetilde{\nabla} = (\nabla, t_b t_c \xi^n \nabla_n \xi^a) \tag{17}$$

First, we show that this is well-defined, i.e., that it is independent of the choice of ∇. So let $\nabla' \in [\nabla]$, and consider $\widetilde{\nabla}' = (\nabla', t_b t_c \xi^n \nabla'_n \xi^a)$. By Proposition 1, $\nabla' = (\nabla, \eta^a t_b t_c)$ for some η^a such that $\nabla^a \eta^b = 0$. Then for any vector field ζ^a,

$$
\begin{aligned}
\widetilde{\nabla}'_a \zeta^b &= \nabla'_a \zeta^b - \zeta^k(\xi^n \nabla'_n \xi^b) t_a t_k \\
&= \nabla_a \zeta^b - \zeta^k \eta^b t_a t_k - \zeta^k(\xi^n \nabla_n \xi^b - \eta^b) t_a t_k \\
&= \nabla_a \zeta^b - \zeta^k(\xi^n \nabla_n \xi^b) t_a t_k \\
&= \widetilde{\nabla}_a \zeta^b
\end{aligned}
$$

A similar computation shows that for any 1-form ω_a, $\widetilde{\nabla}'_a \omega_b = \widetilde{\nabla}_a \omega_b$. So $\widetilde{\nabla}' = \widetilde{\nabla}$, and so (17) is independent of the choice of ∇.

Second, we show that $\widetilde{\nabla}$ is a Newton-Cartan connection compatible with $[\nabla]$. For this, given Proposition 3, it suffices to observe that $\xi^n \nabla_n \xi^a$ is a spacelike field which is twist-free (by equation (9b)).

Finally, we show that $\langle L, \widetilde{\nabla}, \rho, \xi^a \rangle$ is a model of Newton-Cartan gravitation. First, by equations (11) and (9a),

$$\begin{aligned}\widetilde{R}_{bd} &= -t_b t_d \nabla_a(\xi^n \nabla_n \xi^a) \\ &= 4\pi\rho t_b t_d\end{aligned}$$

Second, by equation (10),

$$\begin{aligned}\xi^n \widetilde{\nabla}_n \xi^a &= \xi^n \nabla_n \xi^a - \xi^n \nabla_n \xi^a \\ &= 0\end{aligned}$$

Thus, equations (8) are satisfied.

We now prove uniqueness. Suppose that $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are two Newton-Cartan connections, compatible with $[\nabla]$, such that $\xi^n \widetilde{\nabla}_n \xi^a = \xi^n \widetilde{\nabla}'_n \xi^a = 0$. By Proposition (3), $\widetilde{\nabla}' = (\widetilde{\nabla}, \eta^a t_b t_c)$, where $\widetilde{\nabla}^{[a}\eta^{b]} = 0$. But then by equation (10), $\xi^n \widetilde{\nabla}_n \xi^a = \xi^n \widetilde{\nabla}'_n \xi^a - \eta^a$. So by our supposition, $\eta^a = 0$, and so $\widetilde{\nabla}' = \widetilde{\nabla}$. □

Extending these proofs to the versions of Maxwell and Newton-Cartan gravitation including the continuity equations is straightforward: a model $\langle L, \widetilde{\nabla}, \rho, \xi^a \rangle$ of Newton-Cartan gravitation will satisfy its associated continuity equation iff the corresponding model $\langle L, [\nabla], \rho, \xi^a \rangle$ of Maxwell gravitation does so.

Proof. By Proposition 3, for any $\nabla \in [\nabla]$, there is some spacelike field η^a such that $\widetilde{\nabla} = (\nabla, \eta^a)$. So $\widetilde{\nabla}_a \xi^a = \nabla_a \xi^a$, and hence equation (4) holds with respect to $\widetilde{\nabla}$ iff it holds with respect to ∇. □

The first direction (that any model of Newton-Cartan gravitation gives rise to a unique model of Maxwell gravitation) is not terribly surprising. What is more surprising, perhaps, is the other direction. It is non-trivial that we are able to give (relatively simple) conditions such that, if a timelike vector field ξ^a on Maxwell spacetime obeys those conditions, there exists a Newton-Cartan connection with respect to which ξ^a is a geodesic field. Moreover, it is non-trivial that the connection so picked out is unique. In general, a single congruence of curves is nowhere near enough data to single out a unique connection; the only reason it is so in this case is because of the extra requirements that the connection obeys the homogeneous Trautman conditions and is compatible with the background Maxwell spacetime.

Armed with these technical results, we can now return to the conceptual issues. As one would have hoped, Saunders' analysis is not dependent on the use of the point-particle rather than field-theoretic formulation of gravitational theory: one can expunge a privileged connection from the latter, just as from the former. Proposition 5 does raise some problems for Saunders' assertion that the connection of Newton-Cartan spacetime "is dispensable, to be derived, if at all, by fixing of gauge."[22] The connection is dispensable, in the sense that one can expunge it from every model of the theory without a loss of empirical content—indeed, without a loss of theoretical content, insofar as Propositions 4 and 5 encode an equivalence between the theories of Newton-Cartan and Maxwell gravitation. However, Saunders' claim that the connection is gauge structure is not correct, at least on the analysis here: for any given model of Maxwell gravitation, a unique connection can be reconstructed, without any appeal to conventional choices of gauge.[23]

That said, I suspect that the availability of such a unique reconstruction may be a happy accident of this particular way of doing things, rather than a really robust feature of theories of this kind. More specifically, I conjecture that if the matter fields (ρ

[22][Saunders, 2013, p. 46]

[23]It is of course true that the reconstruction of a *flat* connection, together with a gravitational field, is something that can only be performed by choosing a gauge (although boundary conditions may make some such conditions especially natural or unnatural).

and ξ^a) vanish somewhere, then one cannot reconstruct a unique connection. In support of this claim, note that the proof of Proposition 5 relies upon the fact that ξ^a is nowhere-vanishing. At any point where $\xi^a = 0$, then contra equation (10), $\xi^n \nabla'_n \xi^a = 0 = \xi^n \nabla_n \xi^a$, and so the proof of the uniqueness part of Proposition 5 fails. This would mean that my account and Saunders' are not in tension, since the point-particle setup contains large regions of empty space(time). Note that if this conjecture is correct, then if such fields were admitted it would no longer be so natural to interpret Maxwell gravitation as equivalent to Newton-Cartan gravitation: one could have several non-isomorphic Newton-Cartan models corresponding to the same Maxwell model. That said, it would not necessarily be impossible. To do so would essentially mean regarding the Newton-Cartan connection as "surplus", non-physical structure, so that models of Newton-Cartan gravitation can be physically equivalent even if they carry different connections.[24]

These reflections gives us a better handle on the question of whether acceleration is absolute or relative. To claim that acceleration is relative in Maxwell gravitation would mean taking the spacetime structure in a model $\langle L, [\nabla], \rho, \xi^a \rangle$ to be given by the Maxwell spacetime $\langle L, [\nabla] \rangle$, rather than the Newton-Cartan structure $\langle L, \widetilde{\nabla} \rangle$ definable within the model. In favour of this interpretation, note that L and $[\nabla]$ are the only primitive geometrical structures in any model of Maxwell gravitation; so on a view which identifies spacetime structure as just the primitive geometrical structure of a theory, it would be very natural to read this theory as a theory with merely relative acceleration. On the other hand, if one has a different conception of spacetime structure, then it may well be that the Newton-Cartan connection is properly identified as spatiotemporal structure—the fact that it is derived from material dynamical structures (i.e., ξ^a) notwithstanding. In particular, Knox's "spacetime functionalism"[25]

[24] cf. [Weatherall, 2015a].
[25] [Knox, 2014]

holds that the spacetime structure in a theory is whatever structure encodes the relevant notion of inertial frame in that theory. There are good grounds for thinking that this role is played by the Newton-Cartan connection—and hence, for the spacetime functionalist to maintain that acceleration in Maxwell gravitation is absolute. Thus, this case provides an (admittedly partial) illustration of the so-called "dynamical approach to spacetime geometry",[26] in which one seeks to characterise spacetime geometry as a codification of the behaviour of dynamical structures.[27]

Note that if the conjecture above is correct (i.e., if one cannot reconstruct a unique Newton-Cartan connection from somewhere-vanishing matter fields), then admitting such fields will shift the dialectic again: under those circumstances, even the spacetime functionalist would admit that acceleration in Maxwell gravitation would be relative rather than absolute. This would be in contrast to the situation in Newton-Cartan gravitation, and so the question of whether acceleration is relative will depend upon which theory the spacetime functionalist prefers as their theory of gravitation: Newton-Cartan or Maxwell gravitation.

[Weatherall, 2015b]'s analysis is very similar to that given here. His key result is the following (where I have modified Weatherall's notation, to match that used in this paper):

> Let $\{\gamma\}_\rho$ be the collection of allowed trajectories for a given mass distribution ρ in Maxwell-Huygens [i.e., Maxwell] space-time $\langle L, [\nabla] \rangle$ [...]. Then there exists a unique derivative operator $\widetilde{\nabla}$ such that (1) $\{\gamma\}_\rho$ consists of the timelike geodesics of $\widetilde{\nabla}$ and (2) $\langle L, \widetilde{\nabla} \rangle$ is a model of Newton-Cartan theory for mass density ρ.[28]

The similarity to Proposition 5 above is obvious. The only difference is that Proposi-

[26] [Brown, 2005], [Stevens, 2015]
[27] [Wallace, MS] discusses these issues in more depth.
[28] [Weatherall, 2015b, Proposition 4]

tion 5 assumes only a single congruence of curves, rather than the whole collection of allowed trajectories; as already discussed, however, there are reasons to think that in general (i.e. if the matter fields vanish somewhere), reconstructing the connection will require the full class of curves, not just a single congruence.

The more significant difference is in how one arrives at the collection of allowed trajectories. Weatherall writes the following (again, with the notation modified to that of this paper):

> [...] given a Maxwell-Huygens [i.e., Maxwell] space-time $\langle L, [\nabla]\rangle$, for any $\nabla \in [\nabla]$, there exists some scalar field ϕ such that (1) $\nabla_a \nabla^a \phi = 4\pi\rho$, where ρ is the mass density distribution of space-time; and (2) the allowed trajectories of bodies are curves γ whose acceleration (relative to ∇) is given by $\xi^n \nabla_n \xi^a = \nabla^a \phi$. [...]
>
> What is the invariant physical structure in this theory? For one, as we have seen, there is the standard of rotation shared between the derivative operators. This gives the sense in which this is a theory in Maxwell-Huygens space-time. The other invariant structure, however, is the collection of allowed trajectories for bodies. These are calculated in different ways depending on which representative one chooses from $[\nabla]$, and the acceleration associated with each such curve varies similarly. So we do not have the structure to say that these curves are accelerating or not. But however they are described, that is, whatever acceleration (if any) is attributed to them, the curves themselves are fixed. Indeed, given some distribution of matter in space-time, it is these curves that form the empirical content of Newtonian gravitational theory.[29]

I interpret Weatherall here as making the following observation (expressed in terms

[29][Weatherall, 2015b, pp. 88–89]

of gravitational fields rather than gravitational potentials): given a Maxwell spacetime equipped with a mass density, $\langle L, [\nabla], \rho\rangle$, there is a collection of curves $\{\gamma\}$ such that for any $\nabla \in [\nabla]$, there exists a spacelike vector field G^a_∇ such that:

- $\langle L, [\nabla], \rho, G^a_\nabla\rangle$ satisfies equations (3) and (3b); and
- $\{\gamma\}$ consists of all and only those curves which satisfy $\xi^n \nabla_n \xi^a = G^a$

This, in turn, leads to Weatherall's method for characterising the models of gravitation on Maxwell spacetime: $\langle L, [\nabla], \rho, \{\gamma\}\rangle$ is a model if and only if (i) for any $\nabla \in [\nabla]$, there is some spacelike field G^a_∇ such that $\langle L, \nabla, G^a_\nabla, \rho, \{\gamma\}\rangle$ is a model of Galilean gravitation; and (ii) $\{\gamma\}$ is appropriately maximal, i.e., if γ' is a curve such that $\xi'^n \nabla_n \xi'^a = G^a_\nabla$, then $\gamma' \in \{\gamma\}$. Thus, it is to models picked out in this fashion that Weatherall's Proposition 4 (quoted above) is addressed. It is not hard to show that $\langle L, [\nabla], \rho, \{\gamma\}\rangle$ satisfies these conditions just in case for any vector field ξ^a whose integral curves are all members of $\{\gamma\}$, $\langle L, [\nabla], \rho, \xi^a\rangle$ is a model of Maxwell gravitation (i.e., satisfies equations (9)).

Hence, I am getting at essentially the same point as Weatherall—but, I claim, there is value to having a set of equations which more simply and directly pick out the models of Maxwell gravitation. In particular, it helps us see a little more clearly the reason why the theory may be set on Maxwell spacetime, but not on anything weaker. If the game is just that of picking out a certain class of models, then we can set a gravitational theory on *Leibniz* spacetime just as easily as upon Maxwell spacetime. For consider the following theory, of "Leibniz gravitation": a triple $\langle L, \rho, \{\gamma\}\rangle$ is a model of Leibniz gravitation if and only if for some ∇ compatible with L, there is some spacelike field G^a_∇ such that $\langle L, \nabla, G^a_\nabla, \rho, \{\gamma\}\rangle$ is a model of Galilean gravitation; and (ii) $\{\gamma\}$ is appropriately maximal. We can prove a reconstruction theorem for Leibniz gravitation of just the same sort as Weatherall gravitation: given any model of Leibniz gravitation $\langle L, \rho, \{\gamma\}\rangle$, there is a unique derivative operator $\tilde{\nabla}$ such that $\langle L, \tilde{\nabla}, \rho, \{\gamma\}\rangle$

is a model of Newton-Cartan gravitation.[30]

Yet Leibniz gravitation is a blatant pseudo-theory—"arrant knavery", as Belot rightly derides it.[31] Why is it knavery? Because rather than universally quantifying over connections compatible with the background structure, we existentially quantified over them. The fact that Maxwell gravitation is a legitimate theory, whereas Leibniz gravitation is not, is hard to see when both are presented merely as classes of models. By contrast, if we insist that the class of models be picked out by a set of equations, then we can more easily keep ourselves honest.[32]

I conclude by remarking on some directions for further work. First, there is the conjecture mentioned above: that the reconstruction of the connection is only possible if the matter fields vanish nowhere. Second, there is the question of what happens if non-gravitational interactions are included in the theory. Finally, it would be interesting to know how these issues play out in the relativistic case: that is, whether there is some way of presenting general relativity as a theory set upon some appropriate relativistic analogue of Maxwell spacetime. I leave these issues for the future.

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[30] We can only do this because of the presence of *all* members of $\{\gamma\}$, though. As mentioned above, and unlike Maxwell spacetime, Leibniz spacetime has insufficient structure to enable one to infer a unique connection from a single vector field.

[31] [Belot, 2000]

[32] This raises some questions for the semantic view of theories: if a theory may be any class of models, picked out by any means whatsoever, then it is hard to see how the distinction appealed to here might be drawn.

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