Π^0_1 Classes, Peano Arithmetic, Randomness, and Computable Domination

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1 Introduction and Notation

1.1 An overview of this paper

The topics we cover—Π^0_1 classes, computable domination, degrees of models of arithmetic, and randomness—largely grew up over the last fifty years. Lately, they have received considerable attention, as have the many links between them and other topics. We present a different point of view of these timely areas.

The study of Π^0_1 classes (effectively closed classes in Cantor space or Baire space) emerged in the early 1970's with work by Jockusch and Soare, although Kreisel and Shoenfield had obtained some previous results on degrees of models. The use of Π^0_1 classes rapidly spread to many other areas, including model theory, computable combinatorics and Ramsey's theorem, complexity theory and randomness, and models of Peano arithmetic. In Section 3, we consider two of the original theorems in this study (the Low Basis Theorem and the Computably Dominated Basis Theorem), each showing that a nonempty Π^0_1 class has a member of a certain type. The subsequent popularity of this area has lead to many related results. In Section 3, we take a look at some of these, including an *antibasis* theorem, a proper low_n basis theorem, as well as cone avoidance constructions.

A prominent method of constructing members of Π^0_1 classes is known as *forcing with* Π^0_1 *classes*. The name comes from the fact that in such constructions we begin with a certain tree and pass to a smaller subtree in order to *force* a given requirement. This technique is very flexible and can be used to obtain a number of finer results about the members of nonempty Π^0_1 classes, including ones which avoid cones or form minimal pairs with given noncomputable sets. In Section 4, we discuss this notion of forcing in a general setting, defining the appropriate notions of condition, extension, density, and genericity. We also discuss individual forcing modules, and how they can be combined to achieve desired conclusions.

A function f is *computably dominated (hyperimmune-free)* if every function $g \equiv_{\mathrm{T}} f$ (not only f itself) is bounded by a computable function. The Computably Dominated Basis Theorem in Section 3 connects Π^0_1 classes to computable domination because it shows that every nonempty Π^0_1 class contains a computably dominated (c.d.) member. In Section 5 we explore other properties of computably dominated functions and the degrees they can inhabit. In particular, we look at the following questions: Can degrees comparable with $\emptyset'$ be c.d.? Can a Σ^0_2 set be c.d.? Although sparsely distributed and difficult to construct, the c.d. functions have two surprising

downward closure properties. The c.d. functions also play an important role in algorithmic randomness and complexity, and are treated in recent books in these areas by Nies [1999] and Downey and Hirschfeldt [ta]. Cooper [2004] also covers computable domination and hyperimmune degrees.

One of the original motivations for looking at Π^0_1 classes was the study of degrees of complete extensions of Peano Arithmetic (PA), or equivalently, degrees of models of PA (*PA degrees*). Indeed, the complete extensions of PA form a Π^0_1 class, and if f is any complete extension of PA and $\mathcal{C}$ is any nonempty Π^0_1 class, then f can compute a member $g \in \mathcal{C}$. In Section 6, we consider a number of alternative ways of defining the PA degrees, including as those degrees which contain a $\{0,1\}$-valued diagonally noncomputable (d.n.c.) function.

Over the last decade a rapidly growing area has been Kolmogorov complexity and algorithmic randomness. There has been a close connection between Π^0_1 classes and 1-random (Martin-Löf random) reals. For example, there is a nonempty Π^0_1 class all of whose elements are 1-random, as we shall see in Corollary 7.3. In Section 7, we relate randomness to the computably dominated degrees that we study in Section 5, and we examine the relationship between the measure of a Π^0_1 class and the 1-random reals it contains.

1.2 Notation

Using ordinal notation, identify the ordinal 2 with the set of smaller ordinals $\{0,1\}$. We let $2^{<\omega}$ denote the the set of all finite sequences of 0's and 1's. Identify a set $A \subseteq \omega$ with its characteristic function $f : \omega \mapsto \{0,1\}$ and represent the class of these functions as 2^ω. Let ω^ω denote the set of all functions f from ω to ω. We use these definitions and operations on strings.

σ, τ, ρ	strings $\in 2^{<\omega}$
λ	empty string
$\sigma^\frown\tau$ or $\sigma\tau$	concatenation of σ followed by τ
$\sigma^\frown a$ or σa	concatenation of σ followed by a
$\sigma \prec \tau$, $\sigma \prec A$	σ is an initial segment of τ, A
$\sigma \mid \tau$	σ, τ incomparable, $\neg[\sigma \preceq \tau]$ & $\neg[\tau \preceq \sigma]$
$\lvert\sigma\rvert$ or $\mathrm{lh}(\sigma)$	length of σ, also $\lvert\sigma\rvert = (\mu x)[\sigma(x)\uparrow]$
$A\upharpoonright z$, $\sigma\upharpoonright x$	restriction of $A(x)$ or $\sigma(x)$ to those $x < z$
$A\upharpoonright\upharpoonright z$, $\sigma\upharpoonright\upharpoonright x$	restriction of $A(x)$, or $\sigma(x)$, to those $x \leq z$

We state most definitions for 2^ω but with obvious changes they extend to ω^ω. We now deal with *classes* $\mathcal{C} \subseteq 2^\omega$, *i.e.*, second order objects, rather than just *sets* $A \subseteq \omega$ or functions $f \in \omega^\omega$ which are first order objects. It is customary to call a set $A \subseteq \omega$ or function $f \in \omega^\omega$ a *real*.

2 Open and Closed Classes in Cantor Space

2.1 Open Classes

Definition 2.1. (i) *Cantor space* is 2^ω with the following topology. For every $\sigma \in 2^{<\omega}$ define the *basic open class*

$$[\![\sigma]\!] \;=\; \{\; f \;:\; f \in 2^\omega \quad \& \quad \sigma \prec f \;\}.$$

The *open* classes of Cantor space are unions of basic open classes. A set $A \subseteq 2^{<\omega}$ is an *open representation* of the open class

$$[\![A]\!] \quad = \quad \bigcup_{\sigma \in A} [\![\sigma]\!].$$

(We may assume A is closed *upwards*, *i.e.*, $\sigma \in A$ and $\sigma \prec \tau$ implies $\tau \in A$.)

(ii) A class $\mathcal{A} \subseteq 2^\omega$ is *effectively open (computably open)* if $\mathcal{A} = [\![A]\!]$ for a computable set $A \subseteq \omega$.[1]

(iii) A class $\mathcal{A} \subseteq 2^\omega$ is *(lightface)* Σ^0_1 if there is a computable R such that

$$\mathcal{A} = \{\; f \;:\; (\exists x)\, R(\, f \upharpoonright x\,) \;\} \tag{1}$$

(iv) A class $\mathcal{A}$ is *(boldface)* $\mathbf{\Sigma^0_1}$ if (1) holds with R replaced by R^X for R^X computable in some set $X \subseteq \omega$, in which case we also say $\mathcal{A}$ is $\Sigma^{0,X}_1$.

Theorem 2.2 (Effectively Open Classes)**.** *Fix a class $\mathcal{A} \subseteq 2^\omega$.*

(i) $\mathcal{A}$ is effectively open iff $\mathcal{A}$ is (lightface) Σ^0_1.

(ii) $\mathcal{A}$ is open iff $\mathcal{A}$ is (boldface) $\mathbf{\Sigma^0_1}$.

Proof. (i) Let $\mathcal{A}$ be effectively open. Then $\mathcal{A} = [\![B]\!]$ for some B computable. Define $R(\sigma)$ iff $\sigma \in B$. Now $f \in \mathcal{A}$ iff $(\exists x) R(f \upharpoonright x)$. Hence, $\mathcal{A}$ is Σ^0_1. Conversely, assume $\mathcal{A}$ is Σ^0_1 via a computable R satisfying (1). Define $A = \{\sigma : R(\sigma)\}$. Then $\mathcal{A} = [\![A]\!]$.

(ii) Relativize the proof of (1) to a set $X \subseteq \omega$. □

[1] It is easy to show that if $\mathcal{A} = [\![A]\!]$ with A c.e. , then $\mathcal{A} = [\![B]\!]$ for some *computable* set $B \subseteq \omega$. The notation $[\![\sigma]\!]$ for a basic open set is now becoming standard and has also been adopted in the book by Downey and Hirschfeldt [ta].

2.2 Closed Classes

Definition 2.3. (i) A *tree* $T \subseteq 2^{<\omega}$ is a set closed under initial segments, *i.e.*, $\sigma \in T$ and $\tau \prec \sigma$ imply $\tau \in T$. The set of *infinite paths* through T is

$$[\,T\,] = \{\ f\ :\ (\forall n)\,[\ f\restriction n \in T\ ]\ \}. \tag{2}$$

(ii) A class $\mathcal{C} \subseteq 2^\omega$ is (lightface) Π^0_1 if there is a computable relation $R(x)$ such that

$$\mathcal{C} = \{\ f\ :\ (\forall x)\,R(f\restriction x)\ \}. \tag{3}$$

A class $\mathcal{C} \subseteq 2^\omega$ is *boldface* $\mathbf{\Pi^0_1}$ if (3) holds for R^X computable in some $X \subseteq \omega$. The latter is also written as $\mathcal{C}$ is $\Pi^{0,X}_1$.

(iii) A class $\mathcal{C} \subseteq 2^\omega$ is *effectively closed* (*computably closed*) if its complement is effectively open. A class $\mathcal{C} \subseteq 2^\omega$ is *closed* if its complement is open.

Theorem 2.4 (Effectively Closed Classes)**.** *For any class $\mathcal{C} \subseteq 2^\omega$, the following are equivalent:*

(i) $\mathcal{C} = [\,T\,]$ *for some computable tree* T.

(ii) $\mathcal{C}$ *is effectively closed.*

(iii) $\mathcal{C}$ *is a* (lightface) Π^0_1 *class.*

Proof. This follows from Definition 2.3 and Theorem 2.2. □

Corollary 2.5 (Closed Classes and Trees)**.** *For any class $\mathcal{C} \subseteq 2^\omega$, the following are equivalent:*

(i) $\mathcal{C} = [\,T\,]$ *for some tree* T.

(ii) $\mathcal{C}$ *is closed.*

(iii) $\mathcal{C}$ *is a* (boldface) $\mathbf{\Pi^0_1}$ *class.*

Proof. Relativize the proof of Theorem 2.4 to $X \subseteq \omega$. □

Remark 2.6 (Representing Closed Classes)**.** The most convenient way of representing open and closed classes is with trees. If $\mathcal{C}$ is closed we choose a tree T such that $\mathcal{C} = [\, T \,]$. Define $A = \omega - T$. Then T is downward closed, A is upward closed, and A defines the open set $[\![A]\!] = 2^\omega - [\, T \,] = \overline{\mathcal{C}}$. Note that the representations A and T are conveniently complementary in ω and the open class $[\![A]\!]$ and closed class $[\, T \,]$ are also complementary in 2^ω. The only difference between the effective case and general case is whether the tree T is computable or only computable in some set $X \subseteq \omega$.

We may imagine a path $f \in 2^\omega$ trying to climb the tree T without passing though a node $\sigma \in A$. If f succeeds, then $f \in \mathcal{C} = [\, T \,]$. However, if $f \succ \sigma$ for even *one* node $\sigma \in A$, then f falls off the tree forever and $f \notin \mathcal{C}$.

2.3 The Compactness Theorem

One particularly useful feature of Cantor space is the well-known Compactness Theorem (whose proof we omit), and the Effective Compactness Theorem 2.9, both of which lead into the study of our main topic, Π^0_1 classes.

Theorem 2.7 (Compactness Theorem)**.** *The following easy and well-known properties hold for Cantor Space 2^ω. The term* "compactness" *refers to any of them, but particularly to (iv).*

(i) (Weak König's Lemma). *If $T \subseteq 2^{<\omega}$ is an infinite tree, then $[\, T \,] \neq \emptyset$.*

(ii) *If $T_0 \supseteq T_1 \ldots$ is a decreasing sequence of trees with $[\, T_n \,] \neq \emptyset$ for every n, and intersection $T_\omega = \cap_{n \in \omega} T_n$, then $[\, T_\omega \,] \neq \emptyset$.*

(iii) *If $\{\mathcal{C}_i\}_{i \in \omega}$ is a countable family of closed sets such that $\cap_{i \in F} \mathcal{C}_i \neq \emptyset$ for every* finite *set $F \subseteq \omega$, then $\cap_{i \in \omega} \mathcal{C}_i \neq \emptyset$ also.*

(iv) (Finite Subcover Property). *Any open cover $[\![A]\!] = 2^\omega$ has a finite open subcover $F \subseteq A$ such that $[\![F]\!] = 2^\omega$.*

Proof. See Soare [CTA] Chapter 3. □

2.4 Notation For Trees

Since our principal tool with be trees we introduce some notation.

Definition 2.8. Fix a tree $[\, T \,] \subseteq 2^{<\omega}$.

(i) For $\sigma \in T$ define the subtree T_σ of nodes *comparable with* σ,

$$T_\sigma \;=\; \{\, \tau \in T \;:\; \sigma \preceq \tau \quad \text{or} \quad \tau \prec \sigma \,\}. \tag{4}$$

(ii) Define the subtree of *extendible nodes* $\sigma \in T$.

$$T^{\,\mathrm{ext}} \;=\; \{\, \sigma \in T \;:\; (\exists f \succ \sigma)[\, f \in [\, T \,] \,] \,\} \tag{5}$$

(iii) A point (path) $f \in [\, T \,]$ is *isolated* if

$$(\exists \sigma)[\quad [\, T_\sigma \,] \;=\; \{\, f \,\} \quad]. \tag{6}$$

We say that σ *isolates* f because $[\![\, \sigma \,]\!] \cap [\, T \,] = \{\, f \,\}$ and we call σ an *atom* because it cannot be extended to two incomparable nodes ρ and τ on T. If f is isolated we say it has Cantor-Bendixson rank 0. If f is not isolated, then f is a *limit point*. Note that

$$T^{\,\mathrm{ext}} \;=\; \{\sigma \in T \;:\; (\,\forall n \geq\, |\sigma|\,)\,(\,\exists \tau \succeq \sigma\,)[\;|\tau| \;= n \;\;\&\;\; \tau \in T\,]. \tag{7}$$

2.5 Effective Compactness Theorem

For a *computable* tree $T \subseteq 2^{<\omega}$ we can establish the following effective analogues of the Compactness Theorem 2.7.

Theorem 2.9 (Effective Compactness Theorem)**.** *Let $T \subseteq 2^{<\omega}$ be a computable tree.*

(i) $T^{\,\mathrm{ext}}$ *is a* Π^0_1 *set. Hence,* $\overline{T^{\,\mathrm{ext}}}$ *is* Σ^0_1*,* $\overline{T^{\,\mathrm{ext}}} \leq_{\mathrm{m}} \emptyset'$*, and* $T^{\,\mathrm{ext}} \leq_{\mathrm{T}} \emptyset'$*.*

(ii) (Kreisel Basis Theorem) *If* $[\, T \,] \neq \emptyset$*, then* $(\exists f \leq_{\mathrm{T}} \emptyset')[\; f \in [\, T \,] \,]$*. This part (ii) will be generalized in the* Low Basis Theorem 3.8*.*

(iii) If $[\, T \,] \neq \emptyset$*, and* f *is the lexicographically least member, then* f *has c.e. degree.*

(iv) If $f \in [\, T \,]$ *is isolated, then* f *is computable.*

(v) Given an open cover $[\![\, A \,]\!] = 2^\omega$ *with* A *c.e. there is finite subset* $F \subseteq A$ *such that* $[\![\, F \,]\!] = 2^\omega$ *and a canonical index for* F *can be found uniformly in a c.e. index for* A*.*

Proof. (i) The formal definition of $T^{\rm ext}$ in (5) has one function quantifier and is in Σ^1_1 form. Indeed, this the best we can do for Baire space ω^ω. However, for Cantor space 2^ω we can use the Compactness Theorem 2.7 (i) to reduce $T^{\rm ext}$ to the Π^0_1 form of (7). Note that the quantifier τ is bounded by n and therefore only the $(\forall n)$ quantifier counts in the prefix calculation.[2]

(ii) Now use a $\emptyset'$ oracle to choose $f \in [\, T\,]$ such that $f = \cup_n \sigma_n$ defined as follows. Given $\sigma_n \in T^{\rm ext}$ let $\sigma_{n+1} = \sigma_n{}^\frown 0$ if $\sigma_n{}^\frown 0 \in T^{\rm ext}$ and $\sigma_n{}^\frown 1$ otherwise.

(iii) (This gives a stronger conclusion than (ii).) Let f be the the lexicographically least member of $[\, T\,]$, *i.e.*, in the dictionary ordering $<_L$ on the alphabet $\{\ 0{,}1\ \}$. Define the following c.e. set of nodes $M \subseteq \overline{T^{\rm ext}}$ such that $M \equiv_{\rm T} f$.

$$M = \{\ \sigma :\ (\forall \tau)_{\,|\tau| = |\sigma|}\ [\quad [\ \tau \in T\ \ \&\ \ \tau \leq_L \sigma\ ] \quad \Longrightarrow \quad \tau \in \overline{T^{\rm ext}}\ ]\quad \}$$

(Wait until σ and all its predecessors of length $|\sigma|$ have appeared nonextendible. Then put σ into M. In this way we enumerate all nodes $\tau <_L f$.)

(iv) Choose $\sigma \in T$ with $[\, T_\sigma\,] = \{\, f\,\}$. To compute f assume we have computed $\tau = f \upharpoonright n$. Exactly one of $\tau{}^\frown 0$ and $\tau{}^\frown 1$ is extendible. Enumerate $\overline{T^{\rm ext}}$ until one of these nodes appears and take the other one.

(v) Assume $[\![\, A\,]\!] = 2^\omega$ with A c.e. Enumerate A until a finite set $F \subseteq A$ is found with $[\![\, F\,]\!] = 2^\omega$ by the Compactness Theorem (iv). We can search until we find it. $\square$

Remark 2.10. Note that the conclusions in the Effective Compactness Theorem 2.9 have various levels of effectiveness even though the hypotheses are all effective. In (v) if $[\![\, A\,]\!]$ covers 2^ω then the passage from A to F is *computable* because we simply enumerate A until F appears (as for any Σ_1 process). However, if $[\![\, A\,]\!]$ *fails* to cover 2^ω then the complementary closed class $[\, T\,] = 2^\omega - [\![\, A\,]\!]$ is nonempty. Then (ii) gives a path $f \in [\, T\,]$ with $f \leq_{\rm T} \emptyset'$ and (iii) even produces a path of c.e. degree, but neither produces a *computable* path f because given an extendible string σ the process for the proof of König's Lemma in Theorem 2.7 (i) does not computably determine whether to extend to $\sigma{}^\frown 0$ or $\sigma{}^\frown 1$. In Theorem 3.4 we shall construct a computable tree with paths but no computable paths.

[2] See Soare [CTA] Theorem 4.1.3 (vi) on quantifier manipulation.

3 Basis and Nonbasis Theorems for Π^0_1 Classes

The motivating question of this section is the following: given a nonempty Π^0_1 class $\mathcal{C}$, what can we say about the Turing degrees of $\mathcal{C}$? To address this question, we need to fix some terminology.

Definition 3.1. A Π^0_1 class $\mathcal{C}$ is *special* if it contains no computable member.

It follows that if $T \subseteq 2^{<\omega}$ is a computable tree such that $[\,T\,]$ is special then T^{ext} must be a *perfect* tree, meaning that every $\sigma \in T^{\text{ext}}$ admits incompatible extensions in T^{ext} because any isolated path would be computable. Therefore, every special Π^0_1 class has $2^{\aleph_0}$ members.

Definition 3.2. Let $\mathbf{D}$ be a class of Turing degrees.

(i) We call $\mathbf{D}$ a *basis for* Π^0_1 *classes* if every nonempty Π^0_1 class has a member f with $\deg(f) \leq \mathbf{d}$ for some $\mathbf{d} \in \mathbf{D}$. Otherwise, we call $\mathbf{D}$ a *nonbasis*.

(ii) We call $\mathbf{D}$ an *antibasis*[3] *for* Π^0_1 *classes* if whenever a Π^0_1 class contains a member of every degree in $\mathbf{d} \in \mathbf{D}$, it contains a member of every degree $\mathbf{d} \geq \mathbf{0}$.

We extend the above definitions from classes of degrees to classes $\mathcal{S}$ of subsets of ω, by calling $\mathcal{S}$ a basis, nonbasis, or antibasis if $\{\deg(S) : S \in \mathcal{S}\}$ is, respectively, a basis, nonbasis, or antibasis.

An alternative definition of bases is the following. First, for a Π^0_1 class $\mathcal{C}$, define its *degree spectrum* as $\mathcal{S}(\mathcal{C}) = \{\deg(f) : f \in \mathcal{C}\}$. Then a class of degrees $\mathbf{D}$ is a *basis for* Π^0_1 *classes* if $\mathbf{D} \cap \mathcal{S}(\mathcal{C}) \neq \emptyset$ for all Π^0_1 classes $\mathcal{C} \neq \emptyset$. (We will not refer to degree spectra in the sequel; see Kent and Lewis [2009] for a thorough investigation of their algebraic properties.)

3.1 Nonbasis Theorems For Π^0_1

Next in Theorem 3.4 we show that we cannot always find f computable. Therefore, the class of computable sets is a nonbasis for Π^0_1.

Definition 3.3. If A and B are disjoint sets, then S is a *separating set* if $A \subseteq S$ and $B \cap S = \emptyset$. This class is written $S(A, B)$.

Theorem 3.4.

[3]Some authors (e.g., Cenzer [1999]) use the term *antibasis* as a synonym for *nonbasis*. Our use here follows that of Kent and Lewis [2009].

(i) If W_e and W_i are disjoint c.e. sets, then the class of separating sets $S(W_e, W_i)$ is a nonempty Π^0_1 class.

(ii) There is a nonempty Π^0_1 class with no computable members.

Proof. (i) Define a Π^0_1 class $\mathcal{C}$ by putting f in $\mathcal{C}$ if:

$$(\forall x)(\forall s)[\ [\ x \in W_e \implies f(x) = 1\ ] \quad \& \quad [\ x \in W_i \implies f(x) = 0\ ]\ ].$$

(ii) Let W_e and W_i be disjoint, computably inseparable c.e. sets. □

Corollary 3.5. *The class of computable sets is not a basis for Π^0_1 classes (i.e., $\{\mathbf{0}\}$ is a nonbasis).*

We can generalize the preceding corollary as follows.

Theorem 3.6 (Jockusch and Soare, (1972a), Theorem 4)**.** *The class of incomplete c.e. sets is not a basis for Π^0_1 classes (i.e., the class of c.e. degrees $\mathbf{d} < \mathbf{0}'$ is a nonbasis).*

Proof. Let A be the standard effectively immune set of Post. Recall that this set is constructed by enumerating into $\overline{A}$ at stage $s \geq 0$, for each $e < s$, the least $x > 2e$ such that $x \in W_{e,s} \subseteq A_s$. Then A is co-c.e. and it is easy to see that for every $e \in \omega$, $|\overline{A} \upharpoonright 2e| \leq e$, meaning $A \cap [2^{e+1} - 2, 2^{e+2} - 3] \neq \emptyset$. Thus, if we define

$$\mathcal{C} = \{f \in 2^\omega : f \subseteq A \ \ \& \ \ (\forall e)[f \cap [2^{e+1} - 2, 2^{e+2} - 3] \neq \emptyset]\},$$

then $\mathcal{C}$ is a Π^0_1 class consisting entirely of infinite subsets of A. In particular, since A is effectively immune and so is every infinite subset of an effectively immune set, every member of A is effectively immune. It is a well-known result of Martin that no incomplete c.e. set can compute an effectively immune set, so no incomplete c.e. set can compute any element of $\mathcal{C}$. □

3.2 The Kreisel-Shoenfield Basis Theorem

By the Kreisel Basis Theorem 2.9 (ii) we can always find $f \leq_{\mathrm{T}} \emptyset'$ and by (iii) even f of c.e. degree, although by Theorem 3.6 not necessarily *incomplete* c.e. degree. Can we do better?

Theorem 3.7 (Kreisel-Shoenfield Basis Theorem)**.** *Every nonempty Π^0_1 class $\mathcal{C}$ has a member $f <_T \emptyset'$.*

Shoenfield [1960] improved the Kreisel Basis Theorem to f *strictly* below $\emptyset'$, namely $f <_T \emptyset'$, by considering, for a given Π^0_1 class $\mathcal{C}$, the Π^0_1 class $\mathcal{D}$ of all $\langle f, g\rangle$ such that $f \in \mathcal{C}$ and

$$(\forall e)[\ \Phi^f_e(e)\!\downarrow \quad \Longrightarrow \quad \Phi^f_e(e)\ \neq\ g(e)\ ].$$

He then applied Kreisel's result to $\mathcal{D}$. The following Low Basis Theorem substantially generalizes these results by Kreisel and Shoenfield.

3.3 The Low Basis Theorem

The previous two subsections prove that given a Π^0_1 class $\mathcal{C}$ we cannot always find a computable member $f \in \mathcal{C}$ but we can find a member $f <_{\mathrm{T}} \emptyset'$. The next theorem says we can do much better and always produce f which is *low i.e.*, $f' \equiv_{\mathrm{T}} \emptyset'$ and therefore close to $\emptyset$ in information content and structure. Downey and Hirschfeldt's new book [ta, p. 73] states, "The following is the most famous and widely applicable basis theorem."

Theorem 3.8 (Low Basis Theorem (LBT), Jockusch and Soare, 1972b)**.** *If $\mathcal{C} \subseteq 2^\omega$ is a nonempty Π^0_1 class, then it contains a low member f.*

Proof. Let T be a computable tree such that $[T] = \mathcal{C}$. Use $\emptyset'$ to define a sequence of infinite computable trees $T = T_0 \supseteq T_1 \supseteq \ldots$ as follows. Define

$$U_e \quad = \quad \{\ \sigma :\ \Phi^\sigma_{e,\,|\sigma|}\,(e)\!\uparrow\ \} \tag{8}$$

which is also a computable tree. Given T_e: (1) define $T_{e+1} = T_e \cap U_e$ if $T_e \cap U_e$ is infinite; and (2) define $T_{e+1} = T_e$ otherwise. If (1) then $\Phi^g_e(e)\!\uparrow$ for all $g \in [T_{e+1}]$, and if (2) then $\Phi^g_e(e)\!\downarrow$ for all $g \in [T_{e+1}]$. Note that $\emptyset'$ can decide whether a computable tree is finite by (7).

We say that T_{e+1} *forces the jump* as described in Section 4 because no matter which clause holds in the definition of T_{e+1} we know that either,

$$(\,\forall\, g\ \in\ [\,T_{e+1}\,]\,)\,[\ \ \Phi^g_e(e)\!\downarrow\ \ ] \qquad \text{or} \tag{9}$$

$$(\,\forall\, g\ \in\ [\,T_{e+1}\,]\,)\,[\ \ \Phi^g_e(e)\!\uparrow\ \ ] \tag{10}$$

Choose $f \in \bigcap_{e\in\omega}\,[\,T_e\,]$. This is an intersection of a descending sequence of nonempty closed sets and hence nonempty, by the Compactness Theorem 2.7 (ii). Now $\emptyset'$ can decide using (7) which of (9) or (10) holds at stage $e+1$ in the definition of T_{e+1}.

Therefore, we have $\emptyset'$-computably determined at stage $e+1$ the convergence or divergence of $\Phi_e^f(e)$ even though very little of f has yet been defined by stage $e+1$. This is the nature of forcing, to decide $\Phi_e^f(e)$ even though f is yet undetermined. □

In Section 5.3 we use this method to prove a similar basis theorem that any nonempty Π^0_1 class contains a member f such that every $g \leq_{\mathrm{T}} f$ is dominated by a computable function.

We shall see in Section 6, Theorem 6.4, that there exists a nonempty Π^0_1 class $\mathcal{P}$ such that for every other nonempty Π^0_1 class $\mathcal{C}$ and every $f \in \mathcal{P}$, f computes a member of $\mathcal{C}$ (i.e., for every $f \in \mathcal{P}$, the singleton $\{f\}$ is a basis for Π^0_1 classes). Applying the Low Basis Theorem to $\mathcal{P}$ then yields the following:

Theorem 3.9 (Second Low Basis Theorem, Jockusch and Soare (1972b))**.** *There is a low set A such that every nonempty Π^0_1 class $\mathcal{C} \subseteq 2^\omega$ has a member $f \leq_{\mathrm{T}} A$.*

3.4 Superlow Basis Theorem

The proof of the Low Basis Theorem 3.8 gives even more information about the jump f' than was explicitly claimed but explaining it requires some definitions.

3.4.1 Defining Superlow

Definition 3.10. Assume we are given a set $A \leq_{\mathrm{T}} \emptyset'$.

(i) The set A is *ω-c.e.* if there is a computable sequence $\{A_s\}_{s\in\omega}$ with $A_0 = \emptyset$ and $A_s(x) \in \{0,1\}$, and a computable function $g(x)$ such that

$$(11) \quad A = \lim_s A_s \quad \& \quad | \{ s : A_s(x) \neq A_{s+1}(x) \} | \leq g(x).$$

(ii) If (11) holds for $g(x)$ then A is $g(x)$-c.e. If $g(x) = n$ then A is *n-c.e.*

(iii) A is *truth-table* reducible to B, written $A \leq_{\mathrm{tt}} B$, if there is a *total* Turing reduction Φ_e with $A = \Phi_e^B$.

(iv) A set A is *superlow* if $A' \leq_{\mathrm{tt}} \emptyset'$ or equivalently if A' is ω-c.e.

Theorem 3.11 (Superlow Basis Theorem (SLBT))**.** *Every nonempty Π^0_1 class $\mathcal{C} \subseteq 2^\omega$ has a member A which is superlow and indeed A' is 2^{e+1}-c.e.*

3.4.2 The First Proof of the Superlow Basis Theorem

Proof. (*The first proof (c. 1969) by making A' ω-c.e.*) The first proof[4] (unpublished) of the Low Basis Theorem 3.8 (LBT) was not the $\emptyset'$ oracle proof above but a limit computable proof that A' is 2^{e+1}-c.e. We construct a computable a sequence of strings $\{\sigma_s\}_{s\in\omega}$ such that $A := \lim_s \sigma_s$ is superlow. Fix a computable tree T with $[\,T\,] = \mathcal{C}$. Define the computable tree,

$$U_{e,s} \quad = \quad \{\ \sigma :\ \Phi^{\sigma}_{e,\,s}\ (e)\!\uparrow\ \} \tag{12}$$

Let $T_{0,\,s} = T$ for all s. For every s given $T_{e,s}$: (1) define $T_{e+1,\,s} = T_{e,\,s} \cap U_{e,s}$ if the latter contains a string σ of length s and (2) define $T_{e+1,\,s} = T_{e,\,s}$ otherwise. Let σ_s be the lexicographically least string of length s in $T_{s,\,s}$. Choose a stage s after which the trees $T_{i,s}$ have stopped changing from (1) to (2) for all $i < e$. After s the tree $T_{e,t}$ changes from (1) to (2) at most once at some $t > s$ when $\Phi^{\sigma}_{e,t}(e)\uparrow$ and $\Phi^{\sigma}_{e,t}(e)\downarrow$. For stages $v > t$ the tree $T_{e,v}$ never changes away from (2) and $\Phi^{\sigma}_{e,v}(e)$ remains defined forever. Therefore, $\Phi^{\sigma}_{e,s}(e)$ changes between defined and undefined at most 2^{e+1} times and converges to $A'(e)$. □

3.4.3 A Second Proof of the Superlow Basis

Proof. (*Second proof using a total reduction.*)
Let $I = \{i \in \omega : i$ is an index for a finite Π^0_1 class$\}$. Then I is definable by a Σ^0_1 formula, meaning, since $\emptyset'$ is Σ^0_1-complete, that there is a computable function h such that for all i, $i \in I$ if and only if $h(i) \in \emptyset'$. Fix a nonempty Π^0_1 class $\mathcal{C}$ and a computable tree $T_0 \subseteq 2^{<\omega}$ with $\mathcal{C} = [\,T_0\,]$. We define a total reduction Ψ as follows. Fix X and $e \in \omega$, and assume inductively that for all $x < e$, $\Psi^X(x)$ has been defined, along with (indices for) trees $T_0 \supseteq \cdots \supseteq T_e$. From an index for T_e we can effectively find an index i for the tree $U_e = \{\sigma \in T_e : \Phi^{\sigma}_e(e)\uparrow\}$, and we define $\Phi^X(e)$ to be 0 or 1 depending as $h(i)$ is or is not in X. We then let T_{e+1} be either T_e or U_e, respectively. By comparison with the proof of the Low Basis Theorem, we see that $\Psi^{\emptyset'} = f'$ for some $f \in \mathcal{C}$. □

[4] Jockusch and Soare never stated the Superlow Basis Theorem because the notion of superlow did not exist in 1972. Several people later noticed that the original proof of the Low Basis Theorem can easily be converted to a proof of the Superlow Basis Theorem. The limit computable proof presented here does more since it gives the conclusion at once. The two proofs of the LBT and SLBT illustrate the tradeoff between the two approaches to proofs concerning Π^0_1 classes, an oracle proof, versus a computable approximation.

3.5 The Computably Dominated Basis Theorem

Jockusch and Soare showed that a variation on the proof of the Low Basis Theorem 3.8 yielded a member $f \in \mathcal{C}$ such that every $g \leq_{\mathrm{T}} f$ is bounded by a computable function. Such functions are called *computably dominated* and play an important role.

Definition 3.12.

(i) A function h *bounds (majorizes)* a function g, written $g < h$, if $(\forall x)[\ g(x) < h(x)\ ]$.

(ii) Function h *dominates* g, written $g <^* h$, if $(\forall^\infty x)[\ g(x) < h(x)\ ]$. where $(\forall^\infty x)$ denotes "for almost all x," *i.e.*, for all but finitely many x.

(iii) If $A = \{a_0 < a_1 < \ldots\}$ is an infinite set, the *principal function* of A is p_A where $p_A(n) = a_n$. Extend the definitions of bounds and dominates to an infinite *set* A by using the principal function p_A.

(iv) A function g (or set A) is *computably bounded* if it is bounded by some computable function h.

(v) A (Turing) degree $\mathbf{d}$ is *computably dominated (c.d.)* if f is computably bounded for every $f \in \mathbf{d}$.

(vi) A function f or set A is *computably dominated (c.d.)* if the degree of f (respectively A) is a computably dominated degree.

The key point is that for a function f or set A to be *computably bounded* simply requires a single bounding function, while being *computably dominated* imposes computable bounding on *every* function g in the same degree, a very strong condition. For example we shall prove in Theorem 5.8 that if f is computably dominated then g is computably bounded for all $g \leq_{\mathrm{T}} f$ not only $g \equiv_{\mathrm{T}} f$. In Section 5.3 we shall see that a computably dominated set is also hyperimmune-free although we and Nies use the term *computably dominated.*[5]

The key idea in the next theorem is to use a $\emptyset''$ oracle to build a member f of a given Π^0_1 class with the property that we can decide whether Φ^f_e is

[5] After the term *"hyperimmune-free"* degree was introduced by Miller and Martin [1968] it was often used in the literature, even though it conveys little intuition about the meaning. Recently, Soare introduced the term *"computably dominated"* which better suggests the meaning, and used it in his new book Soare [CTA]. He suggested this to Nies who adoped it in his book, *Computability and Randomness* [2009]. We use this term here.

total or not at a definite stage of the construction. This differs from the proof of the Low Basis Theorem, where we needed only a $\emptyset'$ oracle to similarly decide whether $\Phi_e^f(e)$ converges or not. In both cases, however, we use the same technique (known as *forcing with* Π^0_1 *classes*) of continually pruning an infinite computable tree while preserving certain desired properties. We shall give a more systematic treatment of this technique in Section 4 below.

Theorem 3.13 (Computably Dominated Basis Theorem, Jockusch and Soare, 1972b)**.** *If* $\mathcal{C} \subseteq 2^\omega$ *is a nonempty* Π^0_1 *class, then it contains a low*$_2$ *member such that*

$$(\forall g \leq_{\mathrm{T}} f)\,[\ g \text{ is computably bounded }]. \tag{13}$$

Proof. For any computable tree T and any $e, x \in \omega$, we can $\emptyset'$-uniformly effectively decide whether the following set is infinite:

$$U_{e,x} \quad = \quad \{\ \sigma \in 2^{<\omega}\ :\ \sigma \in T \quad \& \quad \Phi^\sigma_{e,\,|\sigma|}\,(x)\!\uparrow\ \}. \tag{14}$$

Choose a computable tree T_0 such that $\mathcal{C} = [\,T_0\,]$, and assume by induction that T_e is defined for some $e \geq 0$. Use a $\emptyset''$ oracle to determine whether there is an x such that $U_{e,x}$ is infinite.

Case 1. If so, then choose the least such x, and define $T_{e+1} = U_{e,x}$.

Case 2. If not, then define $T_{e+1} = T_e$.

Note that in either case, $T_{e+1} \subseteq T_e$. As before, the intersection $\bigcap_e T_e$ is nonempty by the Compactness Theorem 2.7 (ii), so we can choose some f in it.

Lemma 3.14. *If* $g = \Phi_e^f$ *is total then* g *is computably bounded.*

Proof. Assume $g = \Phi_e^f$ is total. Then T_{e+1} must have been defined by Case 2 since otherwise Φ_e^f could not be total. Then for every x, we can effectively find a level n such that $\Phi_e^\sigma(x)\downarrow$ for all $\sigma \in T_{e+1}$ of length n. Then the function $h(x) = \max\{\ \Phi_e^\sigma(x)\ :\ \sigma \in T_{e+1} \quad \& \quad |\sigma| = n\ \}$ bounds g. □

Lemma 3.15. f *is low*$_2$*.*

Proof. Fix e. If Case 1 holds for T_{e+1} then Φ_e^f is not total. If Case 2 holds for T_{e+1} then Φ_e^f is total. In either case, we are forcing at the finite stage $e+1$ to decide whether or not $e \in \mathrm{Tot}^f = \{\ i : \Phi_i^f \text{ total }\}$. The construction is $\emptyset''$-computable. Hence, $\mathrm{Tot}^f \leq_{\mathrm{T}} \emptyset''$ so $f'' \leq_{\mathrm{T}} \emptyset''$. □

□

Note we cannot always produce a low_1 f in the previous theorem. We shall show in Theorem 5.13. that no noncomputable f with the computable bounding property (13) can be computable in $\emptyset'$.

Theorem 3.16. *Every nonempty special Π^0_1 class has a perfect subclass of computably dominated members.*

Proof. Let $\mathcal{C}$ be a nonempty special Π^0_1 class, and let $T \subseteq 2^{<\omega}$ be an infinite tree with $\mathcal{C} = [T]$. We define an infinite subtree S of T such that every $\sigma \in S$ is extendible and has a pair of incomparable extensions in S, and such that every $f \in [S]$ is computably dominated. We obtain S as $\bigcup_e S_e$ where $S_0 \subset S_1 \subset \cdots$ are finite subtrees of T constructed inductively as follows. Let $S_0 = \{\lambda\}$, and suppose S_e has been defined for some $e \geq 0$ and that the leaves of S_e all have the same length and are extendible. Fix a leaf σ of S_e, and define

$$U_{\sigma,e,x} = \{\ \tau \in T\ :\ \tau \preceq \sigma\ \vee\ \tau \succ \sigma\ \ \&\ \ \Phi^{\tau}_{e,|\tau|}(x)\!\uparrow\ \}.$$

Note that this is a computable subtree of T, and hence, if it is infinite, it cannot have any infinite computable path. We define extensions σ_0, σ_1 of σ as follows.

Case 1. If $U_{\sigma,e,x}$ is infinite for some x, fix the least such x, and let σ_0 and σ_1 be the first incompatible, extendible extensions of σ in $U_{\sigma,e,x}$.

Case 2. If not, let σ_0 and σ_1 be the first incompatible, extendible extensions of σ in T.

Note that in either case σ_0 and σ_1 must exist, since every extendible node in a computable tree with no infinite computable path must have two incompatible extendible extensions in that tree. We let S_{e+1} be the set of all $\tau \in T$ such that either $\tau \preceq \sigma_0$ or $\tau \preceq \sigma_1$ for some leaf σ of S_e.

It is readily seen that S is a perfect subtree of T. Suppose $g = \Phi^f_e$ for some $f \in [S]$. Let σ be the leaf of D_e that is an initial segment of f. Then σ_0 and σ_1 must have been found according to Case 2 in the definition of D_{e+1}, meaning $U_{\sigma,e,x}$ is finite. Thus, given x, find n such that $\Phi^{\tau}_e(x)\!\downarrow$ for all $\tau \succ \sigma$ in T of length n. Let $h(x) = \max\{\ \Phi^{\tau}_e(x)\ :\ \tau \in T\ \ \&\ \ \tau \succ \sigma\ \ \&\ \ |\tau| = n\ \}$. Then h is computable function bounding f. We conclude that $[\,S\,]$ is a perfect subclass of $\mathcal{C}$ all members of which are computably dominated. $\square$

Theorem 3.17 (Kučera and Nies)**.** *Let $\mathcal{P}$ be a nonempty Π^0_1 class and $B >_T \emptyset'$ a Σ^0_2 set. Then there is a computably dominated $f \in \mathcal{P}$ with $f' \leq_{\mathrm{T}} B$.*

For a sketch of the proof see [Nies, 1995, p. 61]. The idea is to fix a c.e. enumeration $\{B_s\}_{s\in\omega}$ of B relative to $\emptyset'$, to build a $\Delta_2^{0,B}$ enumeration $\{f_s\}_{s\in\omega}$, and to allow $f_s(x) \neq f_{s+1}(x)$ only if $B_s \upharpoonright x \neq B_{s+1} \upharpoonright x$.

Theorem 3.18. *The class of sets A which are computably dominated is Π_4^0.*

Proof. The following predicate with free set variable A holds iff A is c.d.

$$(\forall e)[\ \Phi_e^A \text{ total} \implies (\exists i)[\ \varphi_i \text{ total} \ \ \& \ \ (\forall x)[\ \Phi^A(x) < \varphi_i(x)\ ]\ ].$$

Note that "φ_i total" is Π_2^0 and the entire predicate is Π_4^0. □

3.6 Low Antibasis Theorem.

For the purposes of the following theorem, we will say that a set $S \subseteq 2^{<\omega}$ is *isomorphic to* $2^{<\omega}$ provided there is a bijection $g : 2^{<\omega} \to S$ such that for all $\sigma, \tau \in 2^{<\omega}$, $\sigma \preceq \tau$ if and only if $g(\sigma) \preceq g(\tau)$. Notice that if a tree T has a subset isomorphic to $2^{<\omega}$ via a computable such bijection, then $[T]$ has a member of every degree. Indeed, for every real X, we have $Y = \bigcup_x g(X \upharpoonright n) \in [\,T\,]$. Clearly, $Y \leq_T X$, while to compute $X(n)$ from Y for a given n we search for a $\sigma \in 2^{<\omega}$ until we find one of length greater than n with $g(\sigma) \subset Y$, and then $\sigma(n) = X(n)$.

Theorem 3.19 (Kent and Lewis [11])**.** *Every Π_1^0 class that has a member of every nonzero low degree has one of every degree.*

Proof. Fix a nonempty Π_1^0 class $\mathcal{C}$ not containing a member of every degree and let $T \subseteq 2^{<\omega}$ be a computable tree such that $\mathcal{C} = [\,T\,]$. We define a noncomputable low set A such that for all $e \in \omega$,

$$(15) \qquad \Phi_e^A = g \in 2^\omega \qquad \Longrightarrow \qquad [\ g \leq_{\mathrm{T}} \emptyset \quad \vee \quad g \notin [\,T\,]\ ].$$

In particular, $g \notin [\,T\,]$.

We obtain A as $\bigcup_s \sigma_s$ where $\sigma_0 \preceq \sigma_1 \preceq \cdots$ are constructed as follows. Let $\sigma_0 = \emptyset$ and suppose that for some $s \geq 0$, σ_s is given. If $s = 3e$ for some e, ask whether $\varphi_e(|\sigma_s|)$ converges, and define σ_{s+1} to be $\sigma_s(1-\varphi_e(|\sigma_s|))$ if it does and $\sigma_s{}^\frown 0$ otherwise. If $s = 3e+1$ for some e, ask whether there exists any $\sigma \succeq \sigma_s$ with $\Phi_e^\sigma(e)\!\downarrow$ and define σ_{s+1} to be the least such σ if there does and σ_s otherwise. In this way, we ensure that A is noncomputable and low.

Finally, suppose $s = 3e+2$ for some e. Search for a $\sigma \succeq \sigma_s$ such that one of the following cases occurs:

Case 1. For all $\tau \succeq \sigma$, $\Phi_e^\tau(x)$ diverges or is not $\{0,1\}$-valued for some $x < |\tau|$.

Case 2. $\Phi_e^\sigma(x)\!\downarrow$ for all $x < |\sigma|$ and string $\Phi_e^\sigma(0)\Phi_e^\sigma(1)\cdots\Phi_e^\sigma(|\sigma|-1)$ is not in T.

Case 3. σ has no extensions which e-split.

We claim the search must succeed. If not, we can define computable functions $f : 2^{<\omega} \to \{\rho \in 2^{<\omega} : \rho \succeq \tau\}$ and $g : 2^{<\omega} \to T$ as follows. By the failure of Case 1, there exists $\sigma \succeq \sigma_s$ with $\Phi_e^\sigma(x)\!\downarrow \in \{0,1\}$ for all $x < |\sigma|$, and we let $f(\emptyset) = \sigma$ for the least such σ. By the failure of Case 2, we know that the string $\Phi_e^\sigma(0)\Phi_e^\sigma(1)\cdots\Phi_e^\sigma(|\sigma|-1)$ belongs to T, and we let $g(\emptyset)$ be this string. We can thus assume by induction that $f(\sigma)$ has been defined for some σ, and that

$$g(\sigma) = \Phi_e^{f(\sigma)}(0)\Phi_e^{f(\sigma)}(1)\cdots\Phi_e^{f(\sigma)}(|f(\sigma)|-1)$$

belongs to T. By the failure of Case 3, there exist proper extensions τ_0 and τ_1 of $f(\sigma)$ such that $\Phi_e^{\tau_0}(x)\!\downarrow \neq \Phi_e^{\tau_1}(x)$ for some $x < \min\{|\tau_0|, |\tau_1|\}$, and using the failure of Case 1 we can assume that $\Phi_e^{\tau_i}(x)\!\downarrow$ for all $x < |\tau_i|$ and all $i < 2$. For $i < 2$, we let $f(\sigma^\frown i) = \tau_i$ and $g(\sigma^\frown i) = \Phi_e^{\tau_i}(0)\Phi_e^{\tau_i}(1)\cdots\Phi_e^{\tau_i}(|\tau_i|-1)$. Then g is a computable bijection from $2^{<\omega}$ to a subset of T isomorphic to $2^{<\omega}$. Therefore, $[\,T\,]$ has a member of every degree by our opening remarks. This proves our claim.

Now take the least σ satisfying one of Cases 1–3 above and let $\sigma_{s+1} = \sigma$. Under Case 1, this ensures that Φ_e^A is not total or $\{0,1\}$-valued, under Case 2 that Φ_e^A is not a member of $[\,T\,]$, and under Case 3 that Φ_e^A, if total, is computable. This completes the proof. □

3.7 Proper Low$_n$ Basis Theorem

The following generalization of the Low Basis Theorem says that, up to degree, the restriction of the jump operator to any special Π^0_1 class is surjective. The trick used for pushing the jump of the member up to the desired set is not unlike that used in the standard proof of the Friedberg Completeness Criterion.

Theorem 3.20 (Cenzer [1])**.** *For every set $A \geq_T \emptyset'$, every special Π^0_1 class has a member f satisfying $f \oplus \emptyset' \equiv_T f' \equiv_T A$.*

Proof. Fix a nonempty Π^0_1 class $\mathcal{C}$ and a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{C} = [T]$. We build a sequence of infinite computable trees $T = T_0 \supseteq T_1 \supseteq \cdots$ as follows. Let T_e be given. If e is even, define T_{e+1} from T_e as in the proof of the Low Basis Theorem. If e is odd, say $e = 2i+1$, note that T_e^{ext} must

be perfect since $\mathcal{C}$ is special, so $\emptyset'$ can find the smallest extendible nodes $\sigma, \tau \in T_e$ such that $\sigma(x) = 0$ and $\tau(x) = 1$ for some x. Let T_{e+1} consist of all the nodes in T_e comparable with σ or τ, depending as $A(i) = 0$ or $A(i) = 1$, respectively.

Take $f \in \bigcap_{e \in \omega} [T_e]$. If e is even, T_{e+1} can be obtained from T_e $\emptyset'$-effectively, and hence both $f \oplus \emptyset'$-effectively and A-effectively because $A \geq_T \emptyset'$. If e is odd, say $e = 2i + 1$, then to obtain T_{e+1} from T_e we need an oracle for $\emptyset'$ to find the extendible nodes σ and τ and the position x on which they disagree, and then an oracle for A since we need to know $A(i)$. But in this case, $i \in A$ iff $f(x) = 1$, so an oracle for f suffices to determine whether to let T_{e+1} consist of the nodes comparable with σ or the nodes comparable with τ. Since f' is decided during the construction, we consequently have that $f \oplus \emptyset' \leq_T f' \leq_T A \leq_T f \oplus \emptyset'$, as desired. □

The above theorem fails to fully generalize the Friedberg Completeness Criterion because the latter actually produces, for every $A \geq_T \emptyset'$, a 1-*generic* set G with $G \oplus \emptyset' \equiv_T G' \equiv_T A$ (i.e., a set G such that for every Σ^0_1 subset of $2^{<\omega}$, either some initial segment of G lies in the set, or no extension of some initial segment of G does). We cannot reproduce this extra property in the preceding theorem because there exist special Π^0_1 classes with no members of 1-generic degree (e.g., the Π^0_1 class all of whose members have degree $\gg \mathbf{0}$, since, by a result of Kučera [13] and others, no such degree can bound a 1-generic one).

Theorem 3.21. *For every $n \geq 0$, every special Π^0_1 class has a member that is low_{n+1} but not low_n.*

Proof. We proceed by induction. If $n = 0$, the result follows simply by the Low Basis Theorem and the fact that we are dealing with special Π^0_1 classes. Since the Low Basis Theorem easily relativizes, we thus assume that the desired result holds, along with all of its relativizations, for some $n \geq 0$. Fix an arbitrary set A and a nonempty $\Pi^{0,A}_1$ class $\mathcal{C}$. Let $\mathcal{D}$ be a nonempty $\Pi^{0,A'}_1$ class all of whose members have degree strictly above $\deg(A)'$. By the inductive hypothesis relative to A', $\mathcal{D}$ has an element B such that $B^{(n)} \not\leq_T (A')^{(n)} = A^{(n+1)}$ but $B^{(n+1)} \leq_T (A')^{(n+1)} = A^{(n+2)}$. Since $B >_T A'$, it follows by Theorem 3.20, relativized to A, that $\mathcal{C}$ has an element f satisfying $f' \equiv_T B$. Then $f^{(n+1)} \equiv_T B^{(n)} \not\leq_T A^{(n+1)}$ but $f^{(n+2)} \equiv_T B^{(n+1)} \leq_T A^{(n+2)}$, so f is low_{n+2} relative to A and not low_{n+1} relative to A, as desired. This completes the induction and the proof. □

Of course the condition "special" in the preceding two theorems is unavoidable since there exist nonempty Π^0_1 classes all of whose members are

computable.

One interesting topic we do not have time to cover here is the relation between the Cantor-Bendixson rank of a point f in a Π^0_1 class and its Turing degree. For example, any rank 0 (isolated) point must be computable. Cenzer, Clote, Smith, Soare, and Wainer extend this to finite and even computable ordinals. Other results on rank are discussed in Cenzer [1999].

4 Forcing with Π^0_1 classes

The method of constructing the path in the proof of the Low Basis Theorem is known as *forcing with* Π^0_1 *classes*, or sometimes also as *Jockusch-Soare forcing*. It is highly modular and can be used to obtain a wide array of results, and we will encounter it repeatedly in subsequent results about Π^0_1 classes. The purpose of this section is to outline this method, and to give several examples of how it is used.

4.1 Conditions, dense sets, and generics

Forcing in mathematical logic is a technique which traces its roots back to Paul Cohen's celebrated proof of the independence of the Continuum Hypothesis from Zermelo-Fraenkel Set Theory, and, in slightly different form, even further back to the proof by Kleene and Post of the existence of incomparable degrees below $\mathbf{0}'$. The basic idea is to decide, or "force", certain properties or requirements of an object we are building at a definite stage of our building it. Intuitively, we build the object using approximations called *conditions*, which we extend one by one in such a way as to preserve any information we have already decided or committed to. In the case of forcing with Π^0_1 classes, this takes the following form. It should be familiar to the reader who has seen forcing in other contexts.

Definition 4.1.

(i) A *condition* is an infinite computable subtree of $2^{<\omega}$. A condition $\widetilde{T}$ *extends* a condition T if $\widetilde{T} \subseteq T$.

(ii) A collection D of conditions is *dense* if every infinite computable tree T has an extension $\widetilde{T} \in D$.

(iii) For a degree $\mathbf{d}$, a collection D of conditions is $\mathbf{d}$-*effectively dense* if every infinite computable tree T has an extension $\widetilde{T} \in D$, and if an index for $\widetilde{T}$ can be found $\mathbf{d}$-effectively from an index for T.

(iv) Given a family of dense sets $\mathcal{D} = \{D_e : e \in \omega\}$, we call a real $f \in 2^\omega$ *$\mathcal{D}$-generic* if for all e, $f \in [\, T \,]$ for some $T \in D_e$.

The usefulness of forcing comes from the following theorem which says that generics always exist. In practice, this means we can obtain a generic possessing some property or properties simply by adjusting the family $\mathcal{D}$ we are working with.

Theorem 4.2 (Existence of generics for forcing with Π^0_1 classes)**.**

(i) For any family $\mathcal{D} = \{D_e : e \in \omega\}$ of dense sets, there exists a $\mathcal{D}$-generic real f.

(ii) If the D_e are $\mathbf{d}$-effectively dense for some $\mathbf{d}$, then there is a function p with $\deg(p) \leq \mathbf{d}$ such that for each e, $p(e)$ is an index for some $T \in D_e$ with $f \in [\, T \,]$.

Proof. To prove (i), let $T_0 = 2^{<\omega}$. By density of the D_e, we can obtain a chain

$$T_0 \supseteq T_1 \supseteq \cdots$$

of infinite computable trees such that $T_{e+1} \in D_e$ for all e. Since Cantor space is compact and $[\, T_0 \,], [\, T_1 \,], \ldots$ is a nested sequence of nonempty closed sets, $\bigcap_{e \in \omega} [\, T_e \,]$ must be nonempty. Clearly, any member of this intersection is $\mathcal{D}$-generic. To prove (ii), note that if the D_e is $\mathbf{d}$-effectively dense than an index for each T_e can be found $\mathbf{d}$-effectively. □

Let us translate the proof of the Low Basis Theorem into the language of families of dense sets and generic reals. Let $\{T_{t_0}, T_{t_1}, \ldots\}$ be an effective enumeration of all computable trees (finite and infinite) and such that t_i is an index for T_{t_i} for all i. For all $e, i \in \omega$ let $U_{e,t_i} = \{\sigma \in T_{t_i} : \Phi_e^\sigma(e) \uparrow\}$. Then, for each e define

$$D_e = \{\ U_{e,t_i} \ : \ i \in \omega \quad \& \quad |U_{e,t_i}| = \infty \ \} \ \cup \ \{\ T_{t_i} \ : \ i \in \omega \quad \& \quad |U_{e,t_i}| < \infty \ \}$$

and let $\mathcal{D} = \{D_e : e \in \omega\}$. It is not difficult to see that each D_e is $\mathbf{0}'$-effectively dense (notice that we can find an index for U_{e,t_i} computably from an index for T_{t_i}). The Low Basis Theorem is obtained by taking a $\mathcal{D}$-generic f and a function $p \leq_T \emptyset'$ according to Theorem 4.2(ii), and pointing out that $e \in f'$ if and only if $|U_{e,p(e)}| = \infty$ (which $\emptyset'$ can decide).

4.2 Forcing Modules

In the literature, authors customarily refer to dense sets only implicitly, preferring instead, as we did in our original proof of the Low Basis Theorem, to describe the strategy for obtaining T_{e+1} from T_e in the proof of Theorem 4.2 directly. We follow this convention below, but of course we could always translate any argument employing forcing with Π^0_1 classes into the (more formal) language of the previous subsection.

We can obtain a wide array of basis results by modifying the strategy for obtaining T_{e+1} from T_e in Theorem 4.2, or, in more complicated constructions, by varying it depending on e. We think of each such strategy as a *module* for forcing with Π^0_1 classes. As we will see, these modules can then be variously combined to produce different basis results. So as to have as much flexibility as possible in doing this, we describe each of these modules separately. We keep track of the effectiveness in each module, *i.e.*, of how much oracle strength was needed in its proof, so as to gauge how effective any basis result employing this module will be.

Let us illustrate the module concept by looking at a module we have already seen implemented, namely that used in the proof of the Low Basis Theorem. A quick examination of that proof reveals that it consists just of iterations of this module for all $e \in \omega$.

Lemma 4.3 (Lowness Module)**.** *Let T be an infinite computable tree and let $e \in \omega$. There exists an infinite computable subtree $\widetilde{T} \subseteq T$ such that either $\Phi^f_e(e)\!\downarrow$ for all $f \in [\widetilde{T}]$, and hence $e \in f'$; or else $\Phi^f_e(e)\!\uparrow$ for all $f \in [\widetilde{T}]$, and hence $e \notin f'$. Moreover, an index for $\widetilde{T}$ can be obtained $\emptyset'$-uniformly from e and an index for T.*

The utility of having the lowness module by itself is that we can intersperse it with others, and obtain paths that, in addition to satisfying other properties, are low. However, we must be careful that those other modules are also uniform in $\emptyset'$: mixing the lowness module together with, for example, one requiring a $\emptyset''$ oracle will produce a path f satisfying only $f' \leq_T \emptyset''$.

4.3 Examples of Modules

What follows are several examples of forcing modules, and details about how they can be combined to generate ever more sophisticated basis theorems for Π^0_1 classes.

4.3.1 Cone Avoidance Modules

Cone avoidance and Turing incomparability, like lowness, are measures of the complexity of a set. The modules we present next, and the basis theorems they yield, show that no nonempty Π^0_1 class exists whose every member must compute or be computable from a given noncomputable set. We begin with the following module for obtaining upper cone avoidance. The subsequent theorem is obtained by a straightforward iteration.

Lemma 4.4. *Let C be a noncomputable set, T an infinite computable tree, and $i \in \omega$. There exists an infinite computable subtree $\widetilde{T} \subseteq T$ such that $C \neq \Phi_i^f$ for any $f \in [\widetilde{T}]$. An index for $\widetilde{T}$ can be found $(\emptyset' \oplus C)$-effectively from i and an index for T.*

Proof. For each $n \in \omega$, define

$$U_n = \{\sigma \in T : \Phi_i^\sigma(n) \uparrow \vee \Phi_i^\sigma(n) \downarrow \neq C(n)\},$$

noting that each of these is a computable tree whose index as such can be found C-effectively from i and an index for T. We claim that some U_n must be infinite. If not, then for each n we could find a level m and value k such that $\Phi_i^\sigma(n) \downarrow = k$ for all σ of length m, whence it would have to be that $C(n) = k$, so C would be computable. We $\emptyset'$-computably search for the least n such that U_n is infinite, and let $\widetilde{T} = U_n$. Clearly, $\Phi_i^f C$ for all $f \in \widetilde{T}$, as desired. □

Theorem 4.5 (Upper Cone Avoidance Basis Theorem)**.** *Let C be a noncomputable set. Every nonempty Π^0_1 class has a member that does not compute C.*

Since every set computes every computable set, to get an analogous result for lower cone avoidance we must obviously insist that the Π^0_1 classes we deal with be special. The next module and subsequent theorem show that this is the only restriction needed.

Lemma 4.6. *Let C be any set, T an infinite computable tree with no computable paths, and $i \in \omega$. There exists an infinite computable subtree $\widetilde{T} \subseteq T$ such that $f \neq \Phi_i^C$ for any $f \in [\widetilde{T}]$. Moreover, an index for $\widetilde{T}$ can be found C'-effectively from i and an index for T.*

Proof. Since T has no computable paths, it must contain at least two incompatible extendible nodes. Call the least such nodes σ and τ, and say $n < \min\{|\sigma|, |\tau|\}$ is least such that $\sigma(n) \neq \tau(n)$. Ask C' whether $\Phi_i^C(n)$

converges. If so, then one of the two strings, say σ, must disagree with Φ_i^C on n, and we let $\widetilde{T}$ consist of all nodes in T compatible with σ. In this case, $f(n) = \sigma(n)$ for all $f \in [\widetilde{T}]$, so clearly $\Phi_i^C \neq f$ for all such f. Otherwise, we let $\widetilde{T} = T$, and in this case the result follows trivially. □

Theorem 4.7 (Lower Cone Avoidance Basis Theorem)**.** *Let C be any set. Every special Π^0_1 class has a member that is not computable by C.*

We now combine both the preceding modules, as well as the lowness module, into a single basis theorem. This is a standard "weaving argument", where we alternate (in this case, three) strategies depending on e (in this case, depending as e is congruent to 0, 1, or 2 modulo 3) . For completeness, we include the details.

Theorem 4.8 (Incomparability Basis Theorem)**.** *Let $C_0, C_1, \ldots$ be a sequence of noncomputable sets, and let $D = \oplus_{j\in\omega} C'_j$. Every special Π^0_1 class has a member f which is Turing incomparable with C and satisfies $f' \leq_T D$.*

Proof. Fix a nonempty Π^0_1 class $\mathcal{C}$ and a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{C} = [T]$. We build a sequence of infinite computable trees $T = T_0 \supseteq T_1 \supseteq \cdots$ as follows. Let T_e be given. If $e \equiv 0 \pmod 3$, apply the lowness module, Lemma 4.3, with T_e in place of T, and let T_{e+1} be the tree $\widetilde{T}$ obtained there. If $e = 3\langle i, j\rangle + 1$ for some i, j, apply Lemma 4.4 with T_e in place of T and C_j in place of C, and let T_{e+1} be the tree $\widetilde{T}$ obtained there. And if $e = 3\langle i, j\rangle + 2$, apply Lemma 4.6 in a similar fashion. Since $\emptyset' \leq_T D$ and $C'_j \leq_T D$ for all j, in either case, D suffices to find an index for T_{e+1} from an index for T_e. The proof is concluded by taking $f \in \bigcap_{e\in\omega} [T_e]$. The stages congruent to 0 modulo 3 ensure that $f' \leq_T D$, the stages congruent to 1 modulo 3 that $C \not\leq_T f$, and those congruent to 2 modulo 3 that $f \not\leq_T C$. □

The following now easily follow.

Corollary 4.9. *Let $L_0, L_1, \ldots$ be noncomputable low sets such that a lowness index for L_i can be found $\emptyset'$-effectively from i. Every special Π^0_1 class has a member that is low and Turing incomparable with each L_i.*

Corollary 4.10 (Jockusch and Soare, 1972b)**.** *Every special Π^0_1 class has countably many low members that are mutually incomparable.*

4.3.2 Minimal Pair Module

Recall that a pair of degrees $\mathbf{a}$ and $\mathbf{b}$ is said to form a *minimal pair* if $\mathbf{a} \cap \mathbf{b} = 0$. (We do not, as some authors do, insist that $\mathbf{a}$ and $\mathbf{b}$ must also

be nonzero.) The following module prepares for proving the Minimal Pair Basis Theorem.

Lemma 4.11. *Let C be any set, T an infinite computable tree, and $i, j \in \omega$. There exists an infinite computable subtree $\widetilde{T} \subseteq T$ such that if $\Phi_i^f = \Phi_j^C = A$ for some $f \in [\widetilde{T}]$ and some set A, then A is computable. Moreover, an index for $\widetilde{T}$ can be found $(\emptyset'' \oplus C')$-effectively from i, j, and an index for T.*

Proof. We begin by asking whether there exist extendible nodes $\sigma, \tau \in T$ and an $x \in \omega$ such that $\Phi_i^\sigma(x) \downarrow \neq \Phi_i^\tau(x) \downarrow$. This can be done using a $\emptyset''$ oracle, since the question of whether a given node of a computable tree is extendible is Π^0_1. If not, then whenever Φ_i^f is total for some $f \in [T]$ it must be computable, since to figure out the value of $\Phi_i^f(x)$ we have only to find some $\sigma \in T$, such as a sufficiently long initial segment of f, with $\Phi_i^\sigma(x) \downarrow$. In this case, then, we can let $\widetilde{T} = T$. So suppose some such σ and τ exist, and fix the least such. Next, use C' to determine whether $\Phi_j^C(x)$ converges and is $\{0,1\}$-valued. If not, then the conclusion of the lemma holds trivially, and so we can again just let $\widetilde{T} = T$. Otherwise, one of the two computations, say $\Phi_i^\sigma(x)$, must differ from $\Phi_j^C(x)$, and we let $\widetilde{T}$ consist of all nodes of T compatible with σ. In this case, $\Phi_i^f \neq \Phi_j^C$ for any $f \in [\widetilde{T}]$. □

Iterating in the standard way yields the following:

Theorem 4.12 (Minimal Pair Basis Theorem)**.** *Let C be any set. Every Π^0_1 class has a member f such that* $\deg(f)$ *and* $\deg(C)$ *form a minimal pair.*

Proof. Fix a nonempty Π^0_1 class $\mathcal{C}$ and a computable tree $T \subseteq 2^{<\omega}$ such that $\mathcal{C} = [T]$. We build a sequence of infinite computable trees $T = T_0 \supseteq T_1 \supseteq \cdots$ as follows. Let T_e be given, and suppose $e = \langle i, j \rangle$ for some $i, j \in \omega$. Apply the preceding Lemma to the tree T_e, and let T_{e+1} be the tree $\widetilde{T}$ obtained there. Then take $f \in \bigcap_{e \in \omega} [T_e]$. The definition of T_{e+1} from T_e ensures that Φ_i^f and Φ_j^C are only the characteristic function of a set if that set is computable. Hence, $\deg(f)$ and $\deg(C)$ form a minimal pair, as desired. □

By interspersing an additional module into the preceding construction, namely the low$_2$ness module used to control the double jump of f in the proof of the Computably Bounded Basis Theorem 3.13, we obtain the following stronger theorem. Note that that module can be carried out using a $\emptyset''$ oracle.

Theorem 4.13. *Let C be any set. Every Π^0_1 class has a member f such that $f'' \leq_T \emptyset'' \oplus C'$ and* $\deg(f)$ *and* $\deg(C)$ *form a minimal pair.*

The following corollaries are then immediate. The first of these was used by Dzhafarov and Jockusch [6, Theorem 6.2] to show that every computable coloring of pairs has a pair of infinite homogeneous sets whose degrees form a minimal pair.

Corollary 4.14. *Let C be any set with $C' \leq_T \emptyset''$. Every Π^0_1 class has a low$_2$ member f such that* $\deg(f)$ *and* $\deg(C)$ *form a minimal pair.*

Corollary 4.15. *Every Π^0_1 class has a pair of low$_2$ members whose degrees form a minimal pair.*

Of course, we cannot expect the Minimal Pair Basis Theorem to hold in general with a low f, or even a low$_n$ f for any n, since we could always choose the noncomputable set C to be $\emptyset^n$ and thereby get $f \leq_T C$. But more is true: we cannot get a low f even when C is itself low. In Section YY, we will see the existence of a Π^0_1 class all of whose members have diagonally noncomputable (DNC) degree. If we let C be any low member of this Π^0_1 class, then by Kučera's theorem (cf. [**?**], Theorem 2), no Δ^0_2 (let alone low) member of this class can have degree forming a minimal pair with $\deg(C)$. (Note that this also shows that Corollary 4.15 cannot be improved from a minimal pair of low$_2$ members to a minimal pair of low ones.)

We should point out another well-known connection between minimal pairs and Π^0_1 classes. This is not a basis theorem, and is not proved by the methods discussed in this section, but it provides an interesting connection to Corollary 4.15.

Theorem 4.16 (Jockusch and Soare, 1971)**.** *There exist nonempty special Π^0_1 classes $\mathcal{C}_0$ and $\mathcal{C}_1$ such that for all $f \in [\mathcal{C}_0]$ and $g \in [\mathcal{C}_1]$,* $\deg(f)$ *and* $\deg(g)$ *form a minimal pair.*

The proof, which we omit, is based on that of the classical result, due independently to Lachlan and Yates, that there exists a minimal pair of c.e. degrees.

5 Computably Dominated Sets and Degrees

5.1 Computably Bounded Functions and Trees

We can extend the notions of Π^0_1 class and tree by replacing Cantor space 2^ω by Baire space ω^ω and the tree $2^{<\omega}$ by $\omega^{<\omega}$. Theorem 2.4 on effectively closed classes (Π^0_1 classes) still holds but the effective compactness Theorem 2.9 fails. However, we can still obtain the former results if we restrict

to functions and Π^0_1 classes which are *computably bounded*. (The definition of a function $g(x)$ being computably bounded was given in Definition 3.12 above.)

Definition 5.1.

(i) A tree $T \subseteq \omega^{<\omega}$ is *computably bounded (c.b.)* if there is a computable function h such that $(\forall g \in [\, T\,])\,(\forall x)[\; g(x) \;\leq\; h(x)\;]$.

(ii) A Π^0_1 class $\mathcal{C}$ is *computably bounded (c.b.)* if there is a computably bounded computable tree $T \subseteq \omega^{<\omega}$ such that $\mathcal{C} = [\, T\,]$.

Obviously, all Π^0_1 subclasses of 2^ω are computably bounded, their members being bounded by the constant function $h(x) = 1$. But in fact, these $h(x) = 1$ bounded classes suffice for the purposes of studying the computable content of members of computably bounded Π^0_1 classes in general, as the following proposition shows. The next propositions are easy exercises.

Proposition 5.2. *If $T \subseteq \omega^{<\omega}$ is a computably bounded, computable tree then there is a computable tree $S \subseteq 2^{<\omega}$ and a computable map $h : T \to S$ such that h induces a homeomorphism $\widehat{h} : [\, T\,] \to [\, S\,]$ which preserves Turing degree.*

Proposition 5.3. *If $T \subseteq \omega^{<\omega}$ is computable and finite branching (but not necessarily computably bounded) then it is bounded by a function $h \leq_{\mathrm{T}} 0'$.*

Therefore, all results about c.b. computable trees hold for finite branching computable trees if the assertions are relativized to $0'$.

5.2 Post's Hyperimmune Sets

Post's Problem [1944] was the famous problem of finding an incomplete but noncomputable c.e. set W. To accomplish this he defined various notions of thinness on the complement $\overline{W}$. These are described in detail in Soare [CTA] and [1987]. We now describe them only very briefly.

Post called an infinite set A *hyperimmune (h-immune)* if there is no strong array $\{D_{f(x)}\}_{x\in\omega}$ of disjoint finite sets presented by a computable function f and such that the index x effectively specifies the members of finite set D_x. This definition in terms of strong arrays was then related to computably bounding properties.

Theorem 5.4 (Kuznecov, Medvedev, Uspenskii)**.** *An infinite set A is hyperimmune iff its principal function p_A (of Definition 3.12) is not bounded by any computable function.*

Proof. See Soare [1987, Theorem V.2.3]. □

Many authors and we also in the paper will take Theorem 5.4 as a *definition* of hyperimmune (h-immune). A coinfinite c.e. set W is *hypersimple (h-simple)* if $\overline{W}$ is h-immune.

Corollary 5.5. *A coinfinite c.e. set B is h-simple iff $\overline{B}$ is not bounded by any computable function.*

Dekker [1954] connected bounding functions to the computation time of c.e. sets. He proved that for every noncomputable c.e. set B there is a hypersimple set A such that $A \leq_{\rm tt} B \leq_{\rm T} A$. Let f be any one-one computable function with range B. Dekker defined the *deficiency set* to be

$$A = \{\ s : (\exists t > s)\ [\ f(t) < f(s)\ ]\ \}. \tag{16}$$

The rest of the proof is given in Soare [1987, Theorem V.2.5]. The key point is that if a computable function h bounds $\overline{A}$ then A is computable.

Corollary 5.6. *For every noncomputable c.e. set B there is an h-simple set $A \equiv_{\rm T} B$. Hence, every nonzero c.e. degree contains an h-simple set.* □

5.3 Hyperimmune Degrees

The previous work by Post, Dekker, and Martin and Miller [1968] extended these definitions from a single *set* to an entire *degree* by defining a degree to be hyperimmune if it contains a hyperimmune set.

Definition 5.7 (Miller and Martin, 1968)**.**

(i) A degree $\mathbf{d}$ is *hyperimmune (h-immune)* if it contains a hyperimmune set.

(ii) If degree $\mathbf{d}$ contains no hyperimmune set it is called *hyperimmune-free (hi-free)* (since it is *free* of hyperimmune sets). It is also called *computably dominated* according to Definition 3.12 because by Theorem 5.4 every set $B \in \mathbf{d}$ is dominated by a computable function. The second term conveys more intuition about the meaning.

Corollary 5.6 shows that every nonzero c.e. degree is h-immune. The next few results will demonstrate that every nonzero degree comparable with $\mathbf{0}'$ is h-immune, showing that the computably dominated degrees are scarce. However, by Theorem 3.13 at least one nonzero computably dominated degree exists.

5.4 Two Downward Closure Properties of Domination

Recall Definition 3.12 (vi) on computably dominated functions and sets. The key point is that for a function f or set A to be *computably bounded* simply requires a single bounding function, while being *computably dominated* imposes computable bounding on *every* function g in the same degree, a very strong condition. Remarkably, it turns out not to matter whether we insist on having this condition for all $g \leq_{\mathrm{T}} f$ or merely for all $g \equiv_{\mathrm{T}} f$.

Theorem 5.8 (Miller and Martin, 1968)**.** *Suppose A is computably dominated.*

(i) If $B \leq_{\mathrm{T}} A$, then B is computably dominated.

(ii) If $g \leq_{\mathrm{T}} A$, then g is computably dominated.

Proof. (i) Let $B = \Phi_e^A$. Define $g(x)$ as follows. Let $g(0) = 0$. For every $x \in \omega$ define $g(2x+1) = g(2x) + p_B(x) + 1$ and $g(2x+2) = g(2x+1) + p_A(x) + 1$. Now g is strictly increasing and therefore is the principal function of some set $C \equiv_{\mathrm{T}} A$. Therefore, some computable function h dominates $g = p_C$. But then $h(2x+1)$ dominates $p_B(x)$.

(ii) Let $f = \Phi_e^A$. Define $g(0) = 0$. For every $x \in \omega$ define $g(x+1) = g(x) + f(x) + 1$. Now g is strictly increasing and therefore is the principal function of some set $C \equiv_{\mathrm{T}} f$. Therefore, some computable function h dominates $g = p_C$. But then $h(x+1)$ dominates $f(x)$. □

Corollary 5.9. *The hyperimmune degrees are closed upwards and the computably dominated degrees are closed downwards.*

Proof. By Theorem 5.8. □

Theorem 5.10. *A set A is computably dominated iff for every f,*

$$f \leq_{\mathrm{T}} A \quad \Longrightarrow \quad f \leq_{\mathrm{tt}} A. \tag{17}$$

Proof. ($\Longrightarrow$). Suppose that A is computably dominated and $f = \Phi_e^A$. Define $g(x) = (\mu s)\Phi_{e,s}^A(x)$ converges and $g(x) = 0$ if $\Phi_{e,s}^A(x)$ diverges. Now $g \leq_{\mathrm{T}} A$. Therefore, by Theorem 5.8 there exists some computable function h which bounds g. Define a Turing functional $\Psi^X(x)$ which, on any input x and oracle X, runs Φ^X on input x for $h(x)$ many steps, and outputs $\Phi_e^X(x)$ if the latter converges and 0 otherwise. Then Ψ^X is total for every X and $\Psi^A = \Phi_e^A$.

($\Longleftarrow$). Assume (17). Let $f \leq_{\mathrm{T}} A$. Then $f \leq_{\mathrm{tt}} A$. It suffices to prove that f is computably dominated. Fix a total reduction Φ_e such that $f = \Phi_e^A$. Define a computable function h as follows. Given x, we search for a level n such that $\Phi_e^\sigma(x)\downarrow$ for all σ of length n. Such a level must necessarily exist, as otherwise $\{\sigma : \Phi_e^\sigma(x)\uparrow\}$ would be an infinite tree, and $\Phi_e^g(x)$ would not converge for any path g through it. Let $h(x) = \max\{\Phi_e^\sigma(x) : |\sigma| = n\}$. Clearly, h dominates f. □

5.5 Δ_2^0 Degrees are Hyperimmune

Every nonzero c.e. degree is hyperimmune because it contains a hypersimple set. We now prove that nonzero degrees $\mathbf{d} < \mathbf{0}'$ are also hyperimmune. We also explore the Σ_2^0 and other degrees with respect to computable domination.

Definition 5.11. Let A be a Δ_2^0 set and let $\{A_s\}_{s\in\omega}$ be a computable sequence such that $A = \lim_s A_s$. The *computation function* is

$$c_A(x) = (\mu s > x)\,[\, A_s \upharpoonright x = A \upharpoonright x \,], \tag{18}$$

where $A \upharpoonright x$ denotes the restriction of A to elements $y \leq x$.

Theorem 5.12. *Let A be Δ_2^0 and $\{A_s\}_{s\in\omega}$ a Δ_2^0 approximation to A with computation function $c_A(x)$.*

(i) $c_A \equiv_{\mathrm{T}} A$.

(ii) If $g(x)$ dominates $c_A(x)$ then $A \leq_{\mathrm{T}} g$. Therefore, A is computable iff a computable function g dominates $c_A(x)$.

Proof. (i) $A \leq_{\mathrm{T}} c_A$ because $A(x) = A_s(x)$ for $s = c_A(x)$. Also $c_A(x) \leq_{\mathrm{T}} A$ because we generate $A_s(x)$ until the first s with $A \upharpoonright x = A_s \upharpoonright x$.

(ii) If A is computable then c_A is computable by (i). Conversely, assume $c_A(x) < g(x)$ for all x. Define

$$y \;=\; (\mu z > x)\,(\forall t)_{z\leq t\leq g(z)}\,[\, A_t \upharpoonright x \;=\; A_z \upharpoonright x \,]. \tag{19}$$

By the definition of $c_A(x)$ and the fact that $c_A(x) < g(x)$ we know that for all $z \geq x$ the interval $[\,z,\, g(z)\,]$ (called a *frame*) must contain at least *one* stage t which is *z-true* in the sense that $A_t \upharpoonright z = A \upharpoonright z$. But $A = \lim_s A_s$ implies that $A_s \upharpoonright x = A \upharpoonright x$ for almost all s. Therefore, almost all z-frames must contain only stages t which are x-true. This proves that $A(x) = A_y(x)$ because all values for x in the y-frame agree by (19) and one must agree with $A(x)$ because $y > x$ and $c_A(y) \geq c_A(x)$. □

Corollary 5.13 (Miller and Martin, 1968)**.** *If $\emptyset <_{\mathrm{T}} A \leq_{\mathrm{T}} \emptyset'$ then deg(A) is hyperimmune.*

Proof. Let $\emptyset <_{\mathrm{T}} A \leq_{\mathrm{T}} \emptyset'$. Hence, $A \in \Delta^0_2$. By Theorem 5.12 no computable function can dominate $c_A \equiv_{\mathrm{T}} A$. Therefore, by Theorem 5.8 A cannot have computably dominated degree. □

Corollary 5.14. *If a degree* $\mathbf{d}$ *is comparable with* $\mathbf{0}'$ *then* $\mathbf{d}$ *is hyperimmune (not computably dominated).*

Proof. By Corollary 5.13 all degrees $\mathbf{d} \leq \mathbf{0}'$ are hyperimmune. By the upward closure of hyperimmune degrees in Theorem 5.8 all degrees $\mathbf{d} > \mathbf{0}'$ are also hyperimmune. □

The next result generalizes Corollary 5.13 and show that most degrees obtained by iterating the jump are hyperimmune.

Corollary 5.15 (Miller and Martin, 1968)**.** *If $B <_{\mathrm{T}} A \leq_{\mathrm{T}} B'$ then deg(A) is hyperimmune.*

Proof. The set A is $\Delta^{0,B}_2$ and there is a B-computable sequence $\{A_s\}_{s\in\omega}$ such that $A = \lim_s A_s$ by Definition 5.11 relativized to B. Define the computation function c_A as there. If any computable (or even B-computable) function h dominates c_A then $A \leq_{\mathrm{T}} B$, contrary to hypothesis. □

Theorem 5.16. *If A is computably dominated, then $A'' \leq_{\mathrm{T}} A' \oplus \emptyset''$. (In particular, since $A' \oplus \emptyset'' \leq_T (A \oplus \emptyset')'$, A is GL_2.)*

Note that it is easy to prove the weaker fact that A is GL_2. Indeed, by Martin's High Domination Theorem, there exists $f \leq_T \emptyset'$ which dominates every computable function. Since A is computably dominated, every A-computable function is dominated by a computable function, and hence by f. Thus, by Martin's High Domination Theorem relative to A, $A \oplus \emptyset'$ is high relative to A, meaning $A'' \leq_T (A \oplus \emptyset')'$.

Proof. We prove that $\mathrm{Tot}^A \leq_T A' \oplus \emptyset''$. Given $e \in \omega$, we can computably find an index e_0 of the partial A-computable function

$$g(x) = (\mu s)[\, \Phi^A_{e,s}(x) \downarrow \,],$$

noting that this function has the same domain as Φ^A_e and so, in particular, is total if and only if Φ^A_e is. We then search for the least $\langle i, n\rangle$ such that $i \in \mathrm{Tot}$ and either

$$(\exists x < n)(\forall s)[\, \Phi^A_{e_0}(x) \uparrow \,] \tag{20}$$

or

$$(\forall x \geq n)(\forall s)[\ \Phi^A_{e_0,s}(x) \uparrow \quad \& \quad \Phi^A_{e_0}(x) \leq \varphi_i(x)\ ], \tag{21}$$

which we can do using an oracle for $A' \oplus \emptyset''$. Furthermore, the search necessarily terminates. Indeed, if $\Phi^A_{e_0}$ is not total, then (20) holds for any n such that $\Phi^A_{e_0}(x) \uparrow$ for some $x < n$. And if $\Phi^A_{e_0}$ is total then it is dominated by some computable function φ_i by virtue of A being computably dominated, and so (21) holds for all sufficiently large n. By choice of e_0 and $\langle i, n\rangle$, we have that Φ^A_e is total if and only if

$$(\forall x < n)[\ \Phi^A_e(x) \downarrow\ ] \quad \& \quad (\forall x \geq n)(\exists s \leq \varphi_i(x))[\ \Phi^A_{e,s}(x) \downarrow\ ],$$

which can be determined using A'. This completes the proof. □

5.6 Degrees of Σ^0_2 Sets

There are other classes of sets below $\mathbf{0}''$ such as the Σ^0_2 sets. We now show that these include additional sets of hyperimmune degree.

Definition 5.17.

(i) A computable sequence $\{A_s\}_{s\in\omega}$ is a Σ^0_2 *approximation* to a Σ^0_2 set A if

$$x \in A \iff (\forall^\infty s)[\ x\ \in\ A_s\ ]. \tag{22}$$

(ii) For such a Σ^0_2 sequence define the Σ^0_2 *estimation function*

$$E_A(x) = (\mu s \geq x)(\forall z \leq x)[\ z \in \overline{A} \implies (\exists t)_{x\leq t\leq s}[z \notin A_t\ ]]. \tag{23}$$

(This estimation function plays the same role as the computation function played for Δ^0_2 sets.)

Theorem 5.18. *Let A be a Σ^0_2 set and let $\{A_s\}_{s\in\omega}$ be a Σ^0_2 approximation to A with $E_A(x)$ the Σ^0_2 estimation function. If a computable function dominates $E_A(x)$ then A is computably enumerable.*

Proof. Let A be Σ^0_2. Now assume that $(\forall x)[g(x) \geq E_A(x)]$. From (22) and (23) we know,

$$(\forall x)\,[\ x \in \overline{A} \implies (\forall y > x)(\exists t)_{y\leq t\leq g(y)}\ [\ x \notin A_t\ ]\ ], \tag{24}$$

$$(\forall x)\,[\ x \in A \iff (\exists y > x)(\forall t)_{y\leq t\leq g(y)}\ [\ x \in A_t\ ]\ ]. \tag{25}$$

Therefore, A is Σ^0_1 in g. If g is computable, then A is Σ^0_1 and hence c.e. □

Corollary 5.19. *If A is Σ^0_2 and noncomputable then deg(A) is hyperimmune.*

The results here that various degrees cannot be computably dominated are based on the fact that in the Δ^0_2 and Σ^0_2 cases, we have an approximation to A and the fact that a computable function dominating the computation function shows that A is computable or c.e. In contrast, the Computably Bounded Basis Theorem 3.13 produces a computably bounded set $A \leq_{\mathrm{T}} \emptyset''$ (in fact, uncountably many). Hence, A is Δ^0_3 but it cannot be Σ^0_2 or comparable with $\emptyset'$. Therefore, we can find computably dominated degrees but they are not abundant. Cooper [2004, p. 271] considers a slightly different computation function for a Σ^0_2 approximating sequence but his computation function for a Δ^0_2 approximating sequence is the same as here and in other papers.

6 Peano Arithmetic and Π^0_1 Classes

6.1 Logical Background

One of the earliest purposes of computability theory was the study of logical systems and theories. We consider theories in a computable language: one which is countable, and the function, relation, and constant symbols and their arities are effectively given. We also assume that languages come equipped with an effective coding for formulas and sentences in the language, i.e. a *Gödel numbering*, and identify sets of formulas with the corresponding set of Gödel numbers. We can then speak of the Turing degree of a theory in a computable language. Here we will examine the language $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ of arithmetic, and theories extending PA, the theory of Peano arithmetic.

Definition 6.1. Let $\mathcal{D}_{\mathrm{PA}}$ be the set of (Turing) degrees of complete consistent extensions of Peano arithmetic; such a degree is called a *PA degree*.

The following is surely the best known theorem in mathematical logic:

Theorem 6.2 (Gödel, 1931; Rosser 1936)**.**

(i) The theory of Peano arithmetic is incomplete.

(ii) Furthermore, any consistent computably axiomatizable extension of PA is also incomplete.[6]

[6]There are weaker hypotheses which suffice for the incompleteness theorem, but this version of the theorem is all that is needed here.

Corollary 6.3. $\mathbf{0} \notin \mathcal{D}_{PA}$.

Thus, there is no complete consistent extension of PA which is computable. However, there are many ways to extend PA to a complete theory, and there is a very nice way of describing them. We identify a completion of Peano Arithmetic with the set of Gödel numbers of its sentences.

6.2 Π^0_1 classes and completions of theories

Theorem 6.4. *There exists a Π^0_1 class whose members are precisely the completions of Peano Arithmetic. Thus $\mathcal{D}_{PA}$ is the degree spectrum of a Π^0_1 class.*

Proof. (Sketch). Fix a bijective Gödel numbering $G : \omega \to \text{Sent}_{\mathcal{L}}$ for sentences of arithmetic. Given $\sigma \in 2^{<\omega}$, we identify σ with the sentence

$$\theta(\sigma) = \bigwedge_{\sigma(i)=1} G(i) \quad \& \quad \bigwedge_{\sigma(j)=0} \neg G(j).$$

We say that a sentence θ "appears to be consistent at stage t" if there is no derivation of $\neg\theta$ from the first t axioms of PA in fewer than t lines. Since there are finitely many such derivations, the relation $R(\sigma, t) =$ "$\theta(\sigma)$ appears to be consistent at stage t" is computable. Thus the class

$$\mathcal{C} \quad = \quad \{\, f \in 2^\omega : (\forall n)(\forall t < n) R(f \upharpoonright t, n) \,\}$$

is a Π^0_1 class. Some f is an element of this class if and only if the corresponding set of sentences $G(\{n : f(n) = 1\})$ is a complete consistent extension of PA. □

Remark 6.5. This theorem follows from an analysis of Lindenbaum's lemma. Note that no special properties of PA were used, beyond the fact that it is a computably axiomatizable theory in a computable language. Thus the same theorem applies to all such theories.

We defined a PA degree as a degree of a completion of Peano Arithmetic. From this definition, it may be surprising that the class of degrees is closed upwards. This is true, however, and to demonstrate it we will need an important fact arising from Gödel's incompleteness theorem: the proof actually constructs a "Gödel sentence" which is independent of the axioms.

Theorem 6.6 (Gödel's incompleteness theorem, effective version)**.**

From a description of a consistent, computably axiomatizable theory T extending PA, we can effectively find a sentence, called the Gödel sentence of *T, which is independent from T.*

6.3 Equivalent properties of PA degrees

The PA degrees arise naturally in a variety of contexts, especially those relating to trees and weak König's lemma. This is because the PA degrees are exactly those degrees which can carry out weak König's lemma by finding paths through trees. For this reason, there are several equivalent properties which all serve to define the PA degrees; we will highlight a few of these properties.

Definition 6.7. A function $f : \omega \to \omega$ is *diagonally noncomputable (d.n.c.)* if, for all e, if $\varphi_e(e)\downarrow$, then $f(e) \neq \varphi_e(e)$.

Definition 6.8. A function is *n-valued* if $f(e) < n$ for each $e \in \omega$.

The name "diagonally noncomputable" derives from the particular way that d.n.c. functions are noncomputable. We see that if f is d.n.c., f cannot be computable, because then f would be φ_e for some e, but f and φ_e differ on argument e; thus d.n.c. functions diagonalize against all the list of all (partial) computable functions. We will be primarily interested in 2-valued d.n.c. functions.

Theorem 6.9 (Scott, 1962; Jockusch and Soare, 1972b; Solovay, unpublished[7])**.** *For a Turing degree* $\mathbf{d}$*, the following are equivalent:*

(i) $\mathbf{d}$ *is the degree of a complete consistent extension of Peano arithmetic.*

(ii) $\mathbf{d}$ *computes a complete consistent extension of Peano arithmetic.*

(iii) $\mathbf{d}$ *computes a 2-valued d.n.c. function.*

(iv) Every partial computable 2-valued function has a total $\mathbf{d}$*-computable extension.*

(v) Every nonempty Π^0_1 *class has a member of degree at most* $\mathbf{d}$*.*

(vi) Every computably inseparable pair has a separating set of degree at most $\mathbf{d}$*.*

[7] Scott [1962] proved the equivalence of conditions (i) and (v). Jockusch and Soare [1972b] proved the equivalence of conditions (ii) and (vi); the equivalence with (iii) and (iv) is also implicit in their work. Jockusch and Soare left the equivalence of (i) and (ii) as an open question, which was answered by Solovay (unpublished).

Proof. (i) $\implies$ (ii). This implication is trivial.

(ii) $\implies$ (iii). Let $\mathbf{d}$ compute a complete consistent extension T of PA, and let f be the (partial computable) diagonal function $f(e) = \varphi_e(e)$. By results of Gödel and Kleene, there is a formula ψ representing f, in the sense that

$$f(x)\downarrow = y \quad \iff \quad PA \vdash \psi(x, y), \text{ and}$$

$$f(x)\downarrow \neq y \quad \iff \quad PA \vdash \neg\psi(x, y).$$

Since $PA \vdash \psi(x, y)$ implies that $\psi(x, y) \in T$, and T is complete and $\mathbf{d}$-computable, the function

$$\widehat{f}(e) = \begin{cases} 1 & \psi(e, 0) \in T \\ 0 & \neg\psi(e, 0) \in T \end{cases}$$

is a $\mathbf{d}$-computable 2-valued d.n.c. function.

(iii) $\implies$ (iv). Suppose g is a 2-valued d.n.c. function, and let f be a partial computable 2-valued function. There is a computable function $\widehat{f}$ such that $f(x) = \varphi_{\widehat{f}(x)}(\widehat{f}(x))$ for all x. Then $1-(g\circ\widehat{f})$ is a total $\mathbf{d}$-computable 2-valued function extending f.

(iv) $\implies$ (v). Let $\mathcal{P}$ be a nonempty Π^0_1 class, and T a computable tree with $\mathcal{P} = [\,T\,]$. Fix a computable bijection $h : \omega \to 2^{<\omega}$. Let f be the function

$$f(e) = \begin{cases} 0 & \begin{array}{l} h(e) \in T \text{ and there is a level } l \text{ such that } h(e)^\frown 0 \\ \text{has a descendent at level } l \text{ in } T, \text{ but } h(e)^\frown 1 \text{ does not} \end{array} \\ 1 & \begin{array}{l} h(e) \in T \text{ and there is a level } l \text{ such that } h(e)^\frown 1 \\ \text{has a descendent at level } l \text{ in } T, \text{ but } h(e)^\frown 0 \text{ does not.} \end{array} \end{cases}$$

This function f is partial computable, since to compute $f(e)$ one simply searches for a level l such that one case or the other holds. If $h(e) \in T$ is extendible, then either both $h(e)^\frown 0$ and $h(e)^\frown 1$ are extendible, in which case $f(e)\uparrow$, or only one is, so $f(e)\downarrow$, and $h(e)^\frown f(e)$ is extendible. Let $\widehat{f}$ be a 2-valued $\mathbf{d}$-computable extension of f. Then using $\widehat{f}$, we can find an element of $[\,T\,]$ as follows: starting with any string $\sigma \in T^{\text{ext}}$, apply $\widehat{f} \circ h^{-1}$ to get either 0 or 1, which we can append to σ to get a longer string still in T^{ext}. Starting with the empty string, we can iterate this process to get an infinite $\mathbf{d}$-computable path through $[T]$, i.e. an element of $\mathcal{P}$.

(v) $\implies$ (vi). If A, B is a computably inseparable pair, the class of separating sets is a Π^0_1 class by Theorem 3.4. If property 4 holds, this has a $\mathbf{d}$-computable member.

(vi) $\implies$ (i). Fix some order of $\mathcal{L}$-sentences, and some order for generating proofs. Let A be the set of pairs (F, ψ), where F is a finite set of $\mathcal{L}$-sentences and ψ is an $\mathcal{L}$-sentence, such that a proof of a contradiction is found from $\mathrm{PA} \cup F \cup \{\psi\}$ before (if ever) finding a proof of a contradiction from $\mathrm{PA} \cup F \cup \{\neg\psi\}$. Similarly, let B be the set of pairs (F, ψ), such that a proof of contradiction is found from $\mathrm{PA} \cup F \cup \{\neg\psi\}$ before (if ever) finding one from $\mathrm{PA} \cup F \cup \{\psi\}$. Clearly A and B are disjoint c.e. sets. Suppose the pair A, B has a $\mathbf{d}$-computable separating set C. Let $D \in \mathbf{d}$. We will construct a completion T of PA, of degree $\mathbf{d}$, in stages, along with a bijective function $g : \omega \to \mathrm{Sent}_{\mathcal{L}}$, also defined in stages. At stage n we will determine $g(n)$, and decide whether $g(n) \in T$. Define the set of sentences,

$$F_n \quad = \quad (T \cap g[0 \dots n-1]) \ \cup \ \{\neg\psi : \psi \in g[0 \dots n-1] \ \setminus T\}.$$

In other words, F_n keeps track of every sentence we decided by the beginning of stage n. It contains those sentences we have declared to be in T, together with the negations of those sentences we have declared not to be in T. At stage n, do the following:

1. If n is even, let $g(n)$ be the Gödel sentence of $\mathrm{PA} \cup F_n$. If n is odd, let $g(n)$ be the first $\mathcal{L}$-sentence not yet in the range of g.

2. If $n = 2s$ is even, consider whether s is an element of D. If $s \in D$, then $g(n) \in T$; otherwise, $g(n) \notin T$.

3. If n is odd, consider the pair $(F_n, g(n))$. If this pair is in C, then $g(n) \notin T$; otherwise, $g(n) \in T$.

We will show that T is a complete consistent extension of PA, of degree $\mathbf{d}$. Assume (for the sake of induction) that F_n is consistent with PA. (Since $F_0 = \emptyset$, it is consistent with PA.) Note that F_{n+1} is either $F_n \cup \{g(n)\}$ or else $F_n \cup \{\neg g(n)\}$. Since F_n is consistent with PA, at least one of $F_n \cup \{g(n)\}$ and $F_n \cup \{\neg g(n)\}$ must be consistent with PA. Furthermore, if n is even, both are consistent since $g(n)$ is the Gödel sentence for $\mathrm{PA} \cup F_n$. If both are consistent with PA, then clearly F_{n+1} is as well. Suppose instead only one of the two is consistent (so we know n is odd). If only $F_n \cup \{g(n)\}$ is consistent with PA, then a proof of contradiction will be found from $PA \cup F_n \cup \{\neg g(n)\}$ before finding one from $PA \cup F_n \cup \{g(n)\}$, so $(F_n, g(n)) \in B$. Thus $(F_n, g(n)) \notin C$; by the construction, $g(n) \in T$, and F_{n+1} is consistent with PA. Similarly, if only $F_n \wedge \neg g(n)$ is consistent with PA, then the construction goes the opposite way and again F_{n+1} is consistent with PA. By induction, F_n is consistent with PA for all n, so $T = \bigcup_n F_n$ is consistent with PA. Since F_n decides $g(0) \dots g(n-1)$, T is complete. Therefore, T is a complete consistent extension of PA.

In order to show that T has degree $\mathbf{d}$, we first show that $g \leq_T T$. To see this, note that $g(n)$ is either the first $\mathcal{L}$-sentence which is not one of $g(0) \dots g(n-1)$, if n is odd, or else $g(n)$ is the Gödel sentence of $\mathrm{PA} \cup F_n$, where F_n is determined entirely by T and the values $g(0) \dots g(n-1)$. Thus $g(n)$ can be computed from n, $g(0) \dots g(n-1)$, and T, so $g \leq_T T$. From the construction, we see that $s \in D$ if and only if $g(2s) \in T$, so we have $D \leq_T g \oplus T \leq_T T$. However, the entire construction was $\mathbf{d}$-computable, so $T \in \mathbf{d}$. □

7 Applications to randomness

In this section, we explore some of the interactions between, on the one hand, Π^0_1 classes and computably dominated degrees, and, on the other hand, the study of algorithmic randomness and complexity

7.1 Martin-Löf Randomness

Let μ be Lebesgue measure on Cantor space, which we assume the reader to be familiar with. For completeness, we define the measure of an open class $U \subseteq 2^\omega$. Let $V \subset 2^{<\omega}$ be any set with $U = [\![V]\!]$ which is prefix-free (i.e., if $\sigma \in V$ and $\tau \prec \sigma$ then $\tau \notin V$). Such a V can be seen to exist for example as follows. Since U is open, its complement is a $\mathbf{\Pi^0_1}$ class and hence is equal to $[T]$ for some (not necessarily computable) tree $T \subseteq 2^{<\omega}$. Then V can be taken to consist of all elements of $\overline{T}$ whose predecessors all belong to T. Now the measure of U is defined as

$$\mu(U) = \sum_{\sigma \in V} 2^{-|\sigma|}.$$

Lebesgue measure on Cantor space has all the same properties we are familiar with from Lebesgue measure on the real line. In fact, another definition of Lebesgue measure on Cantor space is that the measure of a class $U \subseteq 2^\omega$ is the same as that of the subset $\{r(f) : f \in U\}$ of the closed unit interval $[0, 1]$, where $r(f)$ for $f \in 2^\omega$ is the the real number with binary expansion $0.f(0)f(1)f(2)\cdots$. Recall that a sequence of c.e. sets $V_0, V_1, \dots$ is *uniformly c.e.* (abbreviated u.c.e.) if there exists a computable function f such that $V_n = W_{f(n)}$ for all n.

Definition 7.1.

(i) A sequence $S_0, S_1, \dots$ of subclasses of 2^ω is *uniformly* Σ^0_1 if there exists a u.c.e. sequence $V_0, V_1, \dots$ of subsets of $2^{<\omega}$ such that $S_n = [\![V_n]\!]$ for all n.

(ii) A *Martin-Löf test* is a uniformly Σ^0_1 sequence $S_0, S_1, \ldots$ of subclasses of 2^ω such that $\mu(S_n) \leq 2^{-n}$ for all n.

(iii) A set $X \in 2^\omega$ *passes* a Martin-Löf test $S_0, S_1, \ldots$ if $X \notin \bigcap_{n\in\omega} S_n$.

(iv) A set $X \in 2^\omega$ is *Martin-Löf random* or *1-random* if it passes every Martin-Löf test.

7.2 A Π^0_1 Class of 1-Randoms

A Martin-Löf test $U_0, U_1, \ldots$ is called *universal* if $\bigcap_{n\in\omega} U_n \supseteq \bigcap_{n\in\omega} S_n$ for every other Martin-Löf test $S_0, S_1, \ldots$. Thus, if X passes a universal test, it must pass every test, and hence

$$\bigcap_{n\in\omega} U_n \;=\; \{\; X \in 2^\omega \;:\; X \text{ is not 1-random } \}.$$

The following proposition, whose proof we omit, is thus useful when trying to show that a given set is not 1-random.

Proposition 7.2. *There exists a universal Martin-Löf test.*

Notice that this implies that the class of 1-randoms has measure 1. Indeed, each member of a universal Martin-Löf test $U_0, U_1, \ldots$ is an open set covering $\{X \in 2^\omega \;:\; X \text{ is not 1-random}\}$, implying that

$$\mu(\{X \in 2^\omega : X \text{ is not 1-random}\}) \;\leq\; \mu(U_n) \;\leq\; 2^{-n}$$

for all n. Essentially the same argument, in reverse, yields the following:

Corollary 7.3. *There is a nonempty Π^0_1 class all of whose elements are 1-random.*

Proof. Let $U_0, U_1, \ldots$ be a universal Martin-Löf test. For every $n > 0$, U_n is a proper Σ^0_1 subclass of 2^ω, implying that $\overline{U_n}$ is a nonempty Π^0_1 class. By definition of a universal Martin-Löf test,

$$\overline{U}_n \subseteq \bigcup_{n\in\omega} \overline{U_n} = \overline{\bigcap_{n\in\omega} U_n} = \{X \in 2^\omega : X \text{ is 1-random}\},$$

as desired. □

Thus we can easily obtain the existence of low 1-random sets, of hyperimmune-free 1-random sets, etc.

We mention briefly that Corollary 7.3 does not hold if 1-randomness is replaced by n-randomness for $n > 1$ (a set being *n-random* if it is 1-random relative to $\emptyset^{(n-1)}$). That is, for $n > 1$, there is no nonempty Π^0_1 class all of whose members are n-random.[8] One (easy to see) reason for this is that no n-random set for $n > 1$ can be Δ^0_2, or even contain an infinite Δ^0_2 subset.

7.3 Π^0_1 Classes and Measure

Given the measure-theoretic definition of 1-randomness, it is natural to ask about the measure of Π^0_1 classes containing 1-randoms. The following theorem gives a full answer to this question.

Theorem 7.4. *Let $\mathcal{C}$ be a Π^0_1 class.*

(i) If $\mu(\mathcal{C}) = 0$, then $\mathcal{C}$ contains no 1-random sets.

(ii) If $\mu(\mathcal{C}) > 0$, then every 1-random set computes a member of $\mathcal{C}$.

Proof. (i). Suppose $\mathcal{C}$ has measure 0. Let $T \subseteq 2^{<\omega}$ be a tree such that $\mathcal{C} = [T]$, and for each $n \in \omega$, let $S_n = [\![\{\sigma \in T : |\sigma| = n\}]\!]$. Then $S_0, S_1, \ldots$ is a nested sequence of open classes whose intersection is the measure 0 class $\mathcal{C}$, so it must be that $\mu(S_n) \to 0$. As the sequence (S_n) is given by a strong array of finite sets of strings, the map $n \mapsto \mu(S_n) \in \mathbb{Q}$ is computable, so we can find a computable function p such that $\mu(S_{p(n)}) \leq 2^{-n}$ for all n. Now since $S_0, S_1, \ldots$ is uniformly Σ^0_1, $S_{p(0)}, S_{p(1)}, \ldots$ is a Martin-Löf test. But for all $f \in \mathcal{C}$, $f \in \bigcap_{n\in\omega} S_{p(n)}$, so f is not 1-random.

(ii). Suppose $\mathcal{C}$ has positive measure and let X be a 1-random set. Let V_0 be a prefix-free c.e. subset of $2^{<\omega}$ such that $\overline{\mathcal{C}} = [\![V_0]\!]$. For each $n \in \omega$, let $V_{n+1} = [\![\{\sigma\tau : \sigma \in V_n \ \ \& \ \ \tau \in V_0\}]\!]$, and let $S_n = [\![V_n]\!]$. Notice that for all n, V_n is prefix-free since V_0 is, so we have

$$\mu(S_{n+1}) = \sum_{\sigma\in V_{n+1}} 2^{-|\sigma|} = \sum_{\sigma\in V_n}\sum_{\tau\in V_0} 2^{-|\sigma\tau|} = \sum_{\sigma\in V_n} 2^{-|\sigma|} \sum_{\tau\in V_0} 2^{-|\tau|} = \mu(S_n)\mu(S_0).$$

It follows that $\mu(S_n) = \mu(S_0)^{n+1} = \mu(\overline{\mathcal{C}})^{n+1}$, and hence that $\mu(S_n) \to 0$ since $\mu(\overline{\mathcal{C}}) = 1 - \mu(\mathcal{C}) < 1$. Since $S_0, S_1, \ldots$ is uniformly Σ^0_1, and the measures $\mu(S_n)$ converge to zero faster than the (computable) function $p(n) = q^n$,

[8] This should not be confused with saying that Corollary 7.3 does not relativize, which it does: for each $n > 1$, there is a $\Pi^{0,0^{n-1}}_1$ class all of whose members are n-random.

where $q > \mu(S_0)$ is rational, there is some subsequence of the sequence (S_n) which is a Martin-Löf test. Since X is 1-random, it is not in the intersection of this test, so $X \notin S_n$ for some least n. If $n = 0$, then $X \notin \overline{\mathcal{C}}$ and hence $X \in \mathcal{C}$. If $n > 0$, since $X \in S_{n-1}$, we can choose $\sigma \in V_{n-1}$ such that $\sigma \prec X$. Since no $\tau \in V_0$ can satisfy $\sigma\tau \prec X$, it follows that $Y = \{x - |\sigma| : x \in X \ \ \& \ \ x \geq |\sigma|\} \notin S_0$ as $X = \sigma Y$. Thus, $Y \in \mathcal{C}$, which, since $Y \equiv_T X$, completes the proof. □

We saw in Section 6 that the PA degrees are precisely those which, for every nonempty Π^0_1 class, bound the degree of a member of that class. Part (ii) of the preceding theorem can be seen as saying that the degrees of 1-random sets are precisely the analogs of PA degrees with respect to Π^0_1 classes of positive measure. This is a surprising fact because, in most other settings, the PA degrees and degrees of 1-random sets behave very differently. The following result, whose proof is not difficult but would nonetheless take us too far astray, is an example of this phenomenon.

Theorem 7.5. *If a set X is both 1-random and of PA degree, then $X \geq_T \emptyset'$.*

7.4 Randomness and Computable Domination

We conclude by looking at applications of some of the ideas from Section 5 to two other notions studied in the area of algorithmic randomness. We begin with the following.

Definition 7.6. A set X is *c.e. traceable* if there is a computable function p such that, for each $f \leq_T X$, there is a computable function h with $|W_{h(n)}| \leq p(n)$ and $f(n) \in W_{h(n)}$ for all n. If this holds with $D_{h(n)}$ in place of $W_{h(n)}$, then X is called *computably traceable*.

Clearly, every computably traceable set is c.e. traceable, and it can be shown, though we do not do so here, that this implication is strict (see Downey and Hirschfeldt [ta]). On the other hand, the following theorem shows that the implication reverse if we restrict to sets of computably dominated degree.

Theorem 7.7 (Kjos-Hanssen, Nies, and Stephan, 2005)**.** *If X is a set of computably dominated degree, then X is c.e. traceable if and only if it is computably traceable.*

Proof. Let X be a c.e. traceable set of computably dominated degree, and let p be as in Definition 7.8 (ii). Given $f \leq_T X$, let h_0 be a computable

function with $|W_{h_0(n)}| \leq p(n)$ and $f(n) \in W_{h_0(n)}$ for all n. Define a function g by

$$g(n) = (\mu s)[\ f(n) \in W_{h_0(n),s}\ ],$$

so that g is total and X-computable. By Theorem 5.8 (ii), there exists a computable function h_1 with $h_1(n) \geq g(n)$ for all n. Then if we define h by letting $h(n)$ be the canonical index of the finite set $W_{h_0(n),h_1(n)}$, we have

$$|D_{h(n)}| = |W_{h_0(n),h_1(n)}| \leq |W_{h_0(n)}| \leq p(n)$$

and $f(n) \in W_{h_0(n),h_1(n)} = D_{h(n)}$. Hence, X is computably traceable. □

We obtain a similar result by looking at the following notion of randomness due to Kurtz. In view of Theorem 7.4 (i), it is implied by 1-randomness, and, as above, it can be shown that this implication is strict.

Definition 7.8. A set X is *Kurtz random* or *weakly 1-random* if it is contained in every Σ^0_1 class of measure 1.

Theorem 7.9 (Nies, Stephan, and Terwijn, 2005)**.** *If X is a set of computably dominated degree, then X is 1-random if and only if it is weakly 1-random.*

Proof. Let X be a set of computably dominated degree which is not 1-random. Let $S_0, S_1, \ldots$ be a Martin-Löf test which X does not pass, and let f be a computable function such that $S_n = [\![W_f(n)]\!]$ for all n. Define a function g by

$$g(n) = (\mu s)(\exists \sigma \prec X)[\ \sigma \in W_{f(e),s}\ ],$$

noting that since $X \in [\![W_{f(n)}]\!]$ for all n, g is total and X-computable. By Theorem 5.8 (ii), there exists a computable function h with $h(n) \geq g(n)$ for all n. Then if we let

$$\mathcal{C} = \bigcap_{n \in \omega} W_{f(n),h(n)},$$

we have that $\mathcal{C}$ is a Π^0_1 class with $X \in \mathcal{C}$ and

$$\mu(\mathcal{C}) \leq \mu([\![W_{f(n),h(n)}]\!]) \leq \mu(S_n) = 2^{-n}$$

for all n. Hence, $\overline{\mathcal{C}}$ is a Σ^0_1 class of measure 1 not containing X, so X is not weakly 1-random. □

In fact, it follows by a result of Kurtz (unpublished; see Downey and Hirschfeldt [ta]) that every hyperimmune degree contains a set which is weakly 1-random but not 1-random (or even *Schnorr random*, which is a much weaker notion of randomness). Thus, the degrees separating these randomness notions are *precisely* the hyperimmune degrees.

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