

De Finetti Coherence and Logical Consistency

James M. Dickey, Morris L. Eaton, and William D. Sudderth

*Eaton and Sudderth dedicate this paper to the
memory of their friend and colleague, Jim Dickey.*

Abstract The logical consistency of a collection of assertions about events can be viewed as a special case of coherent probability assessments in the sense of de Finetti.

1 Introduction

Suppose that every event A in a collection \mathcal{D} is assigned a truth value $v(A) = 1$ if A is considered to be certain and $v(A) = 0$ if A is considered to be impossible. These assignments are called “consistent” if no contradiction follows from them, in the sense of having different truth values for equivalent events. (Precise definitions are given below.)

An assignment of probability numbers to the events in \mathcal{D} is said to be “coherent” if, when they are used to set prices, or odds, for bets, a gambler cannot construct a sure win by combining bets at such odds.

If the truth values assigned to the events in \mathcal{D} are viewed as probabilities, then they are consistent if and only if they are coherent. This is a rough statement of the main result of this note.

The result is not surprising. Indeed, de Finetti [2] viewed his theory as “The Logic of the Probable.” However, the precise relationship of coherence to consistency has not been given heretofore—as far as we know.

Various theories of “probabilistic logic” have a history extending back almost as far as the history of probability theory itself. An exposition of this history is given in the book by Hailperin [5].

In Sections 2 and 3 we give careful definitions of the notions of consistency and coherence and present a few of their properties. The main result is given in Section 4.

2 Consistency

Let Ω be the nonempty set of all outcomes of some experiment, and let \mathcal{D} be a nonempty collection of subsets of Ω . By a *truth function* we mean a mapping v from \mathcal{D} to $\{0, 1\}$. For $A \in \mathcal{D}$, we interpret $v(A) = 1$ (respectively, $v(A) = 0$) to mean that the event A is considered certain to occur (respectively, certain not to occur).

If either A or B is certain to occur, then so is $A \cup B$. But, if A and B are both certain not to occur, the same holds for $A \cup B$. Thus we set

$$v(A \cup B) = \max\{v(A), v(B)\}. \quad (2.1)$$

For similar reasons, we set

$$v(A \cap B) = \min\{v(A), v(B)\}, \quad (2.2)$$

$$v(A^c) = 1 - v(A), \quad (2.3)$$

where A^c is the complement of A .

The rules (2.1)–(2.3) are just the expressions in our framework of standard rules of deductive inference in sentential logic. For example, (2.1) corresponds to the rule that the disjunction of two statements is true if and only if either of the statements is true. (See, for example, Section 1.1 of Lightstone [9] or Section 1.3 of Enderton [4].)

The rules (2.1)–(2.3) can be used to extend the truth function v to additional sets. By a slight abuse of notation, we write v for this extension of v as well as for the original function on \mathcal{D} . Furthermore, we allow for the rules to be used repeatedly so that, for example, v can be extended to the collection \mathcal{C} of all finite intersections of elements of \mathcal{D} and their complements. A typical set B in \mathcal{C} has the form

$$B = A_1^{i_1} \cap A_2^{i_2} \cap \cdots \cap A_n^{i_n}, \quad (2.4)$$

where n is a positive integer, $A_1, A_2, \dots, A_n \in \mathcal{D}$, the superscripts i_1, i_2, \dots, i_n are each 0 or 1, and we set

$$A^1 = A, \quad A^0 = A^c$$

for $A \in \mathcal{D}$. Clearly, v can be extended to such a B by repeated use of (2.2) and (2.3).

Let \mathcal{A} be the collection of all finite unions of elements of \mathcal{C} . Then v can be further extended to each element of \mathcal{A} by repeated use of (2.1).

Lemma 2.1 *The collection \mathcal{A} is an algebra of sets; that is, \mathcal{A} is closed under finite unions, finite intersections, and the taking of complements. Also, \mathcal{A} is the smallest algebra containing \mathcal{D} .*

This elementary lemma is doubtless well known. Since we failed to find a reference, a proof is given in the [Appendix](#).

As explained above, we can always extend v from its original domain \mathcal{D} to the algebra \mathcal{A} using (2.1)–(2.3). However, the values of v on \mathcal{A} need not be uniquely determined. This nonuniqueness can occur because an event in \mathcal{A} may have more than one representation as a finite union of elements of \mathcal{C} . Here is a simple example.

Example 2.2 Let $\Omega = \{a, b, c, d\}$; $\mathcal{D} = \{A_1, A_2, A_3\}$ where $A_1 = \{a, b\}$, $A_2 = \{b, c\}$, $A_3 = \{b, d\}$. Suppose $v(A_1) = v(A_2) = 1$, $v(A_3) = 0$. Then the event $B = \{b\}$ can be written $B = A_1 \cap A_2$. So one extension of v has $v(B) = 1$ by (2.2). But also $B = A_1 \cap A_3$, and, by (2.2), we can also have $v(B) = 0$.

Whenever, as in the example, there is an event $B \in \mathcal{A}$ to which v can be extended to have either value 0 or 1, then we can also have $v(B) = 1$ and $v(B^c) = 1$. This means that we can derive from the truth values assumed for sets in \mathcal{D} that B is certain to occur and we can also derive that B^c is certain to occur.

Definition 2.3 The truth function v on \mathcal{D} is *consistent* if there is no A in \mathcal{A} such that both $v(A) = 1$ and $v(A^c) = 1$ can be derived using the rules of inference (2.1)–(2.3).

Our definition corresponds to the classical notion that a set of axioms is inconsistent if, from them, one can derive both some proposition and its negation. (See, for example, Section 6.5 of Lightstone [9].)

Lemma 2.4 *The truth function v on \mathcal{D} is consistent if and only if there is a unique extension of v to the algebra \mathcal{A} using (2.1)–(2.3).*

Proof If v is inconsistent, then we can derive both $v(A) = 1$ and $v(A^c) = 1$. But, by (2.3), we can also derive $v(A) = 1 - v(A^c) = 0$. So the extension is not unique.

Suppose now that v is consistent. If the extension of v to \mathcal{A} is not unique, then there exists $A \in \mathcal{A}$ for which both $v(A) = 1$ and $v(A) = 0$ can be deduced. But then, we can also derive $v(A) = 1$ and $v(A^c) = 1 - v(A) = 1$, contradicting consistency. \square

Often the rules of inference are defined purely syntactically in terms of a formal language, and then models are made to fit the rules. We have chosen to work directly with the model of subsets of Ω , or *events*, in accordance with de Finetti's formulation of coherence presented in the next section.

3 Coherence

Let \mathcal{D} be a collection of subsets of Ω as in the previous section. A *price function* p is a mapping from \mathcal{D} to the unit interval $[0, 1]$. The interpretation is that, for each event $A \in \mathcal{D}$, a bookie assigns the price $p(A)$ in dollars to a ticket worth one dollar if A occurs and worth nothing otherwise. Thus the net payoff to a gambler who purchases such a ticket is, for the outcome $\omega \in \Omega$,

$$A(\omega) - p(A),$$

where $A(\omega) = 1$ or 0 accordingly as $\omega \in A$ or $\omega \in A^c$. To encourage “fair” prices, the bookie is required to be willing to buy as well as sell the tickets at the same prices. Thus a gambler can also have

$$-[A(\omega) - p(A)]$$

as payoff. We also require that the bookie be willing to buy or sell arbitrary quantities of a given ticket. The resulting payoffs have the form

$$a \cdot [A(\omega) - p(A)]$$

where a is a real number corresponding to the quantity of tickets purchased by a gambler. (Negative values of a correspond to sales.) The total payoff to a gambler who buys a_i tickets on $A_i \in \mathcal{D}$ for $i = 1, 2, \dots, n$ is

$$f(\omega) = \sum_{i=1}^n a_i \cdot [A_i(\omega) - p(A_i)] \quad (3.1)$$

when the outcome is ω .

Definition 3.1 The price function p (or the bookie) is *coherent* if there is no function f of the form (3.1) such that $f(\omega) > 0$ for all $\omega \in \Omega$.

Thus a bookie is coherent if there is no sure win for a gambler. Note that if $f(\omega) > 0$ for all $\omega \in \Omega$, then

$$\inf\{f(\omega) : \omega \in \Omega\} > 0. \quad (3.2)$$

This is because a payoff function f as in (3.1) has only finitely many values.

Theorem 3.2 (de Finetti) *The price function p is coherent if and only if there is a finitely additive probability measure μ on the algebra \mathcal{A} such that μ agrees with p on \mathcal{D} .*

Proof First suppose that μ is a finitely additive probability measure on \mathcal{A} that agrees with p on \mathcal{D} . Then the expectation $E_\mu f$ of a payoff f as in (3.1) is

$$E_\mu f = \sum_{i=1}^n a_i \cdot [E_\mu(A_i) - p(A_i)] = \sum_{i=1}^n a_i \cdot 0 = 0.$$

So it cannot be the case that f has a positive infimum as in (3.2). Hence, p is coherent.

Next assume that p is coherent. Let \mathcal{L} be the linear space of all payoff functions f of the form (3.1). Then there exists a finitely additive probability measure μ on \mathcal{A} such that the expectation $E_\mu f \leq 0$ for all $f \in \mathcal{L}$. (The existence of μ follows from Lemma 1 of Heath and Sudderth [6]. The proof of the lemma uses a separating hyperplane argument together with the fact that a positive linear functional on the space of bounded real-valued function on Ω can be represented as an integral with respect to a finitely additive measure.) But $-f \in \mathcal{L}$. So $E_\mu f = -E_\mu(-f) \geq 0$. Hence, $E_\mu f = 0$ for all f . In particular,

$$E_\mu[A - p(A)] = E_\mu(A) - p(A) = 0$$

for all $A \in \mathcal{D}$. So $\mu(A) = p(A)$ for $A \in \mathcal{D}$. □

Suppose that we allow the gambler to place bets on countably many sets A_1, A_2, \dots rather than restricting the gambler to a finite number of bets. We can then define a strengthened notion of coherence in an obvious way. It is also possible to prove versions of Theorem 3.2 in which the measure μ becomes countably additive. (See pages 21–23 of Skyrms [10] and Theorem 6 of Heath and Sudderth [7] for examples.) However, de Finetti regarded the proofs of countable additivity in such theorems to be circular. (See his discussion on pages 91–92 in de Finetti [3].) The circularity arises if we evaluate the payoff from countably many bets as the sum of an infinite series, which is equivalent to an integral with respect to the countably additive measure that assigns mass one to each positive integer. If instead we evaluate the bets

using a purely finitely additive measure on the integers, the argument for countable additivity breaks down.

It is possible to formulate a theory of coherence or probabilism without using the metaphor of bets and gambling. For example, in the statistical literature, there is Buehler's [1] theory of "Coherent preferences" and, in the philosophical literature, there is Joyce's [8] "A nonpragmatic vindication of probabilism."

4 Consistency and Coherence

Let $v : \mathcal{D} \mapsto \{0, 1\}$ be a truth function. Of course, v can also be regarded as a price function for a (very confident) bookie.

Theorem 4.1 *The function v is consistent if and only if it is coherent.*

Proof First suppose that v is consistent. Let v also denote the unique extension of v to \mathcal{A} guaranteed by Lemma 2.2. By Theorem 3.1, it suffices to show that v is a finitely additive probability measure on \mathcal{A} .

Let $A \in \mathcal{A}$. Then either $v(A) = 1$ and $v(A^c) = 0$ or $v(A) = 0$ and $v(A^c) = 1$. (By consistency, $v(A)$ and $v(A^c)$ cannot have the same value.) In either case, $v(\Omega) = v(A \cup A^c) = 1$ by (2.1). Next assume that A and B are disjoint members of \mathcal{A} . We must check that

$$v(A \cup B) = v(A) + v(B). \quad (4.1)$$

Consider three cases.

Case (i) $v(A) = v(B) = 0$.

In this case, $v(A \cup B) = 0$ by (2.1) and (4.1) holds.

Case (ii) $v(A) = 1, v(B) = 0$ (or the reverse)

Again (4.1) follows from (2.1).

Case (iii) $v(A) = v(B) = 1$.

We will show that this is impossible. From $v(A) = 1$, we have, by (2.3), that $v(A^c) = 0$. But $B \subseteq A^c$. So, by (2.2), $v(B) = v(B \cap A^c) = 0$, contradicting (iii).

Now suppose that v is coherent. Then, by Theorem 3.2, there is a finitely additive probability measure μ on \mathcal{A} such that μ agrees with v on \mathcal{D} . Notice that μ satisfies the rules (2.1)–(2.3). For example, $\mu(A \cup B) = 1$ if either $\mu(A) = 1$ or $\mu(B) = 1$ and $\mu(A \cup B) = 0$ if $\mu(A) = \mu(B) = 0$. Likewise (2.2) and (2.3) hold for μ . Consequently, μ must agree with v on \mathcal{A} . Since μ is a probability measure, it is not possible that both $\mu(A)$ and $\mu(A^c)$ are equal to 1. Hence, v is consistent. \square

The unique extension of a consistent truth function from \mathcal{D} to the algebra \mathcal{A} using (2.1)–(2.3) results in a finitely additive probability measure taking only the values 0 and 1. Thus a consequence of Theorems 3.2 and 4.1 is that a truth function v on \mathcal{D} is consistent if and only if it extends to be a finitely additive $\{0, 1\}$ -valued probability measure on \mathcal{A} .

In Section 1, we interpreted the truth assignment $v(A) = 0$ as meaning the event A is certain not to occur. In the language of probability, it would be said that A is *almost certain* not to occur. As a referee pointed out, an event of probability 0 can easily occur. (Indeed, if the set Ω of possible outcomes is infinite and each individual outcome has probability 0, then some one of them must occur.) This change in interpretation does not affect the truth of Theorem 4.1.

Appendix A Proof of Lemma 2.1

Let \mathcal{B} be the smallest algebra of subsets of Ω containing \mathcal{D} . Since \mathcal{B} is closed under finite unions, finite intersections, and complementation, it is clear that $\mathcal{A} \subseteq \mathcal{B}$. By construction, $\mathcal{D} \subseteq \mathcal{A}$. So we need only check that \mathcal{A} is an algebra.

Plainly, \mathcal{A} is closed under finite unions. It is also clear that the collection \mathcal{C} is closed under finite intersections. To see that \mathcal{A} is closed under finite intersections, let

$$E = B_1 \cup B_2 \cup \cdots \cup B_m, \quad E' = B_1' \cup B_2' \cup \cdots \cup B_n'$$

be two elements of \mathcal{A} where the sets B_i and B_j' belong to \mathcal{C} . Then

$$E \cap E' = \bigcup_{i,j} (B_i \cap B_j')$$

is a finite union of sets in \mathcal{C} and hence belongs to \mathcal{A} .

It remains to be shown that \mathcal{A} is closed under the taking of complements. Since

$$(B_1 \cup B_2 \cup \cdots \cup B_m)^c = B_1^c \cap B_2^c \cap \cdots \cap B_m^c,$$

it is enough to show that $B \in \mathcal{C}$ implies $B^c \in \mathcal{A}$. So let B be as in (2.4). Observe that B is one of the sets in the partition of Ω consisting of all sets of the form

$$A_1^{j_1} \cap A_2^{j_2} \cap \cdots \cap A_n^{j_n}$$

where $j_k = 0$ or $j_k = 1$ for all $k = 1, 2, \dots, n$. Thus B^c is the union of all the sets in this partition different from B , and, in particular, is a finite union of elements of \mathcal{C} .

The proof of Lemma 2.1 is now complete.

References

- [1] Buehler, R. J., "Coherent preferences," *The Annals of Statistics*, vol. 4 (1976), pp. 1051–64. [Zbl 0347.60002](#). [MR 0423618](#). [137](#)
- [2] de Finetti, B., "La prévision : ses lois logiques, ses sources subjectives," *Annales de l'Institut Henri Poincaré*, vol. 7 (1937), pp. 1–68. English translation in *Studies in Subjective Probability*, edited by H. Kyburg and H. Smokler, Krieger Publishing, Huntington, 1980. [Zbl 0017.07602](#). [MR 1508036](#). [133](#)
- [3] de Finetti, B., *Probability, Induction and Statistics. The Art of Guessing*, Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York, 1972. [Zbl 0275.60001](#). [MR 0440638](#). [136](#)
- [4] Enderton, H. B., *A Mathematical Introduction to Logic*, Academic Press, New York, 1972. [Zbl 0298.02002](#). [MR 0337470](#). [134](#)
- [5] Hailperin, T., *Sentential Probability Logic. Origins, Development, Current Status, and Technical Applications*, Associated University Presses, Cranbury, 1996. [Zbl 0922.03026](#). [MR 1437603](#). [133](#)
- [6] Heath, D., and W. Sudderth, "On finitely additive priors, coherence, and extended admissibility," *The Annals of Statistics*, vol. 6 (1978), pp. 333–45. [Zbl 0385.62005](#). [MR 0464450](#). [136](#)
- [7] Heath, D. C., and W. D. Sudderth, "On a theorem of de Finetti, oddsmaking, and game theory," *Annals of Mathematical Statistics*, vol. 43 (1972), pp. 2072–77. [Zbl 0256.90068](#). [MR 0351472](#). [136](#)

- [8] Joyce, J. M., “A nonpragmatic vindication of probabilism,” *Philosophy of Science*, vol. 65 (1998), pp. 575–603. [MR 1668874](#). [137](#)
- [9] Lightstone, A. H., *Mathematical Logic. An Introduction to Model Theory*, edited by H. B. Enderton, vol. 9 of *Mathematical Concepts and Methods in Science and Engineering*, Plenum Press, New York, 1978. [Zbl 0382.03002](#). [MR 497355](#). [134](#), [135](#)
- [10] Skyrms, B., *Pragmatics and Empiricism*, Yale University Press, New Haven, 1984. [136](#)

Acknowledgments

We are grateful to an anonymous referee who made several helpful suggestions.

University of Minnesota
224 Church Street SE
Minneapolis MN 55455
USA
eaton@stat.umn.edu
bill@stat.umn.edu