Dynamically rational judgment aggregation*

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22 April 2022

Abstract

Judgment-aggregation theory has always focused on the attainment of rational collective judgments. But so far, rationality has been understood in static terms: as coherence of judgments at a given time, defined as consistency, completeness, and/or deductive closure. This paper asks whether collective judgments can be dynamically rational, so that they change rationally in response to new information. Formally, a judgment aggregation rule is dynamically rational with respect to a given revision operator if, whenever all individuals revise their judgments in light of some information (a learnt proposition), then the new aggregate judgments are the old ones revised in light of this information, i.e., aggregation and revision commute. We prove an impossibility theorem: if the propositions on the agenda are non-trivially connected, no judgment aggregation rule with standard properties is dynamically rational with respect to any revision operator satisfying some basic conditions on revision. Our theorem is the dynamic-rationality counterpart of some well-known impossibility theorems for static rationality. We also explore how dynamic rationality might be achieved by relaxing some of the conditions on the aggregation rule and/or the revision operator. Notably, premise-based aggregation rules are dynamically rational with respect to so-called premise-based revision operators.

*We presented this work at a Computational Social Choice online seminar (COMSOC), 2/2021, the Economic Theory Seminar of Centre d'Economie de la Sorbonne, 3/2021, an MCMP seminar at LMU Munich, 4/2021, a seminar of the Düsseldorf Center for Logic and Philosophy of Science, 5/2021, the LSE workshop on "Freedom and Reason", 6/2021, and the 2021 conference on Theoretical Aspects of Rationality and Knowledge (TARK), 6/2021. We thank the audiences as well as the referees for *Social Choice and Welfare* and TARK for detailed comments. An abridged summary appeared in the TARK Proceedings. Franz Dietrich gratefully acknowledges support from the French National Research Agency through three grants (ANR-17-CE26-0003, ANR-16-FRAL-0010, and ANR-17-EURE-0001). Christian List gratefully acknowledges support from the London School of Economics through a research sabbatical in the autumn of 2020 and through his continuing affiliation as a visiting professor.

1 Introduction

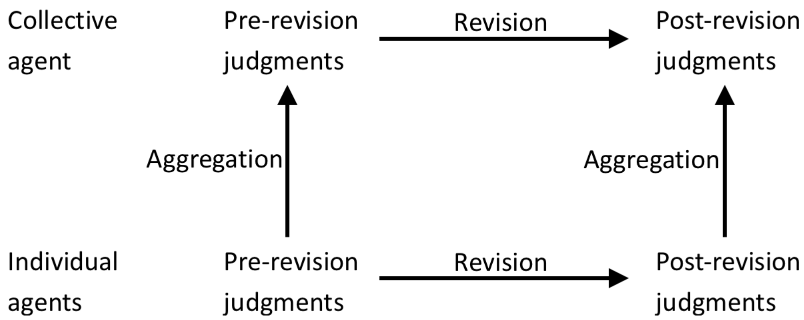
Suppose a group of individuals – say, a committee, expert panel, multi-member court, or other decision-making body – makes collective judgments on some propositions by aggregating its members' individual judgments on those propositions. And now suppose the group learns some new information – in the form of the truth of some proposition – that prompts a rational revision of the judgments held. At first sight, there seem to be two ways in which the group might go about incorporating this new information:

Either (1) the group members first revise their individual judgments based on the newly learnt information, and the group then aggregates its members' post-revision judgments.

Or (2) the group first aggregates its members' pre-revision judgments and then revises the resulting collective judgments based on the new information.

It would be ideal, however, if both approaches led to the same outcome: revision followed by aggregation and aggregation followed by revision. In this case, the group would not only avoid having to choose between the two approaches, but more importantly it would be able to aggregate its members' judgments *at every point in time*, for instance both before and after the receipt of new information, while thereby achieving a form of "dynamically rational" agency at the collective level. The group's aggregated judgments would evolve rationally over time in accordance with a given revision method, provided the group members' individual judgments do so. Figure 1 shows the desired scenario.

Figure 1: Aggregation and revision commute



In this paper, we investigate whether we can find reasonable aggregation rules that allow a group to achieve such dynamic rationality: aggregation rules which commute with reasonable revision methods. Surprisingly, this question has not been studied in the judgment-aggregation framework where judgments are binary verdicts on some propositions: "yes"/"no", "true"/"false", "accept"/"reject". (On judgment-aggregation theory, see List and Pettit 2002, Dietrich and List 2007a, Nehring and Puppe 2010, Dokow and Holzman 2010a, List and Puppe 2009.) The focus in judgment-aggregation theory has generally been on *static* rationality, namely on whether properties such as consistency, completeness, and deductive closure are preserved when individual judgments are aggregated into collective ones at a single point in time.[1]

By contrast, the question of dynamic rationality has received much attention in the distinct setting of probability aggregation, where judgments aren't binary but take the form of subjective probability assignments to the elements of some algebra. In that context, a mix of possibility and impossibility results has been obtained (e.g., Madansky 1964, Genest 1984, Genest et al. 1986, Dietrich 2010, 2019, Russell et al. 2015). These show that some familiar methods of aggregation – notably, the arithmetic averaging of probabilities – fail to commute with belief revision, understood in broadly Bayesian terms, while other methods – particularly geometric averaging – do commute with revision. An investigation of the parallel question in the case of binary judgments is therefore overdue.

Our primary result in this paper is, unfortunately, a negative one. We show that, for a large class of judgment aggregation rules, dynamic rationality is unachievable relative to a large class of reasonable judgment revision methods. However, we also show that if we relax some of our main theorem's conditions on the aggregation rule, dynamically rational aggregation becomes possible. While some of the identified possibilities are primarily technical and of limited substantive interest, we show that so-called "premise-based" aggregation rules, which are more plausible, are in fact dynamically rational relative to corresponding premise-based revision methods. These are quite special, however, and come at a certain cost, and an open question for future research is whether there might be other reasonable ways to avoid our impossibility result.

Our results reinforce a point that has already been defended in the theory of group agency, namely that it is difficult to achieve rational collective agency merely through the aggregation of individual attitudes and without any *sui generis* deliberative processes at the collective level itself (List and Pettit 2011). Previously, this point has been made primarily in relation to static rationality, where impossibility results have been used to show that rational group attitudes cannot generally supervene on rational individual attitudes in a propositionwise manner. Our results establish a similar point in relation to dynamic rationality. Most of our formal proofs are given in an appendix.

[1]The revision of judgments has been investigated only in a different sense in judgment-aggregation theory, namely in peer-disagreement contexts, where revision is prompted not by the learning of some new information but by the fact that others hold distinct judgments. See Pettit (2006) and List (2011).

2 The formal setup

We begin with the basic setup from judgment-aggregation theory (following List and Pettit 2002 and Dietrich 2007). We assume that there is a set of individuals who hold judgments on some set of propositions, and we are looking for a method of aggregating these judgments into resulting collective judgments. The key elements of this setup are the following:

Individuals. These are represented by a finite and non-empty set N. Its members are labelled 1, 2, ..., n. We assume $n \geq 2$.

Propositions. These are represented in formal logic. For our purposes, a thin notion of "logic" will suffice. Specifically, a *logic*, $\mathbf{L}$, is a non-empty set of formal objects called "propositions", which is endowed with two things:

- a *negation operator*, denoted $\neg$, so that, for every proposition p in $\mathbf{L}$, its negation $\neg p$ is also in $\mathbf{L}$; and
- a well-behaved notion of *consistency*, which specifies, for each set of propositions $S \subseteq \mathbf{L}$, whether S is consistent or inconsistent.[2]

Standard propositional, predicate, modal, and conditional logics all fall under this definition, as do Boolean algebras.[3] We call a proposition p *contradictory* if $\{p\}$ is inconsistent, and *tautological* if $\{\neg p\}$ is inconsistent. Any non-contradictory and non-tautological proposition is called *contingent*. A set of propositions $S \subseteq \mathbf{L}$ *entails* another proposition $p \in \mathbf{L}$ if $S \cup \{\neg p\}$ is inconsistent.

Agenda. The *agenda* is the set of those propositions from $\mathbf{L}$ on which judgments are to be made. Formally, this is a finite non-empty subset $X \subseteq \mathbf{L}$, which can be partitioned into proposition-negation pairs. Sometimes it is useful to make this partition explicit. We write $\mathcal{Z}$ to denote the set of proposition-negation pairs into which X is partitioned,

[2] *Well-behavedness* is a three-part requirement: (i) any proposition-negation pair $\{p, \neg p\}$ is inconsistent; (ii) any subset of any consistent set is still consistent; and (iii) the empty set is consistent, and any consistent set S has a consistent superset $S' \supseteq S$ which contains a member of every proposition-negation pair $\{p, \neg p\}$.

[3] Readers familiar with probability theory could take $\mathbf{L}$ to be a Boolean algebra on a non-empty set Ω of possible worlds, e.g., $\mathbf{L} = 2^{\Omega}$, with propositions defined as subsets of Ω, negation defined as set-theoretic complementation, and consistency of a set of propositions defined as non-empty intersection. The Boolean algebra could also be an abstract rather than set-theoretic Boolean algebra.

each of which is of the form $\{p, \neg p\}$ or abbreviated $\{\pm p\}$. The elements of $\mathcal{Z}$ can be interpreted as the *binary issues* under consideration. Then the agenda X is their disjoint union, formally $X = \bigcup_{Z \in \mathcal{Z}} Z$. Throughout this paper, we assume that double-negations cancel out in agenda propositions.[4]

Non-trivial logical connections. Our focus will be on agendas satisfying a non-triviality condition. To define it, call a set of propositions *minimal inconsistent* if it is inconsistent but all its proper subsets are consistent. For example, proposition-negation pairs of the form $\{p, \neg p\}$ (with p contingent) are minimal inconsistent, and so are sets of the form $\{p, q, \neg(p \wedge q)\}$, where "$\wedge$" stands for logical conjunction ("and") (with p and q contingent). We call an agenda *non-simple* if it has at least one minimal inconsistent subset of size greater than two. An example of a non-simple agenda is the set $X = \{\pm p, \pm(p \rightarrow q), \pm q\}$, where p might be the proposition "Current atmospheric CO_2 is above 407 ppm", $p \rightarrow q$ might be the proposition "If current atmospheric CO_2 is above 407 ppm, then the Arctic iceshield will melt by 2050", and q might be the proposition "The Arctic iceshield will melt by 2050". The conditional $p \rightarrow q$ can be formalized in standard propositional logic or in a suitable logic for conditionals. A three-member minimal inconsistent subset of this agenda is $\{p, p \rightarrow q, \neg q\}$.

Judgments. Each individual's (and subsequently the group's) judgments on the given propositions are represented by a *judgment set*, which is a subset $J \subseteq X$, consisting of all those propositions from X that its bearer "accepts" (e.g., affirms or judges to be true). A judgment set J is

- *complete* if it contains a member of each proposition-negation pair from X, i.e., $J \cap Z \neq \emptyset$ for every $Z \in \mathcal{Z}$,
- *consistent* if it is a consistent set in the sense of the given logic, and
- *classically rational* if it has both of these properties.

Any classically rational judgment set J is, by implication, *deductively closed* within X, i.e., it contains any proposition $p \in X$ that is entailed by J. We write $\mathcal{J}$ to denote the set of *all* classically rational judgment sets on the agenda X. A list of judgment sets $\langle J_1, ..., J_n \rangle$ across the individuals in N is called a *profile (of individual judgment sets)*.

[4]To be precise, henceforth, by the *negation* of any proposition $p \in X$ we shall mean the *agenda-internal negation* of p, i.e., the "opposite" proposition (the one distinct from p) in the binary issue $Z \in \mathcal{Z}$ to which p belongs. This is logically equivalent to the ordinary negation of p and will simply be denoted $\neg p$, as a notational shortcut. This convention ensures that $\neg\neg p = p$.

Aggregation rule. A *(judgment) aggregation rule* is a function, F, which maps each profile $\langle J_1, ..., J_n \rangle$ in some domain $\mathcal{D}$ of admissible profiles (often the "universal domain" $\mathcal{D} = \mathcal{J}^n$) to a collective judgment set $J = F(J_1, ..., J_n)$. A standard example is *(propositionwise) majority rule*, which is defined as follows: for each $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$,

$$F(J_1, ..., J_n) = \{p \in X : |\{i : p \in J_i\}| > \frac{n}{2}\}.$$

A typical research question in judgment-aggregation theory is whether we can find aggregation rules that satisfy certain requirements of democratic responsiveness to the individual judgments and collective rationality. Usually, the focus is on the attainment of *static* rationality at the collective level, i.e., rationality of the collective judgments at a particular point in time, especially their consistency and perhaps their completeness. Here, by contrast, our focus will be on requirements of *dynamic* rationality. To introduce these, we must first introduce the notion of judgment revision.

3 Judgment revision

The idea we wish to capture is that whenever any individual (or subsequently the group) learns some new information, in the form of the truth of some proposition, this individual (or the group) must incorporate the learnt information in the judgments held – an idea familiar from belief revision theory in the tradition of Alchourrón, Gärdenfors, and Makinson ("AGM") (1985) (see also Rott 2001, Peppas 2008). Our central concept is that of a *judgment revision operator*. This is a function which assigns to any pair (J, p) of an initial judgment set $J \subseteq X$ and a learnt proposition $p \in X$ a new judgment set $J|p$. We can interpret this as the revised judgment set, given p. It is convenient not to restrict the domain of admissible inputs and outputs of a revision operator, so that it can take any *logically possible* pair (J, p) as input, with $J \subseteq X$ and $p \in X$, and produce any subset of X as output. Formally, it is a function from $2^X \times X$ into 2^X.

We call a revision operator *regular* if it satisfies the following two minimal conditions:

(i) it is *successful*, i.e., $p \in J|p$ for any pair (J, p), and

(ii) it is *conservative*, i.e., $J|p = J$ for any pair (J, p) such that $p \in J$.

Condition (i) ensures that any learnt proposition p is indeed incorporated in the post-revision judgment set ("accept what you learn"). Condition (ii) ensures that if the learnt proposition is already accepted, then nothing changes ("no news, no change").

We further call a revision operator *rationality-preserving* if whenever $J \in \mathcal{J}$, we have $J|p \in \mathcal{J}$ for all non-contradictory propositions $p \in X$.

These definitions are well-illustrated by the class of *distance-based revision operators*, familiar from belief revision theory (on distance-based belief revision, see, among others, Katsuno and Mendelzon 1991 and Lehmann, Magidor, and Schlechta 2001). Such operators require that when a judgment set is revised in light of some new information, the post-revision judgments remain as "close" as possible to the pre-revision judgments, subject to the constraint that the learnt information be incorporated and no inconsistencies be introduced. Different distance-based operators spell out the notion of "closeness" in different ways.

To make this precise, we first consider a *distance metric* on judgment sets (such metrics have been introduced in the area of judgment aggregation by Konieczny and Pino Pérez 2002 and Pigozzi 2006). This is a function d that assigns to any pair of judgment sets $J, J' \subseteq X$ a non-negative real number $d(J, J')$ interpreted as the "distance" between J and J', subject to the minimal condition that $d(J, J') = 0$ if and only if $J = J'$. A simple example of a distance metric is the *Hamming distance*, according to which $d(J, J')$ is the number of propositions in X on which J and J' disagree, i.e.,

$$d(J, J') = |\left\{p \in X : p \in J \not\Leftrightarrow p \in J'\right\}|.$$

Now, given a distance metric d, we can define a corresponding judgment revision operator. For any (J, p), let $J|p$ be a judgment set J' that has minimal distance from J subject to the following constraints:

- J' contains p,
- J' is classically rational, except possibly when J is not classically rational or p is contradictory.[5]

By construction, any distance-based revision operator is successful (because of the first bullet point), rationality-preserving (because of the second), and conservative (because of the distance minimization). We will later construct several other revision operators, and emphatically, our analysis is not restricted to distance-based revision; distance-based revision operators just serve as an illustration.

Before we move on, we should briefly compare our notion of judgment revision with the notion of belief revision in the AGM tradition, though readers unfamiliar with the

[5]Insofar as there need not be a unique such distance-minimizing J', the choice of J' may require a tie-breaking criterion.

AGM theory may skip to the next section. The first point to note is that a judgment revision operator is defined only for judgment sets on the agenda, of the form $J \subseteq X$, while an AGM belief revision operator is usually defined for beliefs over an entire logic of propositions (closed under Boolean operations). Secondly, the inputs and outputs of a judgment revision operator – judgment sets of the form J and $J|p$ – need not be deductively closed, while belief sets in AGM belief revision theory are by definition deductively closed (albeit not necessarily consistent or complete). Thirdly, the two regularity conditions we have imposed on a judgment revision operator, namely successfulness and conservativeness, are much weaker than the full set of AGM conditions on belief revision. Fourthly, however, our condition of rationality-preservation goes beyond the AGM requirements. The AGM theory merely requires *consistency preservation*: if the pre-revision belief set is consistent, then so is the post-revision belief set. The AGM conditions allow a complete and consistent belief set to be revised into an incomplete, albeit consistent one. From the AGM perspective, rationality-preservation may be viewed as too demanding. In the context of judgment-aggregation theory, however, it is a natural condition because of the central status that classical rationality – the conjunction of consistency and completeness – enjoys. Indeed, the universal-domain condition requires that individuals hold classically rational judgment sets. If judgment revision could transform classically rational judgment sets into ones violating that condition, then the (universal) domain of most aggregation rules would fail to be closed under revision.

4 Can aggregation and revision commute?

We are now ready to turn to this paper's question. As noted, we would ideally want any decision-making group to employ a judgment aggregation rule and a revision operator that generate the same collective judgments irrespective of whether revision takes place before or after aggregation. This requirement (an analogue of the classic "external Bayesianity" condition in probability aggregation theory, as in Madansky 1964, Genest 1984, and Genest et al. 1986) is captured by the following condition on the aggregation rule F and the revision operator $|$:

Dynamic rationality: For any profile $\langle J_1, ..., J_n\rangle$ in the domain of F and any learnt proposition $p \in X$ where the revised profile $\langle J_1|p, ..., J_n|p\rangle$ is also in the domain of F, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$.

To see that this condition is surprisingly hard to satisfy, consider an example. Suppose a three-member group is making judgments on the agenda $X = \{\pm p, \pm(p \to q), \pm q\}$, where $p \to q$ is understood as a subjunctive conditional. That is, apart from the subsets of X that include a proposition-negation pair, the only inconsistent subset of X is $\{p, p \to q, \neg q\}$.[6] (We could alternatively use an agenda in classical logic.[7]) Suppose, further, the group members' initial judgments are as shown on the left-hand side of Table 1, where "yes" stands for the acceptance of a proposition and "no" for the acceptance of its negation.

Table 1: A simple example

	Before learning p			After learning p		
	p	$p \to q$	q	p	$p \to q$	q
Individual 1	No	No	Yes	Yes	No	Yes
Individual 2	No	Yes	No	Yes	Yes	Yes
Individual 3	No	No	No	Yes	No	No
Majority	No	No	No	Yes	No	Yes

Suppose now that the aggegation rule is majority rule and the revision operator is based on the Hamming distance, with some tie-breaking provision such that, in the case of a tie, one is more ready to change one's judgment on p or q than on $p \to q$. If the individuals learn the truth of p and revise their judgments, they arrive at the post-revision judgments shown on the right-hand side of Table 1. Aggregating those judgments yields the collective judgment set $\{p, \neg(p \to q), q\}$. By contrast, if the individuals first aggregate their pre-revision judgments, they arrive at the majority judgment set $\{\neg p, \neg(p \to q), \neg q\}$, and its revision in response to learning p yields the judgment set $\{p, \neg(p \to q), \neg q\}$. Thus the group arrives at a different collective judgment set depending on whether aggregation precedes revision or the other way round: the combination of majority rule and distance-based revision is not dynamically rational.

At first sight, one might think that this problem is just an artifact of majority rule or our specific distance-based revision operator, or that it is somehow unique to our example.[8] However, our first formal result shows that the problem is more general. Define a

[6] This subjunctive understanding of $p \to q$ contrasts with the material one, where $p \to q$ is understood less realistically as $\neg p \vee q$. On the material understanding, the subsets $\{p, \neg(p \to q), q\}$, $\{\neg p, \neg(p \to q), q\}$, and $\{\neg p, \neg(p \to q), \neg q\}$ would also be deemed inconsistent.

[7] We could, for instance, use the agenda $X = \{\pm a, \pm b, \pm((a \wedge b) \vee c)\}$, for atomic propositions a, b, and c. This agenda is isomorphic to the one in the main text, because it has only one inconsistent subset, namely $\{a, b, \neg((a \wedge b) \vee c)\}$, besides sets including proposition-negation pairs. One could interchange this agenda for $\{\pm p, \pm(p \to q), \pm q\}$, replacing p with a, $p \to q$ with b, and q with $((a \wedge b) \vee c)$. Our entire discussion would remain valid.

[8] But note that our example would work with other revision operators too. For instance, judgment

uniform quota rule, with acceptance threshold $m \in \{1, 2, ..., n\}$, as the aggregation rule with domain $\mathcal{J}^n$ such that, for each $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$,

$$F(J_1, ..., J_n) = \{p \in X : |\{i : p \in J_i\}| \geq m\}.$$

Majority rule is a special case of a uniform quota rule, namely the one where m is the smallest integer greater than $\frac{n}{2}$.[9] We have:

Theorem 1. If the agenda X is non-simple, then no uniform quota rule whose threshold is below the unanimity threshold n is dynamically rational with respect to any regular rationality-preserving revision operator.

In short, replacing majority rule with some other uniform quota rule with threshold less than n wouldn't solve our problem of dynamic irrationality, and neither would replacing our distance-based revision operator with some other regular rationality-preserving revision operator. (As illustrated later, both the uniformity constraint on the quota rule and the non-unanimitarian constraint are needed for the present result.[10]) In fact, the problem identified by our example and Theorem 1 generalizes further, as shown in the next section.

5 A general impossibility theorem

We will now abstract away from the details of any particular aggregation rule, and suppose instead we are looking for an aggregation rule F that satisfies the following general conditions:

Universal domain: The domain of admissible inputs to the aggregation rule F is the set of all classically rational profiles, i.e., $\mathcal{D} = \mathcal{J}^n$.

Non-imposition: F does not always deliver the same antecedently fixed output judgment set J, irrespective of the individual inputs, i.e., F is not a constant function.

revision could proceed as follows. Upon learning a not-yet-accepted proposition, one first forms revised judgments within the "premise subagenda" $\{\pm p, \pm(p \to q)\}$ which are complete within this subagenda and consistent with the learnt proposition; and one then extends the revised "premise judgments" by adding a "conclusion judgment" such that the additional proposition (q or $\neg q$) is consistent with the revised premise judgments and with the learnt proposition.

[9] On quota rules in judgment aggregation, see Dietrich and List (2007b).

[10] Specifically, in Sections 7.5 and 7.4, we construct examples of a non-uniform quota rule (namely an asymmetric unanimity rule, where the acceptance thresholds are either n or 1, depending on the proposition in question) and a unanimitarian quota rule (a special case of an oligarchy) that are dynamically rational with respect to some regular rationality-preserving revision operators.

Monotonicity: Additional individual support for an accepted proposition does not overturn the proposition's acceptance, i.e., for any profile $\langle J_1, ..., J_n \rangle \in \mathcal{D}$ and any proposition $p \in F(J_1, ..., J_n)$, if any J_i not containing p is replaced by some J_i' containing p and the modified profile $\langle J_1, ..., J_i', ..., J_n \rangle$ remains in $\mathcal{D}$, then $p \in F(J_1, ..., J_i', ..., J_n)$.

Non-oligarchy: There is no non-empty set of individuals $M \subseteq N$ (a set of "oligarchs") such that, for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{D}$, $F(J_1, ..., J_n) = \cap_{i \in M} J_i$.

Systematicity: The collective judgment on each proposition is determined fully and neutrally by individual judgments on that proposition. Formally, for any propositions $p, p' \in X$ and any profiles $\langle J_1, ..., J_n \rangle, \langle J_1', ..., J_n' \rangle \in \mathcal{D}$, if, for all $i \in N$, $p \in J_i \Leftrightarrow p' \in J_i'$, then $p \in J \Leftrightarrow p' \in J'$, where $J = F(J_1, ..., J_n)$ and $J' = F(J_1', ..., J_n')$.

Why are these conditions initially plausible? The reason is that, for each of them, a violation would entail a cost. Violating universal domain would mean that the aggregation rule is not fully robust to pluralism in its inputs; it would be undefined for some classically rational judgment profiles. Violating non-imposition would mean that the collective judgments are totally unresponsive to the individual judgments, which is completely undemocratic. Violating monotonicity could make the aggregation rule erratic in some respect: an individual could come to accept a particular collectively accepted proposition and thereby overturn its acceptance. Violating non-oligarchy would mean two things. First, the collective judgments would depend only on the judgments of the "oligarchs", which is undemocratic when $M \neq N$; and second, the collective judgments would be incomplete with respect to any binary issue on which there is the slightest disagreement among the oligarchs, which would lead to widespread indecision, except when M is singleton. Important special cases of oligarchic rules are *dictatorships of one individual* (where M is singleton) and *unanimity rule* (where $M = N$). Violating systemacity, finally, would mean that the collective judgment on each proposition is no longer determined as a proposition-independent function of individual judgments on that proposition. It may then either depend on individual judgments on other propositions too (a lack of *propositionwise independence*), or the pattern of dependence may vary from proposition to proposition (a lack of *neutrality*). Systematicity – the conjunction of propositionwise independence and neutrality – is the most controversial condition among the five, and it is therefore the first condition that we later consider relaxing. But it's worth noting that it is satisfied by majority rule and all uniform quota rules. Indeed, majority rule and uniform quota rules (except the unanimity rule) satisfy all five conditions.[11]

[11]More generally, the five conditions are satisfied by any non-oligarchic uniform *committee rule* (as

Our main theorem shows that, for non-simple agendas, the present five conditions are incompatible with dynamic rationality:

Theorem 2. If the agenda X is non-simple, then no aggregation rule satisfying universal domain, non-imposition, monotonicity, non-oligarchy, and systematicity is dynamically rational with respect to any regular rationality-preserving revision operator.

So, the problem identified by Theorem 1 is not restricted to uniform quota rules, but extends to all aggregation rules satisfying our conditions. Moreover, since practically all non-trivial agendas are non-simple, the impossibility applies very widely. Notably, the impossibility is not driven by the imposition of any static rationality condition (such as consistency of collective judgments), as no such condition is used in the theorem. In the paper's penultimate section (Section 8), however, we consider the implications of requiring static *and* dynamic rationality.

In the next two sections, we consider possible escape routes from the present impossibility result. Specifically, we run through all of the conditions of Theorem 2 – not only those on the aggregation rule but also those on the revision rule and on the agenda – and show that as soon as any one of the conditions is dropped, while the other conditions are retained, the impossibility ceases to hold: there will then exist some aggregation rules for at least some suitable agendas that are dynamically rational with respect to some revision operators. Thus, from a mathematical perspective, none of the theorem's conditions is redundant. Since this is also true for the condition on the agenda – non-simplicity – we note that we could amend the theorem's antecedent clause by writing "If, and only if, the agenda X is non-simple".

We also note that most of the identified possibilities of dynamically rational aggregation are somewhat contrived and more of theoretical rather than practical interest. Thus the present impossibility result is harder to avoid than the familiar impossibility results concerning statically rational judgment aggregation. However, we begin with one escape route that we consider substantively interesting, even though it is not without costs (Section 6), and we then turn to the more theoretical and contrived escape routes (Section 7).

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discussed in Nehring and Puppe 2002 and Dietrich and List 2007b), defined by a non-empty and superset-closed family $\mathcal{C} \subseteq 2^N$ of *winning coalitions* such that, for each $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, $F(J_1, ..., J_n) = \{p \in X : \{i : p \in J_i\} \in \mathcal{C}\}$. Uniform quota rules, including majority rule, additionally satisfy *anonymity*, i.e., invariance of the collective judgments under permutations of the individuals in a profile. Indeed, the conditions of universal domain, non-imposition, monotonicity, systematicity, and anonymity jointly characterize the class of uniform quota rules. Non-oligarchy excludes the unanimity rule.

6 Dynamic rationality through premise-based aggregation and revision

As systematicity is the most controversial one among our theorem's conditions, we begin with an escape route from our impossibility result that involves giving up systematicity. We will show in this section that the best-known aggregation rules violating systematicity – the *premise-based rules* – are dynamically rational if revision is defined in a corresponding premise-based way.

This possibility is interesting for at least two reasons. First, it shows that dynamic rationality is achievable by at least one prominent and plausible class of judgment aggregation rules (so dynamic rationality is not completely unrealistic from the outset), and second, it illustrates that in order to achieve dynamic rationality, the aggregation rule and the revision operator must interact in the right way: premise-based aggregation is dynamically rational with respect to a matching premise-based revision operator. While premise-based revision is of interest in its own right, the cost of the present possibility is that this revision operator is somewhat special and satisfies only weakened versions of our conditions on revision. In the next section, we show that relaxing systematicity alone, without relaxing any conditions on revision, suffices for a possibility, albeit a more contrived one.

Let us begin by introducing the idea of premise-based aggregation (for related earlier definitions, see List and Pettit 2002, Dietrich 2006, and Dietrich and Mongin 2009; for other discussions, see Kornhauser and Sager 1986, Pettit 2001, Chapman 2002, and Bovens and Rabinowicz 2006). Suppose the agenda X can be partitioned into a subagenda of premises and a subagenda of conclusions. Formally, we represent this partition by partitioning the set $\mathcal{Z}$ of binary issues into a set $\mathcal{Z}_{\text{prem}}$ of "premise issues" and a set $\mathcal{Z}_{\text{conc}}$ of "conclusion issues". Then the subagendas of premises and conclusions are $X_{\text{prem}} = \bigcup_{Z \in \mathcal{Z}_{\text{prem}}} Z$ and $X_{\text{conc}} = \bigcup_{Z \in \mathcal{Z}_{\text{conc}}} Z$. As an illustration, consider again the agenda $X = \{\pm p, \pm(p \to q), \pm q\}$. Here, the premise issues might be $\{\pm p\}$ and $\{\pm(p \to q)\}$, and the conclusion issue might be $\{\pm q\}$. The intuition is that the former might somehow be more fundamental than the latter, so that an agent's judgments on the latter may be derived from the agent's judgments on the former.

To define a *premise-based aggregation rule*, we require two preliminary definitions. For each premise issue $Z \in \mathcal{Z}_{\text{prem}}$, we introduce a *local aggregation rule ("premise aggregator")* F_Z which assigns to each combination of individual judgments on Z a collective judgment on Z. Formally, F_Z is a function from $\mathcal{J}_Z^n$ to $\mathcal{J}_Z$, where $\mathcal{J}_Z$ is the set of all *locally complete and consistent* judgments on Z, i.e., $\mathcal{J}_Z = \{\{p\}, \{\neg p\}\}$ for the binary issue $Z = \{\pm p\}$, assuming p is contingent. In the classical premise-based aggregation rule, each F_Z is majority rule, if n is odd.

To derive the judgments on all conclusion issues, we employ a *consequence operator*, defined as a function Cn that assigns to each set of (already accepted) propositions $J \subseteq X$ another set $Cn(J) \subseteq X$ of propositions that are the "consequences" of J. In the classical case, $Cn(J)$ simply consists of all propositions p in X that are logically entailed by J in the sense that the negation of p is inconsistent with J.

For any profile of individual judgment sets, we now arrive at the overall collective judgment set by

- first aggregating the individual judgments on all the premises, using the given premise aggregators, and
- then deriving their consequences for all other propositions, using the given consequence operator.

Formally, we define our premise-based aggregation rule on the domain of all profiles of judgment sets $J \subseteq X$ that are *classically rational on the premises*, i.e., $J \cap Z \in \mathcal{J}_Z$ for all $Z \in \mathcal{Z}_{\text{prem}}$. Let $\hat{\mathcal{J}}$ be the set of all such judgment sets. (This is a superset of $\mathcal{J}$.) For any profile $\langle J_1, ..., J_n \rangle \in \hat{\mathcal{J}}^n$, we let

$$F(J_1, ..., J_n) = \bigcup_{Z \in \mathcal{Z}} J_Z,$$

where, for each binary issue $Z \in \mathcal{Z}$,

$$J_Z = \begin{cases} F_Z(J_1 \cap Z, ..., J_n \cap Z) & \text{if } Z \in \mathcal{Z}_{\text{prem}}, \\ Cn(\bigcup_{Z' \in \mathcal{Z}_{\text{prem}}} J_{Z'}) \cap Z & \text{if } Z \in \mathcal{Z}_{\text{conc}}. \end{cases}$$

To illustrate this definition, consider the agenda $X = \{\pm p, \pm(p \to q), \pm q\}$ with $\{\pm p\}$ and $\{\pm(p \to q)\}$ designated as the premise issues, and suppose the individual judgments are as shown in Table 2. If the premise-based rule is the classical one, where each premise aggregator F_Z is the majority rule and the consequence operator Cn is the classical one, the collective judgment set will be $\{p, p \to q, q\}$. Propositions p and $p \to q$ will each be accepted by aggregating the individual judgments on those propositions, and proposition q will be accepted by logical inference. It is evident that this aggregation rule violates systematicity, by treating premises and conclusions differently and also by determining the collective judgments on all conclusions in a non-propositionwise-independent way.

Table 2: A premise-based rule illustrated

	p	$p \rightarrow q$	q
Individual 1	Yes	Yes	Yes
Individual 2	Yes	No	No
Individual 3	No	Yes	No
Premise-based rule	Yes	Yes	Yes

Next we introduce the idea of premise-based revision. Here, we also need some preliminary definitions. For each premise issue $Z \in \mathcal{Z}_{\text{prem}}$, we introduce a *local revision operator ("premise revisor")*, denoted $|_Z$, just for that issue. Formally, this is a function (from $2^Z \times X$ into 2^Z) which assigns to any pair (L, p) of an initial local judgment L on issue Z (formally $L \subseteq Z$) and a learnt proposition $p \in X$ a new local judgment on issue Z, denoted $L|_Z p$. As issue Z is of the form $\{\pm p\}$, any local judgment on Z must be of the form $\emptyset, \{p\}, \{\neg p\}, \{p, \neg p\}$. Of these, the first would correspond to withholding judgment on Z, the last would be inconsistent, and only the middle two would encode a locally complete and consistent judgment on issue Z (assuming neither p nor $\neg p$ is contradictory). To derive the revised judgments on all conclusion issues, we employ again our consequence operator Cn, which allows us to assign to each set of (already revised) propositions $J \subseteq X$ the set of propositions that are its consequences, $Cn(J) \subseteq X$.

For any initial judgment set and any newly learnt proposition, the premise-based revision operator now arrives at the revised judgment set by

- first revising the judgments on all the premises, using the given premise revisors, and
- then deriving their consequences for all other propositions, using the given consequence operator.

Formally, for any initial judgment set $J \subseteq X$ and any learnt proposition $p \in X$, the revised judgment set $J|p$ is the union

$$J|p = \bigcup_{Z \in \mathcal{Z}} J_Z$$

of revised local judgment sets $J_Z \subseteq Z$ for all binary issues $Z \in \mathcal{Z}$, where

$$J_Z = \begin{cases} (J \cap Z)|_Z p & \text{if } Z \in \mathcal{Z}_{\text{prem}}, \\ Cn(\bigcup_{Z' \in \mathcal{Z}_{\text{prem}}} J_{Z'}) \cap Z & \text{if } Z \in \mathcal{Z}_{\text{conc}}. \end{cases}$$

Premise-based revision operators are often neither regular nor rationality-preserving, as defined earlier, but many plausible premise-based revision operators satisfy weaker versions of these conditions. In particular, they satisfy *regularity on premises*, in the sense that they satisfy our two regularity conditions, successfulness and conservativeness, restricted to the premises. *Successfulness on premises* means that $p \in J|p$ whenever $p \in X_{\text{prem}}$, and *conservativeness on premises* means that if $p \in J \cap X_{\text{prem}}$, then J and $J|p$ coincide on the premises, i.e., $J \cap X_{\text{prem}} = (J|p) \cap X_{\text{prem}}$. The first of these properties permits that one does not incorporate a newly learnt conclusion proposition in one's revised judgments; rather, one always builds up one's revised judgments from one's judgments on the premises. And the second property permits that if one learns – or is reminded of – an already known premise, one might still change one's judgments on some conclusion, for instance by recognizing certain hitherto unacknowledged consequences of one's existing premise judgments. Premise-based revision operators may fail to be rationality-preserving insofar as a complete and consistent pre-revision judgment set does not always guarantee a complete and consistent post-revision judgment set. Whether or not it does depends on the nature of the subagenda of premises and the nature of the consequence operator. If consequence is defined classically, for instance, then the completeness of the revised judgments depends on whether complete judgments on the premises always logically settle all conclusion propositions; and if the premise issues are logically dependent, then the consistency of the revised judgments may be threatened by the fact that premise-based revision operates independently on each premise issue. However, if (i) the premises are mutually independent and suffice to settle the entire agenda, (ii) the premise revisors are locally rationality-preserving, and (iii) consequence is classical, then premise-based revision is indeed rationality-preserving.[12]

We are now in a position to state our possibility result. Call a revision operator *idempotent* if $(J|p)|p = J|p$ for all $J \subseteq X$ and all $p \in X$ ("learning the same information again does not change one's judgments"). Idempotence is much less demanding than full-blown regularity.

Theorem 3. If the revision operator is premise-based and idempotent, then all premise-based aggregation rules with unanimity-preserving premise aggregators (and with the same premises and consequence operator as in revision) are dynamically rational.

Here, a premise aggregator F_Z is *unanimity-preserving* if $F_Z(L, ..., L) = L$ for any unan-

[12] The premises *settle the entire agenda* if whenever $J \subseteq X_{\text{prem}}$ is complete and consistent within the premise subagenda X_{prem}, then the set of consequences $Cn(J)$ is a complete and consistent judgment set on X. A premise reviser $|_Z$ for premise issue Z is *locally rationality-preserving* if, for any $L \in \mathcal{J}_Z$, and any non-contradictory proposition $p \in X$, $L|_Z p \in \mathcal{J}_Z$.

imous local judgment profile $(L, ..., L)$ on the premise issue Z (i.e., $L \subseteq Z$).

In fact, we can go beyond Theorem 3 and show that, in important special cases, premise-based rules are the *only* dynamically rational aggregation rules with respect to a premise-based revision operator. To state this uniqueness result, we need to introduce two other conditions on the aggregation rule, which replace our original monotonicity and systematicity conditions, neither of which is generally satisfied by a premise-based rule. The first condition is a global version of monotonicity which replaces the focus on accepted propositions with a focus on accepted judgment sets:

Global monotonicity: Additional individual support for a "winning" judgment set does not overturn the outcome, i.e., if any profile $\langle J_1, ..., J_n \rangle \in \mathcal{D}$ is modified into another profile $\langle J_1, ..., J, ..., J_n \rangle \in \mathcal{D}$ by replacing one of the J_is with $J = F(J_1, ..., J_n)$, then $F(J_1, ..., J, ..., J_n) = J$.

To state the second condition, note that any given subagenda of premises X_{prem} induces a *relevance relation* between propositions: premises are relevant to conclusions, but not vice versa. More precisely, the only proposition relevant to any premise $p \in X_{\text{prem}}$ is p itself, while the propositions relevant to any conclusion $p \in X_{\text{conc}}$ are all premises. Formally, if $\mathcal{R}(p)$ denotes the set of propositions relevant to p, we have

$$\mathcal{R}(p) = \begin{cases} \{p\} & \text{if } p \in X_{\text{prem}}, \\ X_{\text{prem}} & \text{if } p \in X_{\text{conc}}. \end{cases}$$

Now our condition that replaces systematicity is the following (Dietrich 2015):

Independence of irrelevant propositions: The collective judgment on each proposition depends only on individual judgments on relevant propositions. Formally, for any proposition $p \in X$ and any profiles $\langle J_1, ..., J_n \rangle, \langle J'_1, ..., J'_n \rangle \in \mathcal{D}$, if, for all $i \in N$, $J_i \cap \mathcal{R}(p) = J'_i \cap \mathcal{R}(p)$, then $p \in J \Leftrightarrow p \in J'$, where $J = F(J_1, ..., J_n)$ and $J' = F(J'_1, ..., J'_n)$.

Global monotonicity and independence of irrelevant propositions jointly weaken the conjunction of monotonicity and systematicity used in our impossibility theorem.[13] Here is the uniqueness theorem:

[13]To be precise, independence of irrelevant propositions weakens the propositionwise independence part of systematicity provided the relevance relation satisfies *non-underdetermination*. This requires that the truth-value of any proposition $p \in X$ is settled by the truth-values of all of the propositions in $\mathcal{R}(p)$, formally, any consistent set of propositions which, for each $q \in \mathcal{R}(p)$, contains one of q or $\neg q$ entails p or entails $\neg p$.

Theorem 4. If the revision operator is premise-based, idempotent, and regular on premises, then the premise-based aggregation rules with unanimity-preserving premise aggregators (and with the same premises and consequence operator as in revision) are the only dynamically rational aggregation rules F from $\hat{\mathcal{J}}^n$ into $\hat{\mathcal{J}}$ satisfying independence of irrelevant propositions and global monotonicity.

Insofar as premise-based judgment aggregation has been prominently discussed in the literature, the present possibility and uniqueness results should be interesting. Indeed, Theorem 4's conditions on the aggregation rule seem eminently reasonable. In particular, independence of irrelevant propositions is arguably much more plausible than systematicity, and global monotonicity is a very plausible condition too. The conditions on the revision operator – idempotence and regularity on premises – are reasonable as well. However, the cost of the present possibility, as already noted, is that the revision operator satisfies only weaker versions of our earlier conditions. We leave it an open question for further discussion how serious this cost is.

7 More theoretical possibilities of dynamic rationality

Having presented an escape route from our impossibility result that should be of substantive interest, we now turn to several more theoretical escape routes. As anticipated, these mostly serve to prove the point that none of the conditions of Theorem 2 is redundant. Nonetheless, at least the first of the subsequent escape routes, which involves giving up universal domain, is of independent interest.

We proceed as follows. We first consider relaxing the theorem's conditions on the aggregation rule. We then consider relaxing the conditions on the revision operator. And we finally consider relaxing the theorem's agenda condition.

7.1 Dynamic rationality without universal domain: majority rule on single-plateaued and other restricted domains

To see that Theorem 2 would fail to hold without the condition of universal domain, we take any non-simple agenda X and construct two examples of restricted domains $\mathcal{D} \subseteq \mathcal{J}^n$ on which majority rule is dynamically rational with respect to some regular rationality-preserving revision operator. Since majority rule satisfies the rest of our conditions (non-imposition, monotonicity, non-oligarchy, and systematicity), the examples establish our point. While the first example is somewhat trivial, the second should be of greater interest.

For our first example, we define the domain as follows:

$$\mathcal{D} = \left\{ \langle J_1, ..., J_n \rangle \in \mathcal{J}^n : |\{i \in N : J_i = J\}| > \frac{n}{2} \text{ for some } J \in \mathcal{J} \right\},$$

i.e., $\mathcal{D}$ consists of all rational judgment profiles in which a majority of individuals hold the same judgment set. It is easy to verify that, whenever a profile $\langle J_1, ..., J_n \rangle$ is in $\mathcal{D}$, then the revised profile $\langle J_1|p, ..., J_n|p \rangle$ is still in $\mathcal{D}$, for any non-contradictory proposition $p \in X$ and any regular rationality-preserving revision operator. Moreover, if F is majority rule on $\mathcal{D}$, then $F(J_1, ..., J_n)$ is simply the judgment set J held by a majority of individuals in $\langle J_1, ..., J_n \rangle$, so that $F(J_1, ..., J_n)|p = J|p$. The revised profile $\langle J_1|p, ..., J_n|p \rangle$ has the property that the majority of individuals who previously held the judgment set J come to hold the judgment set $J|p$, so that the latter is also the majority outcome. Hence $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, as required.

Our second example invokes the idea that the propositions in X can be ordered from "left" to "right" on some cognitive or ideological dimension, in such a way that all individuals' judgments are structured by that order. Specifically, consider a linear order $\leq$ on X, where, for any two propositions $p, q \in X$, $p \leq q$ means that "p is (weakly) to the left of q". We call a profile $\langle J_1, ..., J_n \rangle$ *single-plateaued* relative to $\leq$ if, for every individual $i \in N$,

$$J_i = \{p \in X : p_{left} \leq p \leq p_{right}\} \text{ for some } p_{\text{left}}, p_{\text{right}} \in X,$$

i.e., the individual's judgment set forms a connected interval (a "plateau" of accepted propositions) with respect to $\leq$, ranging from p_{left} to p_{right}. It is already known that single-plateauedness, combined with individual-level consistency, is sufficient for consistent majority judgments (Dietrich and List 2010). To explain how single-plateauedness can also help with dynamic rationality, let $\mathcal{J}_{\leq}$ denote the subset of $\mathcal{J}$ consisting of all classically rational judgment sets that are single-plateaued relative to $\leq$. Define a judgment revision operator as follows: for any pair (J, p),

- if $J \in \mathcal{J}_{\leq}$ and $\mathcal{J}_{\leq}$ contains at least one judgment set containing p, let $J|p$ be the (unique) judgment set $J' \in \mathcal{J}_{\leq}$ containing p whose Hamming distance from J is minimal (so that judgment revision simply shifts the plateau of accepted propositions minimally until it contains p while remaining classically rational);

- otherwise, let $J|p$ be any judgment set $J' \subseteq X$ containing p whose Hamming distance from J is minimal, subject to the constraint that if $J \in \mathcal{J}$ and p is non-contradictory, then $J' \in \mathcal{J}$.

One can now show that, on the domain $\mathcal{D} = \mathcal{J}^n_{\leq}$, majority rule (in a group N with odd-numbered size n) is dynamically rational with respect to the revision operator just defined. The reason is that whenever a classically rational profile $\langle J_1, ..., J_n \rangle$ is single-plateaued relative to $\leq$, the majority judgments will coincide with the individual judgments of a particular profile-specific individual (technically, the median individual relative to some left-right order of the individuals that can be suitably constructed for the given profile), and even if all individuals revise their judgments based on learning a proposition p in line with the first bullet point, the majority judgments will still coincide with the revised judgments of that same individual. Details are given in the appendix.

7.2 Dynamic rationality without non-imposition: an absurd rule accepting all propositions

To see that Theorem 2 would fail to hold without the condition of non-imposition, we take any non-simple agenda X and any regular rationality-preserving revision operator, and note that the following, rather absurd aggregation rule is dynamically rational while satisfying the rest of our conditions: for any profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$,

$$F(J_1, ..., J_n) = X,$$

i.e., the collective judgment set is always identical to the agenda in its entirety. Of course, this aggregation rule is completely unresponsive to the individual judgments and produces totally inconsistent collective judgments. Nonetheless, it satisfies universal domain, monotonicity, non-oligarchy, and systematicity, while also satisfying dynamic rationality. (Note that, for any regular revision operator and any proposition p, $X|p = X$.) Let's call this aggregation rule the *absurd rule.*

One might wonder whether there are any less absurd examples of dynamically rational aggregation rules when we drop non-imposition. In fact, there are none. Our proof of Theorem 2 shows that, for any non-simple agenda X and any regular rationality-preserving revision operator, the absurd rule is the unique dynamically rational aggregation rule satisfying the rest of our conditions. Thus Theorem 2 would continue to hold if we were to replace non-imposition with the requirement that the aggregation rule should not be the absurd rule.

Some readers might wonder why we didn't use the condition of non-absurdity instead of non-imposition in our original statement of the theorem. There are three reasons why we chose not to do so. First, non-imposition, which only requires a non-constant aggregation rule, is already an extremely weak condition. Secondly, non-imposition is a standard condition, familiar from social choice theory, while non-absurdity is not. Thirdly, and perhaps most importantly, non-absurdity has the flavour of a static rationality condition (indeed, non-absurdity requires that the collective judgment set is not always maximally inconsistent), and we would like to avoid any static rationality conditions in what is supposed to be an impossibility theorem concerning dynamic rationality. Just as the impossibility theorems concerning static rationality do not impose any dynamic rationality conditions, we have sought to keep our present theorem free from any static rationality conditions.

7.3 Dynamic rationality without monotonicity: parity rules

To see that Theorem 2 would fail to hold without the condition of monotonicity, we show that, for some non-simple agendas, one can construct non-monotonic aggregation rules that are dynamically rational with respect to some regular rationality-preserving revision operator, while satisfying the rest of our conditions. Specifically, we consider a non-simple agenda X with the following properties:

- X is *affine*, in the sense that every minimal inconsistent subset $Y \subseteq X$ remains inconsistent after negating any two (or any even number) of its members.[14]
- For each contingent proposition $p \in X$, there exists a subagenda X_p (a non-empty subset of X closed under negation) which contains p and shares an even number of propositions with any minimal inconsistent subset of $Y \subseteq X$, i.e., $|Y \cap X_p| \in \{0, 2, 4, 6, ...\}$.[15]

An example of such an agenda is $X = \{\pm p, \pm q, \pm(p \leftrightarrow q)\}$, where p and q are logically independent and $\leftrightarrow$ is the material biconditional. This agenda is clearly non-simple. To see that it is affine, note that its minimal inconsistent subsets, besides all proposition-negation pairs, are $\{\neg p, q, p \leftrightarrow q\}$, $\{p, \neg q, p \leftrightarrow q\}$, $\{p, q, \neg(p \leftrightarrow q)\}$, $\{\neg p, \neg q, \neg(p \leftrightarrow q)\}$. Negating any two members of any one of these sets yields another one of them. Furthermore, for each $p \in X$, we can take X_p to be any subagenda of X that includes $\{\pm p\}$ and exactly one other proposition-negation pair. Then X_p shares an even number of propositions with any minimal inconsistent subset of X.

[14] The negation of *affiness* is *non-affineness* or *pair-negatability*, which is the condition that X has at least one minimal inconsistent subset Y in which we can find two (or an even number of) distinct propositions whose negation renders Y consistent. The name "affineness" is due to Dokow and Holzman (2010a), who introduced this condition in an explicitly algebraic form.

[15] Whether this second property is independent of the first (affineness) or indirectly implied by it is a non-trivial combinatorial question that we need not settle here.

Let us now define a judgment revision operator as follows: for any pair (J, p), let

$$J|p = \begin{cases} J & \text{if } p \in J, \\ (X_p \backslash J) \cup (J \backslash X_p) & \text{if } p \notin J \text{ and } J \in \mathcal{J}, \\ \text{any judgment set containing } p & \text{if } p \notin J \text{ and } J \notin \mathcal{J}, \end{cases}$$

where X_p is the above-defined subagenda if p is contingent and is $\{\pm p\}$ if p is non-contingent. Although this revision operator is admittedly a bit contrived, one can verify that it is both regular and rationality-preserving (details are in the appendix). It now turns out that an even more contrived kind of aggregation rule – a so-called *parity rule* (as introduced by Dokow and Holzman 2010a) – is dynamically rational with respect to this revision operator, while satisfying all of our conditions except monotonicity. To define it, let M be any odd-sized subset of N, and for any profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, let

$$F(J_1, ..., J_n) = \{p \in X : |\{i \in M : p \in J_i\}| \text{ is odd}\},$$

i.e., the set of collectively accepted propositions consists of all propositions that are accepted precisely by an odd number of individuals in M. Clearly, this aggregation rule is non-monotonic. However, we show in the appendix that, for any agenda of the specified kind – such as $X = \{\pm p, \pm q, \pm(p \leftrightarrow q)\}$ – the present aggregation rule is dynamically rational with respect to the revision operator just defined. Furthermore, a parity rule satisfies universal domain, non-imposition, systematicity, and non-oligarchy (assuming $|M| \geq 3$). To be sure, this possibility is of no substantive interest and only illustrates the mathematical point that the monotonicity condition is needed in Theorem 2.

7.4 Dynamic rationality without non-oligarchy: dictatorial and other oligarchic rules

To see that Theorem 2 would fail to hold without the condition of non-oligarchy, we give two examples of oligarchic aggregation rules that are dynamically rational with respect to some (or even any) regular rationality-preserving revision operator for some (or even any) non-simple agenda. These examples suffice to illustrate the non-redundancy of the non-oligarchy condition in our theorem because oligarchic rules always satisfy universal domain, non-imposition, monotonicity, and systematicity. Recall that an oligarchy (as discussed by Gärdenfors 2006, Dietrich and List 2008, and Dokow and Holzman 2010b) is defined by fixing some non-empty set $M \subseteq N$ of individuals (the "oligarchs") such that, for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, we have

$$F(J_1, ..., J_n) = \cap_{i \in M} J_i.$$

Our first example is a trivial one, namely a dictatorship of one individual; here the set of oligarchs is singleton, i.e., $M = \{i\}$ for some fixed $i \in N$. Clearly, if we have $F(J_1, ..., J_n) = J_i$ for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, then it trivially follows that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, irrespective of the agenda X and the revision operator.

For a less trivial example, which permits more than one oligarch and thereby shows that the condition of non-oligarchy in our theorem cannot simply be replaced by non-dictatorship, consider an agenda of the form $X = \{\pm p_1, \pm p_2, \pm p_3\}$, where the set of all classically rational judgment sets is

$$\mathcal{J} = \{\{p_1, p_2, p_3\}, \{\neg p_1, \neg p_2, p_3\}, \{p_1, \neg p_2, \neg p_3\}, \{\neg p_1, p_2, \neg p_3\}\},$$

i.e., the set of all complete subsets of X in which an even number of propositions (either zero or two) is negated. Such an agenda is non-simple: a minimal inconsistent subset of X of size three is $\{\neg p_1, \neg p_2, \neg p_3\}$. An example is once again $X = \{\pm p, \pm q, \pm(p \leftrightarrow q)\}$, where p and q are logically independent and $\leftrightarrow$ is the material biconditional. Now we first define a regular rationality-preserving judgment revision operator for X, and we then show that *any* oligarchic aggregation-rule (including the unanimity rule, where $M = N$) is dynamically rational with respect to it.

To construct the desired revision operator, we start from an assignment of a revised judgment set $J_p \in \mathcal{J}$ for every pair (J, p), where $J \in \mathcal{J}$ and $p \in X$. We construct this assignment such that

- for any $p \in X$ and any $J \in \mathcal{J}$, if $p \in J$, then $J_p = J$, and
- for any $p \in X$ and any $J, J' \in \mathcal{J}$, if J and J' are distinct and do not contain p (i.e., they are the two distinct judgment sets in $\mathcal{J}$ containing $\neg p$), then J_p and J'_p are distinct and contain p (i.e., they are the two distinct judgment sets in $\mathcal{J}$ containing p).

These properties jointly imply that $J_p \in \mathcal{J}$ and $p \in J_p$. We can think of the assignment of a judgment set J_p to each pair (J, p) as the restriction of the desired judgment revision operator to the domain $\mathcal{J} \times X$. Our goal is to extend this operator to *all* pairs (J, p) with $J \subseteq X$ and $p \in X$.

For the purposes of our example, we fully define the revision operator for all pairs (J, p) where J belongs to the set $\mathcal{J}^+$ of all consistent and deductively closed subsets of X (a superset of $\mathcal{J}$). For all other pairs, the operator can be defined arbitrarily, subject only to the restrictions of regularity (i.e., $p \in J|p$, and if $p \in J$ then $J|p = J$). Now, for any pair (J, p) with $J \in \mathcal{J}^+$ and $p \in X$, we define

$$J|p = \bigcap_{J' \in \mathcal{J} : J \subseteq J'} J'_p,$$

i.e., $J|p$ is the intersection of all revised judgment sets of the form J'_p, where J' is a complete and consistent extension of J. To give an intuition for this definition, note that any consistent and deductively closed judgment set J can be expressed as the intersection of all its complete and consistent extensions $J' \supseteq J$. So, our definition says that J is revised by revising all its complete and consistent extensions and taking the intersection of the revised judgment sets. As a special case of this, we have $J|p = J_p$ whenever J is complete and consistent.

This completes the definition of our revision operator. Note that this operator is rationality-preserving and regular. It is rationality-preserving because, for any $J \in \mathcal{J}$, we have $J|p = J_p$, and the latter is in $\mathcal{J}$. It is successful – the first part of regularity – because, for any $J \in \mathcal{J}^+$ and any $p \in X$, the revised judgment set $J|p$ is the intersection of sets of the form J'_p, which each contain p, so that $J|p$ also contains p. Moreover, for any $J \notin \mathcal{J}^+$, $J|p$ contains p by stipulation.

The operator is conservative – the second part of regularity – because, for any $J \in \mathcal{J}^+$ and any $p \in J$, we have

$$J|p = \bigcap_{J' \in \mathcal{J}: J \subseteq J'} J'_p = \bigcap_{J' \in \mathcal{J}: J \subseteq J'} J' = J.$$

The first identity holds by the definition of the revision operator. The second identity holds because we have $J'_p = J'$ whenever $p \in J'$. The third identity holds because J, being consistent and deductively closed, is identical to the intersection of all its complete and consistent extensions. Once again, for any $J \notin \mathcal{J}^+$, conservativeness (if $p \in J$ then $J|p = J$) holds by stipulation.

In the appendix we prove that, on the given non-simple agenda X, every oligarchic aggregation rule is dynamically rational with respect to the constructed revision operator. Let F be any oligarchic aggregation rule with the set $M \subseteq N$ of oligarchs. Our proof establishes that, for every $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and every $p \in X$,

$$F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p),$$

i.e.,

$$\left(\cap_{i \in M} J_i\right)|p = \cap_{i \in M}(J_i|p). \tag{1}$$

To give an intuition for this result, we briefly explain why the judgment set on the left-hand side of identity (1) is included in the judgment set on the right-hand side. The converse inclusion is harder to show, and we refer the reader to the appendix for the full proof. We begin by noting that $\cap_{i \in M} J_i$ is consistent and deductively closed, being the intersection of several consistent and complete judgment sets. Therefore, the definition of our revision operator allows us to rewrite the judgment set on the left-hand side of identity (1) as follows:

$$(\cap_{i\in M} J_i)\,|p = \bigcap_{J'\in\mathcal{J}:(\cap_{i\in M} J_i)\subseteq J'} J'_p.$$

Further, the judgment set on the right-hand side of identity (1), $\cap_{i\in M}(J_i|p)$, can be re-expressed as $\cap_{i\in M}(J_i)_p$, since each $J_i \in \mathcal{J}$, and thus as

$$\bigcap_{J'\in\{J_i:i\in M\}} J'_p. \tag{2}$$

Since each J_i is a complete and consistent extension of the intersection $\cap_{i\in M} J_i$, expression (2) includes

$$\bigcap_{J'\in\mathcal{J}:(\cap_{i\in M} J_i)\subseteq J'} J'_p, \tag{3}$$

because expression (3) is simply an intersection of more sets than expression (2): the sets being intersected in (3) include all those being intersected in (2). This establishes that

$$(\cap_{i\in M} J_i)\,|p \subseteq \cap_{i\in M}(J_i|p),$$

as desired. As noted, in the appendix, we show that the two judgment sets are in fact identical.

7.5 Dynamic rationality without systematicity: non-neutral and non-independent rules

To see that Theorem 2 would fail to hold without the condition of systematicity, even if we don't relax our conditions on the revision operator, we show that, for some non-simple agendas, one can construct aggregation rules which satisfy our other conditions and are dynamically rational with respect to some regular rationality-preserving revision operators.

We begin with a possibility that preserves the independence part of systematicity while giving up the neutrality part. We then consider a possibility that gives up the independence part, while, however, retaining a limited form of neutrality, namely a form of equal treatment of each proposition and its negation. Together, these possibilities illustrate that the systematicity condition is really needed in Theorem 2 and cannot simply be weakened to propositionwise independence or the mere requirement that propositions and their negations be treated equally.

For the first possibility, consider an agenda X of the form $X = \{\pm p : p \in Y\}$, where Y is the only minimal inconsistent subset of X apart from the proposition-negation pairs

$\{p, \neg p\} \subseteq X$ and where Y has three or more elements. An example is the earlier agenda $X = \{\pm p, \pm(p \to q), \pm q\}$, where $p \to q$ is a subjunctive conditional, so that the only minimal inconsistent subsets of X are the proposition-negation pairs and $\{p, p \to q, \neg q\}$. Now we define a revision operator as follows:

$$J|p = \begin{cases} J & \text{if } p \in J, \\ \{p\} \cup (J \backslash \{\neg p\}) & \text{if } p \notin J \text{ and } p \notin Y, \\ \{p\} \cup \{\neg q : q \in Y \backslash \{p\}\} & \text{if } p \notin J \text{ and } p \in Y. \end{cases}$$

It is easy to see that this revision operator is regular. To see that it is also rationality-preserving, take any $J \in \mathcal{J}$ and any $p \in X$. (In the present agenda, all propositions are non-contradictory.) The revised judgment set $J|p$ is complete because J itself is complete and Y contains a member of every proposition-negation pair $\{p, \neg p\} \subseteq X$. Furthermore, $J|p$ is consistent because it includes neither Y itself nor any proposition-negation pair $\{p, \neg p\} \subseteq X$, and so it includes no minimal inconsistent set.

We now show that the following aggregation rule is dynamically rational with respect to this revision operator. For any profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, let $F(J_1, ..., J_n)$ consist of

- all $p \in Y$ such that every J_i contains p, and
- all $p \in X \backslash Y$ such that at least one J_i contains p.

We can think of this aggregation rule as an asymmetric unanimity rule. Propositions in Y are collectively accepted if and only if they are unanimously accepted, while propositions outside Y are collectively accepted if and only if they are not unanimously rejected. It is evident that this aggregation rule satisfies universal domain, non-imposition, monotonicity, and non-oligarchy. It is also evident that it violates systematicity: the collective acceptance criterion is not the same for all propositions (a lack of neutrality). To see that it is dynamically rational with respect to the constructed revision operator, we distinguish between three cases.

- *Case 1*: $p \in Y$ and $p \in J_i$ for all $i \in N$. Then $p \in F(J_1, ..., J_n)$. Because the revision operator is conservative, $J_i|p = J_i$ for every $i \in N$ and $F(J_1, ..., J_n)|p = F(J_1, ..., J_n)$. So, $F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p)$.
- *Case 2*: $p \in Y$ and $p \notin J_i$ for some $i \in N$. Then $J_i|p = \{p\} \cup \{\neg q : q \in Y \backslash \{p\}\}$. This means that, in the profile $\langle J_1|p, ..., J_n|p \rangle$, p is unanimously accepted (because the revision operator is conservative), while all propositions outside Y (namely, those in $\{\neg q : q \in Y \backslash \{p\}\}$) are accepted by at least one individual (namely, individual i). So, $F(J_1|p, ..., J_n|p) = \{p\} \cup \{\neg q : q \in Y \backslash \{p\}\}$. Meanwhile, since

$p \in Y$ and p is not unanimously accepted in the profile $\langle J_1, ..., J_n \rangle$, we have $p \notin F(J_1, ..., J_n)$, and so $F(J_1, ..., J_n)|p = \{p\} \cup \{\neg q : q \in Y \setminus \{p\}\}$. This shows that $F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p)$.

- *Case 3*: $p \notin Y$. Here, revision of any judgment set simply leads to the acceptance of p and the non-acceptance of $\neg p$, while nothing else changes. So, the profile $\langle J_1|p, ..., J_n|p \rangle$ displays unanimous acceptance of p and coincides with the profile $\langle J_1, ..., J_n \rangle$ on all proposition-negation pairs distinct from $\{p, \neg p\}$. Then $F(J_1|p, ..., J_n|p)$ contains p and coincides with $F(J_1, ..., J_n)$ on all other proposition-negation pairs. Furthermore, $F(J_1, ..., J_n)|p$ also contains p and concides with $F(J_1, ..., J_n)$ on all other proposition-negation pairs. Hence, $F(J_1, ..., J_n)|p = F(J_1|p, ..., J_n|p)$.

For our second possibility, consider an agenda X for which there exists a classically rational judgment set $K \in \mathcal{J}$ whose complement is also classically rational, i.e., $\overline{K} = X \backslash K \in \mathcal{J}$. Define the revision operator as follows:

$$J|p = \begin{cases} K & \text{if } p \in K \text{ and } p \notin J, \\ \overline{K} & \text{if } p \in \overline{K} \text{ and } p \notin J, \\ J & \text{if } p \in J. \end{cases}$$

This operator is clearly regular and rationality-preserving. We now consider an aggregation rule F which produces as its output K or $\overline{K}$, depending on whether K or $\overline{K}$ receives greater "sum-total support" among the individual judgments, with a tie-breaking rule for the case in which K and $\overline{K}$ receive equal support. Formally, for any profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, let

$$n_K = \sum_{p \in K} n_p \text{ and } n_{\overline{K}} = \sum_{p \in \overline{K}} n_p,$$

where, for each $p \in X$, $n_p = |\{i \in N : p \in J_i\}|$, and define

$$F(J_1, ..., J_n) = \begin{cases} K & \text{if } n_K > n_{\overline{K}} \\ \overline{K} & \text{if } n_{\overline{K}} > n_K, \\ K & \text{if } n_K = n_{\overline{K}} \text{ and } \cap_i J_i \subseteq K \text{ and } \cap_i J_i \neq \emptyset, \\ \overline{K} & \text{if } n_K = n_{\overline{K}} \text{ and } \cap_i J_i \subseteq \overline{K} \text{ and } \cap_i J_i \neq \emptyset, \\ \cap_i J_i & \text{if } n_K = n_{\overline{K}} \text{ and } \cap_i J_i \nsubseteq K, \overline{K}, \\ K \text{ or } \overline{K} \text{ or } \emptyset & \text{if } n_K = n_{\overline{K}} \text{ and } \cap_i J_i = \emptyset. \end{cases}$$

Manifestly, this aggregation rule violates propositionwise independence (assuming the agenda X is non-trivial). Moreover, if $F(J_1, ..., J_n)$ is defined as $\emptyset$ when $n = n_{\overline{K}}$ and $\cap_i J_i = \emptyset$ in the last line of the displayed formula, then F is neutral between any proposition $p \in X$ and its negation $\neg p \in X$, and thus the rule retains one aspect of neutrality. The rule satisfies the other conditions of Theorem 2, with the exception of (propositionwise) monotonicity. However, it does satisfy the weaker condition of global monotonocity, introduced earlier in the discussion of premise-based aggregation, and this is a more plausible monotonicity requirement in the current case of a non-propositionwise-independent aggregation rule, insofar as it replaces the focus on accepted propositions with a focus on accepted judgment sets. In the appendix, we show that, under an additional symmetry assumption about the agenda (namely that $|J \cap K| = \left|J \cap \overline{K}\right|$ for all $J \in \mathcal{J} \setminus \left\{K, \overline{K}\right\}$), the present aggregation rule is dynamically rational with respect to the revision operator defined above.

7.6 Dynamic rationality with weaker conditions on revision: a wide class of possible rules

We have seen that all of Theorem 2's conditions on the aggregation rule are needed for the impossibility result. We now turn to the theorem's conditions on the revision operator. We will show that, if the revision operator is not required to be successful, or not required to be conservative, or not required to be rationality-preserving, then there exist dynamically rational aggregation rules for non-simple agendas satisfying the rest of our theorem's conditions.

Let us begin with the relaxation of successfulness. The simplest non-successful revision operator is the *constant* one, defined by

$$J|p = J \text{ for all } (J, p).$$

This operator is clearly conservative and rationality-preserving. All aggregation rules are trivially dynamically rational with respect to it.

Next, consider the relaxation of conservativeness. For each proposition $p \in X$, fix a judgment set J_p which contains p and is classically rational (i.e., in $\mathcal{J}$) as long as p is non-contradictory. Consider the revision operator given by

$$J|p = J_p \text{ for all } (J, p).$$

This operator is successful and rationality-preserving. It is easy to see that every unanimity-preserving aggregation rule is dynamically rational with respect to this revision operator, where *unanimity preservation* means that for every unanimous profile $\langle J, ..., J\rangle$ in the domain of F, we have $F(J, ..., J) = J$. To show this, assume that F is unanimity-preserving. Consider any proposition $p \in X$ and any profile $\langle J_1, ..., J_n\rangle$ such that $\langle J_1, ..., J_n\rangle$ and $\langle J_1|p, ..., J_n|p\rangle$ are in the domain of F. Then $F(J_1|p, ..., J_n|p) = F(J_p, ..., J_p) = J_p$ by unanimity-preservation, and $F(J_1, ..., J_n)|p = J_p$. So, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, as required.

Thirdly, consider the relaxation of rationality-preservation. Consider a revision operator with the following property: for any (J, p),

$$J|p = \begin{cases} J & \text{if } p \in J, \\ \text{any judgment set containing } p \text{ that is not classically} & \\ \qquad \text{rational (e.g., consistent but incomplete)} & \text{if } p \notin J. \end{cases}$$

This operator is successful and conservative but sometimes produces less than classically rational (though possibly still consistent) judgment sets as output. One can then show that every aggregation rule satisfying universal domain and *propositionwise unanimity preservation* (i.e., the requirement that if all individuals accept $p \in X$, then so does the collective) is dynamically rational with respect to the present operator. To demonstrate this, suppose F satisfies both conditions. Consider any proposition $p \in X$ and any profile $\langle J_1, ..., J_n\rangle$ such that $\langle J_1, ..., J_n\rangle$ and $\langle J_1|p, ..., J_n|p\rangle$ are in the domain of F, which is $\mathcal{J}^n$. For each i, we then have $J_i|p \in \mathcal{J}$. In particular, $J_i|p$ is complete, and so we can infer from the definition of the revision operator that $p \in J_i$, and hence $J_i|p = J_i$. So $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)$. Meanwhile, $p \in F(J_1, ..., J_n)$ by propositionwise unanimity preservation, and thus $F(J_1, ..., J_n)|p = F(J_1, ..., J_n)$. So, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, as required.

7.7 Dynamic rationality for simple agendas: majority rule and other propositionwise rules

Let us finally consider Theorem 2's condition on the agenda. Recall that the theorem asserts that the impossibility arises if the agenda is non-simple: it has at least one minimal inconsistent subset with more than two propositions. We will show that, if the agenda is simple, there exist aggregation rules that satisfy the required conditions while being dynamically rational with respect to a natural kind of revision operator.

Consider any simple agenda X. Let the revision operator be as follows. For any $J \subseteq X$ and any $p \in X$,

- if $J \in \mathcal{J}$, then $J|p = \{q_p : q \in J\}$, where

$$q_p = \begin{cases} q & \text{if } \{q, p\} \text{ is consistent,} \\ \neg q & \text{otherwise;} \end{cases}$$

- if $J \notin \mathcal{J}$, then $J|p$ can be defined arbitrarily, subject to the regularity conditions that (i) $p \in \mathcal{J}|p$ and (ii) if $p \in J$, then $J|p = J$.

By definition, this operator is regular, and since X is simple, it can also be seen to be rationality-preserving. Moreover, the operator has the special feature of being *local*: the revised judgment on any proposition $q \in X$ depends only on the initial judgment on q and on the learnt proposition p.[16] The following result holds:

Proposition 1. If the agenda X is simple, then every aggregation rule satisfying universal domain, collective rationality, propositionwise independence (or systematicity), and unanimity preservation is dynamically rational with respect to the revision operator just defined.

Here *universal domain* is as before; *collective rationality* is the requirement that $F(J_1, ..., J_n) \in \mathcal{J}$ for every profile $\langle J_1, ..., J_n \rangle \in \mathcal{D}$; *propositionwise independence* is a weakened version of systematicity, where the quantification is restricted to pairs of propositions p, p' with $p = p'$; and *unanimity preservation* is the requirement that $F(J, ..., J) = J$ for every unanimous profile $\langle J, ..., J \rangle \in \mathcal{D}$. Unanimity preservation strengthens non-imposition. An example of an aggregation rule satisfying all of these conditions (if X is simple and n is odd) is majority rule, which of course also satisfies all of the conditions of Theorem 2.

This shows that non-simplicity of the agenda is not only sufficient for our impossibility result, but also necessary. In fact, this is true not just in the case of Theorem 2, but also in the case of Theorem 1.

8 Combining static and dynamic rationality

So far, we have deliberately refrained from imposing any static rationality conditions on collective judgments. Just as previous impossibility results concerning static rationality did not require any dynamic rationality at the collective level, so our present results do

[16] The present definition has the interesting implication that, for a simple agenda X, if the initial judgment set J is classically rational and the learnt proposition p is non-contradictory, then $J|p$ is the unique classically rational judgment set that minimizes the Hamming distance from J subject to containing p. For general agendas, revision by minimizing Hamming distance is neither unique, nor local.

not require any collective rationality of the static sort. However, before we conclude, we would briefly like to look at the implications of requiring both static and dynamic rationality. An exhaustive analysis of this issue is beyond the scope of this paper, and we will state only three simple impossibility results and one possibility result.

Our first impossibility result is an immediate corollary of Theorem 2 above. As should be evident from the discussion in Section 7.2, non-imposition can be replaced with *collective consistency*, i.e., the requirement that collective judgments be consistent. The reason is that, as shown, our theorem continues to hold with non-imposition weakened to non-absurdity, and consistency of collective judgments clearly implies non-absurdity.

Corollary 1. If the agenda X is non-simple, then no aggregation rule satisfying universal domain, monotonicity, non-oligarchy, and systematicity is collectively consistent and dynamically rational with respect to any regular rationality-preserving revision operator.

However, while collective consistency is the most basic static rationality requirement at the collective level, it is common in judgment-aggregation theory to require collective judgments to satisfy the full classical rationality conditions of consistency *and* completeness. So, static rationality is typically formalized as follows:

Static rationality (often simply called "collective rationality"): For any profile $\langle J_1, ..., J_n\rangle$ in the domain of F, $F(J_1, ..., J_n)$ is consistent and complete.

Our next two results show that the conjunction of static and dynamic rationality is extraordinarily restrictive. For the first result, call an agenda X *weakly connected* if it is both non-simple and non-affine (the negation of affine, as defined in Section 7.3). Note that, since non-simplicity is a weak condition and affine agendas are extremely special, weak connectedness is itself a relatively undemanding condition, which is met in practically all standard examples of judgment aggregation problems.[17]

Proposition 2. If the agenda X is weakly connected, then the only aggregation rules satisfying universal domain and systematicity that are statically and dynamically rational (with respect to a regular revision operator) are the dictatorships of one individual.

This result is a direct corollary of a theorem from Dietrich and List (2007a). It is shown there that the remaining conditions of Proposition 2 – without dynamic rationality – are satisfied only by dictatorships and inverse dictatorships, where an *inverse dictatorship* always takes the collective judgment set to be the propositionwise negation of the individual judgment set of some antecedently fixed individual. It is easy to see,

[17] For an agenda to be affine, it has to be isomorphic to an agenda in standard propositional logic with negation and the material biconditional as the only logical connectives (Dokow and Holzman 2010a).

however, that inverse dictatorships violate dynamic rationality. One way to verify this is to observe that inverse dictatorships violate a key property that is implied by dynamic rationality, namely propositionwise unanimity preservation, and so only dictatorships remain as possibilities. The relevant fact is the following:

Fact 1. Any aggregation rule that is dynamically rational with respect to a regular revision operator is propositionwise unanimity-preserving, i.e., for any profile $\langle J_1, ..., J_n \rangle$ in the domain of F, if $p \in J_i$ for all $i \in N$, then $p \in F(J_1, ..., J_n)$.

To prove this, let F be dynamically rational with respect to a regular revision operator, and consider any profile $\langle J_1, ..., J_n \rangle$ in the domain of F and any proposition $p \in X$ where $p \in J_i$ for all $i \in N$. Then $J_i|p = J_i$ for all $i \in N$ by conservativeness of the revision operator, and so $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)$. Furthermore, we must have $p \in F(J_1, ..., J_n)|p$ by successfulness of the revision operator. But by dynamic rationality, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, and so $F(J_1, ..., J_n)|p = F(J_1, ..., J_n)$, from which we can infer that $p \in F(J_1, ..., J_n)$.

Fact 1 also allows us to derive a second result from a theorem in Dietrich and List (2007a), namely from the judgment-aggregation variant of Arrow's classic impossibility theorem (the latter was also proved in a slightly different formalism by Dokow and Holzman 2010a and is closely related to prior results in Nehring and Puppe 2002, 2010). To state this second result, call an agenda X *strongly connected* if it is both path-connected and non-affine, where *path-connectedness* is a strengthening of non-simplicity, requiring that any contingent propositions $p, q \in X$ can be connected by a path of conditional entailments.[18] An example of a strongly connected agenda is $X = \{\pm p, \pm q, \pm(p \wedge q), \pm(p \vee q)\}$ in standard propositional logic, where p and q are distinct atomic propositions. Also, any agenda that has the form of a non-trivial Boolean algebra (i.e., that is closed under conjunction or equivalently under disjunction, with two or more contingent proposition-negation pairs) is strongly connected.

Proposition 3. If the agenda X is strongly connected, then the only aggregation rules satisfying universal domain and propositionwise independence that are statically and dynamically rational (with respect to a regular revision operator) are the dictatorships of one individual.

Aside from the agenda condition, Proposition 3 matches Proposition 2, except that

[18]Proposition $p \in X$ *conditionally entails* proposition $q \in X$, written $p \vdash^* q$, if there exists some $Y \subseteq X$, consistent with each of p and $\neg q$, such that $\{p\} \cup Y$ entails q. A *path of conditional entailments* from p to q is a sequence of propositions $p_1, ..., p_k \in X$ with $p_1 = p$ and $p_k = q$ such that $p_1 \vdash^* p_2$, $p_2 \vdash^* p_3$, ..., $p_{k-1} \vdash^* p_k$.

the requirement of systematicity is weakened to propositionwise independence. The static precursor of Proposition 3 – "Arrow's theorem in judgment aggregation" – asserts that, for strongly connected agendas, the only aggregation rules satisfying universal domain, static rationality, propositionwise independence, and propositionwise unanimity preservation are the dictatorships of one individual. Given Fact 1, propositionwise unanimity preservation can be dropped from this list once dynamic rationality is added, and so Proposition 3 follows.

Note that while Theorems 1 and 2 use only the agenda condition of non-simplicity, Propositions 2 and 3 add some further agenda conditions: non-affineness in the case of Proposition 2 and path-connectedness in the case of Proposition 3. Without these additional agenda conditions, the two propositions wouldn't hold. The kinds of parity rules discussed in Section 7.3 could be used as counterexamples to Proposition 2 if non-affineness were dropped, and the asymmetric unanimity rules discussed in Section 7.5 could be used as counterexamples to Proposition 3 if path-connectedness were dropped.

At the same time, it is worth noting that Propositions 2 and 3 impose weaker conditions on the revision operator than Theorems 1 and 2 (and Corollary 1). For Propositions 2 and 3, we must only require revision to be regular (so, any revision operator with standard AGM properties would meet this requirement). The operator need not be rationality-preserving, though, from a substantive perspective, we may still find it congenial to require some form of rationality-preservation.

We close this section with a possibility result. In special cases, the combination of static and dynamic rationality can be achieved through premise-based aggregation. In particular, if in Theorem 3 we make two additional assumptions about the premises, then we get the result that suitable premise-based aggregation rules are statically and dynamically rational. The assumptions are, first, that the premises are logically independent from each other (i.e., there are no logical interconnections between distinct premise issues in $\mathcal{Z}_{\text{prem}}$); and second, the premises settle all propositions on the agenda via the consequence operator Cn (i.e., whenever a set $J \subseteq X_{\text{prem}}$ is complete and consistent within the premise subagenda X_{prem}, the set of consequences $Cn(J)$ is a complete and consistent judgment set on X). If consequence is classical, the latter just means that any complete and consistent judgments on premises logically entail complete judgments on the conclusions. For our result, consequence need not be classical, but it needs to be *inclusive*, meaning that $J \subseteq Cn(J)$ for all $J \subseteq X$. We now state our corollary of Theorem 3.

Corollary 2. If the revision operator is premise-based and idempotent, and the premises are logically independent and settle the agenda with respect to the consequence operator

Cn (which is classical or more generally inclusive), then all premise-based aggregation rules with unanimity-preserving premise aggregators (and with the same premises and consequence operator as in revision) are statically and dynamically rational.

9 Concluding remarks

We have shown that, for any non-simple agenda, no aggregation rule satisfying some standard conditions on aggregation is dynamically rational with respect to any revision operator satisfying some basic conditions on revision. And we have shown that if we also require static rationality, then, for a large class of agendas, dynamic rationality becomes impossible under even weaker conditions. The impossibility of dynamically rational judgment aggregation is harder to avoid than earlier impossibility results concerning static rationality. In the case of those earlier results, plausible escape routes tend to become available as soon as we relax one of the relevant theorems' conditions. In contrast, by relaxing merely one of our main result's conditions, we appear to open up only some relatively theoretical and contrived escape routes – with the exception of the route of dropping universal domain, where majority rule becomes dynamically (and in fact also statically) rational on suitably restricted domains. More substantial escape routes appear to become available only if we jointly relax several conditions, as illustrated by the possibility of dynamic rationality through premise-based aggregation and revision.

In light of this, one might be tempted to relax the condition of dynamic rationality itself. This would mean that revision and aggregation no longer commute, and the group in question would not function as a dynamically rational agent over time that aggregates its members' judgments *at every point in time.* Instead, the group would have to embrace irrational changes of collective judgments in response to new information or choose whether to aggregate its members' judgments before or after the receipt of such information. Irrational changes of collective judgments might be avoided, for instance, by aggregating judgments only before the receipt of any information, i.e., at some initial time, and deriving all subsequent collective judgments by revising the pre-information collective judgments without considering any revised individual judgments at all. Call this the "ex-ante approach" to aggregation. Alternatively, one might aggregate the individual judgments only after all relevant information has been received and the group members have revised their individual judgments in response to it. The group would then refrain from forming any collective judgments at the initial stage. Only the individuals' post-information judgments would be aggregated. Call this the "ex-post approach".

However, both approaches, which involve aggregation at only one point in time, have their problems. First, they violate a certain kind of democratic principle according to which the judgments of the group should be determined, at every point in time, as a function of the individual judgments at that time. We might call this "timepointwise", "temporally local", or "intratemporal supervenience". The ex-ante approach disconnects the collective judgments from the individual ones at any subsequent time, after the receipt of any information, and the ex-post approach generates no collective judgments at all at the earlier time. Secondly, if revision and aggregation do not commute, this opens the door to manipulation by anyone who can influence the time at which information is received and the time at which aggregation takes place. Thirdly, there does not generally seem to be a privileged time at which aggregation should take place, because groups can be subject to a flow of incoming information in the form of an open-ended sequence of learning events. The group will then have to respond rationally to each item of incoming information, and it seems arbitrary to aggregate judgments only after the k^{th} learning event for some fixed number k (say, the 7th), while not forming collective judgments beforehand and ignoring individual judgments afterwards. It seems much more systematic to take an approach that allows individual and collective judgments to co-evolve rationally over time, while retaining an aggregative connection between them at each point in time, and this is, precisely, what dynamic rationality in our present sense requires. In any event, our results need not be interpreted as merely addressing the question of whether, and when, such a rational co-evolution of individual and collective judgments across time is possible, but they can equally be interpreted as addressing the conditions under which the "ex-ante" and "ex-post" approaches do or do not yield the same outcome.

Our analysis of dynamically rational judgment aggregation has been a first step. The picture gets more complicated if different individuals and/or the group could use different revision operators, if propositions outside the agenda could be learnt, or if revision could make individual judgment sets incomplete and the aggregation rule permits incomplete inputs, to mention just three possible avenues for further research. That said, our analysis reinforces the point that it is surprisingly difficult to achieve rationality at the collective level merely through the aggregation of individual judgments. While this point is well known in the case of static rationality, our results extend the point to dynamic rationality.

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A Proof of the impossibility theorems

This appendix gives proofs of both impossibility theorems, contained in Sections 4 and 5, respectively.

A.1 Theorem 1

We first prove Theorem 1. One could easily prove Theorem 1 as a corollary of Theorem 2, but we here provide a direct, self-contained proof. The proof begins with a lemma. Recall that an aggregation rule F *preserves unanimity* if $F(J,...,J) = J$ for all unanimous profiles $\langle J,...,J\rangle$ in its domain. A judgment set is *weakly consistent* if it contains no pair $p, \neg p \in X$ (i.e., is not 'drastically inconsistent'). An aggregation rule F *guarantees* some condition on judgment sets (e.g., weak consistency) if $F(J_1,...,J_n)$ satisfies the condition for each profile $\langle J_1,...,J_n\rangle$ in the domain.

Lemma 1 *If a unanimity-preserving systematic aggregation rule with universal domain (e.g., a uniform quota rule) is dynamically rational with respect to a regular rationality-preserving revision operator, then it guarantees weak consistency.*

Proof. Let F be as specified. We may assume without loss of generality that X contains a contingent proposition; otherwise there would exist only one (unanimous) profile in $\mathcal{J}^n$, and weak consistency would follow from unanimity preservation.

Consider a profile $\langle J_1,...,J_n\rangle \in \mathcal{J}^n$ and a $p \in F(J_1,...,J_n)$. We show that $\neg p \notin F(J_1,...,J_n)$. By assumption, there is a contingent $q \in X$. As $\neg q$ is non-contradictory, some $J \in \mathcal{J}$ contains $\neg q$. Meanwhile $q \in J|q$ by successfulness, and $J|q \in \mathcal{J}$ by rationality-preservation and q's consistency. Construct the profile $\langle J_1',...,J_n'\rangle \in \mathcal{J}^n$ in which

$$J_i' = \begin{cases} J|q & \text{if } p \in J_i \\ J & \text{if } p \notin J_i. \end{cases}$$

Note that, for all individuals i, $p \in J_i \Leftrightarrow q \in J_i'$, or equivalently, $\neg p \in J_i \Leftrightarrow \neg q \in J_i'$. So, by systematicity, it suffices to show that $\neg q \notin F(J_1',...,J_n')$. By rationality-preservation, $\langle J_1'|q,...,J_n'|q\rangle \in \mathcal{J}^n$. So, by dynamic rationality, $F(J_1'|q,...,J_n'|q) = F(J_1',...,J_n')|q$. In this equation, the left side equals $F(J|q,...,J|q)$ (because $(J|q)|q = J|q$ by regularity), which in turn equals $J|p$ by unanimity preservation; and the right side equals $F(J_1',...,J_n')$, by conservativeness and the fact that $q \in F(J_1',...,J_n')$. So, $J|q = F(J_1',...,J_n')$. Hence, $\neg q \notin F(J_1',...,J_n')$. ■

Proof of Theorem 1. Let X be non-simple; so we may pick a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$. Let F be a uniform quota rule on $\mathcal{J}^n$ with some acceptance threshold

$m < n$. Fix a regular rationality-preserving revision operator. For a contradiction, assume F is dynamically rational. Then, by Lemma 1, F guarantees weak consistency; so $m > \frac{n}{2}$.

For each $y \in Y$, fix a rational judgment set $J_{\neg y} \in \mathcal{J}$ such that $Y \backslash \{y\} \subseteq J_{\neg y}$. Pick a $p \in Y$. Since $J_{\neg p}|p$ cannot contain all $y \in Y$ (as revision preserves rationality) but contains p (as revision is successful), there is some $q \in Y \backslash \{p\}$ such that $q \notin J_{\neg p}|p$. As $|Y| \geq 3$, we may pick a third proposition $r \in Y \backslash \{p, q\}$.

Let $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ be a profile in which some $n - m$ individuals i hold $J_i = J_{\neg p}$, other $n - m$ individuals i hold $J_i = J_{\neg q}$, and all remaining individuals i hold $J_i = J_{\neg r}$. As p and q are each accepted by m individuals, $p, q \in F(J_1, ..., J_n)$. Consider the revised profile $\langle J_1|p, ..., J_n|p \rangle$. As $q \notin J_{\neg p}|p$, and as by regularity $J_{\neg q}|p = J_{\neg q}$ and $J_{\neg r}|p = J_{\neg r}$, in the new profile only the individuals who used to hold $J_{\neg r}$ accept q; so q is accepted by $n - 2(n - m) = 2m - n < m$ individuals. So $q \notin F(J_1|p, ..., J_n|p)$. Now, $F(J_1, ..., J_n)$ equals $F(J_1, ..., J_n)|p$ because it contains p and revision is regular, and differs from $F(J_1|p, ..., J_n|p)$ because it contains q while $F(J_1|p, ..., J_n|p)$ does not. Therefore $F(J_1|p, ..., J_n|p) \neq F(J_1, ..., J_n)|p$. ■

A.2 Theorem 2

We now prove Theorem 2, in a slightly stronger version that weakens the (already weak) condition of non-imposition to non-absurdity. Non-imposition forbids that the collective judgment set is always the same. Non-absurdity merely forbids that the collective judgment set is always the entire agenda X (an absurd judgment set).

The proof of Theorem 2 uses again Lemma 1, but it also uses the following additional lemma.

Lemma 2 *The aggregation conditions in Theorem 2 with non-imposition weakened to non-absurdity (and with dynamic rationality defined with respect to a regular rationality-preserving revision operator) imply unanimity-preservation.*

Proof. Let F be an aggregation rule satisfying these conditions, with regular rationality-preserving revision. We show unanimity-preservation. By systematicity, it suffices to show that N is a winning coalition and $\varnothing$ is a losing coalition.

Claim 1: The full coalition N is winning.

Consider any rational profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and any p contained in all J_i. We must show that $F(J_1, \ldots, J_n)$ contains p. As revision is conservative, $\langle J_1|p, ..., J_n|p \rangle = \langle J_1, ..., J_n \rangle$. So, by dynamic rationality, $F(J_1, \ldots, J_n)|p = F(J_1, \ldots, J_n)$. The left side contains p by successfulness of revision. So $F(J_1, \ldots, J_n)$ contains p. Q.e.d.

Claim 2: There is a non-tautological $p \in X$ and a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ such that $p \notin F(J_1, ..., J_n)$.

By non-absurdity, there is a $p \in X$ and a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ such that $p \notin F(J_1, ..., J_n)$. Assume for a contradiction that p is tautological. Then p belongs to all J_i. So $\langle J_1, ..., J_n \rangle = \langle J_1|p, ..., J_n|p \rangle$, as revision is conservative. Hence, by dynamic rationality $F(J_1, \ldots, J_n)|p = F(J_1, \ldots, J_n)$. Noting that $p \in F(J_1, \ldots, J_n)|p$ (as revision is successful), it follows that $p \in F(J_1, ..., J_n)$, a contradiction. Q.e.d.

Claim 3: The empty coalition $\varnothing$ is not winning.

Pick p and $\langle J_1, ..., J_n \rangle$ as in Claim 2. As p is non-tautological, there is a rational profile $J \in \mathcal{J}$ such that $p \notin J$. We prove that $p \notin F(J, ..., J)$, which establishes that the empty coalition $\varnothing$ $(= \{i : p \in J\})$ is not winning. Since $p \notin F(J_1, ..., J_n)$, and since everyone who accepts p in the profile $\langle J, ..., J \rangle \in \mathcal{J}^n$ (namely, no-one) accepts p in $\langle J_1, ..., J_n \rangle$, we have $p \notin F(J, ..., J)$. More precisely, this follows by replacing the judgment sets in $\langle J, ..., J \rangle$ by those in $\langle J_1, ..., J_n \rangle$ one by one, and using monotonicity whenever the replacing judgment set J_i (unlike the replaced one J) contains p while using systematicity whenever the replacing judgment set does (like the replaced one) not contain p. ■

Proof of Theorem 2. Let X be non-simple. For a contradiction, assume F is an aggregation rule satisfying all mentioned conditions, with dynamic consistency defined with respect to a given regular rationality-preserving revision operator. By Lemmas 1 and 2, F is unanimity-preserving and guarantees weak consistency.

By non-simplicity, we may pick a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$. For each $y \in Y$, fix a $J_{\neg y} \in \mathcal{J}$ such that $Y \backslash \{y\} \subseteq J_{\neg y}$. Pick a $p \in Y$. Since $J_{\neg p}|p$ cannot contain all $y \in Y$ (as revision preserves rationality) but contains p (as revision is successful), there is some $q \in Y \backslash \{p\}$ such that $q \notin J_{\neg p}|p$. As $|Y| \geq 3$, we may pick a third proposition $r \in Y \backslash \{p, q\}$.

By systematicity, F is given by its winning coalitions. Note the following:

- N is winning while $\varnothing$ is not winning, by unanimity-preservation.
- Supersets of winning coalitions are winning, i.e., whenever $C \subseteq N$ is winning, so is any $C' \subseteq N$ such that $C \subseteq C'$. This follows from monotonicity.
- Any two winning coalitions C, C' have non-empty intersection. Otherwise $N \backslash C \supseteq C'$, so that $N \backslash C$ would be winning by monotonicity; but then we would have two complementary winning coalitions (C and $N \backslash C$), which would contradict weak consistency.
- There exist at least two *minimal* winning coalitions. Otherwise the set of winning coalitions would be a filter over N, implying oligarchy.

Pick two distinct minimal winning coalition C and C'. Construct a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ by letting

$$J_i = \begin{cases} J_{\neg p} & \text{if } i \in N \backslash C \\ J_{\neg q} & \text{if } i \in C \backslash C' \\ J_{\neg r} & \text{if } i \in C \cap C'. \end{cases}$$

As p and q are accepted by winning coalitions (namely by C and by $N \backslash (C \backslash C') \supseteq C'$, respectively), $p, q \in F(J_1, ..., J_n)$. Consider the revised profile $\langle J_1|p, ..., J_n|p \rangle$. As $q \notin J_{\neg p}|p$ and as (by conservativeness of revision) $J_{\neg q}|p = J_{\neg q}$ and $J_{\neg r}|p = J_{\neg r}$, in the new profile q is accepted only by those individuals who used to submit $J_{\neg r}$, hence by the coalition $C \cap C'$. This coalition is not winning because (as $C \cap C' \neq \varnothing$) it is a *strict* subset of a *minimal* winning coalition (i.e., of C or C'). So $q \notin F(J_1|p, ..., J_n|p)$. Now, $F(J_1, ..., J_n)$ equals $F(J_1, ..., J_n)|p$ (as it contains p and as revision is conservative) and it differs from $F(J_1|p, ..., J_n|p)$ because it contains q while $F(J_1|p, ..., J_n|p)$ does not. Therefore, $F(J_1|p, ..., J_n|p) \neq F(J_1, ..., J_n)|p$. ■

B Proof of the possibility theorems and corollary about the premise-based approach

We now prove our three possibility results about premise-based aggregation and revision – the two theorems in Section 6 and the corollary in Section 8. Earlier definitions and notation apply. In particular, recall that

- any premise subagenda X_{prem} induces a conclusion subagenda $X_{\text{conc}} = X \backslash X_{\text{prem}}$, a set of premise issues $\mathcal{Z}_{\text{prem}} \subseteq \mathcal{Z}$, and a set of conclusion issues $\mathcal{Z}_{\text{conc}} \subseteq \mathcal{Z}$,
- any premise subagenda X_{prem}, premise aggregators $(F_Z)_{Z \in \mathcal{Z}_{\text{prem}}}$, and consequence operator Cn jointly induce a premise-based rule F on $\hat{\mathcal{J}}^n$,
- any premise subagenda X_{prem}, premise revisors $(|_Z)_{Z \in \mathcal{Z}_{\text{prem}}}$, and consequence operator Cn jointly induce a premise-based revision operator.

B.1 Theorem 3

To prove Theorem 3, fix a proper (premise) subagenda X_{prem}, a consequence operator Cn, and an idempotent premise-based revision operator. Let $F : \hat{\mathcal{J}}^n \to \hat{\mathcal{J}}$ be a premise-based rule with unanimity-preserving premise aggregators $F_Z : \mathcal{J}_Z^n \to \mathcal{J}_Z$ ($Z \in \mathcal{Z}_{\text{prem}}$). To show that F is dynamically rational, consider any profile $\langle J_1, ..., J_n \rangle \in \hat{\mathcal{J}}^n$ and learnt proposition $p \in X$ such that $\langle J_1|p, ..., J_n|p \rangle \in \hat{\mathcal{J}}^n$. We must show that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$. This is done by proving that, for all issues $Z \in \mathcal{Z}$,

$$F(J_1|p, ..., J_n|p) \cap Z = [F(J_1, ..., J_n)|p] \cap Z. \tag{1}$$

Claim 1: Equation (1) holds for all premise issues $Z \in \mathcal{Z}_{\text{prem}}$.

Consider any $Z \in \mathcal{Z}_{\text{prem}}$. For each individual i, let q_i be the single member of $J_i \cap Z$. Also, let q_0 be the single member of $F_Z(J_1 \cap Z, ..., J_n \cap Z)$. Then,

$$F_Z(\{q_1\}, ..., \{q_n\}) = \{q_0\}. \tag{2}$$

The left side of the desired equation (1) is rewritable as follows:

$$\begin{aligned} F(J_1|p, ..., J_n|p) \cap Z &= F_Z((J_1|p) \cap Z, ..., (J_n|p)) \cap Z) \\ &= F_Z(\{q_1\}|^Z p, ..., \{q_n\}|^Z p), \end{aligned}$$

where the first and second equation holds by definition of premise-based aggregation and revision, respectively. Meanwhile the right side of the desired equation equals $\{q_0\}|^Z p$, by definition of premise-based revision. So the desired equation reduces to

$$F_Z(\{q_1\}|^Z p, ..., \{q_n\}|^Z p) = \{q_0\}|^Z p. \tag{3}$$

In other words, we must show that F_Z is (in the obvious sense) dynamically rational at the local profile $(\{q_1\}, ..., \{q_n\})$ and the learnt proposition p. There are two cases:

- *Case 1*: learning p does not lead to revision of any judgments on Z, i.e., $\{q\}|^Z p = \{q\}$ for each $\{q\} \in \mathcal{J}_Z$. Then the desired equation (3) reduces to the known equation (2), hence is true.
- *Case 2*: learning p leads to revision of some judgment on Z, i.e., there is a $\{q\} \in \mathcal{J}_Z$ such that $\{q\}|^Z p \neq \{q\}$.
  - *Subcase 2.1*: $\{q\}|^Z p \in \mathcal{J}_Z$. Here, as $\{q\}|^Z p \neq \{q\}$, we must have $\{q\}|^Z p = \{\neg q\}$, and thus, by idempotence of revision, $\{\neg q\}|^Z p = \{\neg q\}$. So the desired equation (3) reduces to

    $$F_Z(\{\neg q\}, ..., \{\neg q\}) = \{\neg q\},$$

    which holds because F_Z preserves unanimity.
  - *Subcase 2.2*: $\{q\}|^Z p \notin \mathcal{J}_Z$. Recall that, by assumption, for each individual i we have $J_i|p \in \hat{\mathcal{J}}$; hence $\{q_i\}|^Z p \in \mathcal{J}_Z$, which (because $\{q\}|^Z p \notin \mathcal{J}_Z$) implies that $q_i \neq q$. So, $q_i = \neg q$. Hence, by (2), $F_Z(\{\neg q\}, ..., \{\neg q\}) = \{q_0\}$. By unanimity preservation it follows that $q_0 = \neg q$. Hence the desired equation (3) reduces to

    $$F_Z(\{\neg q\}|^Z p, ..., \{\neg q\}|^Z p) = \{\neg q\}|^Z p,$$

    which holds because F_Z preserves unanimity. (In fact, $\{\neg q\}|^Z p$ must equal $\{\neg q\}$; otherwise $\{\neg q\}|^Z p$ would equal $\{q\}$, whence $(\{\neg q\}|^Z p)|^Z p = \{q\}|^Z p \neq \{\neg q\}$, contradicting idempotence.)

Claim 2: Equation (1) holds for all conclusion issues $Z \in \mathcal{Z}_{\text{conc}}$.

Consider any $Z \in \mathcal{Z}_{\text{conc}}$. By definition of premise-based aggregation, the left side of the desired equation (3) equals

$$Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1|p, ..., J_n|p) \cap Z']) \cap Z,$$

while by definition of premise-based revision the right side of the desired equation equals

$$Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1, ..., J_n)|p \cap Z']) \cap Z.$$

So the desired equation becomes

$$Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1|p, ..., J_n|p) \cap Z']) \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} [F(J_1, ..., J_n)|p \cap Z']) \cap Z.$$

This holds by Claim 1. ■

B.2 Theorem 4

To prove Theorem 4, we consider a premise-based revision operator, given by a premise subagenda X_{prem}, a consequence operator Cn, and premise revisors $(|_Z)_{Z \in \mathcal{Z}_{\text{prem}}}$. We assume revision is idempotent, and regular on premises.

Part 1. First consider a premise-based rule F. It obviously maps from $\hat{\mathcal{J}}^n$ to $\hat{\mathcal{J}}$. By Theorem 3, it is dynamically rational, provided its premise aggregators F_Z preserve unanimity. It is independent of irrelevant propositions, because the collective judgment on a proposition $p \in X$ is entirely fixed by the individual judgments on the *relevant* proposition (in $\mathcal{R}(p)$), whether p is a premise or a conclusion, as is clear from the definition or premise-based rules. Finally, provided each F_Z is monotonic, F is globally monotonic, by the following argument: if in a profile in $\hat{\mathcal{J}}^n$ one replaces someone's judgment set by the collective judgment set, then that individual's judgment on each premise issue $Z \in \mathcal{Z}_{\text{prem}}$ is replaced by the collective judgment on Z, which by monotonicity of each F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) has no effect on collective judgments on premises, and thus has no effect on conclusions either since aggregation is premise-based.

Part 2. Conversely, assume $F : \hat{\mathcal{J}}^n \to \hat{\mathcal{J}}$ is a dynamically rational aggregation rule that is independent of irrelevant propositions and globally monotonic. Let G be the premise-based rule whose premise aggregators $F_Z : \mathcal{J}_Z^n \to \mathcal{J}_Z$ ($Z \in \mathcal{Z}_{\text{prem}}$) are defined as follows. For any premise issue $Z \in \mathcal{Z}_{\text{prem}}$ and any local profile $(L_1, ..., L_n) \in \mathcal{J}_Z^n$, let $F_Z(L_1, ..., L_n) = F(J_1, ..., J_n) \cap Z$ for some (hence, by independence of irrelevant propositions, *any*) profile $\langle J_1, ..., J_n \rangle \in \hat{\mathcal{J}}^n$ such that $L_1 \subseteq J_1, ..., L_n \subseteq J_n$. Since F maps into $\hat{\mathcal{J}}$, each F_Z indeed maps into $\mathcal{J}_Z$.

We must prove that $F = G$ and that each F_Z is unanimity-preserving and monotonic.

Claim 1. $F = G$.

Fix a profile $\langle J_1, ..., J_n \rangle \in \hat{\mathcal{J}}^n$ and write $J_F := F(J_1, ..., J_n)$ and $J_G := G(J_1, ..., J_n)$. We prove that $J_F = J_G$ by showing that, for all issues $Z \in \mathcal{Z}$,

$$J_F \cap Z = J_G \cap Z. \tag{4}$$

Firstly, equation (4) holds for all premise issues $Z \in \mathcal{Z}_{\text{prem}}$, because each side then equals $F_Z(J_1 \cap Z, ..., J_n \cap Z)$. Now fix a conclusion issues $Z \in \mathcal{Z}_{\text{conc}}$. By definition of premise-based rules,

$$J_G \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} (J_G \cap Z')) \cap Z. \tag{5}$$

Turning to J_F, and using repeatedly that F is globally monotonic,

$$\begin{aligned} J_F &= F(J_1, ..., J_n) \\ &= F(J_F, J_2, ..., J_n) \\ &= F(J_F, J_F, J_3, ..., J_n) \\ &\cdots \\ &= F(J_F, ..., J_F), \end{aligned}$$

where we have used in each step that the new profile still lies in the domain of F since $J_F \in \hat{\mathcal{J}}$. Now pick any $p \in J_F \cap X_{\text{prem}}$ (noting that $J_F \cap X_{\text{prem}} \neq \varnothing$ because $J_F \in \hat{\mathcal{J}}$). Now $J_F \cap X_{\text{prem}} = J_F|p \cap X_{\text{prem}}$, since revision is conservative on premises. So, since F is independent of irrelevant propositions (and only premises are relevant to any propositions),

$$F(J_F, ..., J_F) = F(J_F|p, ..., J_F|p) = F(J_F, ..., J_F)|p.$$

where the second equality holds by dynamic rationality. Therefore, since $J_F = F(J_F, ..., J_F)$, we have shown that $J_F = J_F|p$. Meanwhile, since revision is premise-based,

$$(J_F|p) \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} ((J_F|p) \cap Z') \cap Z.$$

Replacing $J_F|p$ by J_F, we obtain

$$J_F \cap Z = Cn(\cup_{Z' \in \mathcal{Z}_{\text{prem}}} (J_F \cap Z') \cap Z. \tag{6}$$

By (5), (6), and Claim 1, we can deduce (4). Q.e.d.

Claim 2. Each premise aggregator F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) preserves unanimity.

Consider any $Z \in \mathcal{Z}_{\text{prem}}$ and any unanimous local profile $(L, ..., L) \in \mathcal{J}_Z^n$, say $L = \{p\}$. We must show that $F_Z(L, ..., L) = L$, or equivalently (as $F_Z(L, ..., L)$ belongs to $\mathcal{J}_Z$

and is thus singleton) that $p \in F_Z(L, ..., L)$. Choose any extension $J \supseteq L$ in $\hat{\mathcal{J}}$. Since revision is conservative on premises, $(J|p) \cap X_{\text{prem}} = J \cap X_{\text{prem}}$. Hence, not just J, but also $J|p$ is a member of $\hat{\mathcal{J}}$ that extends L. So, $F_Z(L, ..., L)$ equals $F(J|p, ..., J|p) \cap Z$, which equals $F(J, ..., J)|p$ by dynamic rationality. As revision is successful on premises, p belongs to $F(J, ..., J)|p$, hence to $F_Z(L, ..., L)$. Q.e.d.

Claim 3. Each premise aggregator F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) is monotonic.

The argument is simple. Consider any F_Z ($Z \in \mathcal{Z}_{\text{prem}}$) and any local profile $(L_1, ..., L_n) \in \mathcal{J}_Z^n$. Note that ordinary and global monotonicity are equivalent given the local nature of the agenda (and the fact that F_Z maps into the same set $\mathcal{J}_Z$ to which also individual judgment sets belong, so that we can substitute collective for individual judgment sets without leaving the domain of F_Z). So let us show global monotonicity of F_Z. Let $(L_1, ..., L, ..., L_n)$ arise from $(L_1, ..., L_n)$ by replacing some individual i's judgment set L_i by $L = F_Z(L_1, ..., L_n)$. We must show that $F(L_1, ..., L, ..., L_n) = L$. Pick extensions $J_1 \supseteq L_1, ..., J_n \supseteq L_n$ in $\hat{\mathcal{J}}$. Define $J = F(J_1, ..., J_n)$. Note that $L = J \cap Z$, and so $L \subseteq J$. Now

$$\begin{aligned} L &= F_Z(L_1, ..., L_n) \\ &= F(J_1, ..., J_n) \cap Z \text{ as } L_1 \subseteq J_1, ..., L_n \subseteq J_n \\ &= F(J_1, ..., J, ..., J_n) \cap Z \text{ as } F \text{ is monotonic} \\ &= F_Z(L_1, ..., L, ..., L_n) \text{ as } L_1 \subseteq J_1, ..., L \subseteq J, ..., L_n \subseteq J_n. \ \blacksquare \end{aligned}$$

B.3 Corollary 2

Consider a revision operator and an aggregation rule F which satisfy all assumptions in Corollary 2. By Theorem 3, F is dynamically rational. It remains to show static rationality. To this end, consider any profile $(J_1, ..., J_n) \in \hat{\mathcal{J}}^n$. We prove that $J = F(J_1, ..., J_n)$ is rational. As the premises are logically independent, $J \cap X_{\text{prem}}$ is rational in the subagenda X_{prem}. So, since the premises settle the agenda with respect to Cn, $Cn(J \cap X_{\text{prem}})$ is rational in full agenda X. It thus remains to show that $J = Cn(J \cap X_{\text{prem}})$. This holds for two reasons:

- J and $Cn(J \cap X_{\text{prem}})$ share the same conclusion propositions, i.e., $J \cap X_{\text{conc}} = Cn(J \cap X_{\text{prem}}) \cap X_{\text{conc}}$, by definition of premise-based aggregation.
- J and $Cn(J \cap X_{\text{prem}})$ share the same premise propositions, i.e., $J \cap X_{\text{prem}} = Cn(J \cap X_{\text{prem}}) \cap X_{\text{prem}}$. Why? For one, as Cn is inclusive, $J \cap X_{\text{prem}}$ is included in $Cn(J \cap X_{\text{prem}})$, hence in $Cn(J \cap X_{\text{prem}}) \cap X_{\text{prem}}$. For another, $Cn(J \cap X_{\text{prem}}) \cap X_{\text{prem}}$ cannot be a *strict* superset of $J \cap X_{\text{prem}}$; otherwise, as $J \cap X_{\text{prem}}$ contains a member of each proposition-negation pair in X_{prem}, $Cn(J \cap X_{\text{prem}}) \cap X_{\text{prem}}$ would contain both members of some such pair, violating the consistency of $Cn(J \cap X_{\text{prem}})$. $\blacksquare$

C Proof of the possibility claims in Section 7

This appendix turns to the various escape routes discussed in Section 7.

C.1 Possibilities without universal domain

In the main text, we have discussed two types of aggregation rules satisfying all conditions in Theorem 2 except universal domain. The second type requires formal elaboration. There, we consider a fixed linear order $\leq$ of the propositions, representing for instance a political left-to-right order or the propositions. Recall that $\mathcal{J}_{\leq}$ denotes the set of those rational judgment sets $J \in \mathcal{J}$ which are *single-plateaued* with respect to $\leq$, as defined earlier. Recall also that the order $\leq$ induces a natural revision rule, as defined above.[1] This revision operator is obviously regular and rationality-preserving. As long as n is odd, majority rule restricted to $\mathcal{J}_{\leq}^n$ satisfies all aggregation conditions of Theorem 2 except universal domain. This is obvious for most aggregation conditions, but requires a proof for dynamic rationality.

Proposition 4 *If n is odd, majority rule on the restricted domain $\mathcal{J}_{\leq}^n$ is dynamically rational with respect to the above revision operator.*

Proof. Assume n is odd, F is majority rule on $\mathcal{J}_{\leq}^n$, and revision is defined as above. To prove dynamic rationality, consider any $\langle J_1, ..., J_n \rangle \in \mathcal{J}_{\leq}^n$ and $p \in X$ such that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}_{\leq}^n$. For all $J \in \mathcal{J}$, write $\min J$ for J's minimal element with respect to $\leq$. For simplicity, assume that $i < j \Rightarrow \min J_i \leq \min J_j$. Assuming this condition is no loss of generality, because the condition can always be enforced by reordering the profile appropriately (reordering makes no difference since majority rule is anonymous).

As the profile is single-plateaued and rational, it is unidimensionally aligned (see Dietrich and List 2010). Unidimensional alignment means that there exists a permutation $(i_1, ..., i_n)$ of the individuals such that each proposition $q \in X$ is accepted either by a 'left-segment' of individuals (i.e., $\{i : q \in J_i\} = \{i_1, ..., i_k\}$ for some $k \in \{0, ..., n\}$) or by a 'right-segment' of individuals (i.e., $\{i : q \in J_i\} = \{i_k, ..., i_n\}$ for some $k \in \{1, ..., n+1\}$); a consequence is that the majority judgment set is the judgment set of the median individual, i.e.,

$$F(J_1, ..., J_n) = J_{i_{(n+1)/2}}$$

[1] Our above definition of $J|p$ could be generalised slightly, as follows. If $J \in \mathcal{J}_{\leq}$ and some $J' \in \mathcal{J}_{\leq}$ contains p (first bullet point), the definition of $J|p$ remains unchanged. Otherwise (second bullet point), $J|p$ can be defined arbitrarily, only subject to respecting the regularity conditions that $p \in J|p$ and that $J|p = J$ if $p \in J$, and the rationality-preservation condition that $J|p \in \mathcal{J}$ if $J \in \mathcal{J}$ and p is non-contradictory. Our proof holds for this general definition.

(List 2003). A permutation $(i_1, ..., i_n)$ with the mentioned property is called a *structuring order*, and the profile $\langle J_1, ..., J_n \rangle$ is more explicitly called unidimensionally aligned 'with respect to $(i_1, ..., i_n)$'. Our initial assumption on the order of the judgment sets in $\langle J_1, ..., J_n \rangle$ yields a natural structuring order:

Claim 1: $\langle J_1, ..., J_n \rangle$ is unidimensionally aligned with respect to the structuring order $(1, ..., n)$. In particular,

$$F(J_1, ..., J_n) = J_{(n+1)/2}. \tag{7}$$

Write $X = \{p_1, ..., p_{|X|}\}$ where $p_1 < p_2 < \cdots < p_{|X|}$. Consider any $q \in X$. There are two cases.

- *Case 1:* $q \in \{p_1, ..., p_{|X|/2}\}$, i.e., q is 'more to the left'. We show that $\{i : q \in J_i\} = \{1, ..., k\}$ for some $k \in \{0, ..., n\}$. To prove this, we consider an individual i such that $q \in J_i$, and show for any given other individual $j < i$ that again $q \in J_j$. This follows from three facts. First, $\min J_j \leq q$, because $\min J_j \leq \min J_i$ (as $j < i$) and $\min J_i \leq p$ (as $p \in J_i$). Second, $q \leq \max J_j$, because J_j contains $\frac{|X|}{2}$ propositions (by rationality) while there are less than $\frac{|X|}{2}$ propositions to the left of q (as $q \in \{p_1, ..., p_{|X|/2}\}$). Third, J_j is an 'interval' or 'plateau', i.e., contains all propositions between $\min J_j$ and $\max J_j$, by single-plateauedness.
- *Case 2:* $q \in \{p_{|X|/2+1}, ..., p_{|X|}\}$, i.e., q is 'more to the right'). We show that $\{i : q \in J_i\} = \{k, ..., n\}$ for some $k \in \{1, ..., n+1\}$. To this end, we consider an individual i such that $q \in J_i$, and show for any other individual $j > i$ that again $q \in J_j$. This holds because, for reasons analogous to those in Case 1, $q \leq \max J_j$, $\min J_j \leq q$, and J_j is an 'interval'. Q.e.d.

Claim 2: The set $\mathcal{J}_{\leq,p} := \{J \in \mathcal{J}_{\leq} : p \in J\}$ is non-empty.

Recall that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}^n_{\leq}$ and each $J_i|p$ contains p. So, $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}^n_{\leq,p}$, whence $\mathcal{J}_{\leq,p} \neq \varnothing$. Q.e.d.

Claim 3: The revised judgment profile is

$$(J_1|p, ..., J_n|p) = \begin{cases} (J_1, ..., J_k, J, ..., J) & \text{if } p \in \{p_1, ..., p_{|X|/2}\} \\ (J', ..., J', J_{k'}, ..., J_n) & \text{if } p \in \{p_{|X|/2+1}, ..., p_{|X|}\} \end{cases}$$

where

- $k = \max\{i : p \in J_i\}$, interpreted as 0 if $\{i : p \in J_i\} = \varnothing$,
- J is the right-most judgment set in $\mathcal{J}_{\leq,p}$ $(\neq \varnothing)$, i.e., $J \in \mathcal{J}_{\leq,p}$ and $\min \tilde{J} \leq \min J$ for all $\tilde{J} \in \mathcal{J}_{\leq,p}$,
- $k' = \min\{i : p \in J_i\}$, interpreted as $n + 1$ if $\{i : p \in J_i\} = \varnothing$,
- J' is the left-most judgment set in $\mathcal{J}_{\leq,p}$ $(\neq \varnothing)$, i.e., $J' \in \mathcal{J}_{\leq,p}$ and $\min J' \leq \min \tilde{J}$ for all $\tilde{J} \in \mathcal{J}_{\leq,p}$.

This claim follows from the definition of the revision operator and the fact that in the profile $\langle J_1, ..., J_n \rangle$ the judgment sets are ordered such that $i < j \Rightarrow \min J_i \leq \min J_j$. For instance, assume $p \in \{p_1, ..., p_{|X|/2}\}$. Then no judgment set J_i is 'to the left' of p (as $|J_i| = |X|/2$). Those J_i which already contain p are unchanged: $J_i|p = J_i$. Those J_i which do not contain p lie 'to the right' of p, so that their revision 'shifts' them minimally to the left such that p is accepted: $J_i|p = J$. Q.e.d.

Claim 4: $\langle J_1|p, ..., J_n|p \rangle$ is again unidimensionally aligned with respect to the structuring $(1, ..., n)$. In particular,

$$F(J_1|p, ..., J_n|p) = J_{(n+1)/2}|p. \tag{8}$$

Claim 3 implies that the judgment sets in the revised profile $\langle J_1|p, ..., J_n|p \rangle$ have the analogous property to that of in the original profile: $i < j \Rightarrow \min(J_i|p) \leq \min(J_j|p)$. So, by an argument analogous to that used to prove Claim 1, $\langle J_1|p, ..., J_n|p \rangle$ is a unidimensionally aligned profile with structuring order $(1, ..., n)$. Q.e.d.

By (7) in Claim 1 and (8) in Claim 4, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$. ■

C.2 Possibilities without non-imposition

As noted, there is a single aggregation rule satisfying all conditions of Theorem 2 except non-imposition: the *absurd* rule, which maps each $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ to the judgment set $F(J_1, ..., J_n) = X$. The absurd rule obviously satisfies all other conditions. In particular, dynamic rationality is trivially satisfied with respect to any conservative revision operator, because conservativeness prevents the collective from ever revising its degenerate judgment set $J = X$.

But why do other constant aggregation rules on $\mathcal{J}^n$ fail to satisfy all other conditions? While this fact already follows from our proof in Appendix A.2, let us now give an intuition. Consider a constant rule on $\mathcal{J}^n$ which always generates some given judgment set J, and let $J \neq X$. Pick a $p \in X \backslash J$, and let p be contingent for the sake of this illustration. Since p is collectively rejected *regardless* of which individuals accept p, no coalition whatsoever is winning for p. Therefore, supposing systematicity, no coalition is winning for any proposition in X. So no proposition is ever collectively accepted: $J = \varnothing$. But then dynamic rationality fails, because whenever the individuals learn some (contingent) proposition q (for instance $q = p$), then the revised judgment profile still aggregates into $J = \varnothing$, although dynamic rationality would have required the collective to come to acquire the judgment set $\varnothing|q$, which contains q, assuming revision is successful.

C.3 Possibilities without monotonicity

In identifying a non-monotonic escape route, we have limited attention to agendas with two structural properties, and introduced a particular revision operator for such agendas. We have claimed that these revision operators obey our requirements, and that so-called *parity rules* satisfy all conditions of Theorem 2 except monotonicity. Both claims are now established formally.

Proposition 5 *For agendas with both properties, the revision operator defined earlier is regular and rationality-preserving.*

Proof. Let X satisfy both conditions. The relevant revision operator is obviously regular. To show that it preserves rationality, assume J is rational and p is non-contradictory. We must show that $J|p$ is rational. This is obvious if $p \in J$, as then $J|p = J$. Henceforth let $p \notin J$. So $J|p = (X_p \backslash J) \cup (J \backslash X_p)$, where '$X_p$' is the earlier-defined subagenda. Since J contains exactly one member of each pair $q, \neg q \in X$, so does $J|p$. It thus remains to show that $J|p$ is consistent. For a contradiction, let $J|p$ be inconsistent. Pick a minimal inconsistent subset Y of $J|p$. Noting that p is contingent (it is non-contradictory by assumption and non-tautological by $p \notin J$), it follows that $|Y \cap X_p| \in \{0, 2, 4, ...\}$. So, since X is affine (i.e., not pair-negatable), the set Y' arising from Y by negating the members of $Y \cap X_p$ is again inconsistent. So, as $Y' \subseteq J$, also J is inconsistent, a contradiction. ∎

Proposition 6 *If the agenda X has both properties (and contains at least one contingent proposition, e.g., is non-simple), then all parity rules with $|M| \neq 1$ satisfy each condition of the theorem except monotonicity.*

Proof. Let X be as specified. Consider the parity rule F whose (odd-sized) subgroup $M \subseteq N$ satisfies $|M| \neq 1$. Clearly, F is universal, non-oligarchic, non-constant, systematic, and non-monotonic, where non-oligarchy and non-monotonicity hold because $|M| \neq 1$ (and because X contains a contingent proposition). To prove dynamic rationality, consider a profile $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and a $p \in X$ such that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}^n$ (p is non-contradictory because $J_1|p$ is rational and contains p). We fix a $q \in X$ and must show that

$$q \in F(J_1|p, ..., J_n|p) \Leftrightarrow q \in F(J_1, ..., J_n)|p. \tag{9}$$

Note that $F(J_1, ..., J_n) \in \mathcal{J}$ since parity rules guarantee rationality for affine, i.e., non-pair-negatable, agendas (Dokow and Holzman 2010).

Case 1: $q \in X \backslash X_p$. Then, as $J_1, ..., J_n$ are rational, by definition of revision we have

$$q \in J_i \Leftrightarrow q \in J_i|p \text{ for } i = 1, ..., n,$$

which by independence of parity rules implies

$$q \in F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1|p, ..., J_n|p).$$

Analogously, as $F(J_1, ..., J_n)$ is rational, by definition of revision we have

$$q \in F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1, ..., J_n)|p.$$

These two equivalences together imply (9).

Case 2: $q \in X_p$. By definition of revision, for all $i \in M$, as $J_i \in \mathcal{J}$,

$$q \in J_i|p \Leftrightarrow \begin{cases} q \in J_i & \text{if } p \in J_i \\ q \notin J_i & \text{if } p \notin J_i, \end{cases} \tag{10}$$

and analogously, as $F(J_1, ..., J_n) \in \mathcal{J}$,

$$q \in F(J_1, ..., J_n)|p \Leftrightarrow \begin{cases} q \in F(J_1, ..., J_n) & \text{if } p \in F(J_1, ..., J_n) \\ q \notin F(J_1, ..., J_n) & \text{if } p \notin F(J_1, ..., J_n). \end{cases} \tag{11}$$

It will prove useful to prove a simple combinatorial fact:

$$\text{finite sets } S \text{ and } S' \text{ have same parity if and only if } \left|S \bigtriangleup S'\right| \text{ is even.} \tag{12}$$

Here, the parity of a set is 'even' or 'odd', depending on whether its cardinality is even or odd, and $S \bigtriangleup S'$ denotes the symmetric difference $(S \backslash S') \cup (S' \backslash S)$. The equivalence (12) holds because, for any finite sets S and S', firstly S and S' have same parity if and only if $|S| + |S'|$ is even, and secondly $|S| + |S'|$ is even if and only if $|S \bigtriangleup S'|$ is even as

$$|S| + \left|S'\right| = \left|S \cap S'\right| + \left|S \backslash S'\right| + \left|S \cap S'\right| + \left|S' \backslash S\right| = 2\left|S \cap S'\right| + \left|S \bigtriangleup S'\right|.$$

For all $r \in X$, let $M_r := \{i \in M : r \in J_i\}$ and $M'_r = \{i \in M : r \in J_i|p\}$. By (10),

$$M_q \bigtriangleup M'_q = M \backslash M_p. \tag{13}$$

We consider two subcases.

Subcase 2.1: $|M_p|$ is odd. Then, by definition of parity rules, $p \in F(J_1, ..., J_n)$, so that by (11)

$$q \in F(J_1, ..., J_n)|p \Leftrightarrow q \in F(J_1, ..., J_n). \tag{14}$$

As $|M|$ and $|M_p|$ are odd, $|M \backslash M_p|$ $(= |M| - |M_p|)$ is even. Hence, by (13), $\left|M_q \bigtriangleup M'_q\right|$ is even, so that by (12) M_q and M'_q have same parity. Thus, by definition of parity rules,

$$q \in F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1|p, ..., J_n|p).$$

This equivalence and the equivalence (14) imply (9).

Subcase 2.2: $|M_p|$ is even. Then, firstly, $p \notin F(J_1, ..., J_n)$, so that by (11)

$$q \in F(J_1, ..., J_n)|p \Leftrightarrow q \notin F(J_1, ..., J_n). \tag{15}$$

As $|M|$ is odd and $|M_p|$ is even, $|M \backslash M_p|$ ($= |M| - |M_p|$) is odd. Hence, by (13), $\left|M_q \bigtriangleup M'_q\right|$ is odd, and so by (12) M_q and M'_q have opposed parity. Therefore, by definition of parity rules,

$$q \notin F(J_1, ..., J_n) \Leftrightarrow q \in F(J_1|p, ..., J_n|p).$$

Combining this equivalence with (15), we again obtain (9). ■

C.4 Possibilities without non-oligarchy

We have specified two types of aggregation rule that satisfy all conditions of Theorem 2 except non-oligarchy. The first of these oligarchic escape routes are trivial: they are the dictatorships. We here focus on the second, less trivial, oligarchic possibility. This possibility was restricted to a special agenda (of the form $X = \{\pm p_1, \pm p_2, \pm p_3\}$ with certain logical interconnections) and a special revision operator, as defined above. We now formally establish that this revision operator indeed has the desirable properties, and that oligarchies become dynamically rational (they obviously satisfy the other conditions in Theorem 2 except non-oligarchy).

The sets $J_p \in \mathcal{J}$ (for $p \in X$ and $J \in \mathcal{J}$) are defined as before, and $\mathcal{J}^+ \subseteq \mathcal{J}$ still denotes the set of consistent and deductively closed judgment sets. Recall that for all $J \in \mathcal{J}^+$

$$J|p = \cap_{J' \in \mathcal{J}: J \subseteq J'} J'_p, \tag{16}$$

which in the special case of a rational $J \in \mathcal{J}$ implies

$$J|p = J_p \text{ if } J \in \mathcal{J}. \tag{17}$$

Proposition 7 *The specified revision operator for the special agenda X is regular and rationality-preserving.*

Proof. Revision is rationality-preserving by definition. To show that revision is successful, consider any $p \in X$ and $J \subseteq X$. We show that $p \in J|p$. If $J \notin \mathcal{J}^+$, then $p \in J|p$ by assumption. If $J \in \mathcal{J}^+$, then $p \in J|p$ because in (16) each J'_p contains p.

To finally show that revision is conservative, consider any $p \in J \subseteq X$. We prove that $J|p = J$. If $p \notin \mathcal{J}^+$, then this again holds by assumption. If $J \in \mathcal{J}^+$, then it holds because

$$J|p = \cap_{J' \in \mathcal{J}: J \subseteq J'} J'_p = \cap_{J' \in \mathcal{J}: J \subseteq J'} J' = J,$$

where the first equality holds by definition of $J|p$, the second because each J'_p equals J' (as $p \in J'$), and the third by Lemma 3 below. ■

The following is a general logical fact about deductively closed judgment sets, which does not depend on our specific agenda.

Lemma 3 *For an arbitrary agenda X, the consistent and deductively closed judgment sets are the intersections of one or more rational judgments:*

$$\mathcal{J}^+ = \{\cap_{J \in S} J : S \subseteq \mathcal{J}, S \neq \varnothing\}.$$

In particular, each $H \in \mathcal{J}^+$ is the intersection of its rational extensions:

$$H = \cap_{J \in \mathcal{J}: H \subseteq J} J.$$

Proof. First, any intersection $\cap_{J \in S} J$ with $S \subseteq \mathcal{J}$ is deductively closed, and if $S \neq \varnothing$ also consistent, hence in $\mathcal{J}^+$. Conversely, consider any $H \in \mathcal{J}^+$ and define $S = \{J \in \mathcal{J} : H \subseteq J\}$. As H is consistent, $S \neq \varnothing$. We show $H = \cap_{J \in S} J$. Clearly, $H \subseteq \cap_{J \in S} J$. To see why $\cap_{J \in S} J \subseteq H$, note that an $p \in \cap_{J \in S} J$ is entailed by H, hence belongs to H by deductive closure. ■

The following lemma tells us which judgment sets are consistent and deductively closed for our specific agenda:

Lemma 4 *For the special agenda X considered here, the set of consistent and deductively closed judgment sets is*

$$\mathcal{J}^+ = \mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\}.$$

Proof. Consider the given agenda. First, $\mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\} \subseteq \mathcal{J}^+$, since each rational or singleton or empty judgment set is consistent and moreover deductively closed, for agenda in question. Conversely, consider a judgment set $H \notin \mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\}$. We show that $H \notin \mathcal{J}^+$. We can exclude that H contains both members of some issue $\{\pm p_k\}$, as otherwise H is obviously inconsistent, hence outside $\mathcal{J}^+$. As $H \notin \mathcal{J} \cup \{\{p\} : p \in X\} \cup \{\varnothing\}$, the number of issues with which H intersects (defined by $|\{k \in: H \cap \{\pm p_k\} \neq \varnothing\}|$) is not 3, not 1, and not 0. So that number is 2, i.e., H is a two-proposition set. Thus H is not deductively closed, since any two-proposition subset of X entails a third proposition from the remaining issue (e.g., $\{p_1, p_2\}$ entails p_3, and $\{p_1, \neg p_2\}$ entails $\neg p_3$). Hence, $H \notin \mathcal{J}^+$. ■

Proposition 8 *For the special agenda X considered here, every oligarchy is dynamically rational with respect to the above revision operator.*

Proof. Consider the given agenda X and revision operator. Let F be an oligarchy, with set of oligarchs M $(\neq \varnothing)$. To prove dynamic rationality, let $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and $p \in X$. We show that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, i.e., that

$$\cap_{i \in M}(J_i|p) = (\cap_{i \in M} J_i)|p.$$

On the left, each $J_i|p$ reduces to $(J_i)_p$ by (17), as $J_i \in \mathcal{J}$. On the right, $(\cap_{i \in M} J_i)|p$ reduces to $\cap_{J \in \mathcal{J}: \cap_{i \in M} J_i \subseteq J} J_p$ by (16), as $\cap_{i \in M} J_i \in \mathcal{J}^+$ by Lemma 3. So we must show that

$$\cap_{i \in M}(J_i)_p = \cap_{J \in \mathcal{J}: \cap_{i \in M} J_i \subseteq J} J_p. \tag{18}$$

By Lemmas 3 and 4, $\cap_{i \in M} J_i$ belongs to $\mathcal{J}^+ = \mathcal{J} \cup \{\{q\} : q \in X\} \cup \{\varnothing\}$. This leads to three cases.

Case 1: $\cap_{i \in M} J_i \in \mathcal{J}$. Here all J_i $(i \in M)$ coincide, say with $J^* \in \mathcal{J}$. Hence (18) holds, since both sides equal J^*_p.

Case 2: $\cap_{i \in M} J_i = \{q\}$ for some $q \in X$. Here the sets J_i $(i \in M)$ are rational extensions of $\{q\}$, but not all same one extension. So, since $\{q\}$ has just two rational extensions, the sets J_i $(i \in M)$ include *all* rational extensions of $\{q\}$; formally, $\{J_i : i \in M\} = \{J \in \mathcal{J} : \{q\} \subseteq J\}$. So each side of (18) equals $\cap_{J \in \mathcal{J}: \{q\} \subseteq J} J_p$, proving (18).

Case 3: $\cap_{i \in M} J_i = \varnothing$. To establish (18), we prove that both sides equal $\{p\}$. The right side of (18) reduces to $\cap_{J \in \mathcal{J}} J_p$, which equals $\{p\}$ by definition of the sets J_p $(J \in \mathcal{J})$. We must show that also the left side equals $\{p\}$. Note that $\cap_{i \in M}(J_i)_p \supseteq \{p\}$, since each $(J_i)_p$ contains p. To prove that $\cap_{i \in M}(J_i)_p \subseteq \{p\}$, we call J' and J'' the two rational extensions of $\{p\}$, and distinguish between three subcases.

Subcase 3.1: J' and J'' are among the sets J_i $(i \in M)$. Then

$$\cap_{i \in M}(J_i)_p \subseteq J'_p \cap J''_p = J' \cap J'' = \{p\},$$

where the second equality holds because $J'_p = J'$ (as $p \in J'$) and $J''_p = J''$ (as $p \in J''$). So $\cap_{i \in M}(J_i)_p \subseteq \{p\}$.

Subcase 3.2: Exactly one of J' and J'' is among the sets J_i $(i \in M)$. Without loss of generality, let J' but not J'' be among these sets, and let $p = p_1$, $J' = \{p_1, p_2, p_3\}$, and $J'' = \{p_1, \neg p_2, \neg p_3\}$ (other cases are handled analogously). Since $\cap_{i \in M} J_i = \varnothing$, among the sets J_i $(i \in M)$ there are sets $\hat{J}$ and $\tilde{J}$ such that $p_2 \notin \hat{J}$ and $p_3 \notin \tilde{J}$. These two sets cannot contain p; otherwise they would be rational extensions of $\{p\}$, so that J' would not be the only rational extension of $\{p\}$ among the sets J_i $(i \in M)$. Moreover, $\hat{J} \neq \tilde{J}$; otherwise both sets would equal $\{p_1, \neg p_2, \neg p_3\} = J''$, so that J'' would be among the sets J_i ($i \in M$). These two facts imply that $\hat{J}_p \neq \tilde{J}_p$, by construction of the sets J_p ($J \in \mathcal{J}$). Since $\hat{J}_p$ and $\tilde{J}_p$ are distinct rational extensions of $\{p\}$, $\hat{J}_p \cap \tilde{J}_p = \{p\}$. Meanwhile $\cap_{i \in M}(J_i)_p \subseteq \hat{J}_p \cap \tilde{J}_p$. So $\cap_{i \in M}(J_i)_p \subseteq \{p\}$.

Subcase 3.3: Neither J' nor J'' is among the sets J_i ($i \in M$). This subcase is in fact impossible, because it would imply that all J_i ($i \in M$) are extensions of $\{\neg p\}$, contradicting that $\cap_{i \in M} J_i = \varnothing$. ■

C.5 Possibilities without systematicity

We have discussed two non-systematic escape routes, one that retains independence and one that does not. We now treat both in turn, providing the missing proofs.

The first non-systematic escape route. Here we have specified a particular asymmetric unanimity rule. To make this rule dynamically rational, we have assumed a special non-simple agenda X and an equally special revision operator. We now establish that the revision operator has the desired properties, and that the rule is indeed dynamically rational with respect to it; the rule obviously satisfies all other conditions of Theorem 2 except systematicity, in fact except the neutrality part of systematicity.

Proposition 9 *The relevant revision operator for the given non-simple agenda is regular and rationality-preserving.*

Proof. Assume this agenda X. Revision is obviously regular. To see that revision preserves rationality, consider any rational J and any $p \in X$. $J|p$ is complete because Y contains a member of each pair $\{\pm q\} \subseteq X$. $J|p$ is consistent because it includes neither Y, nor any pair $\{\pm q\}$, hence includes no minimal inconsistent set. ■

Proposition 10 *The relevant asymmetric unanimity rule for the given non-simple agenda is dynamically rational with respect to the given revision operator.*

Proof. Consider the given agenda, revision operator, and aggregation rule. To verify dynamic rationality, consider any $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and any $p \in X$ such that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}^n$ (in fact, membership of $\langle J_1|p, ..., J_n|p \rangle$ in $\mathcal{J}^n$ already follows from the fact that X contains no contradictory proposition and revision preserves rationality). We distinguish between three cases:

- *Case 1:* $p \in Y$ and $p \in J_1, ..., J_n$. Then $p \in F(J_1, ..., J_n)$. By conservativeness, neither any of $J_1, ..., J_n$ nor $F(J_1, ..., J_n)$ changes by learning p. So $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$.

- *Case 2:* $p \in Y$ and $p \notin J_i$ for some i. Since all of $J_1|p, ..., J_n|p$ contain p by successfulness, and since $J_i|p$ contains $\neg y$ for all $y \in Y \backslash \{p\}$, we have $F(J_1|p, ..., J_n|p) = \{p\} \cup \{\neg y : y \in Y \backslash \{p\}\}$. Meanwhile, as $p \in Y$ and $p \notin J_i$, we have $p \notin F(J_1, ..., J_n)$, so that $F(J_1, ..., J_n)|p = \{p\} \cup \{\neg y : y \in Y \backslash \{p\}\}$. So, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$.
- *Case 3:* $p \notin Y$. Then the revised profile $\langle J_1|p, ..., J_n|p\rangle$ displays unanimous acceptance of p and (as $p \notin Y$) coincides with the initial profile $\langle J_1, ..., J_n\rangle$ outside the issue $\{\pm p\}$. So, $F(J_1|p, ..., J_n|p)$ contains p and coincides with $F(J_1, ..., J_n)$ outside $\{\pm p\}$. Also $F(J_1, ..., J_n)|p$ contains p and (because $p \notin Y$) coincides with $F(J_1, ..., J_n)$ outside $\{\pm p\}$. Hence, $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$. ■

The second non-systematic escape route. Here we have considered any agenda X with a rational judgment set K whose complement $\overline{K} = X \backslash K$ is also rational. For such an agenda, we have defined a particular (regular and rationality-preserving) revision operator, and a special type of aggregation rule. Provided the agenda is non-trivial, this rule is non-independent and satisfies all aggregation assumptions of Theorem 2 except systematicity. We now establish the non-trivial claim that this rule is dynamically rational if the agenda satisfies an additional symmetry assumption:

Proposition 11 *Given an agenda X with a judgment set $K \in \mathcal{J}$ such that $\overline{K} = X \backslash K \in \mathcal{J}$ and $|J \cap K| = |J \cap \overline{K}|$ for all $J \in \mathcal{J} \backslash \{K, \overline{K}\}$, the specified rule F is dynamically rational with respect to the specified revision operator.*

Under the agenda assumption in Proposition 11, the definition of the rule can be restated as follows: for all profiles $\langle J_1, ..., J_n\rangle \in \mathcal{J}^n$, writing $n^J = |\{i : J_i = J\}|$ for all $J \subseteq X$,

$$F(J_1, ..., J_n) = \begin{cases} K & \text{if } n^K > n^{\overline{K}} \\ \overline{K} & \text{if } n^{\overline{K}} > n^K \\ K & \text{if } n^K = n^{\overline{K}} \ \& \ \cap_i J_i \subseteq K \ \& \ \cap_i J_i \neq \varnothing \\ \overline{K} & \text{if } n^K = n^{\overline{K}} \ \& \ \cap_i J_i \subseteq \overline{K} \ \& \ \cap_i J_i \neq \varnothing \\ \cap_i J_i & \text{if } n^K = n^{\overline{K}} \ \& \ \cap_i J_i \nsubseteq K, \overline{K} \\ K \text{ or } \overline{K} \text{ or } \cap_i J_i \ (= \varnothing) & \text{if } n^K = n^{\overline{K}} \ \& \ \cap_i J_i = \varnothing. \end{cases}$$

The following lemma justifies this alternative expression for F:

Lemma 5 *Given the agenda in Proposition 11, for all profiles $\langle J_1, ..., J_n\rangle \in \mathcal{J}^n$, using the earlier notation 'n_J' and 'n^J',*

$$n_K > (<, =) \ n_{\overline{K}} \Leftrightarrow n^K > (<, =) \ n^{\overline{K}}.$$

Proof. Let X and $\langle J_1, ..., J_n \rangle$ be as specified. We show the equivalence for '>'; the equivalence for '<' and '=' are provable analogously. The inequality $n_K > n_{\overline{K}}$ means that $\sum_i |J_i \cap K| > \sum_i |J_i \cap \overline{K}|$. By assumption on the agenda, writing $k := |X|/2 = |K| = |\overline{K}|$, the latter inequality can be rewritten as

$$k |\{i : J_i = K\}| + \frac{k}{2} |\{i : J_i \neq K, \overline{K}\}| > k |\{i : J_i = \overline{K}\}| + \frac{k}{2} |\{i : J_i \neq K, \overline{K}\}| .$$

This reduces to $|\{i : J_i = K\}| > |\{i : J_i = \overline{K}\}|$, i.e., to $n^K > n^{\overline{K}}$. ■

Before we can prove Proposition 11, a second lemma is needed.

Lemma 6 *Given the agenda in Proposition 11, for all profiles $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$, using the earlier notation 'n_J' and 'n^J',*
(a) if $n_K > n_{\overline{K}}$ (equivalently $n^K > n^{\overline{K}}$) then $\cap_i J_i \subseteq K$,
(b) if $n_{\overline{K}} > n_K$ (equivalently $n^{\overline{K}} > n^K$) then $\cap_i J_i \subseteq \overline{K}$.

Proof. Let X and $\langle J_1, ..., J_n \rangle$ be as specified. To show (a), assume $n_K > n_{\overline{K}}$, equivalently $n^K > n^{\overline{K}}$. In particular, $n^K > 0$. So someone has judgment set K. Hence, $\cap_i J_i \subseteq K$. The proof of (b) is analogous. ■

Proof of Proposition 11. Let X, F, and the revision operator be as specified. Consider $\langle J_1, ..., J_n \rangle \in \mathcal{J}^n$ and $p \in X$. Note that $\langle J_1|p, ..., J_n|p \rangle \in \mathcal{J}^n$. Write $J = F(J_1, ..., J_n)$ and $J' = F(J_1|p, ..., J_n|p)$. We must show that $J' = J|p$. We assume without loss of generality that $p \in K$; the proof is analogous if $p \in \overline{K}$. For any judgment set $H \subseteq X$, the number of individuals with judgment set H in, respectively, the initial and revised profile are denoted by $n^H = |\{i : J_i = H\}|$ and $(n^H)' = |\{i : J_i' = H\}|$.

Case 1: $n^K > n^{\overline{K}}$. Here $J = K$. Hence, $J|p = K$. Moreover, $(n^{\overline{K}})' = 0$, since $\overline{K}$ does not contain p; and $(n^K)' \geq n^K$ (> 0), since everyone who used to have judgment set K has revised judgment set K. Therefore, $(n^K)' > (n^{\overline{K}})'$, whence $J' = K$. This shows that $J' = J|p$ $(= K)$.

Case 2: $n^K < n^{\overline{K}}$. Here $J = \overline{K}$. Thus, $J|p = K$. Moreover, $(n^{\overline{K}})' = 0$, as in Case 1; and $(n^K)' \geq n^{\overline{K}}$ (> 0), as everyone who used to have judgment set $\overline{K}$ has revised judgment set K. So $(n^K)' > (n^{\overline{K}})'$, whence $J' = K$. Hence again $J' = J|p$ $(= K)$.

Case 3: $n^K = n^{\overline{K}}$ & $\cap_i J_i \subseteq K$ & $\cap_i J_i \neq \varnothing$. Here $J = K$. So $J|p = K$.

Subcase 3.1: $p \in \cap J_i$. Then each J_i contains p, whence each $J_i|p$ equals J_i. So $\langle J_1|p, ..., J_n|p \rangle = \langle J_1, ..., J_n \rangle$, and thus $J' = J$. In sum, $J|p = J'$ $(= J = K)$.

Subcase 3.2: $p \in K \backslash \cap J_i$. Then $(n^K)' > n^K$, because everyone with initial judgment set K has revised judgment set K, and someone (who initially rejected p) newly acquires the judgment set K. Meanwhile $(n^{\overline{K}})' = 0$, as before. So, $(n^K)' > (n^{\overline{K}})'$, implying that $J' = K$. So again, $J|p = J'$ $(= J = K)$.

Case 4: $n^K = n^{\overline{K}}$ & $\cap_i J_i \subseteq \overline{K}$ & $\cap_i J_i \neq \varnothing$. Here $J = \overline{K}$. Thus $J|p = K$. Note that $p \notin \cap_i J_i$, since $p \in K$ while $\cap_i J_i \subseteq \overline{K}$. So, by the argument in Subcase 3.2, $J' = K$. In sum, $J|p = J'$ $(= K)$.

Case 5: $n^K = n^{\overline{K}}$ & $\cap_i J_i \nsubseteq K, \overline{K}$. Here $J = \cap_i J_i$.

Subcase 5.1: $p \in \cap_i J_i$ $(= J)$. Then $J|p = J$. Moreover, as in Subcase 3.1, $J' = J$. In sum, $J|p = J'$ $(= J = \cap_i J_i)$.

Subcase 5.2: $p \in K \backslash (\cap_i J_i)$. Then $J|p = K$. As in Subcase 3.2, $J' = K$. In sum, $J|p = J'$ $(= K)$.

Case 6: $n^K = n^{\overline{K}}$ & $\cap_i J_i = \varnothing$. Here $J = K$ or $J = \overline{K}$ or $J = \cap_i J_i$ $(= \varnothing)$. In all three cases $J|p = K$. Moreover, since $p \notin \cap_i J_i$, by the argument in Subcase 3.2 we have $J' = K$. In sum, $J|p = J'$ $(= K)$. ■

C.6 Possibilities for simple agendas

If the agenda is simple, then plenty of aggregation rules satisfy the conditions of the theorem, in the case of dynamic rationality assuming a particular revision operator defined above. This result was stated as 'Proposition 1'. We now show first that this revision operator satisfies our desiderata, and then that Proposition 1 holds. Notation is as above.

Proposition 12 *The specified revision operator is regular, and for a simple agenda also rationality-preserving.*

Proof. Consider the specified revision operator. Let $J \subseteq X$ and $p \in J$. If $J \notin \mathcal{J}$, then regularity applied to J and p holds by stipulation, and rationality preservation applied to J and p holds vacuously. Now suppose $J \in \mathcal{J}$. We must show three things.

1. (*successfulness*) We have to show that $p \in J|p$. Note that J contains p or $\neg p$, as $J \in \mathcal{J}$. In the first case, $J|p$ contains p_p, which equals p because $\{p, p\}$ is consistent (as $J \in \mathcal{J}$). In the second case, $J|p$ contains $(\neg p)_p$, which equals p as $\{\neg p, p\}$ is inconsistent. So, in any case, $p \in J|p$.

2. (*conservativeness*) If $p \in J$, then $J|p = J$ because for each $q \in J$ we have $q_p = q$ (since $\{q, p\}$ is consistent, being included in the rational judgment set J).

3. (*rationality preservation*) Assume X is simple. For a contradiction, let p be non-contradictory and let $J|p \notin \mathcal{J}$. Then J is inconsistent, hence has a minimal inconsistent subset Y. By simplicity of X, $|Y| \leq 2$, say $Y = \{q_p, q'_p\}$ for some $q, q' \in J$. By

definition of q_p and q'_p (and by p's non-contradictoriness), the sets $\{q_p, p\}$ and $\{q'_p, p\}$ are consistent. Since $\{q_p, p\}$ is consistent and q_p entails $\neg q'_p$, also $\{\neg q'_p, p\}$ is consistent. Similarly, since $\{q'_p, p\}$ is consistent and q'_p entails $\neg q_p$, $\{\neg q_p, p\}$ is consistent. Now, as $\{q_p, p\}$ and $\{\neg q_p, p\}$ are consistent, $q_p = q$ by definition. Analogously, as $\{q'_p, p\}$ and $\{\neg q'_p, p\}$ are consistent, $q'_p = q'$. So, $\{q_p, q'_p\} = \{q, q'\}$, a subset of the consistent set J. This contradicts the inconsistency of $\{q_p, q'_p\}$. ■

Proof of Proposition 1. Consider a simple agenda X, a unanimity-preserving independent rule $F : \mathcal{J}^n \to \mathcal{J}$, and the revision operator defined above. To prove dynamic rationality, consider a profile $\langle J_1, ..., J_n\rangle \in \mathcal{J}^n$ and a learnt proposition $p \in X$ such that $\langle J_1|p, ..., J_n|p\rangle \in \mathcal{J}^n$ (i.e., such that p is non-contradictory). To prove that $F(J_1|p, ..., J_n|p) = F(J_1, ..., J_n)|p$, we fix a $q \in X$ and show that $F(J_1|p, ..., J_n|p)$ and $F(J_1, ..., J_n)|p$ coincide on q. We distinguish between three cases.

- *Case 1*: $\{q, p\}$ and $\{\neg q, p\}$ are both consistent. Then $J|p$ coincides with J on q for all $J \in \mathcal{J}$. In particular, $J_i|p$ coincides with J_i on q for all i, and hence (by independence) $F(J_1|p, ..., J_n|p)$ coincides with $F(J_1, ..., J_n)$ on q. Meanwhile $F(J_1, ..., J_n)|p$ also coincides with $F(J_1, ..., J_n)$ on q. So, $F(J_1|p, ..., J_n|p)$ coincides with $F(J_1, ..., J_n)|p$ on q.
- *Case 2*: $\{q, p\}$ is consistent and $\{\neg q, p\}$ is inconsistent. Then $q \in J|p$ for all $J \in \mathcal{J}$. In particular, $q \in J_i|p$ for all i, whence by unanimity preservation and independence $q \in F(J_1|p, ..., J_n|p)$. Meanwhile $q \in F(J_1, ..., J_n)|p$. So $F(J_1|p, ..., J_n|p)$ and $F(J_1, ..., J_n)|p$ coincide on q.
- *Case 3*: $\{q, p\}$ is inconsistent and $\{\neg q, p\}$ is consistent. Then $\neg q \in J|p$ for all $J \in \mathcal{J}$. In particular, $\neg q \in J_i|p$ for all i, whence by unanimity preservation and independence $\neg q \in F(J_1|p, ..., J_n|p)$. Meanwhile $\neg q \in F(J_1, ..., J_n)|p$. So $F(J_1|p, ..., J_n|p)$ and $F(J_1, ..., J_n)|p$ coincide on $\neg q$, hence on q. ■