The Generic Filter Property in Nonstandard Analysis

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Abstract

In this paper two new combinatorial principles in nonstandard analysis are isolated and applications are given. The second principle provides an equivalent formulation of Henson's isomorphism property.

Introduction.

In his 1974 paper [He], C. Ward Henson introduced the *isomorphism property* (IP) for nonstandard models of analysis, giving a number of interesting applications. He showed that IP gives a nice picture of internal sets and applied it to investigate nonstandard objects in functional analysis. Just to mention two results in that paper, it follows from IP that any two infinite internal sets have the same cardinality, and that the nonstandard hull of any Banach space is isometrically isomorphic to the nonstandard hull of some separable Banach space.

Later on, David Ross [Ro] introduced the *special model axiom*, a property strictly stronger than IP, and used it to investigate internal orders and certain questions concerning Loeb measures. Recently, several papers appeared in the literature (see [J3] for a survey) where the strength of these principles is investigated and applications are given. In particular, Renling Jin solved almost all the problems left open in [Ro] and, jointly with Saharon Shelah, found an equivalent formulation of IP, namely the *resplendency property* (RP). With respect to IP, in several applications the use of RP considerably simplifies proofs (see [JS]).

All the above principles are formulated as model-theoretic properties, and in their known applications a heavy use of the formalism of first-order logic seems to be essential. The goal of this paper is to show that formulating and applying the isomorphism property is possible using only the basics of nonstandard analysis. To this end, we will show that what makes IP stronger than the usual κ-saturation, is a purely combinatorial property of internal sets, namely the principle Δ_1.

One of the consequences of the isomorphism property is the usual saturation property. Renling Jin for countable languages, and subsequently James Schmerl

in the general case, showed that the isomorphism property for languages of cardinality less than κ is equivalent to IP for finite languages plus κ-saturation. The main result in this paper is the equivalence between the isomorphism property for finite languages and our combinatorial principle Δ_1.

In our opinion, the material presented here should suggest that the isomorphism property is in fact an easy-to-handle principle for anybody working in nonstandard analysis. To justify this claim, we will assume Δ_1 and give new proofs of several applications, including almost all results appeared so far in the literature, which are consequences of IP. With the only exception of the last section, everything in this paper is formulated within the usual language of nonstandard analysis. In particular, any technical notion and result from first order logic, such as the theory of a structure, the consistency of a set of formulas, the compactness theorem, the Löwenheim-Skolem theorem etc., is avoided.

Our basic definitions and notations in nonstandard analysis follow [CK] §4.4. In particular, by a nonstandard model we mean a triple $\langle V(X), V(Y), * \rangle$, where $V(X)$ and $V(Y)$ denote the superstructures of height ω over the infinite base sets X and Y respectively, and where the nonstandard embedding $* : V(X) \to V(Y)$ is a bounded elementary extension with $^*X = Y$. [1] For simplicity, we shall directly assume X and Y to be sets of atoms, and that the set of natural numbers $\mathbb{N} \subseteq X$. Internal sets are elements of the internal submodel $\mathcal{I}_* = \{y \in V(Y) : y \in {}^*x \text{ for some } x \in V(X)\}$. If A is an internal set, we denote by $\mathcal{P}_I(A)$ the internal collection of its internal subsets. A *finite set is an internal set in 1-1 internal bijection with some initial segment $[0, \xi] = \{0, \dots, \xi\} \subset {}^*\mathbb{N}$. A *infinite set is an internal set which is not *finite. For unexplained notions and results in nonstandard analysis, good references are the surveys in [ACH]. For model-theoretic topics, we refer to [CK].

§1. The principle Δ_0.
Let us recall some basic algebraic notions. Let $\langle P, \leq \rangle$ be a partially ordered set (*poset*). [2] A subset $D \subseteq P$ is *dense* if for each $p \in P$ there exists $d \in D$ with $d \leq p$. A subset $F \subseteq P$ is a *filter* if (*i*) $p' \geq p \in F$ implies $p' \in F$ (upward closure) and (*ii*) $p, p' \in F$ implies $q \leq p, p'$ for some $q \in F$ (compatibility). We now borrow a combinatorial notion from the forcing tecnique in set theory, and transfer it into the nonstandard analysis context. Let P be an internal poset. We say that a filter $G \subseteq P$ is *generic* if $G \cap D \neq \emptyset$ for each internal dense $D \subseteq P$. Now, consider the following *generic filter property* Δ_0 for nonstandard models.

$$\Delta_0 : \quad \textit{Each internal poset has a generic filter.}$$

In §3, we will show that there is plenty of nonstandard models where Δ_0 holds.

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[1] Recall that a base set A is a set which behaves as a set of atoms with respect to its superstructure, i.e. $a \cap x = \emptyset$ for all $a \in A$ and $x \in V(A)$.

[2] Abusing notation, sometimes we shall identify P with $\langle P, \leq \rangle$.

Before going to applications, let's see a sample of poset we will be considering throughout the paper. For internal sets A and B, denote by

$$\mathcal{F}(A, B) = \{f : f \text{ is an internal partial function from } A \text{ to } B\}$$

and set $f \leq g$ if f is an extension of g. [3] Clearly, if A and B are internal, the poset $\langle \mathcal{F}(A, B), \leq \rangle$ is internal as well. One can think of each $f \in \mathcal{F}(A, B)$ as a piece of information concerning a total function $\Phi : A \to B$. In this sense, $f \leq g$ means that f gives more information about Φ than g does. Now, let F be any filter on $\mathcal{F}(A, B)$. By definition of filter, it is easily seen that $\bigcup F = \bigcup \{f : f \in F\}$ is (the graph of) some partial function from A to B. In other words, pieces of information provided by elements of a filter $F \subseteq \mathcal{F}(A, B)$ are compatible with each other, so that they can be consistently put together. We now give a number of applications of Δ_0. These results have been already proved by assuming the isomorphism property IP. However, as Δ_0 is strictly weaker than IP, our proofs yield stronger theorems. [4] Also, note that in all the proofs appeared so far in the literature an essential use of the formalism of first order logic is made, and technical results from model theory are employed. On the contrary, proofs from Δ_0 have an algebraic flavour, they are often definitely shorter and (our opinion) they seem to be easier and closer to intuition.

In the following, when we say that Δ_0 implies a property ψ, we mean that ψ is true in any nonstandard model where Δ_0 holds. References to proofs of related results are given for each of the presented applications.

Let $\langle A, \leq \rangle$ be a poset. For each $a \in A$, denote by $S_a = \{x \in A : x < a\}$ the *initial segment* generated by a and denote by $T_a = \{x \in A : x \not\geq a\} \supseteq S_a$. Note that $a \leq a'$ implies $S_a \subseteq S_{a'}$ and $T_a \subseteq T_{a'}$. If $\langle A, \leq \rangle$ is linearly ordered, then trivially $S_a = T_a$.

Proposition 1.1 ([Ro] Th.5.1; [J1] Th.6; [JS] Appl.3)
Assume Δ_0. Let $\langle A, < \rangle$ be an internal poset without right end-point. Then there exists an external set $X \subset A$ such that $X \cap S_a$ is internal for each $a \in A$.

PROOF. Let $P = \{f \in \mathcal{F}(A, \{0, 1\}) : \operatorname{dom}(f) \subseteq T_a \text{ for some } a \in A\}$. For each $a \in A$ let $\Lambda_a = \{f \in P : S_a \subseteq \operatorname{dom}(f)\}$, and for each internal $B \subseteq A$ let $\Gamma_B = \{f \in P : \exists \, a \in \operatorname{dom}(f) \; f(a) \neq \chi_B(a)\}$, where $\chi_B : B \to \{0, 1\}$ is the characteristic function of B. If $f \in P$ with $\operatorname{dom}(f) \subseteq T_{a'}$, define $g = f \cup \{\langle x, 0 \rangle : x \in S_a \setminus \operatorname{dom}(f)\}$. $g \supseteq f$ is an internal function with $\operatorname{dom}(g) = \operatorname{dom}(f) \cup S_a \subseteq T_{a''}$ where $a'' = a$ if $a \geq a'$, and $a'' = a'$ if $a \not\geq a'$ respectively. Thus $g \in \Lambda_a$ and $g \leq f$. This proves that each Λ_a is dense in P. Proceed similarly to show that also each Γ_B is dense. If G is a generic filter on P, then $F = \bigcup G$ is a partial function. For each $a \in A$, take $a' > a$. As $G \cap \Lambda_{a'} \neq \emptyset$,

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[3] In the usual axiomatic set theory, a function *is* its graph. Thus, a partial function f from A to B is a subset $f \subseteq A \times B$ such that if $\langle a, b \rangle \in f$ and $\langle a, b' \rangle \in f$, then $a = a'$. f is an extension of g if $f \supseteq g$.

[4] In [DH] it is proved that, for any given cardinal κ, there are nonstandard models where $\Delta_0 + \kappa$-saturation holds but IP fails.

clearly $a \in S_{a'} \subseteq \text{dom}(f)$. Thus $\text{dom}(F) = A$ and $F : A \to \{0,1\}$ is the characteristic function of some $X \subseteq A$. X is external because for every internal set B, $\Gamma_B \cap G \neq \emptyset$ implies $B \neq X$. For each $a \in A$, take $f \in G \cap \Lambda_a$. Then $X \cap S_a = \{x \in \text{dom}(f) : x \in S_a \text{ and } f(x) = 1\}$ is internal. $\quad \dashv$

Corollary 1.2
Assume Δ_0. There exists an external $X \subset {}^*\mathbb{N}$ such that $X \cap [0,\xi]$ is internal for each $\xi \in {}^*\mathbb{N}$.

If f is any function, and $X \subseteq \text{dom}(f)$, we denote by $f \lceil X = \{\langle x, f(x) \rangle : x \in X\}$ its restriction to X. If f is 1-1 and $Y \subseteq \text{ran}(f)$, we denote by $f^{-1} \lceil Y = \{\langle x, f(x) \rangle : f(x) \in Y\}$ its co-restriction to Y.

Proposition 1.3 ([HE] Th.1.8; [RO] Th.6.1; [JS] Appl.4)
Assume Δ_0. Let A and B be internal *infinite sets. Then there exist bijections $\varphi, \psi : A \to B$ such that:
(i) for all *finite $a \subset A$ and $b \subset B$, the restrictions $\varphi \lceil a$ and $\varphi^{-1} \lceil b$ are internal functions;
(ii) for all $a \subset A$ and $b \subset B$ such that $A \backslash a$ and $B \backslash b$ are *infinite, the restrictions $\psi \lceil a$ and $\psi^{-1} \lceil b$ are internal functions.

PROOF. Let $P = \{f \in \mathcal{F}(A,B) : f \text{ is } {}^*\text{finite and 1-1}\}$. For each *finite $a \subset A$ let $\Lambda_a = \{f \in P : a \subseteq \text{dom}(f)\}$, and for each *finite $b \subset B$ let $\Gamma_b = \{f \in P : b \subseteq \text{Range}(f)\}$. Consider the following standard property.

> "Let X and Y be infinite sets and let $\pi = \{f \in \mathcal{F}(X \times Y) : f \text{ is a finite 1-1 function}\}$ be partially ordered by reverse inclusion. Then for all finite $x \subset X$ and $y \subset Y$, $\lambda(x) = \{f \in P : x \subseteq \text{dom}(f)\}$ and $\gamma(y) = \{f \in P : y \subseteq \text{Range}(f)\}$ are dense in π".

By transfer, one gets that all Λ_a and Γ_b are dense in P. Take G a generic filter on P. Then $\varphi = \bigcup G : A \to B$ is a bijection which satisfies *(i)*. To prove *(ii)*, proceed similarly by considering $P' = \{f \in \mathcal{F}(A,B) : f \text{ is 1-1 and both } A \backslash \text{dom}(f) \text{ and } B \backslash \text{ran}(f) \text{ are } {}^*\text{infinite}\}$ instead of P. $\quad \dashv$

Corollary 1.4 ([RO] Lemma 3.1)
Assume Δ_0. Any two *infinite internal sets have the same external cardinality.

Proposition 1.5 ([RO] Th.4.5; [J1] Th.5; [JS] Appl.2)
Assume Δ_0. Let $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ be internal linear orders without right endpoints. Then there is an order-preserving cofinal map $\varphi : A' \to B$ where $A' \subseteq A$ is cofinal. Moreover, if $\langle B, \leq \rangle$ is dense, one can take $A' = A$.

PROOF. By a straightforward application of the transfer principle, one gets the following fact. For every $a \in A$ and for every $b \in B$, the sets $\Lambda_a = \{f \in P : a' \geq a \text{ for some } a' \in \text{dom}(f)\}$ and $\Gamma_b = \{f \in P : b' \geq b \text{ for some } b \in \text{ran}(f)\}$ are dense in the internal poset $P = \{f \in \mathcal{F}(A,B) : f \text{ is } {}^*\text{finite and order-preserving}\}$. Moreover, if $\langle B, \leq \rangle$ is dense without end-points, then also each $\Lambda'_a = \{f \in P :$

$a \in \text{dom}(f)\}$ is dense in P. If G is a generic filter on P, then $F = \bigcup G$ is a map with the desired properties. $\quad \dashv$

Corollary 1.6
Assume Δ_0. If $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ are internal linear orders without right endpoint, then they have the same cofinality.

Proposition 1.7
Assume Δ_0. Let $\langle B, \leq \rangle$ be an internal linear order without right end-point. Then there is an order-preserving cofinal map $F : {}^*\mathbb{N} \to B$.

PROOF. Consider the internal poset $P = \{f \in \mathcal{F}({}^*\mathbb{N}, B) : f$ is order-preserving and $\text{dom}(f) = [0, \xi]$ for some $\xi \in {}^*\mathbb{N}\}$. For all $a \in A$ and $b \in B$, the internal sets $\Lambda_a = \{f \in P : a \in \text{dom}(f)\}$ and $\Gamma_b = \{f \in P : b' \geq b$ for some $b' \in \text{ran}(f)\}$ are dense in P. In fact, let $f \in P$ with $\text{dom}(f) = [0, \xi]$. If $b > f(a)$ for every $a \leq \xi$, then $g = f \cup \{\langle \xi + 1, b \rangle\} \leq f$ and $g \in \Gamma_b$. Now suppose $a \notin [0, \xi]$. Apply transfer to the following standard property. "Let $\langle X, < \rangle$ be a linear order without right end-point. Then for each $x \in X$ there is an order-preserving map $\theta : \mathbb{N} \to X$ with $\theta(0) > x$". Pick an internal order-preserving map $\Theta : {}^*\mathbb{N} \to B$ with $\Theta(0) > f(\xi)$ and define $g = f \cup \{\langle \xi + \eta + 1, \Theta(\eta) \rangle : 0 \leq \eta \leq a - \xi - 1\}$. Then $g \leq f$ and $g \in \Lambda_a$. If G is a generic filter on P, then $\bigcup G$ is the map for which we are looking. $\quad \dashv$

Proposition 1.8
Assume Δ_0. If $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ are dense internal linear orders without end-points, then they are isomorphic.

PROOF. Proceed as in the proof of Proposition 1.5: the density of A and B implies that the internal sets $\Lambda'_a = \{f \in P : a \in \text{dom}(f)\}$ and $\Gamma'_b = \{f \in P : b \in \text{Range}(f)\}$ are dense in P for all $a \in A$ and all $b \in B$. $\quad \dashv$

An initial segment $X \subseteq F$ of an ordered field F is a *regular gap* if it is upper bounded without least upper bound, and for every positive $\varepsilon \in F$, there is some $x \in X$ with $x + \varepsilon \notin X$. An ordered field without regular gaps is called *Scott-complete*.

Proposition 1.9 ([KA] Th.4.5)
Assume Δ_0. For every $\xi, \eta \in {}^*\mathbb{R}$ with $\xi < \eta$, there exists a regular gap X such that $\xi \in X < \eta$. [5]

PROOF. Let the internal set $P = \{[A, B]$ closed (nonempty) interval: $\xi < A < B < \eta\}$ be partially ordered by inclusion, and let G be a generic filter on P. We claim that the initial segment $X = \{x \in {}^*\mathbb{R} : x < A$ for some $[A, B] \in G\}$ is a regular gap with the desired properties. Clearly $\xi \in X$. Note that if $x \geq B$ for some $[A, B] \in G$, then $x \notin X$ (if, by contradiction, $x < A'$ for some $[A', B'] \in G$, then $B \leq x < A' \Rightarrow [A, B] \cap [A', B'] = \emptyset$, against the filter

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[5]By $X < \eta$ we mean $x < \eta$ for all $x \in X$.

property of G). In particular, $X < \eta$ and X is upper bounded. For each $a \in {}^*\mathbb{R}$, $\Lambda_a = \{[A,B] \in P : a \notin [A,B]\}$ is dense in P, so $a \notin [A,B]$ for some $[A,B] \in G$, and a cannot be the least upper bound of X. At last, for every positive $\varepsilon \in {}^*\mathbb{R}$, it is easily seen that $\Gamma_\varepsilon = \{[A,B] \in P : B - A < \varepsilon\}$ is dense, so there is $[A,B] \in G$ with $B - A < \frac{\varepsilon}{2}$. Clearly $x = A - \frac{\varepsilon}{2} \in X$ but $x + \varepsilon \notin X$. This completes the proof that X is regular. $\dashv$

§2. The principle Δ_1.

Sometimes a set which is not dense in a poset P can be dense with respect to some subset of P. For instance, let A and B be infinite *finite sets and consider $P = \{f \in \mathcal{F}(A,B) : f \text{ is 1-1}\}$. If the internal cardinality $|A| < |B|$, then for no $a \in A$, the set $\Lambda_a = \{f \in P : a \in \text{dom}(f)\}$ is dense in P. However, if we consider the subset $Q = \{f \in P : f \text{ is finite }\} \subset P$, then each $\Lambda_a \cap Q$ is dense in Q. We say that a set D is Q-dense if $D \cap Q$ is dense in Q. If $\mathcal{D}$ is any collection, we say that a filter G is $\mathcal{D}$-generic if $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. We now generalize Δ_0 and introduce the principle Δ_1.

> Δ_1 : Let P be an internal poset, and let $\{\mathcal{D}_\xi : \xi \in {}^*\mathbb{N}\}$ be an internal collection. Denote by $\mathcal{D} = \bigcup\{\mathcal{D}_n : n \in \mathbb{N}\}$. If there is a nonempty subset $Q \subset P$ such that every $D \in \mathcal{D}$ is Q-dense, then there exists a $\mathcal{D}$-generic filter G on P.

In particular, a $\mathcal{D}$-generic filter exists for every internal collection $\mathcal{D}$ of Q-dense subsets (trivially, consider the constant internal sequence $\mathcal{D}_\xi = \mathcal{D}$ for all $\xi \in {}^*\mathbb{N}$). We remark that, assuming $\aleph_1$-saturation, any countable sequence $\{\mathcal{D}_n : n \in \mathbb{N}\}$ of internal sets can be extended to an internal sequence $\{\mathcal{D}_\xi : \xi \in {}^*\mathbb{N}\}$, but this fact does not hold not in general. [6] In the last section it is shown that Δ_1 is equivalent to Henson's isomorphism property for finite languages.

By assuming Δ_1, the result in Proposition 1.5 can be improved (no density hypothesis is needed).

Proposition 2.1 ([Ro] Th.4.5; [J1] Th.5; [JS] Appl.2)
Assume Δ_1. If $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ are internal linear orders without right endpoints, then there is an order-preserving cofinal map $\varphi : A \to B$.

PROOF. Consider the internal poset $P = \{f \in \mathcal{F}(A,B) : f \text{ is } {}^*\text{finite and order-preserving}\}$ and let $Q = \{f \in P : \forall a, a' \in \text{dom}(f) \text{ with } a < a', \text{ the open interval } (a,a') \text{ is infinite}\}$. We claim that $\mathcal{D} = \{\Lambda_a : a \in A\} \cup \{\Gamma_b : b \in B\}$ is a family of Q-dense subsets, where $\Lambda_a = \{f \in P : a \in \text{dom}(f)\}$ and $\Gamma_b = \{f \in P : b' \geq b \text{ for some } b' \in \text{ran}(f)\}$. Let $f \in Q$ be given. Since f is *finite, there is a greatest element $\tilde{a}$ in the domain of f. Now let $a \in A$, and assume $a > \tilde{a}$ (otherwise the proof is trivial). Pick an unbounded $\xi \in {}^*\mathbb{N}$. As B has no last element,

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[6]Clearly, in presence of $\aleph_1$-saturation, Δ_1 is conveniently reformulated this way: "Let P be an internal poset and let $\mathcal{D}$ be a countable union of internal sets. If there is a nonempty subset ... etc."

one can find an internal collection $\{f(\widetilde{a}) < b_0 < b_1 < \ldots < b_\xi\}$ of elements of B. Then $f' = f \cup \{\langle a, b_\xi \rangle\} \in \Lambda_a \cap Q$. Now, if $b \in B$, find an internal collection $\{b < b'_0 < b'_1 < \ldots < b'_\xi\}$ similarly as above. Pick $a' > \widetilde{a}$ and let $f' = f \cup \{\langle a', b'_\xi \rangle\}$. Then clearly $f' \in \Gamma_b \cap Q$. If G is a $\mathcal{D}$-generic filter, then $\bigcup G : A \to B$ is the desired cofinal map. $\quad\dashv$

Proposition 2.2 ([HE] Th.1.7)
Assume Δ_1. If A and B are two infinite internal sets then there is a bijection $\varphi : A \to B$ such that, for every subset $a \subseteq A$, a is internal if and only if its image $\varphi[a] = \{\varphi(x) : x \in a\}$ is internal.

PROOF. Consider the internal set

$$P = \{f \in \mathcal{F}(\mathcal{P}_I(A), \mathcal{P}_I(B)) : f \text{ is 1-1},$$
$$\text{dom(f) and ran(f) are partitions of } A \text{ and } B \text{ respectively,}$$
$$\text{and } \forall a \in \text{dom}(f) \; a \text{ is a singleton} \Leftrightarrow f(a) \text{ is a singleton}\}.$$

and define the following internal partial order on P: $f \leq g \Leftrightarrow \text{dom}(f)$ is a refinement of $\text{dom}(g)$ and $g(a) = \bigcup\{f(c) : c \in C\}$ whenever $a = \bigcup C \in \text{dom}(g)$ for some internal $C \subset \text{dom}(f)$. [7] Now, let $Q = \{f \in P : f$ is finite and $\forall a \in \text{dom}(f)$ either a and $f(a)$ are both infinite or they have the same finite cardinality$\}$ and consider sets $\Lambda_a = \{f \in P : a = \bigcup C$ for some internal $C \subseteq \text{dom}(f)\}$ and $\Gamma_b = \{f \in P : b = \bigcup C$ for some internal $C \subseteq \text{ran}(f)\}$. Then all elements in $\mathcal{D} = \{\Lambda_a : a \in \mathcal{P}_I(A)\} \cup \{\Gamma_b : b \in \mathcal{P}_I(B)\}$ are Q-dense. In fact, let $f \in Q$ and $a \in \mathcal{P}_I(A)$ be given. Define a function g with $\text{dom}(g) = \{a' \cap a : a' \in \text{dom}(f)\} \cup \{a' \setminus a : a' \in \text{dom}(f)\}$ as follows. If $a' \cap a$ and $a' \setminus a$ are both infinite, pick an internal $b \subseteq f(a)$ such that b and $f(a) \setminus b$ are both infinite, and define $g(a' \cap a) = b$ and $g(a' \setminus a) = f(a) \setminus b$. If at least one of the two sets is finite, let's say $|a' \cap a| = n$, pick $b \subseteq f(a)$ with $|b| = n$ and define $g(a' \cap a) = b$ and $g(a' \setminus a) = f(a) \setminus b$. Note that $a = \bigcup C$ where $C = \{a' \cap a : a' \in \text{dom}(f)\}$ is an internal subset of $\text{dom}(g)$. Clearly, $g \in \Lambda_a \cap Q$ and $g \leq f$, hence Λ_a is Q-dense. The proof that each Γ_b is Q-dense is similar. Now, let G be a $\mathcal{D}$-generic filter on P. It is straightforwardly proved that $\bigcup G = \Phi : \mathcal{P}_I(A) \to \mathcal{P}_I(B)$ is a bijection. Since a is a singleton if and only if $\Phi(a)$ is a singleton, we can define a map $\varphi : A \to B$ by setting $\varphi(x) = y$ if and only if $\Phi(\{x\}) = \{y\}$. φ is the bijection with the desired properties. $\quad\dashv$

Corollary 2.3
Assume Δ_1. All infinite internal sets have the same cardinality.

Proposition 2.4 ([HE] Th.1.7)
Assume Δ_1. If A and B are infinite internal sets, then there exists a bijection $\varphi : A \to B$ such that, for every subset $a \subseteq A$, a is internal $\Leftrightarrow$ either $\varphi[a]$ or $B \setminus \varphi[a]$ is *finite.

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[7] A partition τ is a *refinement* of π if every $x \in \pi$ is union of elements in τ.

PROOF. Let $\mathcal{P}'(B) = \{b \in \mathcal{P}_I(B) : \text{either } b \text{ or } B \setminus b \text{ is *finite}\}$. The proof goes as in the previous proposition, except that one considers the following poset P' instead of P:

$$P' = \{f \in \mathcal{F}(\mathcal{P}_I(A), \mathcal{P}'(B)) : f \text{ is 1-1,}$$
$$\text{dom(f) and ran(f) are partitions of } A \text{ and } B \text{ respectively,}$$
$$\text{and } \forall a \in \text{dom}(f) \ a \text{ is a singleton} \Leftrightarrow f(a) \text{ is a singleton}\}.$$

$\dashv$

An initial segment $U \subset {}^*\mathbb{N}$ is an *additive cut* if $x + y \in U$ for all $x, y \in U$. Any additive cut $U \subset [0, N]$ yields a topology on the interval $[0, N]$. Precisely, a subset $A \subseteq [0, N]$ is *U-open* iff for every $a \in A$ there is $\xi \in [0, N] \setminus U$ such that $[a - \xi, a + \xi] \subseteq A$. The U-open sets form the U-topology. An *U-meager* set is a set which is meager in the U-topology. An additive cut $U \subseteq [0, N]$ is a *good cut* if there is a U-meager set of positive Loeb measure. A *bad cut* is an additive cut which is not good. We refer the interested reader to [KL], where the above notion of U-topology is introduced and studied. See also [J4].

Proposition 2.5 ([J4] Th.4)
Assume Δ_1. Then for every unbounded $N \in {}^*\mathbb{N}$, there is a bad cut $U \subset [0, N]$.

PROOF. We shall use the following characterization proved in [KL].

> An additive cut $U \subset [0, N]$ is a *bad cut* if and only if for every U-crossing sequence $f : [0, M] \to {}^*\mathbb{N}$, the sum $\sum_{x < M} \frac{f(x)}{f(x+1)}$ is unbounded.

Recall that a U-crossing sequence f is a strictly increasing internal function defined on an initial segment of ${}^*\mathbb{N}$ such that, for every $u \in U$, there is some $\xi \in \text{dom}(f)$ with $u < f(\xi) \in U$. Now let an unbounded N be given, and let $P = \{(A, B) \text{ open interval}: 0 \leq A < B \leq N\}$ be partially ordered by inclusion. Consider $Q = \{(A, B) \in P : A/B \sim 0\}$. We shall show that $\Lambda_a = \{(A, B) \in P : a + a \leq A \text{ or } a \geq B\}$ is Q-dense in P for each $a \in {}^*\mathbb{N}$. Let $(A, B) \in Q$ be given. We need to find $(A', B') \in Q \cap \Lambda_a$ with $(A', B') \subseteq (A, B)$. If $a + a \leq A$, already $(A, B) \in \Lambda_a$; and if $a + a \geq B$, let $A' = A$ and $B' = B/2$. We are left to consider the case $A < a + a < B$. Since $\frac{A}{B} = \frac{A}{a+a} \cdot \frac{a+a}{B} \sim 0$, at least one of the two factors is infinitesimal. If $\frac{A}{a+a} \sim 0$, let $A' = A$ and $B' = a$, and if $\frac{a+a}{B} \sim 0$, let $A' = a + a$ and $B' = B$. It is easily seen that $(A', B') \in Q \cap \Lambda_a$. We now claim that for each strictly increasing internal function $f : [0, M] \to {}^*\mathbb{N}$ such that $\sum_{x < M} \frac{f(x)}{f(x+1)}$ is bounded, the set $\Gamma_f = \{(A, B) \in P : \text{ran}(f) \cap (A, B) = \emptyset\}$ is Q-dense. Let $(A, B) \in Q$ and f be as above. If $\text{ran}(f) \cap (A, B) = \emptyset$ the thesis is trivial. Otherwise, consider $\varepsilon = \min\{\frac{f(x)}{f(x+1)} : [f(x), f(x+1)] \subseteq (A, B)\}$. Such a least element exists because f is *finite, and moreover $\varepsilon \sim 0$. To prove this latter fact, we distinguish two cases. First, let us assume that $\text{ran}(f) \cap (A, B) =$

$\{f(\xi) < f(\xi+1) < \cdots < f(\xi+m)\}$ is finite. Then $m \cdot \varepsilon \leq \frac{f(\xi)}{f(\xi+1)} \cdots \frac{f(\xi+m-1)}{f(\xi+m)} = \frac{f(\xi)}{f(\xi+m)} < A/B \sim 0$ implies $\varepsilon \sim 0$. In case $\mathrm{ran}(f) \cap (A, B)$ is infinite, i.e. if m is unbounded, then $m \cdot \varepsilon \leq \sum_{i<m} \frac{f(\xi+i)}{f(\xi+i+1)} \leq \sum_{x<M} \frac{f(x)}{f(x+1)}$ is bounded, hence $\varepsilon \sim 0$. Pick $\eta \in \{\xi < \xi+1 < \ldots < \xi+m-1\}$ with $\varepsilon = \frac{f(\eta)}{f(\eta+1)}$ and let $A' = f(\eta)$ and $B' = f(\eta+1)$. Then clearly $(A', B') \subseteq (A, B)$ and $(A', B') \in \Gamma_f \cap Q$. Now define $\mathcal{D}_0 = \{\Lambda_a : a \in {}^*\mathbb{N}\}$ and, for $\xi \geq 1$, let $\mathcal{D}_\xi = \{\Gamma_f : f : [0, M] \to {}^*\mathbb{N}$ is a strictly increasing internal function such that $\sum_{x<M} \frac{f(x)}{f(x+1)} \leq \xi\}$. We proved above that every set in $\mathcal{D} = \bigcup\{\mathcal{D}_n : n \in \mathbb{N}\}$ is Q-dense. Thus there is a $\mathcal{D}$-generic filter G on P. We claim that the initial segment $U = \{u \in {}^*\mathbb{N} : u \leq A$ for some $(A, B) \in G\}$ is the desired bad cut. It is straightforwardly seen that $G \cap \Lambda_a \neq \emptyset$ for each $a \in {}^*\mathbb{N}$ implies that U is an additive cut (note that $(A, B) \in G \Rightarrow B \notin U$). Besides, each strictly increasing internal function $f : [0, M] \to {}^*\mathbb{N}$ such that $\sum_{x<M} \frac{f(x)}{f(x+1)}$ is bounded, is in $\mathcal{D}_n$ for some finite $n \geq 1$. Thus $G \cap \Gamma_f \neq \emptyset$ and there is $(A, B) \in G$ with $\mathrm{ran}(f) \cap (A, B) = \emptyset$. Now, clearly $A \in U$ and if $f(\xi) > A$ then also $f(\xi) > B$, hence $f(\xi) \notin U$. We conclude that f cannot be U-crossing. $\dashv$

The next result, first proved by David Ross by assuming the special model axiom, is about Loeb measures.

Proposition 2.6 ([Ro] Th.5.5; [JS] Appl.1)
Assume Δ_1. Let Ω be an internal set, $\mathcal{A}$ an internal algebra of subsets of Ω containing all singletons, and $\mu : \mathcal{A} \to {}^*\mathbb{R}$ an internal finitely additive measure. If $\mu(\Omega)$ is unbounded and $\mu(\{\xi\})$ is infinitesimal for each $\xi \in \Omega$, then there is a subset $X \subset \Omega$ such that:
(i) for all $A \in \mathcal{A}$ with $\mu(A)$ finite, $A \cap X \in \mathcal{A}$ has infinitesimal measure;
(ii) for all $A \in \mathcal{A}$ with $A \supseteq X$, $\mu(A)$ is unbounded.

PROOF. Consider the internal poset $P = \{f \in \mathcal{F}(\Omega, \{0, 1\}) : f^{-1}(0), f^{-1}(1) \in \mathcal{A}\}$ and let $Q = \{f \in P : f^{-1}(0)$ has finite measure and $f^{-1}(1)$ is a finite set\}. [8] For each $\xi \in {}^*\mathbb{N}$, let $\mathcal{D}_\xi = \{\Lambda_\xi(A) : \mu(A) < \xi\} \cup \{\Gamma(A) : \mu(A) < \xi\}$ where $\Lambda_\xi(A) = \{f \in P : A \subseteq \mathrm{dom}(f)$ and $\mu(f^{-1}(1) \cap A) < 1/\xi\}$ and $\Gamma(A) = \{f \in P : f(a) = 1$ for some $a \in \mathrm{dom}(f) \setminus A\}$. Note that $\{\mathcal{D}_\xi : \xi \in {}^*\mathbb{N}\}$ is an internal family. Each set in $\mathcal{D} = \bigcup\{\mathcal{D}_n : n \in \mathbb{N}\}$ is Q-dense in P. In fact, given $f \in Q$ with $\mu(f^{-1}(0)) < n \in \mathbb{N}$, define $g = f \cup \{\langle a, 0 \rangle : a \in A \setminus \mathrm{dom}(f)\}$ and $h = f \cup \{\langle a', 1 \rangle\}$ where $a' \notin A \cup \mathrm{dom}(f)$. Then $g \in \Lambda_n(A) \cap Q$, $h \in \Gamma(A) \cap Q$ and $g, h \leq f$. Now let G be a $\mathcal{D}$-generic filter. Since $G \cap \Lambda_n(\{a\}) \neq \emptyset$ for each $a \in \Omega$, $\bigcup G : \Omega \to \{0, 1\}$ is the characteristic function of some $X \subset \Omega$. Now, let $A \in \mathcal{A}$ be a given set of finite measure, let's say $\mu(A) < m \in \mathbb{N}$. As $G \cap \Gamma(A) \neq \emptyset$, clearly $X \not\subseteq A$ and this proves (ii). As for (i), take any $f \in G \cap \Lambda_m(A)$. Then $A \cap X = A \cap f^{-1}(1) \in \mathcal{A}$ and $\mu(A \cap X) < 1/m$. The latter inequality holds for each standard $k \geq m$ as well, thus $\mu(A \cap X)$ is infinitesimal. $\dashv$

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[8] Abusing notation, we denoted $f^{-1}(0) = \{a \in \mathrm{dom}(f) : f(a) = 0\}$. Similarly for $f^{-1}(1)$.

§3. The equivalence between Δ_1 and the isomorphism property.

Let us first consider the problem of finding (a large class of) nonstandard models where Δ_0 holds. Denote by Z_0 the set theory where the following axioms are postulated:

> *Weak extensionality, empty set, pair, union, power set, bounded separation schema.*

By *weak extensionality* we mean the following version of extensionality which allows atoms.

$$\forall x \forall y \, [\exists z \in x \wedge (\forall t \; t \in x \leftrightarrow t \in y)] \rightarrow x = y$$

The bounded separation schema is the separation schema where only bounded formulas are considered. Note that if $* : V(X) \rightarrow V(Y)$ is any nonstandard embedding, superstructures $V(X)$ and $V(Y)$, as well as the internal submodel $\mathcal{I}_*$, are models of Z_0. In general, the interpretation of the membership relation symbol in a model $\mathcal{M} \models Z_0$ is not the real membership. Thus we need to distinguish between an element $A \in \mathcal{M}$ and the collection $A^{\mathcal{M}} = \{a \in \mathcal{M} : \mathcal{M} \models a \in A\}$ of its $\mathcal{M}$-elements. If $\mathcal{M} \models "\langle P, \preceq \rangle$ is a poset", we denote by $\langle P^{\mathcal{M}}, \leq \rangle$, or simply by $P^{\mathcal{M}}$, the poset where, by definition, $p \leq q \Leftrightarrow \mathcal{M} \models p \preceq q$. We say that a model $\mathcal{M} \models Z_0$ satisfies the *generic filter property* Δ_0 if for every $P \in \mathcal{M}$ with $\mathcal{M} \models "P$ is a poset", there is a filter G on $P^{\mathcal{M}}$ such that $G \cap D^{\mathcal{M}} \neq \emptyset$ whenever $\mathcal{M} \models "D$ is a dense subset of P". We remark that in general $G \notin \mathcal{M}$, i.e. $G \neq A^{\mathcal{M}}$ for all $A \in \mathcal{M}$. In order to make this definition of Δ_0 consistent with the one given in §1, we agree on the following. When we say that a nonstandard model $\langle V(X), V(Y), * \rangle$ satisfies Δ_0, we actually mean that its internal submodel does.

Theorem 3.1

Any model of Z_0 has an elementary extension of the same cardinality where Δ_0 holds.

PROOF. Let a model $\mathcal{N} \models Z_0$ be given (clearly $\mathcal{N}$ is infinite). For every $n \in \mathbb{N}$, we define an elementary extension $\mathcal{M}_n \prec \mathcal{M}_{n+1}$ and an element $G_P^n \in \mathcal{M}_{n+1}$ for each $P \in \mathcal{M}_n$ with $\mathcal{M}_n \models "P$ is a poset", in such a way that the following are satisfied.
(i) $\mathcal{M}_{n+1}$ has the same cardinality as $\mathcal{N}$.
(ii) $\mathcal{M}_{n+1} \models "G_P^n$ is a filter on P".
(iii) If $\mathcal{M}_n \models "D$ is a dense subset of P" then $\mathcal{M}_{n+1} \models "G_P^n \cap D \neq \emptyset$".
(iv) If $\mathcal{M}_{n-1} \models "Q$ is a poset" then $\mathcal{M}_{n+1} \models "G_Q^{n-1} \subseteq G_Q^n$".
Let $\mathcal{M}_0 = \emptyset$ be the empty structure and $\mathcal{M}_1 = \mathcal{N}$. The elementary extension $\emptyset \prec \mathcal{N}$ trivially satisfies the thesis, because there is no G_P^0 to be defined. Now assume $\mathcal{M}_{n-1}$ and $\mathcal{M}_n$ have been defined that satisfy the desired properties. Let L_n be the language containing the membership relation symbol, a constant

symbol $\mathbf{a}$ for each element $a \in \mathcal{M}_n$ and a constant symbol γ_P for each $P \in \mathcal{M}_n$ with $\mathcal{M}_n \models$ "P is a poset". Consider the following set Σ_n of L_n-sentences.

$$\{\varphi(\mathbf{a}_1, \ldots, \mathbf{a}_k) : \mathcal{M}_n \models \varphi(a_1, \ldots, a_k)\} \bigcup$$
$$\{"\gamma_P \text{ is a filter on } \mathbf{P}" : \mathcal{M}_n \models "P \text{ is a poset}"\} \bigcup$$
$$\{"\gamma_P \cap \mathbf{D} \neq \emptyset" : \mathcal{M}_n \models "D \text{ is a dense subset of the poset } P"\} \bigcup$$
$$\{"\mathbf{g} \in \gamma_Q" : \mathcal{M}_n \models "g \in G_Q^{n-1}"\}.$$

Each finite subset of Σ_n is realized in some expansion of $\mathcal{M}_n$ as a consequence of the following fact, which is easily proved.

> Given a poset P, a filter $F \subseteq P$, a finite numeber of elements $p_1, \ldots, p_n \in F$, and a finite number $D_1, \ldots, D_k$ of dense subsets of P, there is a (principal) filter G on P such that $p_1, \ldots, p_n \in G$ and $G \cap D_i \neq \emptyset$ for all $i = 1, \ldots, k$.

By the completeness theorem and the downward Löwenheim-Skolem theorem, there exists a model $\mathcal{M}_{n+1} \models \Sigma_n$ of the same cardinality as $\mathcal{M}_n$. Without loss of generality, we can identify each element $a \in \mathcal{M}_n$ with the interpretation in $\mathcal{M}_{n+1}$ of the corresponding constant symbol $\mathbf{a}$. Let G_P^n be the interpretation in $\mathcal{M}_{n+1}$ of the constant symbol γ_P. Then $\mathcal{M}_n \prec \mathcal{M}_{n+1}$ is an elementary extension and the desired properties are satisfied. We claim that the union $\mathcal{M} = \bigcup\{\mathcal{M}_n : n \in \mathbb{N}\}$ of the elementary chain satisfies Δ_0. By (i), $\mathcal{M}$ and $\mathcal{N}$ have the same cardinality. If $P \in \mathcal{M}$ and $\mathcal{M} \models "P$ is a poset", then there is some $n \in \mathbb{N}$ with $\mathcal{M}_n \models "P$ is a poset". Let $G_P = \bigcup\{(G_P^m)^{\mathcal{M}_{n+1}} : m \geq n\}$. It is easily seen that G is a filter on $P^{\mathcal{M}}$, as a consequence of (ii) and (iv). Moreover, if $\mathcal{M} \models "D$ is a dense subset of P", then, by (iii), $(G_P^m \cap D)^{\mathcal{M}_m} \neq \emptyset$ for some $m \geq n$, hence, by (iv), $G_P \cap D^{\mathcal{M}} \neq \emptyset$. This completes the proof. $\dashv$

Corollary 3.2
For every infinite set of atoms X, there is a nonstandard model $\langle V(X), V(Y), * \rangle$ where Δ_0 holds. Moreover, we can assume that the cardinality of the internal submodel $|\mathcal{I}_*| = |V(X)|$.

PROOF. Every ω-superstructure over a set of atoms X is a model of Z_0. Thus Theorem 3.1 can be applied to the model $\mathcal{S} = \langle V(X), \in \rangle$ to get an elementary extension $\mathcal{N} \succ \mathcal{S}$ having the same cardinality as $\mathcal{S}$ and where Δ_0 holds. Now consider the submodel $\mathcal{N}^t \subset \mathcal{N}$, the so-called *truncation* of $\mathcal{N}$, whose universe is $N^t = \{b \in \mathcal{N} : b \in a \text{ for some } a \in \mathcal{S}\}$. Note that $|\mathcal{N}^t| = |\mathcal{S}|$. It is straightforwardly proved that $\mathcal{N}^t$ is a bounded elementary submodel of $\mathcal{N}$. $\mathcal{N}^t$ is wellfounded because $\mathcal{S}$ only consists of finite levels in the cumulative hierarchy over X. Now let $\pi : N^t \to T$ be a Mostowski collapse where $Y = \pi(X)$ is a set of atoms. We can assume $\pi(x) = x$ for all $x \in X$. One can easily verify that $T \subseteq V(Y)$. The inclusion maps $\jmath : V(X) \hookrightarrow N^t$ and $\imath : T \hookrightarrow V(Y)$ are bounded elementary extensions (recall that bounded formulas are preserved

under transitive models). If $* = \imath \circ \pi \circ \jmath$, then $\langle V(X), V(Y), * \rangle$ is the desired nonstandard universe. $\dashv$

Let us now turn to the strength of the principle Δ_1. Trivially $\Delta_1 \Rightarrow \Delta_0$. The reverse implication does not hold, even in presence of saturation. In fact, it is proved in [DH] that for any cardinal κ, there are κ-saturated nonstandard models where Δ_0 holds but Δ_1 fails.

Recall that an L-structure $\mathcal{A}$ is *internally presented* provided its universe and all $\mathcal{A}$-interpretation of symbols in L are internal sets. For finite languages, an internally presented structure is necessarily internal, but not in general. The *isomorphism property* $IP(\kappa)$, κ an infinite cardinal, was introduced by Ward Henson in his 1974 paper [HE].

> $IP(\kappa)$: Let $\mathcal{A}$ and $\mathcal{B}$ be internally presented L-structures for some first-order language L of cardinality less than κ. If $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent then they are isomorphic.

For any given κ, nonstandard models which satisfy $IP(\kappa)$ can be constructed as direct limits of chains of enlargements. Recently, several papers have appeared in the literature about the isomorphism property and other similar strong saturation principles (see Jin's survey [J3]). In the sequel, we shall use an equivalent formulation of $IP(\kappa)$, namely the *resplendency property* $RP(\kappa)$, which was isolated by Renling Jin and Saharon Shelah in [JS].

> $RP(\kappa)$: Let $\mathcal{A}$ be an internally presented L-structure for some first-order language L of cardinality less than κ, and let $\mathbf{X}$ be a new relation symbol. If $\Sigma(\mathbf{X})$ is a set of $L \cup \{\mathbf{X}\}$-sentences which is consistent with the theory of $\mathcal{A}$, then $\Sigma(\mathbf{X})$ is realized in $\mathcal{A}$, i.e. $\langle \mathcal{A}, X \rangle \models \Sigma(\mathbf{X})$ for some relation X on A.

Theorem 3.3
$IP(\aleph_0) \Rightarrow \Delta_1$.

PROOF. Let P be an internal poset, $\emptyset \neq Q \subseteq P$ and $\{\mathcal{D}_\xi : \xi \in {}^*\mathbb{N}\}$ an internal family such that every set in $\mathcal{D} = \bigcup\{\mathcal{D}_n : n \in \mathbb{N}\}$ is Q-dense. Consider the structure $\mathcal{A} = \langle A; P, {}^*\mathbb{N}, \Theta, \in, \preceq, \leq \rangle$ where the universe $A = P \cup \mathcal{P}_I(P) \cup \mathcal{P}_I(\mathcal{P}_I(P)) \cup {}^*\mathbb{N}$; $\Theta : {}^*\mathbb{N} \to \mathcal{P}_I(\mathcal{P}_I(P))$ is the internal mapping $\xi \mapsto \mathcal{D}_\xi$; $\preceq$ is the partial order on P; and $\leq$ is the usual linear order on ${}^*\mathbb{N}$. Clearly $\mathcal{A}$ is an internally presented structure for a finite language. Note that each standard $n \in \mathbb{N}$ is definable in $\mathcal{A}$. That is, for each $n \in \mathbb{N}$ there is an L-formula $\varphi_n(x)$ (having x as its only free variable) such that $\mathcal{A} \models \varphi_n(a) \Leftrightarrow a = n$. For simplicity, we denote by "$D \in \mathcal{D}_n$" the L-formula "$\forall x \, \varphi_n(x) \to D \in \Theta(x)$". Now, let $\mathbf{X}$ be a new unary relation symbol and let $\Sigma(\mathbf{X})$ be the set containing the following $(L \cup \{\mathbf{X}\})$-sentences:

- "$\mathbf{X}$ is a filter on P"

- "$\forall D \in \mathcal{D}_n,\ D \cap \mathbf{X} \neq \emptyset$" $(n = 0, 1, 2, \ldots)$

Note that $\langle \mathcal{A}, G \rangle \models \Sigma(\mathbf{X})$ iff G is a $\mathcal{D}$-generic filter on P. Thus, by $RP(\aleph_0)$, it suffices to show that theory$(\mathcal{A}) \cup \Sigma(\mathbf{X})$ is consistent. Apply the downward Löwenheim-Skolem theorem and get a countable elementary submodel $\langle \mathcal{A}', Q' \rangle = \langle A'; P', N', \Theta', \in', \preceq, \leq, Q' \rangle$ of $\langle \mathcal{A}, Q \rangle$. For simplicity, we identify each $a \in \mathcal{A}'$ with the set $a^{\mathcal{A}'}$ of its $\mathcal{A}'$-elements. For every $n \in \mathbb{N}$, denote by $\mathcal{D}'_n = \Theta'(n)$ and enumerate the (countably many) elements of $\mathcal{D}' = \bigcup\{\mathcal{D}'_n : n \in \mathbb{N}\} = \{D_i : i \in \mathbb{N}\}$. We now define by induction a (weakly) decreasing sequence $\{q_i : i \in \mathbb{N}\} \subseteq Q'$ where $q_i \in D_i$ for every $i \in \mathbb{N}$. Pick any $q' \in Q'$. Since $D_0 \in \mathcal{D}' \subseteq \mathcal{D}$, by the hypothesis $\langle \mathcal{A}, Q \rangle \models \exists x \in Q \cap D_0\ x \preceq q'$, and so by transfer we get an element $q_0 \in Q' \cap D_0$ with $\mathcal{A}' \models q_0 \preceq q'$. For any given $q_i \in Q'$, this construction can be repeated to find an element $q_{i+1} \in Q' \cap D_i$ such that $\mathcal{A}' \models q_{i+1} \preceq q_i$. Now, let $X = \{p \in P' : \mathcal{A}' \models p \succeq q_i$ for some $i \in \mathbb{N}\}$ be the filter on P' generated by the sequence $\{q_i : i \in \mathbb{N}\}$. Then X realizes $\Sigma(\mathbf{X})$ in $\mathcal{A}'$, hence theory$(\mathcal{A}) \cup \Sigma(\mathbf{X})$ is consistent. $\dashv$

In the proof of the next theorem, we shall use the following general property of nonstandard models.

Lemma 3.4
Every internal structure for a finite language is $\aleph_0$-saturated.

PROOF. Let $\mathcal{A}$ be an internal structure for some finite language L. Let $\{a_1, \ldots, a_n\}$ be a given finite subset of A and $\Sigma(x)$ be a set of formulas in the language $L' = L \cup \{c_1, \ldots, c_n\}$, where $c_1, \ldots, c_n$ are new constant symbols. Suppose that $\Sigma(x)$ is consistent with the theory of $\mathcal{A}' = \langle \mathcal{A}, a_1, \ldots, a_n \rangle$. We have to show that $\Sigma(x)$ is realized in $\mathcal{A}'$ by some element a. By using an appropriate coding, we can identify each symbol in L' and each L'-formula with an element in $V(\emptyset)$, the collection of hereditarily finite sets. Recall that if $*$ is any nonstandard embedding, then $^*a = a$ for each $a \in V(\emptyset)$ (this is easily proved by induction on the finite rank of a). In particular, we can assume $^*\varphi = \varphi$ for each L'-formula. If $\mathcal{F}$ denotes the set of all L'-formulas, by transfer one gets an internal version $\models_i$ of the satisfaction relation for internal L'-structures. By definition, $\models_i$ applies to internal formulas in $^*\mathcal{F}$; however, by the above considerations, we can assume that $\models_i$ and $\models$ agree on standard formulas (equivalently, on internal formulas of finite lenght). Now, $\Sigma(x) = \{\sigma_n(x) : n \in \mathbb{N}\}$ is a (possibly finite) countable set. Let $\chi : \mathbb{N} \to \mathcal{F}$ be the standard function such that $\chi : n \mapsto \sigma_0(x) \wedge \ldots \wedge \sigma_n(x)$, and consider its nonstandard extension $^*\chi : {}^*\mathbb{N} \to {}^*\mathcal{F}$. As a finite tuple of internal sets, $\mathcal{A}'$ is internal, and so the following is an internal property for $\xi \in {}^*\mathbb{N}$. $P(\xi) : "\mathcal{A}' \models_i \exists x\ ^*\chi(\xi)(x)"$. By hypothesis, $P(n)$ holds for each standard $n \in \mathbb{N}$; thus by overspill, there is an unbounded $N \in {}^*\mathbb{N}$ and an element $a \in A$ such that $\mathcal{A}' \models_i {}^*\chi(N)(a)$. By applying *transfer* to the following standard fact:

"For every $n, m \in \mathbb{N}$ and for every L'-structure $\mathcal{B}$,

$$n < m \Rightarrow \mathcal{B} \models \forall x \ [\chi(m)(x) \to \chi(n)(x)]\text{"}$$

one gets $\mathcal{A}' \models \sigma_0(a) \wedge \ldots \wedge \sigma_n(a)$ for all standard $n \in \mathbb{N}$. In particular, $\mathcal{A}' \models \Sigma(a)$ and the proof is completed. $\quad\dashv$

Theorem 3.5
$\Delta_1 \Rightarrow IP(\aleph_0)$

PROOF. Let $\mathcal{A}$ and $\mathcal{B}$ be two internal structures for a finite language L, and suppose they are elementarily equivalent. For each $S \in L$, denote by $S^{\mathcal{A}}$ and $S^{\mathcal{B}}$ its interpretation in $\mathcal{A}$ and $\mathcal{B}$ respectively, and consider the following poset

$$P = \{f \in \mathcal{F}(A,B) : f \text{ is 1-1 and}$$
for each constant symbol $c \in L$ and for every $a \in \text{dom}(f)$, $a = c^{\mathcal{A}} \Leftrightarrow f(a) = c^{\mathcal{B}};$
for each n-ary function symbol $F \in L$ and for every $a_1, \ldots, a_n, a \in \text{dom}(f)$
$$F^{\mathcal{A}}(a_1, \ldots, a_n) = a \Leftrightarrow F^{\mathcal{B}}(f(a_1), \ldots, f(a_n)) = f(a);$$
for each n-ary relation symbol $R \in L$ and for every $a_1, \ldots, a_n \in \text{dom}(f)$
$$\langle a_1, \ldots, a_n \rangle \in R^{\mathcal{A}} \Leftrightarrow \langle f(a_1), \ldots, f(a_n) \rangle \in R^{\mathcal{B}}\}$$

P is internal because L is finite. Now let

$$Q = \{f \in P : \text{dom}(f) = \{a_1, \ldots, a_n\} \text{ is finite and}$$
$$\langle \mathcal{A}, a_1, \ldots, a_n \rangle \equiv \langle \mathcal{B}, f(a_1), \ldots, f(a_n) \rangle\}$$

Note that Q is nonempty. In fact, trivially $\mathcal{A} \equiv \mathcal{B}$ implies that the empty function $\emptyset \in Q$. Let $\mathcal{D} = \{\Lambda_a : a \in A\} \cup \{\Gamma_b : b \in B\}$ where $\Lambda_a = \{f \in P : a \in \text{dom}(f)\}$ and $\Gamma_b = \{f \in P : b \in \text{ran}(f)\}$. $\mathcal{D}$ is an internal family of Q-dense subsets, as a straight consequence of the following *back-and-forth* property.

> If $\langle \mathcal{A}, a_1, \ldots, a_n \rangle \equiv \langle \mathcal{B}, b_1, \ldots, b_n \rangle$, then for each $a \in \mathcal{A}$, there is $b \in \mathcal{B}$ with $\langle \mathcal{A}, a_1, \ldots, a_n, a \rangle \equiv \langle \mathcal{B}, b_1, \ldots, b_n, b \rangle$. Vice versa, for each $b \in \mathcal{B}$, there is $a \in \mathcal{A}$ with $\langle \mathcal{A}, a_1, \ldots, a_n, a \rangle \equiv \langle \mathcal{B}, b_1, \ldots, b_n, b \rangle$.

The above property holds because, by previous lemma, both $\mathcal{A}$ and $\mathcal{B}$ are $\aleph_0$-saturated. Now pick G a $\mathcal{D}$-generic filter on P. The union $\bigcup G : A \to B$ is the desired isomorphism between $\mathcal{A}$ and $\mathcal{B}$. $\quad\dashv$

A proof of $IP(\aleph_0)$ from Δ_1' (a preliminary version of Δ_1), was first indicated to the author by Ward Henson during the Analog Congress and, independently in written form, by Karel Hrbàček. Precisely, Δ_1' is the (apparently) weakened version of Δ_1 where Q is a countable unions of internal sets. The idea that Δ_1' could be strengthened to arbitrary external Q, and that the resulting principle Δ_1 would still follow from $IP(\aleph_0)$, was suggested by Karel Hrbàček.

The proof of $\Delta_1' \Rightarrow IP_{\aleph_0}$ goes as follows. For each $N \in {}^*\mathbb{N}$, let $\mathcal{F}_N$ be the set of all internal L-formulas having quantifier rank at most N and whose free variables are among $x_1, \ldots, x_N$. Denote by $\equiv_N$ the internal relation of elementary equivalence restricted to formulas in $\mathcal{F}_N$. By overspill, there is an

unbounded N with $\mathcal{A} \equiv_N \mathcal{B}$. Let P be defined as in the above proof, and let $Q = \bigcup\{Q_n : n \in \mathbb{N}\} \subseteq P$ where

$$Q_n = \{f \in P : \operatorname{dom}(f) = \{a_1, \ldots, a_k\} \text{ with } k \leq n \text{ and}$$
$$\langle \mathcal{A}, a_1, \ldots, a_k \rangle \equiv_{N-n} \langle \mathcal{B}, f(a_1), \ldots, f(a_k) \rangle\}$$

Note that Q is nonempty because the empty function $\emptyset \in Q_0$. By applying the internal version of the *back-and-forth* property of an Ehrenfeucht-Fraïssé game, it is proved that all sets Λ_a and Γ_b (defined as above) are Q-dense. [9] A generic filter on P yields the desired isomorphism.

Renling Jin [J2] for $\kappa = \aleph_1$ and James Schmerl [Sc] in the general case, proved the equivalence $IP(\kappa) \Leftrightarrow IP(\aleph_0) + \kappa$-saturation. This result, together with Theorems 3.3 and 3.5, yields the

Corollary 3.6
For each infinite cardinal κ, the isomorphism property for languages of cardinality less than κ is equivalent to Δ_1 plus κ-saturation.

Several combinatorial principles in the style of Δ_0 and Δ_1 and their applications, will appear in [DH].

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